COMPLETELY CONTINUOUS MULTILINEAR OPERATORS ON $C(K)$ SPACES

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Abstract. Given a k-linear operator T from a product of $C(K)$ spaces into a Banach space X, our main result proves the equivalence between T being completely continuous, T having an X-valued separately $\omega^* - \omega^*$ continuous extension to the product of the biduals and T having a regular associated polymeasure. It is well known that, in the linear case, these are also equivalent to T being weakly compact, and that, for $k > 1$, T being weakly compact implies the conditions above but the converse fails.

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The purpose of this paper is to present some results concerning vector valued completely continuous operators from a product of $C(K)$ spaces.

First we will explain our notation: if K is a compact Hausdorff space, $C(K)$ will be the space of scalar valued continuous functions on K endowed with the supremun norm, Σ will denote the σ-algebra of the Borel sets of K, and $B(\Sigma)$ will be the space of Σ- measurable functions on K which are the uniform limit of Σ-simple functions. X will denote a Banach space and X^{**} its bidual; we will assume, when necessary, that X is embedded in X^{**}. We shall use the convention $\overset{[i]}{\dots}$ to mean that the i-th coordinate is not involved. If $E_1, \dots, E_k, X$ are Banach spaces, we will denote by $\mathcal{L}^k(E_1, \dots, E_k; X)$ the Banach space of the continuous multilinear operators from $E_1 \times \cdots \times E_k$ into X with the usual operator norm. As is well known, the Riesz representation theorem gives a representation of the operators on $C(K)$ as integrals with respect to Radon measures, and this has been very fruitfully used in the study of the properties of the $C(K)$ spaces and the operators defined on them. In a series of papers (see especially [9], [10]), Dobrakov developed a theory of *polymeasures*, functions defined on a product of σ-algebras which are measures on each variable separately, that can be used to obtain a Riesz-style representation theorem for multilinear operators defined on a product of $C(K)$ spaces.

We will denote the semivariation of a polymeasure γ by $\|\gamma\|$ (for the general theory of polymeasures see [9] or [16]). It seems convenient to recall here that a polymeasure is called *regular* if it is separately regular and it is called *countably additive* if it is separately countably additive.

We now state the previously announced general representation theorem which extends and completes previous results (see [6]).

Theorem 1. *Let $K_1, \dots, K_k$ be compact Hausdorff spaces, let X be a Banach space and let $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k); X)$. Then there is a unique $\tilde{T} \in \mathcal{L}^k(B(\Sigma_1), \dots, B(\Sigma_k), X^{**})$ which extends T and is $\omega^* - \omega^*$ separately continuous (the ω^*- topology that we consider in $B(\Sigma_i)$ is the one induced by the ω^*-topology of $C(K_i)^{**}$). In addition, we have*

1. $\|T\| = \|\tilde{T}\|$.

2. For every $(g_1, \overset{[i]}{\dots}, g_k) \in B(\Sigma_1) \times \overset{[i]}{\dots} \times B(\Sigma_k)$ there is a unique X^{**}-valued bounded ω^*-Radon measure $\gamma_{g_1, \overset{[i]}{\dots}, g_k}$ on K_i (i.e., an X^{**}-valued finitely additive bounded vector measure on the Borel subsets of K_i, such that for every $x^* \in X^*$, $x^* \circ \gamma_{g_1, \overset{[i]}{\dots}, g_k}$ is a Radon measure on K_i), verifying

$$\int g_i d\gamma_{g_1, \overset{[i]}{\dots}, g_k} = \tilde{T}(g_1, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_k), \text{ for all } g_i \in B(\Sigma_i)$$

3. $\tilde{T}$ is $\omega^* - \omega^*$ sequentially continuous (i.e., if $\forall i = 1, \dots, k$, $(g_i^n)_{n \in \mathbb{N}} \subset B(\Sigma_i)$, $g_i^n \overset{\omega^*}{\to} g_i$, then $\lim_{n\to\infty} \tilde{T}(g_1^n, \dots, g_k^n) = \tilde{T}(g_1, \dots, g_k)$ in the $\sigma(X^{**}, X^*)$ topology.

Also, if we define $\gamma : B(\Sigma_1) \times \cdots \times B(\Sigma_k) \mapsto X^{**}$ by

$$\gamma(A_1, \dots, A_k) = \tilde{T}(\chi_{A_1}, \dots \chi_{A_k}),$$

then γ is a polymeasure of bounded semivariation that verifies

(a) $\|T\| = \|\gamma\|$.

(b) $T(f_1, \dots, f_k) = \int (f_1, \dots, f_k) d\gamma$ $(f_i \in C(K_i))$

(c) For every $x^* \in X^*$, $x^* \circ \gamma$ is a separately regular polymeasure and the map $x^* \mapsto x^* \circ \gamma$ is continuous for the topologies $\sigma(X^*, X)$ and $\sigma((C(K_1)\hat{\otimes}\cdots\hat{\otimes}C(K_k))^*, C(K_1)\hat{\otimes}\cdots\hat{\otimes}C(K_k))$.

Conversely, if $\gamma : B(\Sigma_1) \times \cdots \times B(\Sigma_k) \mapsto X^{**}$ is a polymeasure which verifies (c), then it has finite semivariation and formula (b) defines a k-linear continuous operator from $C(K_1) \times \cdots \times C(K_k)$ into X for which (a) holds.

Therefore the correspondence $T \leftrightarrow \gamma$ is an isometric isomorphism.

It is also proved in [6] that, if $E_1, \dots, E_k$ are Banach spaces such that, for every $i \neq j$, every linear operator from E_i into E_j^* is weakly compact, then every operator $T \in \mathcal{L}^k(E_1, \dots, E_k; X)$ can be uniquely extended, with the same norm and separate $\omega^* - \omega^*$ continuity, to an operator $T^{**} \in \mathcal{L}^k(E_1^{**}, \dots, E_k^{**}; X^{**})$. Of course the condition is fulfilled if every E_i is a $C(K_i)$ space and in that case $\tilde{T}$ is the restriction of T^{**} to $B(\Sigma_1) \times \cdots \times B(\Sigma_k)$. It is worth noting that, in case all the E_i's are equal and T is symmetric, T^{**} coincides with the well known Aron-Berner extension of T (see [2], [1]).

It is known, see [6], that if T is weakly compact then T^{**} is X-valued, although the reciprocal is not true in general. The main result in the present paper states that T is completely continuous if and only if T^{**} is X-valued.

If $E_1, \dots, E_k, X$ are Banach spaces, a multilinear operator $T \in \mathcal{L}^k(E_1, \dots, E_k; X)$ is said to be completely continuous if, given weakly Cauchy sequences $(x_i^n)_{n\in\mathbb{N}} \subset E_i$, $(i = 1, \dots, k)$, the sequence $(T(x_1^n, \dots, x_k^n))$ is norm convergent. These operators are studied, among other places, in [12], [13] and [15].

Our first result is a lemma crucial for the proof of the main results.

Lemma 2. *Let $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k); X)$. Then T is completely continuous if and only if for all weak Cauchy sequences $(f_j^n)_{n\in\mathbb{N}} \subset C(K_j)$, $(1 \le j \le k)$, and every $i \in 1, \dots, k$, the measures $\{\gamma_{f_1^n,\overset{[i]}{\dots},f_k^n}; n \in \mathbb{N}\}$ mentioned in Theorem 1 are uniformly countably additive.*

Proof. Let us first suppose T to be completely continuous and let us fix i with $1 \le i \le k$. We define the operators $T_n \in \mathcal{L}(C(K_i); X)$ by

$$T_n(f_i) = T(f_1^n, \dots, f_{i-1}^n, f_i, f_{i+1}^n, \dots, f_k) \ .$$

Since T is completely continuous, we get that for each $n \in \mathbb{N}$, T_n is also completely continuous. By the reciprocal Dunford-Pettis property of $C(K)$ this means that T_n is weakly compact and therefore its associated measure $\gamma_n = \gamma_{f_1^n,\overset{[i]}{\dots},f_k^n}$ is countably additive and regular.

Now, were the measures $(\gamma_n)_{n\in\mathbb{N}}$ not uniformly countably additive, there would exist an $\epsilon > 0$ and a sequence $(A_i^m)_{m\in\mathbb{N}}$ of disjoint open sets of Σ_i such that for all $m \in \mathbb{N}$,

$$\sup_n \|\gamma_n(A_i^m)\| > \epsilon \ .$$

Then we could choose two sequences of indexes $(m(p))_{p\in\mathbb{N}}, (n(p))_{p\in\mathbb{N}}$, where $(n(p))$ is an increasing sequence, such that

$$\left\|\gamma_{n(p)}(A_i^{m(p)})\right\| > \epsilon$$

For simplicity we will write this as

$$\|\gamma_p(A_i^p)\| > \epsilon$$

Since every γ_n is regular, we would get that for each $p \in \mathbb{N}$ there would exist a $f_i^p \in C(K_i)$ such that supp$f_i^p \subset A_i^p$, $\|f_i^p\| \leq 1$, and

$$\left\| \int f_i^p d\gamma_p \right\| > \epsilon \text{ , i.e. } \|T(f_1^p, \dots, f_i^p, \dots, f_k^p)\| > \epsilon \text{ ;}$$

but clearly f_i^p weakly converges to 0 and therefore, T being completely continuous,

$$\|T(f_1^p, \dots, f_i^p, \dots, f_k^p)\| \to 0$$

(the proof of this fact can be found in Lemma 2.4 and Theorem 2.3 in [4]), a contradiction.

For the other implication, let us choose for every $j = 1, \dots, k$, $(f_j^n) \subset C(K_j)$ to be weakly Cauchy sequences, with $\|f_j^n\| \leq 1$, such that at least one of them, say (f_i^n), weakly converges to 0. According to our hypothesis, the measures $\{\gamma_n = \gamma_{f_1^n, \overset{[i]}{\dots}, f_k^n}; n \in \mathbb{N}\}$ are uniformly countably additive. Let λ be a positive measure such that the measures $(\gamma_n)_{n\in\mathbb{N}}$ are uniformly λ-continuous. For every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{n\in\mathbb{N}} \|\gamma_n(A)\| < \epsilon/2 \text{ when } \lambda(A) < \delta$$

Since $f_i^n \xrightarrow{\omega} 0$ then $\forall t \in K_i$, $f_i^n(t) \to 0$. Then, according to Egoroff's theorem there is a compact $K_i' \subset K_i$ such that $f_i^n \to 0$ uniformly on K_i' and $\lambda(K_i \setminus K_i') < \delta$. Let $n_0 \in \mathbb{N}$ be such that for every $n > n_0$,

$$\|f_i^n\|_{K_i'} \leq \epsilon/2 \, \|\gamma\| \text{, where } \|f\|_{K_i'} = \sup_{t\in K_i'} f(t)$$

Then, for every $n > n_0$,

$$\|T(f_1^n, \dots, f_k^n)\| = \left\| \int_{K_i} f_i^n d\gamma_n \right\| \leq \left\| \int_{K_i'} f_i^n d\gamma_n \right\| + \left\| \int_{K_i \setminus K_i'} f_i^n d\gamma_n \right\| \leq$$

$$\|f_i^n\|_{K_i'} \|\gamma_n\| (K_i') + \|\gamma_n\| (K_i \setminus K_i') < \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore

$$\|T(f_1^n, \dots, f_k^n)\| \to 0$$

The hypothesis that $f_i^n \xrightarrow{\omega} 0$ can now be removed in a standard way (see [15], end of the proof of Theorem 2.1) and we conclude that T is completely continuous. □

We will also use another result, whose proof can be found in [6, Corollary 4].

Proposition 3. *Let $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k); X)$. With the notations of Theorem 1, if $\tilde{T}$ is X-valued, then $\tilde{T}$ is sequentially $\omega^* - \|\cdot\|$ continuous.*

We need also the following lemma from [15].

Lemma 4. *Let E, X be Banach spaces and let $T \in \mathcal{L}(E, c_0(X))$, where $T(x) = (T_n(x))_n$. Then T is weakly compact if and only if*

i) For every $n \in \mathbb{N}$, $T_n \in \mathcal{L}(E, X)$ is weakly compact, and

ii) For every $z \in E^{**}$, $\lim_n \|T_n^{**}(z)\| = 0$

Before stating our main result we must observe that, given an operator $T \in \mathcal{L}^k(C(K_1), \ldots, C(K_k); X)$, the fact that T^{**} is separately $\omega^*-\omega^*$ continuous implies that, for every $(f_1, \ldots, f_{k-1}) \in C(K_1) \times \cdots \times C(K_{k-1})$ and for every $z_k \in C(K_k)^{**}$,

$$(T_{f_1,\ldots,f_{k-1}})^{**}(z_k) = T^{**}(f_1, \ldots, f_{k-1}, z_k)$$

where $T_{f_1,\ldots,f_{k-1}} \in \mathcal{L}(C(K_k); X)$ is the operator defined as

$$T_{f_1,\ldots,f_{k-1}}(f_k) = T(f_1, \ldots, f_{k-1}, f_k)$$

and $(T_{f_1,\ldots,f_{k-1}})^{**}$ is its bitranspose. The same fact also implies that

$$(T_{z_k})^{**} = (T^{**})_{z_k} \ ,$$

where $T_{z_k} \in \mathcal{L}^{k-1}(C(K_1), \ldots, C(K_{k-1}); X)$ is the operator defined as

$$T_{z_k}(f_1, \ldots, f_{k-1}) = T^{**}(f_1, \ldots, f_{k-1}, z_k)$$

and $(T^{**})_{z_k} \in \mathcal{L}^{k-1}(C(K_1)^{**}, \ldots, C(K_{k-1})^{**}; X)$ is the operator defined as

$$(T^{**})_{z_k}(z_1, \ldots, z_{k-1}) = T^{**}(z_1, \ldots, z_{k-1}, z_k)$$

Both of these equalities are used in the proof of the next theorem.

We are now ready to state our main result. Part of its proof is a modification of an idea in [11] which in turn is based on [15].

Theorem 5. *Let $K_1, \ldots, K_k$ be compact Hausdorff spaces, let X be a Banach space and let $T \in \mathcal{L}^k(C(K_1), \ldots, C(K_k); X)$. Then, using the notations of Theorem 1, the following are equivalent:*
a) T is completely continuous.
b) T^{**} is X-valued.
c) $\tilde{T}$ is X-valued.
d) γ is countably additive.
e) γ is regular.

Proof. a) $\Rightarrow$ b): We will argue by induction on k. If $k = 1$ the result is well known. Suppose it is true for $k-1$. Let now g belong to $C(K_k)^{**}$. We define the operator $T_g \in \mathcal{L}^{k-1}(C(K_1) \times \ldots \times C(K_{k-1}); X)$ as

$$T_g(f_1, \ldots, f_{k-1}) = T^{**}(f_1, \ldots, f_{k-1}, g)$$

To check that T_g is indeed X-valued, let us observe that if we fix $(f_1, \ldots, f_{k-1}) \in C(K_1) \times \ldots \times C(K_{k-1})$, the operator $T_{f_1,\ldots,f_{k-1}} \in \mathcal{L}(C(K_k); X)$ defined as

$$T_{f_1,\ldots,f_{k-1}}(f) = T(f_1, \ldots, f_{k-1}, f)$$

is completely continuous, therefore also weakly compact, and hence $(T_{f_1,\ldots,f_{k-1}})^{**}$ is X-valued. Since $\forall g \in C(K_k)^{**}$,

$$(T_{f_1,\ldots,f_{k-1}})^{**}(g) = T_g(f_1, \ldots, f_{k-1})$$

we obtain that T_g is X-valued.

T_g is clearly linear and continuous. We will prove now that it is also completely continuous: for this, let us consider $((f_1^n), \ldots, (f_{k-1}^n)) \subset C(K_1) \times \ldots \times C(K_{k-1})$ to be weakly Cauchy sequences, with $\|f_i^n\| \leq 1$, at least one of which, weakly converges to zero. To simplify notation we assume $f_1^n \overset{\omega}{\to} 0$. Then, if we fix $f_k \in C(K_k)$, the sequence $(\|T(f_1^n, \ldots, f_{k-1}^n, f_k)\|)_n$ converges to zero, as can again be seen in [4], Lemma 2.4. and Theorem 2.3. Therefore we can define the operator $S \in \mathcal{L}(C(K_k); c_0(X))$ as

$$S(f) = (T(f_1^n, \ldots, f_{k-1}^n, f_k))_{n\in\mathbb{N}} = (T_n(f_k))_{n\in\mathbb{N}}$$

S is obviously linear and continuous. We will show now that S is completely continuous: let $(f_k^m) \subset C(K_k)$ be a weakly converging to zero sequence. We want to prove that $S(f_k^m) \to 0$, i.e., that

$$\sup_{n\in\mathbb{N}} \left\| T(f_1^n, \ldots, f_{k-1}^n, f_k^m) \right\| \stackrel{m}{\to} 0$$

We must recall here that, with the notation of Theorem 1. 2.,

$$T(f_1^n, \ldots, f_{k-1}^n, f_k^m) = \int f_k^m d\gamma_{f_1^n,\ldots,f_{k-1}^n}$$

According to Lemma 2, the measures $\gamma_n = \gamma_{f_1^n,\ldots,f_{k-1}^n}$ are uniformly countably additive. Take $\epsilon > 0$. Similarly as in the "if" part of Lemma 2, we can obtain, for all n and for m large enough, $\|T(f_1^n, \ldots, f_{k-1}^n, f_k^m)\| < \epsilon$.

Hence, S is completely continuous. Then S is also weakly compact and, according to Lemma 4, this implies that for every $g \in C(K_k)^{**}$,

$$\lim_{n\to\infty} \|T_n^{**}(g)\| = 0$$

Since $T_n^{**}(g) = T^{**}(f_1^n, \ldots, f_{k-1}^n, g) = T_g(f_1^n, \ldots, f_{k-1}^n)$ we get that

$$\lim_{n\to\infty} \left\| T_g(f_1^n, \ldots, f_{k-1}^n) \right\| = 0$$

As in Lemma 2, we conclude that T_g is completely continuous. Now the induction hypothesis tells us that T_g^{**} is X-valued. Since this happens for every $g \in C(K)^{**}$, we get that T^{**} is X-valued.

b) $\Rightarrow$ c) is obvious, and c) $\Rightarrow$ a) is a direct consequence of Proposition 3 above.

c) $\Rightarrow$ d) follows either from Proposition 3, or from the Orlicz-Pettis theorem. To see that d) $\Rightarrow$ e) we only need to consider that γ is separately countably additive and separately ω^*-regular, therefore separately regular. e) $\Rightarrow$ d) follows from Alexandroff theorem.

d) $\Rightarrow$ c). First we observe that if γ is countably additive, so are the measures $\gamma_{g_1,\stackrel{[i]}{\ldots},g_k}$ defined in Theorem 1.2. This is obvious when, for every $j \in \{1, \stackrel{[i]}{\ldots}, k\}$, g_j is a Σ_j-simple function, and it follows easily for general g_j just considering that the Σ_j-simple functions are dense in $B(\Sigma_j)$ and using the Vitali-Hahn-Saks-Nikodým theorem. Let us fix now $f_2, \ldots, f_k \in C(K_2) \times \ldots \times C(K_k)$. $\gamma_{f_2,\ldots,f_k}$ is the Radon measure associated to the operator $T_{f_2,\ldots,f_k} \in \mathcal{L}(C(K_1); X)$ defined as $T_{f_2,\ldots,f_k}(f_1) = T(f_1, f_2, \ldots, f_k)$. It is then well known that the fact that $\gamma_{f_2,\ldots,f_k}$ is countably additive implies that $(T_{f_2,\ldots,f_k})^{**}$ is X-valued, and , therefore, $\tilde{T}(g_1, f_2, \ldots, f_k) \in X$ for all $g_1 \in B(\Sigma_1)$. Next we consider the operator $T_{g_1,f_3,\ldots,f_k} \in \mathcal{L}(C(K_2); X)$ defined as $T_{g_1,f_3,\ldots,f_k}(f_2) = \tilde{T}(g_1, f_2, f_3, \ldots, f_k)$. By analogous reasonings we obtain that $\tilde{T}(g_1, g_2, f_3, \ldots, f_k) \in X$ for all $(g_1, g_2) \in B(\Sigma_1) \times B(\Sigma_2)$. Proceeding likewise we finish the proof. □

The equivalence between c), d) and e) above can be found in [10, Theorem 6], where only the Baire Σ-algebras are considered. The reasonings in that paper do not apply to Borel Σ-algebras.

It is already known, see [14, p. 385], that, contrary to what happens in the linear case, for $k > 1$ there exist completely continuous k-homogeneous polynomials on $C(K)$ which are not weakly compact; we want to point out that our result above seems to indicate that in the case of polynomials and multilinear operators, at least on $C(K)$, the class of the completely continuous operators could be somehow the "right" class for certain applications.

Corollary 5 from [6] states that, if $E_1, \dots, E_k$ are Banach spaces such that, for every $i \neq j$ every linear operator E_i into E_j^* is weakly compact and in addition, for every $i = 1, \dots, k$, every linear operator from E_i into X is weakly compact (if all the E_i's are $C(K_i)$ spaces this is the case, for example, if $X \not\supset c_0$, or if every K_i is stonean and $X \not\supset l_\infty$, or if every $C(K_i)$ is a Grothendieck space and X is separable), then the extension T^{**} mentioned after Theorem 1 of any operator $T \in \mathcal{L}^k(E_1, \dots, E_k; X)$ is X-valued. Using that we get the next corollary (see [11, Corollary 7] where the result is stated, in the polynomial case, in a more general setting).

Corollary 6. *Let $K_1, \dots K_k$ be compact Hausdorff spaces, and X a Banach space. If every linear operator from every $C(K_i)$ into X is weakly compact, then every multilinear operator $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k); X)$ is completely continuous.*

We will now prove a strengthening of Lemma 2 which will be necessary for the proof of the next proposition.

Lemma 7. *Let $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k); X)$, and for $i = 1, \dots, k$ let $\tilde{T}_i \in \mathcal{L}^k(B(\Sigma_1), \dots, B(\Sigma_i), C(K_{i+1}), \dots, C(K_k); X)$ be the restriction of $\tilde{T}$ to $B(\Sigma_1) \times \dots \times B(\Sigma_i) \times C(K_{i+1}) \times \dots \times C(K_k)$. For $j = 1, \dots, i$ let $(g_j^n)_{n\in\mathbb{N}} \subset B(\Sigma_j)$ and for $j = i+1, \dots, k$ let $(g_j^n)_{n\in\mathbb{N}} \subset C(K_j)$ be weakly Cauchy sequences. If $\tilde{T}_i$ is completely continuous then for every $l = 1, \dots, k$, the measures $\{\gamma_{g_1^n, \overset{[l]}{\dots}, g_k^n}; n \in \mathbb{N}\}$ are uniformly countably additive.*

Proof. We only need to realize that in that case T is completely continuous, and therefore, according to Theorem 5, the measures $\{\gamma_{g_1^n, \overset{[l]}{\dots}, g_k^n}; n \in \mathbb{N}\}$ are X-valued and countably additive. Now the proof proceeds exactly like the proof of Lemma 2. □

The next result is an application of Theorem 5.

Proposition 8. *Let $K_1, \dots, K_k$ be compact Hausdorff spaces, X a Banach space, $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k); X)$ and $\tilde{T}$ its extension defined in Theorem 1. Then T is completely continuous if and only if $\tilde{T}$ is completely continuous.*

Proof. One of the implications is clear. For the other, let us suppose T as in the hypothesis, and let us define $\tilde{T}_1$ as the restriction of $\tilde{T}$ to $B(\Sigma_1) \times C(K_2) \times \dots \times C(K_k)$. First we want to prove that $\tilde{T}_1$ is completely continuous. Let $(g_1^n) \subset B(\Sigma_1)$, $(f_2^n) \subset C(K_2)$, …, $(f_k^n) \subset C(K_k)$ be weakly Cauchy sequences all of them in the respective unit ball. All we need to prove is that, if any one of the k weakly Cauchy sequences above chosen weakly converges to 0 then

$$\lim_{n\to\infty} \left\| \tilde{T}_1(g_1^n, f_2^n, \dots, f_k^n) \right\| = 0$$

The proof of this fact is slightly different depending on whether the chosen sequence is the first one or any of the others. In the first case, Lemma 7 or Lemma 2 tell us that the measures $(\gamma_n = \gamma_{f_2^n,\ldots,f_k^n})$ are uniformly countably additive. Let λ be a positive measure such that the measures $(\gamma_n)_{n\in\mathbb{N}}$ are uniformly λ-continuous. For every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\|\gamma_n\|(A) < \epsilon/3 \text{ whenever } \lambda(A) < \delta$$

According to Luzin's theorem, there exists $K_1' \subset K_1$ such that $\forall n \in \mathbb{N}$,

$$g_1^n{}_{|_{K_1'}} = f_1^n \in C(K_1') \text{ and } \lambda(K_1 \setminus K_1') < \delta$$

Let $H = \overline{[(f_1^n)_{n\in\mathbb{N}}]} \subset C(K_1')$. Theorem 1 in [5] tells us that there exists an extension operator $S : H \mapsto C(K_1)$; let us call $S(f_1^n) = h_1^n$. Since (g_1^n) weakly converges to 0, so do (f_1^n) and (h_1^n). This last fact allows us to choose an $n_0 \in \mathbb{N}$ such that for every $n > n_0$,

$$\|T(h_1^n, f_2^n, \ldots, f_k^n)\| < \epsilon/3$$

Then, for every $n > n_0$,

$$\left\|\tilde{T}(g_1^n, f_2^n, \ldots, f_k^n)\right\| = \left\|\int_{K_1} g_1^n d\gamma_n\right\| =$$

$$= \left\|\int_{K_1'} g_1^n d\gamma_n + \int_{K_1\setminus K_1'} g_1^n d\gamma_n\right\| = \left\|\int_{K_1'} h_1^n d\gamma_n + \int_{K_1\setminus K_1'} g_1^n d\gamma_n\right\| =$$

$$= \left\|\int_{K_1'} h_1^n d\gamma_n + \int_{K_1\setminus K_1'} h_1^n d\gamma_n - \int_{K_1\setminus K_1'} h_1^n d\gamma_n + \int_{K_1\setminus K_1'} g_1^n d\gamma_n\right\| =$$

$$= \left\|\int_{K_1} h_1^n d\gamma_n - \int_{K_1\setminus K_1'} h_1^n d\gamma_n + \int_{K_1\setminus K_1'} g_1^n d\gamma_n\right\| \leq$$

$$\leq \|T(h_1^n, f_2^n, \ldots, f_k^n)\| + \left\|\int_{K_1\setminus K_1'} h_1^n d\gamma_n\right\| + \left\|\int_{K_1\setminus K_1'} g_1^n d\gamma_n\right\| < \epsilon.$$

Now, if the sequence which weakly converges to 0 is not the first but any of the others, for example f_2^n, we consider again an $\epsilon > 0$ and λ, δ, K_1', f_1^n and h_1^n to be defined as previously. In this case (h_1^n) does not in general weakly converge to 0, but (f_2^n) does, so again we can choose an $n_0 \in \mathbb{N}$ such that for every $n > n_0$,

$$\|T(h_1^n, f_2^n, \ldots, f_k^n)\| < \epsilon/3$$

Reasoning now as before we get again that if $n > n_0$,

$$\left\|\tilde{T}(g_1^n, f_2^n, \ldots, f_k^n)\right\| < \epsilon$$

This finishes the proof that $\tilde{T}_1 : B(\Sigma_1) \times C(K_2) \times \cdots \times C(K_k) \mapsto X$ is completely continuous. We now consider $\tilde{T}_2$ as the restriction of $\tilde{T}$ to $B(\Sigma_1) \times B(\Sigma_2) \times C(K_3) \times \cdots \times C(K_k)$. To prove that $\tilde{T}_2$ is also completely continuous we consider weakly Cauchy sequences $(g_1^n) \subset B(\Sigma_1)$, $(g_2^n) \subset B(\Sigma_2)$, $(f_3^n) \subset C(K_3)$, …, $(f_k^n) \subset C(K_k)$. Now Lemma 7 tells us that the measures $\gamma_{g_1^n, f_3^n, \ldots, f_k^n} : \Sigma_2 \mapsto X$ are uniformly countably additive, and we can repeat almost exactly the previous reasonings to prove that $\tilde{T}_2$ is completely continuous. Proceeding likewise we finish the proof. $\square$

A couple of comments are probably appropriate here; first, let us recall that $\mathcal{L}^k(E_1,\dots,E_k;X)$ is isometrically isomorphic to $\mathcal{L}(E_1\hat{\otimes}\cdots\hat{\otimes}E_k;X)$, where $E_1\hat{\otimes}\cdots\hat{\otimes}E_k$ is the projective tensor product of $E_1,\dots,E_k$. Given an operator $T\in\mathcal{L}^k(E_1,\dots,E_k;X)$, let us denote by $\hat{T}$ the linear operator associated to it by the previously mentioned isomorphism. Then a natural question arises in this context: is it true that T is completely continuous if and only if so is $\hat{T}$?. It is easy to see that, for spaces with the Dunford-Pettis Property (in particular for $C(K)$ spaces), if $\hat{T}$ is completely continuous then so is T. This follows from a well known result (see [8]) that states that if $E_1,\dots,E_k$ are spaces with the Dunford-Pettis Property and $(x_1^n)\subset E_1$,..., $(x_k^n)\subset E_k$ are weak Cauchy sequences, then the sequence $(x_1^n\otimes\cdots\otimes x_k^n)\subset E_1\hat{\otimes}\cdots\hat{\otimes}E_k$ is also weak Cauchy. For $C(K)$ spaces with K scattered (equivalently $C(K)$ does not contain an isomorphic copy of ℓ_1) the reciprocal is also true: if $T\in\mathcal{L}^k(C(K_1),\dots C(K_k);X)$ is completely continuous, then T is weakly compact (see [12]). This, in turn is equivalent to $\hat{T}$ being weakly compact, and since, for $K_1,\dots,K_k$ scattered, $\big(C(K_1)\hat{\otimes}\cdots\hat{\otimes}C(K_k)\big)^*$ is Schur, we get that $C(K_1)\hat{\otimes}\cdots\hat{\otimes}C(K_k)$ has the Dunford-Pettis Property, which implies that $\hat{T}$ is completely continuous. For general K it seems likely that there exist a Banach space X and a completely continuous operator $T\in\mathcal{L}^k(C(K_1),\dots,C(K_k);X)$ such that its associated linear operator $\hat{T}$ is not completely continuous. Let us note that if this was not the case, then $C(K_1)\hat{\otimes}\cdots\hat{\otimes}C(K_k)$ would always have the Dunford-Pettis Property, and this is at the present time not known, for example for $C[0,1]\hat{\otimes}C[0,1]$, or for $\ell_\infty\hat{\otimes}\ell_\infty$ (see [7]).

One final remark: in [3] a new an interesting class of multilinear *scalar* operators called *regular operators* (regularity which has no relation with the measure-theoretic notion of regularity of a measure or polymeasure) is defined and studied. In $C(K)$ spaces this class is contained in the class of the completely continuous multilinear scalar operators. If K is scattered both classes coincide, otherwise the containment is strict.

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