EXTENSION OF MULTILINEAR OPERATORS ON BANACH SPACES

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Introduction

These notes deal with the extension of multilinear operators on Banach spaces. The organization of the paper is as follows.

In the first Section we study the extension of the product on a Banach algebra to the bidual and some related structures including modules and derivations. The approach is elementary and uses the classical Arens' technique. Actually most of the results of Section 1 can be easily derived from Section 2.

In Section 2 we consider the problem of extending multilinear forms on a given Banach space X to a larger space Y containing it as a closed subspace. First, we consider the case in which $Y = X''$ and we present the Aron-Berner extension as a (linear continuous) extension operator $\mathcal{L}^n(X) \to \mathcal{L}^n(X'')$. Here, $\mathcal{L}^n(Z)$ denotes the Banach space of all n-linear forms $Z \times \cdots \times Z \to \mathbb{K}$. Next, we show that each operator $X' \to Y'$ induces an operator $\mathcal{L}^n(X) \to \mathcal{L}^n(Y)$ by using an idea of Nicodemi. Moreover, if X is a subspace of Y and $X' \to Y'$ extends linear forms, then the induced Nicodemi operators extend multilinear forms. Thus, for instance, the Aron-Berner extension is just the Nicodemi operator associated to the natural embedding $X' \to X'''$. The main result of the Section is that an extension operator $X' \to Y'$ exists if and only if, for some $n \geq 1$, an extension operator $\mathcal{L}^n(X) \to \mathcal{L}^n(Y)$ exists if and only if there is an extension operator $\mathcal{L}^n(X) \to \mathcal{L}^n(Y)$ for all $n \geq 1$. And all this happens if and only if X is locally complemented in Y.

In general, the procedures described in Section 2 do not send symmetric forms into symmetric forms. Since polynomials are in correspondence with symmetric forms via polarization the methods of Section 2 cannot be applied straightforwardly to polynomials. In third Section we shall show that the extension operators of Section 2 preserve the symmetry if (and only) if X is regular (that is, every linear operator $X \to X'$ is weakly compact). Also, we give some applications to the (co)homology of Banach algebras.

Given a multilinear operator $T : X \times \cdots \times X \to Z$, the (vector valued version of the) Aron-Berner extension provides to us with a multilinear extension $\alpha\beta(T) : X'' \times \cdots \times X'' \to Z''$ which, in general takes values in

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Z''. In Section 4 we study some consequences of the fact that the range of $\alpha\beta(T)$ stays in the original space Z. We shall show that those operators whose Aron-Berner extensions are Z-valued play a similar role in the "multilinear theory" that weakly compact operators in the "linear theory", thus obtaining multilinear characterizations of some classical Banach space properties related to weak compactness in terms of operators having Z-valued Aron-Berner extensions.

Finally, in Section 5 we give an application of the Aron-Berner extension to the representation of multilinear operators on spaces of continuous functions by polymeasures.

1. The Arens product in the second dual of a Banach algebra

In this Section we consider some particular, but important examples of "extensions" of multilinear (mainly bilinear) operators. To fix ideas, suppose $\mathcal{A}$ is a (not necessarily commutative nor associative or unital) Banach algebra. Of course this means that one has a bilinear operator $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ possibly with some additional properties. For obvious reasons it would be interesting to have a reasonable "extension" of the product of $\mathcal{A}$ to the second conjugate space $\mathcal{A}''$. In some cases the extension is obvious: for instance, if $\mathcal{A} = c_0$ (multiplication is given coordinatewise), then $\mathcal{A}'' = l_\infty$ and the required extension is given by the usual, coordinatewise product in l_∞. The same can be said about the noncommutative version of c_0: when $\mathcal{K}(H)$ is the algebra of compact operators on the Hilbert space H, then $\mathcal{K}(H)'' = \mathcal{L}(H)$ and the composition of operators in $\mathcal{L}(H)$ obviously extends that of $\mathcal{K}(H)$.

Next, consider $\mathcal{A} = C_0(\Omega)$, where Ω is a locally compact space. It is well-known by those acquainted with the theory of Banach lattices that $\mathcal{A}''$ can be isometrically represented as a $C(K)$ space for a suitable compact space K but unless Ω is dispersed it is unclear whether the natural inclusion map $C_0(\Omega) \to C(K)$ is a homomorphism. (If Ω was dispersed, then $C_0(\Omega)' = l_1(\Omega)$, so $C_0(\Omega)'' = l_\infty(\Omega) = C(\beta\Omega_d)$ and everything is clear. We have written Ω_d for the underlying set Ω viewed as a discrete space.)

Consider now group algebras. Let $(G, \cdot)$ be a (not necessarily abelian) locally compact group with (right) Haar measure dt. The group algebra of G is the Banach space $L_1(G) = L_1(G, dt)$ endowed with the convolution product

$$f * g(s) = \int_G f(t)g(t^{-1} \cdot s)dt.$$

(At first sight the convolution product might look artificial, but note that, when G is a discrete group, it is the only product in $L_1(G) = l_1(G)$ for which one has $e_s * e_t = e_{s \cdot t}$, so that $*$ is simply a sort of coding of the law on the underlying group G.) In this case it is not even clear that $L_1(G)''$ carries a reasonable structure of Banach algebra. Notice that $(L_1(G), *)$ is commutative if and only if G is.

Let us read the master. Even in we are now thinking about Banach algebras it will be convenient (for the sake of clarity) to consider arbitrary bilinear operators, so we follow Arens' paper [2] to understand the previous one [1]. Suppose $m : X \times Y \to Z$ is a bilinear operator acting between Banach spaces. First, define

$$m' : Z' \times X \longrightarrow Y', \quad \langle m'(z', x), y\rangle = \langle z', m(x, y)\rangle.$$

Now, iterate the procedure and define another bilinear operator as

$$m'' : Y'' \times Z' \longrightarrow X', \quad \langle m''(y'', z'), x\rangle = \langle y'', m'(z', x)\rangle.$$

Iterating once again, we arrive to

$$m''' : X'' \times Y'' \longrightarrow Z'', \quad \langle m'''(x'', y''), z'\rangle = \langle x'', m''(y'', z')\rangle.$$

Clearly, $m'''(x, y) = m(x, y)$ for all $x \in X, y \in Y$. This bilinear map m''' will be called the (first) Arens extension of m. An interesting property of m''' is given in the following.

Lemma 1. *With the preceding notations one has* $\|m'''\| = \|m\|$.

Proof. Obviously $\|m'\| = \|m\|$. Iterate. □

Let us consider the case in which $X = Y = Z$ is an associative Banach algebra $\mathcal{A}$ and p is the product of $\mathcal{A}$.

Theorem 1 (Arens [2])**.** *The Banach space* $\mathcal{A}''$ *equipped with* p''' *is an associative Banach algebra which extends* $(\mathcal{A}, p)$.

Proof. That p''' is bilinear is obvious. The point is to show that the operation p''' is associative:

$$p'''(p'''(x'', y''), z'') = p'''(x'', p'''(y'', z'')) \quad (x'', y'', z'' \in \mathcal{A}'').$$

That is, by the very definition of p''',

$$\langle p'''(x'', y''), p''(z'', z')\rangle = \langle x'', p''(p'''(y'', z''), z')\rangle \quad (x'', y'', z'' \in \mathcal{A}'', z' \in A').$$

Since $\langle p'''(x'', y''), p''(z'', z')\rangle = \langle x'', p''(y'', p''(z'', z'))\rangle$ it suffices to show that

$$p''(y'', p''(z'', z')) = p''(p'''(y'', z''), z') \quad (y'', z'' \in \mathcal{A}'', z' \in \mathcal{A}').$$

So, one has to verify that $\langle y'', p'(p''(z'', z'), x)\rangle = \langle p'''(y'', z''), p'(z', x)\rangle$ holds for all x, z', y'', z''. Again, since one has $\langle p'''(y'', z''), p'(z', x)\rangle = \langle y'', p''(z'', p'(z', x))\rangle$ this amounts to verify the relation

$$p'(p''(z'', z'), x) = p''(z'', p'(z', x)),$$

that is, $\langle p'(p''(z'', z')), y\rangle = \langle p''(z'', p'(z', x)), y\rangle$ for all x, y, z', z''. Which can be written as $\langle p''(z'', z'), p(x, y)\rangle = \langle z'', p'(p'(z', x), y)\rangle$. Applying the definition of p'' and "eliminating" z'' this becomes

$$p'(z', p(x, y)) = p'(p'(z', x), y).$$

Applying (twice) the definition of p' and eliminating z' the preceding identity can be rewritten as

$$p(x, p(y, z)) = p(p(x, y), z)$$

that is just the associativity of $\mathcal{A}$. This completes the proof. □

The (covariant) functorial nature of Arens product is given by the following simple result.

Proposition 1 (Arens [2], Civin and Yood [16]). *If $T : \mathcal{A} \to \mathcal{B}$ is a homomorphism (of Banach algebras) so is $T'' : \mathcal{A}'' \to \mathcal{B}''$ when $\mathcal{A}''$ and $\mathcal{B}''$ are equipped with their Arens products.*

Proof. Straightforward computations. □

Corollary 1. *Every homomorphism $\mathcal{A} \to \mathbb{K}$ extends to a homomorphism $\mathcal{A}'' \to \mathbb{K}$.* □

A property which is not preserved by the Arens extension is commutativity. This is because there are two Arens extensions of the product of an algebra rather than one. Given a Banach algebra $\mathcal{A}$, consider the reversed algebra $\mathcal{A}_{rev}$ which is just $\mathcal{A}$ endowed with the reversed product $p_{rev}(a, b) = p(b, a)$. (Clearly, $\mathcal{A}$ is commutative if and only if $\mathcal{A} = \mathcal{A}_{rev}$.) Thus, $((\mathcal{A}_{rev})'')_{rev}$ is clearly an extension of $\mathcal{A}$ (its second Arens extension) which is as natural as $\mathcal{A}''$ is. But, in general, these extensions are different, even if $\mathcal{A}$ was commutative. In this setting, $\mathcal{A}$ is said to be Arens regular if its two Arens extensions coincide, that is, if $(\mathcal{A}_{rev})'' = (\mathcal{A}'')_{rev}$. Clearly, if $\mathcal{A}$ is commutative, then $\mathcal{A}''$ is commutative if and only if $\mathcal{A}$ is Arens regular. Let us remark here that any C^*-algebra is Arens regular and its second dual space is again a C^*-algebra under a natural involution [53, 55, 26]. In particular, every commutative C^*-algebra (= $C_0(\Omega)$ space) is Arens regular and its bidual is again a commutative C^*-algebra. On the negative side, $(L_1(G), *)$ is Arens regular if and only if G is finite. These phenomena will be treated later, in a more general framework; see Section 3. The situation is illustrated by the following.

Example 1 (Arens [1]). *The Banach algebra $(l_1(\mathbb{Z}), *)''$ fails to be commutative.*

Proof. Recall that

$$x * y(n) = \sum_{m=-\infty}^{\infty} x(m)y(n-m).$$

It is well-known that $l_1(\mathbb{Z})' = l_\infty(\mathbb{Z})$ and that the conjugate space of $l_\infty(\mathbb{Z})$ equals the space of all finitely additive measures on (the power set of) $\mathbb{Z}$ with bounded variation. Thus, for $\mu \in l_\infty(\mathbb{Z})'$, we shall write $\int f d\mu$ (or $\int_{\mathbb{Z}} f(n)d\mu(n)$) instead of $\langle \mu, f \rangle$ for the value of μ at $f \in l_\infty(\mathbb{Z})$. It is easily seen that

$$f * x(n) = \sum_m f(m)x(n-m) \quad (f \in l_\infty(\mathbb{Z}), x \in l_1(\mathbb{Z}), n \in \mathbb{Z}).$$

From where it follows that

$$\nu * f(n) = \langle \nu, f * e_n \rangle = \int_{\mathbb{Z}} f(m-n)d\nu(m),$$

for $\nu \in l_1(\mathbb{Z})'', f \in l_\infty(\mathbb{Z}), n \in \mathbb{Z}$. So the first Arens product in $l_1(\mathbb{Z})''$ is given by the "convolution of measures"

$$\langle \mu * \nu, f\rangle = \int_{\mathbb{Z}} \left(\int_{\mathbb{Z}} f(m-n) d\nu(m) \right) d\mu(n) \quad (\mu, \nu \in l_1(\mathbb{Z})'', f \in l_\infty(\mathbb{Z})).$$

To exhibit the noncommutativity of the convolution of finitely additive measures, take free ultrafilters (i.e., zero-one-valued measures vanishing on finite sets) μ and ν such that $\mu(\mathbb{N}) = 1, \nu(\mathbb{N}) = 0$ (hence $\mu(\mathbb{Z}\backslash\mathbb{N}) = 0, \nu(\mathbb{Z}\backslash\mathbb{N}) = 1$) and let $f = 1_{\mathbb{N}}$. Then, clearly, $\langle \mu*\nu, f\rangle = 0$, while $\langle \nu*\mu, f\rangle = 1$, so that $\mu*\nu$ is different from $\nu * \mu$. This example is essentially in Zalduendo [59]. □

Remark 1. Observe that the lack of commutativity of Arens product in $l_1(\mathbb{Z})''$ is essentially the failure of Fubini theorem for finitely additive measures. It would be interesting to known for which pairs of free ultrafilters one has $\mu * \nu = \nu * \mu$.

Unfortunately, we lack the opportunity of treating the bidual algebras in some detail. We refer the reader to the survey paper by Duncan and Hosseiniun [26] for further information on the topic.

1.1. **Extension of modules.** Despite our affect and admiration for Prof. Ángel Rodríguez Palacios, from now on, all algebras are assumed to be associative. As Helemskii observed in [39], contemporary analysis is "swarming with modules". And, fortunately, things are so that. (*Proof.* Any book by Palamodov's book. □) Of course, given an (associative) algebra $\mathcal{A}$, a (left $\mathcal{A}$) module X is a representation $\mathcal{A} \to \mathcal{L}(X)$, or else, a bilinear operator

$$m : \mathcal{A} \times X \longrightarrow X$$

satisfying $m(a \cdot b, x) = m(a, m(b, x))$. If we apply Arens' procedure to m, we obtain a bilinear extension

$$m''' : \mathcal{A}'' \times X'' \longrightarrow X''$$

and since $\mathcal{A}''$ is itself an algebra under the Arens product one may wonder whether m''' defines in X'' a (left) module structure over $\mathcal{A}''$.

Theorem 2. *Let $(\mathcal{A}, p)$ be an associative Banach algebra and let $m : \mathcal{A} \times X \to X$ be a left-module. Then $m''' : \mathcal{A}'' \times X'' \to X''$ makes X'' into a left-$\mathcal{A}''$-module.*

Proof. Since m''' is a bilinear operator, one only has to show that the left action $\mathcal{A}''$ in X'' is compatible with the product of $\mathcal{A}''$ in the sense that

$$(1) \qquad m'''(p'''(a'', b''), x'') = m'''(a'', m'''(b'', x'')) \quad (a'', b'' \in \mathcal{A}'', x'' \in X'').$$

This is a straightforward verification that we sketch. Applying the definition of m''' in both sides of (1) and then that of p''' in the left-hand side, and eliminating a'', we obtain

$$(2) \quad p''(b'', m''(x'', x')) = m''(m'''(b'', x''), x') \quad (b'' \in \mathcal{A}'', x'' \in X'', x' \in X').$$

Now, using the definitions of p'', m'' and m''' to eliminate b'', (2) becomes

$$p'(m''(x'', x'), a) = m''(x'', m'(x', a)) \quad (x'' \in X'', x' \in X', a \in \mathcal{A}). \tag{3}$$

Applying first the definition of p' in the left-hand side of (3) and then that of m'' in both sides, and eliminating x'', this becomes

$$p'(m''(x'', x'), a) = m''(x'', m'(x', a)) \quad (x'' \in X'', x' \in X', a \in \mathcal{A}). \tag{4}$$

That is,

$$m'(x', p(a, b)) = m'(m'(x', a), b) \qquad (x' \in X', a, b \in \mathcal{A}). \tag{5}$$

Which is obvious, since

$$m(p(a, b), x) = m(a, m(b, x)) \qquad (a, b \in \mathcal{A}, x \in X). \tag{6}$$

This completes the proof. □

In what follows, given a left $\mathcal{A}$-module X, when speaking of the left $\mathcal{A}''$-module X'' we understand that $\mathcal{A}''$ is the first Arens extension of $\mathcal{A}$ and the module structure of X'' is that given by Theorem 2.

We now stablish that passing from the left $\mathcal{A}$-module X to the left $\mathcal{A}''$-module X'' is a covariant functor from the category of left $\mathcal{A}$-modules into the category of left $\mathcal{A}''$-modules. Recall that an operator $T : X \to Y$ acting between $\mathcal{A}$-modules is a homomorphism (of left $\mathcal{A}$-modules) if, in addition of being linear, one has

$$T(a \cdot x) = a \cdot T(x)$$

for all $a \in \mathcal{A}$ and $x \in X$. If you are thinking about representations, then homomorphisms are intertwining operators. The proof of the following result is an elementary verification and will be omitted.

Lemma 2. *If $T : X \to Y$ is a morphism of left $\mathcal{A}$-modules, then the bitranspose map $T'' : X'' \to Y''$ is a morphism of left $\mathcal{A}''$-modules.* □

Consider now right-modules. A right-module over an algebra $\mathcal{B}$ is a Banach space X endowed with a right outer multiplication over $\mathcal{B}$, that is, a bilinear operator

$$n : X \times \mathcal{B} \longrightarrow X$$

satisfying $n(x, a \cdot b) = n(n(x, a), b)$ for all $x \in X$ and $a, b \in \mathcal{B}$.

Clearly, every right $\mathcal{B}$-module can be regarded as a left-module over the reversed algebra $\mathcal{B}_{rev}$. Thus, our previous construction implies that X'' admits a structure of left $(\mathcal{B}_{rev})''$-module, that is, X'' is a right $((\mathcal{B}_{rev})'')_{rev}$-module. But, since $((\mathcal{B}_{rev})'')_{rev}$ need not coincide with $\mathcal{B}''$ (unless $\mathcal{B}$ is Arens regular) this construction is useless to obtain a suitable right outer action of $\mathcal{B}''$ on X''.

This problem can be surrounded as follows. Suppose the outer right action of $\mathcal{B}$ given by $n : X \times \mathcal{B} \to X$ and define $n''' : X'' \times \mathcal{B}'' \to X''$ exactly as before, that is, define bilinear operators

$$\begin{array}{rclcrcl} n' & : & X' \times X \longrightarrow \mathcal{B}', & & \langle n'(x', x), b\rangle & = & \langle x', n(x, b)\rangle; \\ n'' & : & \mathcal{B}'' \times X' \longrightarrow X', & & \langle n''(b'', x'), x\rangle & = & \langle b'', n'(x', x)\rangle; \\ n''' & : & X'' \times \mathcal{B}'' \longrightarrow X'', & & \langle n'''(x'', b''), x'\rangle & = & \langle x'', n''(b'', x')\rangle. \end{array}$$

One then has.

Theorem 3. *Let $(\mathcal{B}, q)$ be an (associative) algebra and let $n : X \times \mathcal{B} \to X$ be a right-module action. Then $n''' : X'' \times \mathcal{B}'' \to X''$ makes X'' into a right $\mathcal{B}''$-module.*

Proof. This is more or less as the proof of Theorem 2, so we give only the main steps. Each of the following identities implies the following one. The last one means that n''' defines a right outer action of $\mathcal{B}''$ on X'':

$$\begin{aligned} n(n(x, b), a) &= n(x, q(b, a)), \\ n'(b', n(x, b)) &= q'(n'(b', x), b), \\ n'(n''(b'', b'), x) &= q''(b'', n'(b', x)), \\ n''(a'', n''(b'', b')) &= n''(q'''(a'', b''), b'), \\ n'''(n'''(x'', a''), b'') &= n'''(x'', q'''(a'', b'')). \end{aligned}$$

□

A homomorphism of right $\mathcal{B}$-modules is an operator $T : X \to Y$ satisfying $T(x \cdot b) = T(x) \cdot b$ for all $x \in X$ and $b \in \mathcal{B}$. As before, one has

Lemma 3. *If $T : X \to Y$ is a morphism of right $\mathcal{B}$-modules, then the bitranspose map $T'' : X'' \to Y''$ is a morphism of right $\mathcal{B}''$-modules.* □

Remark 2. Let $m : \mathcal{A} \times X \to X$ (resp. $n : X \times \mathcal{A} \to X$) be a left (resp. right) module over $\mathcal{A}$. Then the dual space X' is a right (resp. left) $\mathcal{A}$-module under the dual product

$$m^* : X' \times \mathcal{A} \longrightarrow X', \quad \langle m^*(x', a), x\rangle = \langle x', m(a, x)\rangle$$

(resp. $n^* : \mathcal{A} \times X' \to X', \langle n^*(a, x'), x\rangle = \langle x', n(x, a)\rangle$. Thus X'' is always a left (resp. right) module over $\mathcal{A}$ under the product

$$\langle m^{**}(a, x''), x'\rangle = \langle x'', m^*(x', a)\rangle$$

(resp. $\langle n^{**}(x'', a), x'\rangle = \langle x'', n^*(a, x')\rangle$). Hence, given $a \in \mathcal{A}$ and $x'' \in X''$, the product $a \cdot x''$ (resp. $x'' \cdot a$) can be understood in two (a priori different) ways, namely $m^{**}(a, x'')$ and $m'''(a, x'')$ (resp. $n^{**}(x'', a)$ and $n'''(x'', a)$). Fortunately, we have the following result, whose easy verification is left to the reader.

Lemma 4. *With the above notations one has $m^{**}(a, x'') = m'''(a, x'')$ and $n^{**}(x'', a) = n'''(x'', a)$ for all $a \in \mathcal{A}, x'' \in X''$.* □

1.2. **Bimodules and derivations.** Bimodules play a major role in homological algebra. Let $\mathcal{A}$ and $\mathcal{B}$ be two (associative) Banach algebras. A bimodule (or, more accurately, an $\mathcal{A}-\mathcal{B}$-module) is a Banach space X which is simultaneously a left $\mathcal{A}$-module, a right $\mathcal{B}$-module and satisfies that

$$(a \cdot x) \cdot b = a \cdot (x \cdot b)$$

for all $a \in \mathcal{A}$, $x \in X$ and $b \in \mathcal{B}$. When $\mathcal{B} = \mathcal{A}$, we speak of an $\mathcal{A}$-bimodule or a bimodule over $\mathcal{A}$ instead of an $\mathcal{A} - \mathcal{A}$-module.

Theorem 4. *If X is an $\mathcal{A} - \mathcal{B}$-module, then X'' is an $\mathcal{A}'' - \mathcal{B}''$-module.*

Proof. The following identities are all equivalent. The first one is our hypothesis. The last one is the conclusion of the Theorem we are in proving.

$$\begin{aligned} n(m(a,x),b) &= m(a,n(x,b)),\\ n'(x',m(a,x)) &= n'(n'(x',a),x),\\ m'(n''(b'',x'),a) &= n''(b'',m'(x',a)),\\ m''(x'',n''(b'',x')) &= m''(n'''(x'',b''),x'),\\ n'''(m'''(a'',x''),b'') &= m'''(a'',n'''(x'',b'')). \end{aligned}$$

□

Corollary 2 (Gourdeau [34])**.** *If X is a bimodule over $\mathcal{A}$, then X'' is a bimodule over $\mathcal{A}''$.*

Remark 3. Corollary 2 was first proved by F. Gourdeau in [34], with a somewhat eccentric proof. It is not clear to us whether Gourdeau's approach can be used to prove Theorem 4.

As probably everybody knows, the most important operators in homology are bimodule homomorphisms and derivations. Let X and Y be two $\mathcal{A}-\mathcal{B}$-modules. An operator $T : X \to Y$ is a homomorphism of $\mathcal{A}-\mathcal{B}$-modules if it is simultaneously a homomorphism of left $\mathcal{A}$-modules and a homomorphism of right $\mathcal{B}$-modules. If X is a bimodule over $\mathcal{A}$, then a derivation $D : \mathcal{A} \to X$ is a linear operator satisfying Leibniz's rule

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b).$$

The simplest derivations have the form

$$\delta_x(a) = a \cdot x - x \cdot a$$

for some $x \in X$ and all $a \in \mathcal{A}$. These are called inner derivations. The interest of derivations in Banach algebras is due in part to their connections with automorphisms (= "intrinsic" symmetries of the algebra). For instance, if $D : \mathcal{A} \to \mathcal{A}$ is a derivation, then $\exp(D) = \sum_{n=0}^{\infty} D^n/n!$ is an automorphism. And, conversely, if U is an automorphism of $\mathcal{A}$ which is close to the identity (say $\|U - 1\| < \sqrt{2} - 1$), then $U = \exp(D)$ for some derivation. Moreover, inner automorphisms (i. e., having the form $a \mapsto u^{-1}au$ for a certain invertible u) correspond to inner derivations and vice-versa.

Lemma 5. (a) *If $T : X \to Y$ is a homomorphism of $\mathcal{A} - \mathcal{B}$-modules, then $T'' : X'' \to Y''$ is a homomorphism of $\mathcal{A}'' - \mathcal{B}''$-modules.*
(b) *If $D : \mathcal{A} \to X$ is a derivation, then so is $D'' : \mathcal{A}'' \to X''$.*

Proof. The first part trivially follows from Theorem 4 and Corollaries 2 and 3. As for the second one, let $D : \mathcal{A} \to X$ be a derivation. Let p, m and n denote respectively the product of $\mathcal{A}$ and the left and right actions of $\mathcal{A}$ on X. One has.

$$\begin{aligned} D(p(a,b)) &= n(Da,b) + m(a,Db), \\ p'(D'x',a) &= n'(x',Da) + D'm'(x',a), \\ p''(b'',D'x') &= D'(n''(b'',x')) + m''(D''b'',x'), \\ D''(p'''(a'',b'')) &= n'''(D''a'',b'') + m'''(a'',D''b'')). \end{aligned}$$

And D'' is a derivation. □

These results open the possibility of linking (co) homological properties of $\mathcal{A}''$ to those of $\mathcal{A}$, as we shall see. Recall from [41, 39] that a Banach algebra $\mathcal{A}$ is amenable (or cohomologically trivial) if every derivation into a dual bimodule X' is inner. (A dual bimodule is a dual Banach space X' whose structure of bimodule is inherited by a bimodule structure on X by the process described in Remark 2.)

Perhaps a few words about amenable algebras are in order. First, group algebras $L_1(G)$ are amenable if and only if the underlying group is amenable in the traditional sense of harmonic analysis (that is, there is an invariant mean for the space $L_\infty(G)$), which is the case if G is either abelian or compact.

Next, if $\mathcal{A} \to \mathcal{B}$ is a homomorphism with dense range and $\mathcal{A}$ is amenable, then so is $\mathcal{B}$. From this it is easily obtained that all algebras $C(K)$ are amenable. It also follows that the algebra $\mathcal{L}(H) \oplus \mathbb{C} \cdot 1_H$ is amenable and that amenability is not hereditary: $C(\mathbb{T})$ is amenable (here $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$) while the disc algebra A is not. To see that A is not amenable, let us make $\mathbb{C}$ into an A-bimodule taking $f \cdot \lambda = \lambda \cdot f = \lambda f(0)$. Obviously $\mathbb{C}$ is a dual bimodule. The map $D : A \to \mathbb{C}$ given by $Df = f'(0)$ is clearly an outer derivation and A is not amenable. (Much more is true: if $\mathcal{A}$ is an amenable subalgebra of $C(K)$ which separates K, then $\mathcal{A} = C(K)$.)

Amenability of C^*-algebras has been completely elucidated by Connes and Haagerup: it turns out that they are exactly the nuclear C^*-algebras. Thus, for instance, $\mathcal{K}(H) \oplus 1_H$ or the Fermion algebra are amenable while $\mathcal{L}(H)$ itself or the Calkin algebra are not. All this can be seen in Helemskii [39]

We close the first part of the paper with a very simple proof of the following result.

Theorem 5 (Gourdeau [34]). *If $\mathcal{A}''$ is amenable, then $\mathcal{A}$ is amenable.*

Proof. Let $D : \mathcal{A} \to X'$ be a derivation. Then $D'' : \mathcal{A}'' \to (X')''$ is a derivation on $\mathcal{A}''$. Even if $(X')''$ is not a dual bimodule over $\mathcal{A}''$, the second conjugate $((X')'')''$ is and, therefore, the derivation $j \circ D'' : \mathcal{A}'' \to ((X')'')''$ must be inner (here j is the natural embedding $(X')'' \to ((X')'')''$. So there is ξ in the fifth dual of X such that

$$j(D''(a'')) = a'' \cdot \xi - \xi \cdot a'' \quad (a'' \in \mathcal{A}'').$$

In particular, we have $D(a) = a \cdot \xi - \xi \cdot a$. Now, observe that the canonical embeddings $X \to X''$ and $X'' \to X''''$ are homomorphisms of $\mathcal{A}$-bimodules (see Remark 2) and so the composition $i : X \to X''''$. Hence, the adjoint projection π from the fifth dual of X onto X' is a homomorphism of $\mathcal{A}$-bimodules as well. Therefore,

$$D(a) = \pi(D(a)) = \pi(a \cdot \xi - \xi \cdot a) = a \cdot \pi(\xi) - \pi(\xi) \cdot a,$$

so that $D = \delta_{\pi(\xi)}$ is inner. This completes the proof. □

Remark 4. Let X be a bimodule on $\mathcal{A}$. Let $\mathcal{H}^1(\mathcal{A}, X)$ denote the quotient of the space of all derivations $\mathcal{A} \to X$ by the subspace of inner derivations ($\mathcal{H}^1(\mathcal{A}, X)$ is often called the first cohomology group of $\mathcal{A}$ with coefficients in X). Amenability then means that $\mathcal{H}^1(\mathcal{A}, X') = 0$ for all dual bimodules X'.

The proof of Theorem 5 together with Lemma 5 and the obvious fact that D'' is inner when D is shows that for every dual bimodule X', the group $\mathcal{H}^1(\mathcal{A}, X')$ may be regarded as a subgroup of $\mathcal{H}^1(\mathcal{A}'', (X')'')$ under bitransposition. One might expect that the same occurs with arbitrary (not necessarily dual) bimodules. Let us smash that hope:

Example 2. *An outer derivation whose bitranspose is inner.*

Proof. Let K be a compact Hausdorff space and consider the Banach space

$$X = \{f \in C(K \times K) : f(t,t) = 0 \text{ for all } t \in K\}.$$

Then X is a bimodule over $C(K)$ under the products

$$a \cdot f(s,t) = a(s)f(s,t), \quad f \cdot a(s,t) = a(t)f(s,t).$$

Define $D : C(K) \to X$ as

$$Da(s,t) = a(s) - a(t).$$

It is easily seen that D is a derivation. Suppose D inner. Then,

$$a(s) - a(t) = (a(s) - a(t))g(s,t) \quad (a \in C(K), s, t \in K),$$

where g is such that $\delta_g = D$. Varying a we see that $g(s,t) = 1$ for all $s \neq t$, so that $g = 1 - 1_\Delta$ (here Δ denotes the diagonal of $K \times K$). Thus 1_Δ is a continuous map on $K \times K$. This implies that every point is isolated in K. So, D is inner if and only if K is finite.

The proof that D'' is inner for any compact K will be postponed until Section 3. We prove that in the case in which $K = \omega$ is the one-point compactification of $\mathbb{N}$. Since both ω and $\omega \times \omega$ are scattered, we have

$C(\omega)'' = l_\infty(\omega)$ and $C(\omega \times \omega)'' = l_\infty(\omega \times \omega)$. Thus, X'' may be identified with the space of all bounded functions on $\omega \times \omega$ vanishing on the diagonal. Now it is easily verified that the bimodule structure of X'' is given by

$$\alpha\cdot\phi(s,t) = \alpha(s)\phi(s,t), \quad \phi\cdot\alpha\ (s,t) = \alpha(t)\phi(s,t), \quad (\alpha \in l_\infty(\omega\times\omega), \phi \in X'').$$

Also, the bitranspose map $D'' : l_\infty(\omega) \to X''$ is nothing but

$$D\alpha(s,t) = \alpha(s) - \alpha(t),$$

and since 1_Δ belongs to $l_\infty(\omega \times \omega)$ we see that $g = 1 - 1_\Delta$ is an element of X'' such that $D'' = \delta_g$ is inner. (Another possibility is to show that X'' is a dual bimodule over $l_\infty(\omega)$ and then recall that $l_\infty(\omega) = C(\beta\omega_d)$ is an amenable algebra). □

2. Extension of multilinear forms

We deal in this Section with the extension of (vector-valued) multilinear operators and forms. Let us first consider the extension to bidual spaces. Let $X_1, \dots, X_n$ and Z be Banach spaces and $T : X_1 \times \cdots \times X_n \to Z$ a (continuous) multilinear operator. Our immediate objective is to extend T to a multilinear operator $\epsilon(T) : X_1'' \times \cdots \times X_n'' \to Z''$. In general one cannot expect to get an extension taking values in Z, even in the linear case: for instance, the identity on c_0 cannot be extended to a linear operator $l_\infty \to c_0$ since c_0 is uncomplemented in l_∞. See the unbearable paper [13].

On the other hand, we are looking for linear methods of extension (i.e., with $\epsilon(T)$ depending linearly on T) which are bounded (i.e., with $\|\epsilon(T)\| \le \text{const.}\|T\|$). In this case, we may restrict our attention to multilinear forms. Indeed, suppose we have a linear bounded method of extension $\epsilon : \mathcal{L}^n(X_1, \dots, X_n) \to \mathcal{L}^n(X_1'', \dots, X_n'')$. Then, given a multilinear operator $T : X_1 \times \cdots \times X_n \to Z$, we obtain a multilinear extension $\epsilon(T) : X_1'' \times \cdots \times X_n'' \to Z''$ taking

$$\langle \epsilon(T)(x_1'', \dots, x_n''), z' \rangle = \epsilon(z' \circ T)(x_1'', \dots, x_n'').$$

Moreover, the extension operator $\mathcal{L}^n(X_1, \dots, X_n; Z) \to \mathcal{L}^n(X_1'', \dots, X_n''; Z'')$ is linear and bounded, with the same norm than $\epsilon : \mathcal{L}^n(X_1, \dots, X_n) \to \mathcal{L}^n(X_1'', \dots, X_n'')$.

2.1. **Aron-Berner extension.** In [3], Aron and Berner found a linear method for the extension of multilinear forms from X to X''. (They use it to extending holomorphic functions of bounded type from X to X'' via their Taylor expansion.) We now describe the Aron-Berner method for an arbitrary collection $X_1, \dots, X_n$ of Banach spaces.

Given $z_i \in X_i''$, define

$$\bar{z}_i : \mathcal{L}^n(X_1, \dots, X_n) \longrightarrow \mathcal{L}^{n-1}(X_1, \dots, X_{i-1}, X_i, \dots, X_n)$$

by $\bar{z}_i(T)(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \langle z_i, T(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n) \rangle$. Here, $T(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n)$ denotes the linear form obtained from

T by fixing the $n-1$ variables $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n$. The map $\bar{z}_i$ is linear, continuous and of norm $\|z_i\|$. Now, given $T\in\mathcal{L}^n(X_1,\dots,X_n)$ we can define the extended n-linear form $\alpha\beta(T)\in\mathcal{L}^n(X_1'',\dots,X_n'')$ by

$$\alpha\beta(T)(z_1,\dots,z_n)=\bar{z}_1\circ\cdots\circ\bar{z}_n(T).$$

This extension $\alpha\beta(T)$ is called the Aron-Berner extension of T. We have in fact $n!$ extensions, one for each choice of the order in the applications $\bar{z}_i$. We shall discuss to what extend these procedures depend on the ordering in next Section 3.

2.2. **The Davie-Gamelin description of the Aron-Berner extension.** There is a somewhat simpler description of the Aron-Berner map, due to Davie and Gamelin [17]. Given $T\in\mathcal{L}^n(X_1,\dots,X_n)$, define $\delta\gamma(T)\in\mathcal{L}^n(X_1'',\dots,X_n'')$ by

$$\delta\gamma(T)(z_1,\dots z_n)=\lim_{x_1\to z_1}\cdots\lim_{x_n\to z_n}A(x_1,\dots,x_n),$$

where the iterated limit is taken for x_j in X_j converging to z_j with respect to the weak* topology of X_j''. Clearly, $\delta\gamma(T)$ is an extension of T, with $\|\delta\gamma(T)\|=\|T\|$. It is also clear that $\delta\gamma(T)$ is separately weakly* continuous in its first variable. In fact, it is the unique extension $\tilde{T}\in^n(X_1'',\dots,X_n'')$ of T such that, for each $1\le i\le n$ and all $x_j\in X_j$ and $z_k\in X_k''$, the linear form

$$z_i\in X_i''\longmapsto\tilde{T}(x_1,\dots,x_{i-1},z_i,z_{i+1},\dots,z_n)$$

is weakly* continuous.

Analogously to the Aron-Berner extension, we have in fact $n!$ possible extensions, one for each ordering in the iterated limit.

Lemma 6. *The Aron-Berner and Davie-Gamelin extensions are identical.*

Proof. Let $X_1,\dots,X_n$ be Banach spaces and $T\in\mathcal{L}^n(X_1,\dots,X_n)$. Given z_i fixed in X_i'', one has

$$\begin{aligned}\bar{z}_i(T)(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)&=\langle z_i,T(x_1,\dots,x_{i-1},\cdot,x_{i+1},\dots,x_n)\rangle\\&=\lim_{x_i\to z_i}T(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n),\end{aligned}$$

where $x_i\in X_i$ converges to $z_i\in X_i''$ in the weak* topology of X_i''. It is now clear that, for every $T\in\mathcal{L}^n(X_1,\dots,X_n)$, we have

$$\begin{aligned}\alpha\beta(T)(z_1,\dots,z_n)&=\bar{z}_1\circ\cdots\circ\bar{z}_n(T)\\&=\lim_{x_1\to z_1}\cdots\lim_{x_n\to z_n}T(x_1,\dots,x_n)\\&=\delta\gamma(T)(z_1,\dots,z_n).\end{aligned}$$

This completes the proof. □

Lemma 7. *Let $m:X\times Y\to Z$ be a bilinear operator. Then the first Arens extension m''' and the (vector-valued version of the) Davie-Gamelin extension $\delta\gamma(m)$ are identical.*

Proof. Of course, we define $\delta\gamma(m) : X'' \times Y'' \to Z''$ by

$$\langle \delta\gamma(m)(x'', y''), z' \rangle = \delta\gamma(z' \circ m)(x'', y'')$$

for all $x'' \in X'', y'' \in Y''$ and $z' \in Z'$. One has

$$\begin{aligned} \langle m'''(x'', y''), z' \rangle &= \langle x'', m''(y'', z') \rangle \\ &= \lim_{x \to x''} \langle m''(y'', z'), x \rangle \\ &= \lim_{x \to x''} \langle y'', m'(z', x) \rangle \\ &= \lim_{x \to x''} \lim_{y \to y''} \langle m'(z', x), y \rangle \\ &= \lim_{x \to x''} \lim_{y \to y''} \langle m(x, y), z' \rangle \\ &= \langle \delta\gamma(m)(x'', y''), z' \rangle. \end{aligned}$$

□

Exercice 1. *Use the preceding Lemma to obtain simpler proofs of Theorems 1, 2, 3 and 4.*

2.3. **Nicodemi operators.** Nicodemi operators where introduced in [50] in a rather algebraical form and then applied to (continuous) multilinear operators by Galindo, García, Maestre and Mujica in [28] (Other applications can be found in [11].) We need some notation. Let $X_1, \dots, X_n$ be Banach spaces. For each $1 \le i \le n$, there is a natural isomorphism

$$(\cdot)_i : \mathcal{L}^n(X_1, \dots, X_n) \longrightarrow \mathcal{L}^{n-1}(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n; X_i')$$

given by

$$\langle A_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), x_i \rangle = A(x_1, \dots, x_n).$$

The inverse isomorphism shall be denoted $(\cdot)^i$. Thus, for every operator $B \in \mathcal{L}^{n-1}(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n; X_i')$, one has

$$B^i(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) = \langle B(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), x_i \rangle.$$

Now, let $Y_1, \dots, Y_n$ another collection of Banach spaces. Suppose there are given linear operators $\phi_i : X_i' \to Y_i'$. It is then possible to construct a linear operator $\Phi : \mathcal{L}^n(X_1, \dots, X_n) \to \mathcal{L}^n(Y_1, \dots, Y_n)$ as follows. For each $1 \le i \le n$, define

$$\phi_{(i)} : \mathcal{L}^n(X_1, \dots, X_{n-i+1}, Y_{n-i}, \dots, Y_n) \longrightarrow \mathcal{L}^n(X_1, \dots, X_{n-i}, Y_{n-i+1}, \dots, Y_n)$$

by

$$\phi_{(i)}(A) = (\phi_i \circ A_i)^i.$$

Now, define an operator $\Phi : \mathcal{L}^n(X_1, \dots, X_n) \to \mathcal{L}^n(Y_1, \dots, Y_n)$ as the composition

$$\Phi = \phi_{(1)} \circ \cdots \circ \phi_{(n)}.$$

As before, there are other possible choices of the ordering of the application of the operators $\phi_{(i)}$.

As a particular case, let X and Y be Banach spaces and let $\phi : X' \to Y'$ be a fixed operator. Nicodemi's procedure generates a sequence of operators

$$\phi^{(n)} : \mathcal{L}^n(X) \longrightarrow \mathcal{L}^n(Y),$$

where we have written $\phi^{(n)}$ instead of Φ for the composition $\phi_{(1)} \circ \cdots \circ \phi_{(n)}$ and $\phi_k = \phi$ for all k.

Remark 5. With the notations above, notice that if $A \in \mathcal{L}^n(X_1, \ldots, X_n)$ has the form $A = x_1' \otimes \cdots \otimes x_n'$, then $\Phi(A) = \phi_1(x_1') \otimes \cdots \otimes \phi_n(x_n')$.

Moreover, it is easily seen that if each operator $\phi_k : X_k' \to Y_k'$ is the adjoint of some operator $\psi_k : Y_k \to X_k$, then

$$\Phi(A)(y_1, \ldots, y_n) = A(\psi_1(y_1), \ldots, \psi_n(y_n))$$

for all $y_i \in Y_i$.

An interesting property of Nicodemi operators is given in the following Proposition.

Proposition 2 (Galindo, García, Maestre and Mujica [28]). *Suppose each X_k is a (closed, linear) subspace of Y_k and that each operator $\phi_k : X_k' \to Y_k'$ extends functionals (that is, for every $x' \in X_k$, the restriction of $\phi_k(x')$ to X_k is x'). Then the associated Nicodemi operator $\Phi : \mathcal{L}^n(X_1, \ldots, X_n) \to \mathcal{L}^n(Y_1, \ldots, Y_n)$ extends multilinear forms.*

Proof. Suppose that $\phi_n : X_n' \to Y_n'$ extends functionals and let $A \in \mathcal{L}^n(X_1, \ldots, X_n)$. We show that $\phi_{(n)}(A) \in \mathcal{L}^n(X_1, \ldots, X_{n-1}, Y_n)$ is an extension of A. Indeed, take $x_1 \in X_i$ for $1 \le i \le n$. One has

$$\begin{aligned}
\phi_{(n)}(A)(x_1, \ldots, x_n) &= (\phi_n \circ A_n)^n(x_1, \ldots, x_n) \\
&= \langle \phi_n(A_n(x_1, \ldots, x_{n-1})), x_n \rangle \\
&= \langle A_n(x_1, \ldots, x_{n-1}), x_n \rangle \\
&= A(x_1, \ldots, x_n).
\end{aligned}$$

Iterate. □

We now show the basic connection between the Arens, Aron-Berner, Davie-Gamelin and Nicodemi operators.

Proposition 3. *Let X_i be Banach spaces. Then the Aron-Berner extension operator $\alpha\beta : \mathcal{L}^n(X_1, \ldots, X_n) \to \mathcal{L}^n(X_1'', \ldots, X_n'')$ (and therefore that of Davie-Gamelin) are the Nicodemi operator associated to the natural inclusion maps $i_k : X_k' \to X_k'''$.*

This is straightforward from next Lemma.

Lemma 8. *Let $\phi_k : X_k' \to Y_k'$ be arbitrary operators for $1 \le k \le n$ and let $\Phi : \mathcal{L}^n(X_1, \ldots, X_n) \to \mathcal{L}^n(Y_1, \ldots, Y_n)$ denote the associated Nicodemi mapping. Then, for each $A \in \mathcal{L}^n(X_1, \ldots, X_n)$ and all $y_i \in Y_i$, one has*

$$\Phi(A)(y_1, \ldots, y_n) = \lim_{x_1 \to \phi_1'(y_1)} \cdots \lim_{x_n \to \phi_n'(y_n)} A(x_1, \ldots, x_n),$$

where the iterated limits are taken for $x_i \in X_i$ converging to $\phi_i'(y_i)$ in the weak topology of X_i''.*

Proof. Let $A \in \mathcal{L}^n(X_1, \dots, X_i, Y_{i+1}, \dots, Y_n)$. One then has

$$\begin{aligned}\phi_{(i)}A(x_1, \dots, x_{i-1}, y_i, y_{i+1}, \dots, y_n) &= (\phi_i \circ A_i)^i(x_1, \dots, x_{i-1}, y_i, y_{i+1}, \dots, y_n)\\ &= \langle (\phi_i \circ A_i)(x_1, \dots, x_{i-1}, y_{i+1}, \dots, y_n), y_i\rangle\\ &= \langle \Phi^i(A_i(x_1, \dots, x_{i-1}, y_{i+1}, \dots, y_n), y_i\rangle\\ &= \langle A_i(x_1, \dots, x_{i-1}, y_{i+1}, \dots, y_n), \phi_i'(y_i)\rangle\\ &= \lim_{x_i \to \phi_i'(y_i)} \langle A_i(x_1, \dots, x_{i-1}, y_{i+1}, \dots, y_n), x_i\rangle\\ &= \lim_{x_i \to \phi_i'(y_i)} A(x_1, \dots, x_{i-1}, x_i, y_{i+1}, \dots, y_n).\end{aligned}$$

Iterate. □

Proposition 3 and Lemma 8 can be rephrased saying that, given operators $\phi_k : X_k' \to Y_k'$, one has

$$\Phi(A)(y_1, \dots, y_n) = \alpha\beta(A)(\phi_1'(y_1), \dots, \phi_n'(y_n))$$

for all $y_i \in Y_i$. In other words, we have a commutative diagram

$$\begin{array}{ccc}\mathcal{L}^n(X_1, \dots, X_n) & \xrightarrow{\Phi} & \mathcal{L}^n(Y_1, \dots, Y_n)\\ \alpha\beta \downarrow & & \uparrow r\\ \mathcal{L}^n(X_1'', \dots, X_n'') & \longrightarrow & \mathcal{L}^n(Y_1'', \dots, Y_n'')\end{array}$$

where r is the restriction map and the lower arrow is plain composition with the operators $\phi_k' : Y_k'' \to X_k''$.

2.4. **Application: multilinear forms on dual-isomorphic spaces.** Suppose X_k and Y_k are Banach spaces whose duals are isomorphic for $1 \le k \le n$ and let $\phi_k : X_k' \to Y_k'$ be the corresponding isomorphisms. It is plain from the definition that each step $\phi_{(k)}$ in the construction of the associated Nicodemi operator is an isomorphism as well and so is $\Phi : \mathcal{L}^n(X_1, \dots, X_n) \to \mathcal{L}^n(Y_1, \dots, Y_n)$. To sum up, we have:

Theorem 6 ([12] also [45, 18])**.** *Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ be Banach spaces such that X_k' and Y_k' are isomorphic for each $1 \le k \le n$. Then the spaces of multilinear forms $\mathcal{L}^n(X_1, \dots, X_n)$ and $\mathcal{L}^n(Y_1, \dots, Y_n)$ are isomorphic. In particular, if X' is isomorphic to Y', then $\mathcal{L}^n(X)$ and $\mathcal{L}^n(Y)$ are isomorphic for all $n \ge 1$.*

This result has some interesting consequences. For instance, taking $X = C[0,1]$ and $Y = c_0(\Gamma, C[0,1])$ (here Γ has the power of continuum) we obtain that $\mathcal{L}^n(X)$ and $\mathcal{L}^n(Y)$ are isomorphic for all $n \ge 1$ in spite of the fact that X is separable and Y is not. Also, taking $X = l_1(l_2^n)$ and $Y = l_1(l_2^n) \oplus l_2$ we see that $\mathcal{L}^n(X)$ and $\mathcal{L}^n(Y)$ are isomorphic, in spite of the fact that every multilinear form on X is weakly sequentially continuous (since X has the Schur property), while Y obviously admits bilinear forms which are not weakly sequentially continuous.

2.5. **Extension to ultraproducts.** Ultrapowers of Banach spaces are of capital importance in the local theory of Banach spaces. Let us briefly sketch their basic properties. (See [54] or [40] for further information.) Let X be a Banach space, S an arbitrary set and U an ultrafilter on S. The ultrapower of X with respect to U is the Banach space obtained taking the quotient of $l_\infty(S,X) = \{x : S \to X : \sup_s \|x(s)\|_X < \infty\}$ by the subspace

$$N_U = \left\{ f \in l_\infty(S,X) : \lim_{U(s)} \|f(s)\|_X = 0 \right\}$$

and will be denoted by X_U. The norm of X_U enjoys the following nice property:

$$\|[f]\|_{X_U} = \lim_{U(s)} \|f(s)\|_X,$$

where $[f]$ denotes the class of $f \in l_\infty(S,X)$ in X_U. Observe that X_U contains a natural copy of the space X regarded as the space of (classes of) constant maps $S \to X$.

Lindström and Ryan [46] obtained a method of extension for multilinear forms from X to its ultrapower X_U as follows. For $A \in \mathcal{L}^n(X)$, define $\lambda\rho(A) \in \mathcal{L}^n(X_U)$ by

$$\lambda\rho(A)([x_1],\dots,[x_n]) = \lim_{U(s_1)} \cdots \lim_{U(s_n)} A(x_1(s_1),\dots,x_n(s_n)).$$

It is easily seen that $\lambda\rho(A)$ is an extension of A, with $\|\lambda\rho(A)\| = \|A\|$. Also, it is clear that $\lambda\rho(A)$ depends linearly on A.

Again, $\lambda\rho$ can be regarded as a Nicodemi operator. To see this, consider the obvious map $\phi : X' \to (X_U)'$ given by $\langle \phi(x'), [x]\rangle = \lim_{U(s)} \langle x', x(s)\rangle$. It is now clear that, for each n, the extension operator $\lambda\rho : \mathcal{L}^n(X) \to \mathcal{L}^n(X_U)$ is the Nicodemi operator $\phi^{(n)}$ induced by ϕ.

Remark 6. There is simpler way of extending multilinear forms to ultrapowers. Let A be a multilinear form on X and X_U an ultrapower of X. We can define an extension of A to X_U by

$$\upsilon(A)([x_1],\dots,[x_n]) = \lim_{U(s)} A(x_1(s),\dots,x_n(s)).$$

It can be proved that this extension $\upsilon(A)$ cannot be obtained by a Nicodemi operator.

2.6. **Extension and locally complemented subspaces.** So far we have seen that sometimes it is possible to extend multilinear forms from a subspace X of Y to the whole of Y in a linear and continuous way. This is so, for instance, if $Y = X''$ or if Y is an ultrapower of X. On the other hand, Proposition 2 shows that the only possible obstruction to the existence of linear extension operators $\mathcal{L}^n(X) \to \mathcal{L}^n(Y)$ stems from the linear case $n = 1$.

It will be convenient to have a more intrinsic criterion on the embedding $X \to Y$ for the existence of linear extension operators $\mathcal{L}^n(X) \to \mathcal{L}^n(Y)$. (We remark here that even the notion of extension depends, not only on the involved spaces, but also on the particular embedding $X \to Y$, that is, on the position of X inside Y.)

The key point turns out to be the by now classical notion of a locally complemented subspace.

Definition 1. *We shall say that a (closed) subspace X of a Banach space Y is locally complemented in Y if there is a constant M such that whenever F is a finite-dimensional subspace of Y there is a linear map (depending on the given finite dimensional subspace) $T : F \to X$ so that $\|T\| \leq M$ and $Tx = x$ for all $x \in F \cap X$.*

Thus, for instance, the Principle of Local Reflexivity of Lindenstrauss and Rosenthal [48] says that every Banach space is locally complemented in its bidual. Also, it is well-known that every Banach space is locally complemented in its ultrapowers.

The following result clarifies the situation in the linear case. For a proof, see [42].

Lemma 9. *Let X be a closed subspace of Y. The following are equivalent.*

(a) *X is locally complemented in Y.*
(b) *X'' is complemented in Y'' under the natural embedding.*
(c) *There is a linear extension operator $E : X' \to Y'$ (that is such that $\langle E(x'), x\rangle = \langle x', x\rangle$ for all $x \in X, x' \in X'$).*

Theorem 7. *Let X be a closed subspace of Y. The following are equivalent.*

(a) *X is locally complemented in Y.*
(b) *For each $n \geq 1$ and every collection $X_2, \ldots, X_n$ of Banach spaces, there is a linear extension operator $\mathcal{L}^n(X, X_2, \ldots, X_n) \to \mathcal{L}^n(Y, X_2, \ldots, X_n)$.*
(c) *For some $n \geq 1$ there exist a collection $X_2, \ldots, X_n$ of nontrivial Banach spaces and an extension operator $\mathcal{L}^n(X, X_2, \ldots, X_n) \to \mathcal{L}^n(Y, X_2, \ldots, X_n)$.*

Proof. That (a) and (b) are equivalent follows from the previous Lemma and Proposition 2. That (b) implies (c) is obvious. It remains to prove that (c) implies (a). Of course, if (c) holds for $n = 1$, then X is locally complemented in Y, by the Lemma. So, we may and do assume that (c) holds for a certain family of Banach spaces $X_2, \ldots, X_n$, with $n \geq 2$. Since $\mathcal{L}^n(X, X_2, \ldots, X_n) = \mathcal{L}^2(X, X_2\widehat{\otimes} \cdots \widehat{\otimes} X_n)$ and $\mathcal{L}^n(Y, X_2, \ldots, X_n) = \mathcal{L}^2(Y, X_2\widehat{\otimes} \cdots \widehat{\otimes} X_n)$, taking $E = X_2\widehat{\otimes} \cdots \widehat{\otimes} X_n$, we can suppose that a linear extension operator $\mathcal{L}^2(X, E) \to \mathcal{L}^2(Y, E)$ exists. Taking into account the universal property of the projective tensor product and the Lemma, this implies that $X\widehat{\otimes} E$ is a locally complemented subspace of $Y\widehat{\otimes} E$ under the obvious map. Now, fix a norm one $e_0 \in E$ and define isometric embeddings $X \to X\widehat{\otimes} E$ by $x \mapsto x \otimes e_0$ and $Y \to X\widehat{\otimes} E$ by $y \mapsto y \otimes e_0$ (see [7]). We have a commutative diagram of inclusion maps

$$\begin{array}{ccc} X\widehat{\otimes}E & \longrightarrow & Y\widehat{\otimes}E \\ \uparrow & & \uparrow \\ X & \longrightarrow & Y \end{array}$$

Since "being a locally complemented subspace of" is transitive, the proof will be complete if we show that X is complemented in $X\widehat{\otimes}E$. To this end, pick a norm-one $f \in E'$ so that $f(e_0) = 1$ and define an operator $P : X\widehat{\otimes}E \to X$ by $P(x \otimes e) = f(e)x$. Clearly, P is a projection. Moreover, $\|P\| = 1$ since $P = \mathrm{Id}_X \otimes f : X\widehat{\otimes}E \to X\widehat{\otimes}\mathbb{K} = X$. This completes the proof. □

Corollary 3. *Suppose that X_i is a subspace of Y_i, for $1 \leq i \leq n$. Then there exists a linear extension operator $\mathcal{L}^n(X_1, \ldots, X_n) \to \mathcal{L}^n(Y_1, \ldots, Y_n)$ if and only if each X_i is locally complemented in Y_i.*

Remark 7. An obvious "symmetrization" argument leads to a very simple proof of the following result (which was motivated by a question of Zalduendo; see [8, Problem 9])

Corollary 4 (Peris [59])**.** *Let X be a subspace of Y. Then there exists a linear extension operator $\mathcal{P}^n(X) \to \mathcal{P}^n(Y)$ for all (or some) $n \geq 1$ if and only if X is locally complemented in Y.*

Exercice 2. *Let X be a subspace of Y. Then there exists a linear continuous extension operator $\epsilon : \mathcal{H}_b(X) \to \mathcal{H}_b(Y)$ for holomorphic functions of bounded type (see* [21] *for definitions) if and only if X is locally complemented in Y. (Hint. Show that the map $x' \in X' \mapsto d(\epsilon(x'))(0) \in Y'$ is a linear extension operator.)*

3. Regularity and permutation of the variables

As we mentioned before, Nicodemi operators (hence Aron-Berner extensions) require a choice in the ordering of the involved variables. In this Section we study to what extent the extended map is independent of that choice.

Let us reconsider bidual algebras in this setting. Suppose $\mathcal{A}$ a Banach algebra with product p. The first Arens product on $\mathcal{A}''$ is given by

$$p'''(a'', b'') = \delta\gamma(p)(a'', b'') = w^* - \lim_{a \to a''} \left(w^* - \lim_{b \to b''} p(a, b) \right).$$

The second Arens product is $((p_{rev})''')_{rev}$. Since

$$\begin{aligned} ((p_{rev})''')_{rev}(a'', b'') &= (p_{rev})'''(b'', a'') \\ &= \delta\gamma(p_{rev})(b'', a'') \\ &= w^* - \lim_{b \to b''} \left(w^* - \lim_{a \to a''} p_{rev}(a, b) \right) \\ &= w^* - \lim_{b \to b''} \left(w^* - \lim_{a \to a''} p(a, b) \right) \end{aligned}$$

we see that $\mathcal{A}$ is Arens regular (that is, the two Arens products coincide on $\mathcal{A}''$) if and only if the (vector-valued version of the) Davie-Gamelin extension of p does not depend on the order in the iterated limit.

Now, suppose $\mathcal{A}$ commutative. Then $\mathcal{A}''$ is commutative if and only if $\mathcal{A}$ is Arens-regular. Observe that $\mathcal{A}$ is commutative if and only if for every $a' \in \mathcal{A}'$, the bilinear form $a' \circ p$ is symmetric. Since $\alpha\beta(a' \circ p) = a' \circ p'''$ the fact that $(l_1(\mathbb{Z}), *)$ is commutative but $(l_1(\mathbb{Z}), *)''$ is not shows that the Aron-Berner extension of a symmetric form need not be symmetric. This has some unpleasant consequences when dealing with polynomials.

From now on, we restrict ourselves to multilinear forms defined on some fixed Banach space X. The consideration of multilinear operators and (or) different spaces does not involve new ideas and would make the notation somewhat confusing.

Given $A \in \mathcal{L}^n(X)$ and $\sigma \in S_n$ (the group of permutations of $\{1, 2, \ldots, n\}$), put

$$A^\sigma(x_1, \ldots, x_n) = A(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

The symmetric group S_n acts by conjugation on the space of operators $\epsilon : \mathcal{L}^n(X) \to \mathcal{L}^n(Y)$ sending ϵ to the operator

$$A \in \mathcal{L}^n(X) \longmapsto (\epsilon(A^\sigma))^{\sigma^{-1}},$$

which will be denoted by ϵ_σ. Thus one may wonder if a given operator is compatible with the action of the symmetric group in the sense that for some (or for every) $A \in \mathcal{L}^n(X)$ one has $\epsilon_\sigma(A) = \epsilon(A)$ for all $\sigma \in S_n$.

Observe that is ϵ is the Aron-Berner extension or a Nicodemi operator, then $\epsilon_{\sigma^{-1}}(A) = \epsilon(A)$ means that one can change the "usual" ordering in the involved variables (from the last to the first) by the new order induced by σ. Let us check this for the Davie-Gamelin extension. Since

$$\begin{aligned} \delta\gamma_{\sigma^{-1}}(A)(x_1'', \ldots, x_n'') &= \delta\gamma(A^{\sigma^{-1}})(x''_{\sigma(1)}, \ldots, x''_{\sigma(n)}) \\ &= \lim_{x_1 \to x''_{\sigma(1)}} \cdots \lim_{x_n \to x''_{\sigma(n)}} A^{\sigma^{-1}}(x_1, \ldots, x_n) \\ &= \lim_{x_1 \to x''_{\sigma(1)}} \cdots \lim_{x_n \to x''_{\sigma(n)}} A(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) \\ &= \lim_{y_{\sigma(1)} \to x''_{\sigma(1)}} \cdots \lim_{y_{\sigma(n)} \to x''_{\sigma(n)}} A(y_1, \ldots, y_n) \\ &= \lim_{x_{\sigma(1)} \to x''_{\sigma(1)}} \cdots \lim_{x_{\sigma(n)} \to x''_{\sigma(n)}} A(x_1, \ldots, x_n), \end{aligned}$$

we see that the condition $\delta\gamma_{\sigma^{-1}}(A) = \delta\gamma(A)$ means

$$\lim_{x_1 \to x_1''} \cdots \lim_{x_n \to x_n''} A(x_1, \ldots, x_n) = \lim_{x_{\sigma(1)} \to x''_{\sigma(1)}} \cdots \lim_{x_{\sigma(n)} \to x''_{\sigma(n)}} A(x_1, \ldots, x_n).$$

Or, which is the same, that one can obtain the same extension taking the iterated limits in the ordering given by σ.

Of course, if $\epsilon : \mathcal{L}^n(X) \to \mathcal{L}^n(Y)$ is any operator, one can obtain an operator invariant by conjugation taking

$$\epsilon_s = \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon_\sigma,$$

but some good properties of ϵ may be lost. (For instance, if ϵ is a Nicodemi operator induced by an isomorphism $\phi : X' \to Y'$, then ϵ is itself an isomorphism, but we do not known whether ϵ_s is or not.) Note, however, that if ϵ is an extension operator, then so is ϵ_s.

Since Nicodemi operators factorize through Aron-Berner extensions (see Lemma 8), it is clear that if $A \in \mathcal{L}^n(X)$ is such that $\alpha\beta_\sigma(A) = \alpha\beta(A)$ for all $\sigma \in S_n$, then one also has $\phi_\sigma^{(n)}(A) = \phi^{(n)}(A)$ for every Banach space Y and every linear operator $\phi : X' \to Y'$. The following "formal" result will be useful in what follows.

Lemma 10. *Let $A \in \mathcal{L}^n(X)$. Then $\alpha\beta_\sigma(A) = \alpha\beta(A)$ for all $\sigma \in S_n$ if and only if $\alpha\beta(A) : X'' \times \cdots \times X'' \to \mathbb{K}$ is separately weakly* continuous in each argument. In this case, $\alpha\beta(A)$ is the unique extension of A to X'' which is separately weakly* continuous in each argument.*

Proof. Suppose $\alpha\beta_{\sigma^{-1}}(A) = \alpha\beta(A)$. Then,

$$\alpha\beta(A)(x_1'', \ldots, x_n'') = \lim_{x_{\sigma(1)} \to x''_{\sigma(1)}} \cdots \lim_{x_{\sigma(n)} \to x''_{\sigma(n)}} A(x_1, \ldots, x_n).$$

Since the second member in the identity above is obviously weakly* continuous in the $\sigma(1)$ variable, varying σ in S_n, we see that $\alpha\beta(A)$ is separately weakly* continuous in each variable.

Clearly, if $\alpha\beta(A) = \delta\gamma(A)$ is separately weakly* continuous in each variable, then for every $1 \le i \le n$ one has

$$\delta\gamma(A)(x_1'', \ldots, x_i'', \ldots, x_n'') = \lim_{x_i \to x_i''} \delta\gamma(A)(x_1'', \ldots, x_i, \ldots, x_n'').$$

Thus, the iterated limit in the Davie-Gamelin extension of A is independent of the order. □

For bilinear forms one has a more satisfactory result:

Lemma 11 (Grothendieck [35])**.** *Let A be a bilinear form on X. The following are equivalent:*

(a) *$\alpha\beta_\sigma(A) = \alpha\beta(A)$ for $\sigma = (1, 2)$.*
(b) *The associated operator $T : X \to X'$ given by $\langle Tx, y\rangle = A(x, y)$ is weakly compact.*

Proof. We give a proof based on the so-called "double limit criterion" of Grothendieck [35]: a linear operator $T : X \to X'$ is weakly compact if and only if, given arbitrary bounded filters $\mathcal{F}$ and $\mathcal{G}$ on X, one has

$$\lim_{\mathcal{F}(x)} \lim_{\mathcal{G}(y)} \langle T(x), y\rangle = \lim_{\mathcal{G}(y)} \lim_{\mathcal{F}(x)} \langle T(x), y\rangle$$

provided both limits exist.

From this, the implication (b) $\Rightarrow$ (a) is immediate.

As for the converse, let $\mathcal{F}$ and $\mathcal{G}$ be bounded filters on X and assume that $\lim_{\mathcal{F}(x)} \lim_{\mathcal{G}(y)} \langle T(x), y\rangle$ and $\lim_{\mathcal{G}(y)} \lim_{\mathcal{F}(x)} \langle T(x), y\rangle$ exist. Take bounded ultrafilters $\mathcal{U}$ and $\mathcal{V}$ in X'' containing, respectively, $\mathcal{F}$ and $\mathcal{G}$. By the Banach-Alaoglu theorem $\mathcal{U}$ (resp. $\mathcal{V}$) must converge to some point x'' (resp. y'') in X. So,

$$\begin{aligned}
\lim_{\mathcal{F}(x)} \lim_{\mathcal{G}(y)} \langle T(x), y\rangle &= \lim_{\mathcal{U}(x)} \lim_{\mathcal{V}(y)} \langle T(x), y\rangle \\
&= \lim_{\mathcal{U}(x)} \lim_{\mathcal{V}(y)} A(x,y) \\
&= \lim_{x\to x''} \lim_{y\to y''} A(x,y) \\
&= \alpha\beta(A)(x'', y'') \\
&= \alpha\beta_\sigma(A)(x'', y'') \\
&= \lim_{y\to y''} \lim_{x\to x''} A(x,y) \\
&= \lim_{\mathcal{V}(y)} \lim_{\mathcal{U}(x)} \langle T(x), y\rangle \\
&= \lim_{\mathcal{G}(y)} \lim_{\mathcal{F}(x)} \langle T(x), y\rangle .
\end{aligned}$$

Hence T is weakly compact. □

This suggests the following definition.

Definition 2. *A Banach space X is said to be regular if all operators $X \to X'$ are weakly compact.*

The most important examples of regular spaces (apart from reflexive spaces) are the $C_0(\Omega)$ spaces, and more generally, $\mathcal{L}_\infty$ spaces and C*-algebras. In fact, Banach spaces having the property (V) of Pełczyński are regular (see [29]). This includes other important Banach spaces such as the disc and polydisc algebras, H^∞ and the Wiener algebra.

We are ready to prove the main result of this section.

Theorem 8. *Let X be a regular Banach space. Then, for all $n \geq 1$, the Aron-Berner extension $\alpha\beta : \mathcal{L}^n(X) \to \mathcal{L}^n(X'')$ is invariant by conjugation under the symmetric group.*

Proof. Since the symmetric group can be generated by traspositions of two consecutive indices, it obviously suffices to show that $\alpha\beta_\sigma(A) = \alpha\beta(A)$ for all $A \in \mathcal{L}^n(X)$ and $\sigma = (i, i{+}1)$, where $1 \leq i \leq n{-}1$. But this case immediately reduces to that of bilinear forms and follows from Lemma 11. □

Corollary 5. *Let X be a regular Banach space. Then, for all $n \geq 1$, all Banach spaces Y and every operator $\phi : X' \to Y'$ the induced Nicodemi operator $\phi^{(n)} : \mathcal{L}^n(X) \to \mathcal{L}^n(Y)$ is invariant by conjugation under the symmetric group.*

Corollary 6. *Let X be a regular Banach space. Then all Nicodemi operators transform symmetric forms on X into symmetric forms. In particular the Aron-Berner extension of every symmetric multilinear form on X is again a symmetric multilinear form on X''.*

Surprisingly enough, the converse of the preceding Corollary does not hold. Recall that an operator $T : X \to X'$ is said to be symmetric if

$$\langle Tx, y\rangle = \langle Ty, x\rangle$$

for all $x, y \in X$. This just means that the induced bilinear mapping $A(x, y) = \langle Tx, y\rangle$ is symmetric.

Definition 3. *A Banach space X is said to be symmetrically regular if all symmetric operators $X \to X'$ are weakly compact.*

Let $\mathcal{L}^n_s(X)$ denote the space of n-linear symmetric forms on X. Using the same arguments that in the multilinear case, one can prove the following.

Theorem 9. *For a Banach space X the following are equivalent:*

(a) *X is symmetrically regular.*
(b) *For every symmetric bilinear form A on X, $\alpha\beta(A)$ is a symmetric form on X''.*
(b) *For each $n \geq 2$, $\alpha\beta(A)$ belongs to $\mathcal{L}^n_s(X'')$ whenever $A \in \mathcal{L}^n_s(X)$.*
(d) *All Nicodemi operators transform symmetric multilinear forms on X into symmetric forms.*

Example 3 (Leung [47])**.** *There are symmetrically regular spaces which are not regular.*

Actually, Leung shows that the dual of James quasi-reflexive space J is symmetrically regular but not regular. This means that, while for every symmetric bilinear form on J' the Aron-Berner extension is symmetric on J''', there is an antisymmetric bilinear form on J' whose Aron-Berner extension is not antisymmetric.

Exercice 3. *Prove that if X is a stable Banach space (that is, X is isomorphic to its square $X \times X$), then X is symmetrically regular if and only if it is regular. So, the presence of James' creature in the above Example is not accidental.*

Exercice 4. *Prove that the Banach algebra $\mathcal{A}$ is Arens regular (as a Banach algebra) if and only if, for every $a' \in \mathcal{A}'$, the operator $T : \mathcal{A} \to \mathcal{A}'$ given by $\langle Ta, b\rangle = \langle a', ab\rangle$ is weakly compact. Prove that, apart from the trivial case in which G is a finite group, no group algebra $L_1(G)$ is Arens regular.*

Exercice 5. *Let $\mathcal{A}$ be a Banach algebra whose underlying Banach space is regular. Show that for all Banach $\mathcal{A}$-bimodules X the $\mathcal{A}''$-bimodule X'' is a dual bimodule. (Hint. Assume, without loss of generality, that $\mathcal{A}$ is unital and define a trilinear operator $\mathcal{A}'' \times X' \times \mathcal{A}'' \to X'$ by*

$$\langle t(a'', x', b''), x\rangle = \lim_{a \to a''} \lim_{b \to b''} \langle x', b \cdot x \cdot a\rangle.$$

Now, use that, for each fixed $x' \in X'$, the map t is a Davie-Gamelin extension and Lemmas 10 and 11 to show that t makes X' into an $\mathcal{A}''$-bimodule and verify that the usual $\mathcal{A}''$-bimodule structure on X'' is just the dual of that of X'.)

Exercice 6. *Show that the bitranspose derivation D'' appearing in the proof of Example 2 is inner for all compact spaces K.*

Exercice 7 ([12])**.** *Let X and Y be dual-isomorphic Banach spaces. Suppose X regular. Prove that Y is regular and that $\mathcal{H}_b(X)$ and $\mathcal{H}_b(Y)$ are isomorphic Fréchet algebras.*

Question 1. *Suppose $\phi : X' \to Y'$ is an isomorphism and let $\phi^{(n)} : \mathcal{L}^n(X) \to \mathcal{L}^n(Y)$ be the associated Nicodemi operator. Must the symmetrized operator $\phi_s^{(n)} : \mathcal{L}_s^n(X) \to \mathcal{L}_s^n(Y)$ be an isomorphism? (An affirmative answer would imply that $\mathcal{P}(^nX)$ and $\mathcal{P}(^nY)$ are isomorphic for each $n \geq 1$.)*

4. The range of the Aron-Berner extensions

Let X_i be subspaces of Y_i. As we already mentioned, if one has a linear extension operator $\epsilon : \mathcal{L}^n(X_1, \ldots, X_n) \to \mathcal{L}^n(Y_1, \ldots, Y_n)$ for multilinear forms, one can obtain a linear extension operator $\mathcal{L}^n(X_1, \ldots, X_n; Z) \to \mathcal{L}^n(Y_1, \ldots, Y_n; Z'')$ by the formula

$$\langle \epsilon(T)(y_1, \ldots, y_n), z' \rangle = \epsilon(z' \circ T)(y_1, \ldots, y_n)$$

for all Banach spaces Z. In particular, we can regard the Aron-Berner extension as an extension operator $\alpha\beta : \mathcal{L}^n(X_1, \ldots, X_n; Z) \to \mathcal{L}^n(X_1'', \ldots, X_n''; Z'')$. Simple examples show that, in general, one cannot expect that the range of the Aron-Berner extensions stay in the original space Z. In fact, in the linear case $\alpha\beta(T) = T''$ and so, $\alpha\beta(T)$ takes values in Z if and only if T is weakly compact, by Gantmacher theorem.

The aim of this section is to show that multilinear operators whose Aron-Berner extensions are Z-valued play the same role in the "multilinear theory" of Banach spaces that weakly compact linear operators in the "linear theory". Let us recall that a multilinear operator is weakly compact if it maps bounded sets into relatively weakly compact sets. We start with the following simple

Proposition 4. *Let $T \in \mathcal{L}^n(X_1, \ldots, X_n; Z)$. If T is weakly compact then $\alpha\beta(T)$ is Z-valued.*

Proof. The hypothesis implies that the weak closure of $T\left(B(X_1) \times \cdots \times B(X_n)\right)$ is a weakly compact set in Z we denote by K. Now, fix $x_i \in B(X_i)$ $(1 \leq i \leq n-1)$ and pick $x'' \in B(X_n'')$. Since $B(X_n)$ is weakly* dense in $B(X_n'')$ one has,

$$\begin{aligned}
\alpha\beta(T)\left(x_1, \ldots, x_{n-1}, x''\right) &= \text{weak}^* - \lim_x T\left(x_1, \ldots, x_{n-1}, x\right) \\
&= \text{weak} - \lim_x T\left(x_1, \ldots, x_{n-1}, x\right),
\end{aligned}$$

as $x \in B(X_n)$ converges weakly* to x''. Hence $\alpha\beta(T)(x_1, \ldots, x_{n-1}, x'')$ belongs to K for points in the corresponding balls. Iterate. □

The polynomial version of this result can be found in [15].

The reciprocal to the previous proposition is not true:

Example 4. *There is a non-weakly compact bilinear operator $l_\infty \times l_\infty \to l_1$ whose Aron-Berner extension is l_1-valued.*

Proof. Let $q : \ell_\infty \to \ell_2$ be a continuous surjective operator and let us define $S : \ell_\infty \times \ell_\infty \to \ell_1$ as the coordinatewise multiplication

$$S(x, y) = q(x) \cdot q(y)$$

It is easily seen that S maps the unit ball of $\ell_\infty \times \ell_\infty$ onto a neighborhood of the origin in l_1. Hence S is not weakly compact. On the other hand, for $x'', y'' \in l''_\infty$, one has

$$\alpha\beta(S)(x'', y'') = q''(x'') \cdot q''(y'')$$

which clearly belongs to ℓ_1. □

Hence, we see that, for multilinear mappings, having an "Z-valued" Aron-Berner extension is a less restrictive condition that being weakly compact.

In the remainder of this Section we shall show that most of the classical Banach space properties related to weak compactness admit characterizations in terms of multilinear operators having Z-valued Aron-Berner extensions.

Let us recall that a Banach space X has the Grothendieck property if every linear operator from X to a separable Banach space, equivalently to c_0, is weakly compact.

Theorem 10 ([14]). *For a Banach space X the following are equivalent:*

(a) *X has the Grothendieck property.*
(b) *For any separable Banach space Z, every n-linear operator $T \in \mathcal{L}^n(X; Z)$ has Z-valued Aron-Berner extensions.*
(c) *Every symmetric bilinear application $S : X \times X \to c_0$ which is separately compact has c_0-valued Aron-Berner extension.*

Proof. It is trivial that (b) implies (c) . We show that (c) implies (a). If $T : X \to c_0$ is a linear operator, we can consider the symmetric bilinear form $S : X \times X \to c_0$ given by $S(x, y) = T(x) \cdot T(y)$. It is easy to see that S is separately compact and that

$$\alpha\beta(S)(x'', y'') = T''(x'') \cdot T''(y''),$$

now the product being that of ℓ_∞. If X lacks the Grothendieck property, there is a linear operator $T : X \to c_0$ that is not weakly compact, which implies that T'' cannot fall into c_0. Thus, there exists $x'' \in X''$ so that $T''(x'') \notin c_0$ and therefore $\alpha\beta(S)(x'', x'') \notin c_0$.

It remains to see that (a) implies (b). This immediately follows from the following result, of independent interest. □

Lemma 12. *Suppose that every operator from X into Z is weakly compact. Then, every multilinear operator $T \in \mathcal{L}^n(X;Z)$ has Z-valued Aron-Berner extensions.*

Proof. Let T be a multilinear operator from X to Z For $1 \le i \le n-1$, let us fix $x_i \in X$ and take $z_n \in X''$. Consider the first limit appearing in the Aron-Berner extension of T

$$\alpha\beta(T)(x_1, \ldots, x_{n-1}, z_n) = \text{weak}^* - \lim_{x_n \to z_n} T(x_1, \ldots, x_{n-1}, x_n).$$

Then $\alpha\beta(T)(x_1, \ldots, x_{n-1}, z_n)$ belongs to Z (instead of Z'') since it is the value at z_n of the bitranspose of the weakly compact operator $T(x_1, \ldots, x_{n-1}, \cdot) : X \to Z$. Now, for $1 \le i \le n-2$ let x_i be fixed in X and take $z_{n-1}, z_n \in X''$. Then we have

$$\alpha\beta(T)(x_1, \ldots, x_{n-2}, z_{n-1}, z_n) = \text{weak}^* - \lim_{x_{n-1} \to z_{n-1}} \alpha\beta(T)(x_1, \ldots, x_{n-2}, x_{n-1}, z_n),$$

which also belongs to Z since it is the value at z_{n-1} of the bitranspose of $\alpha\beta(T)(x_1, \ldots, x_{n-2}, \cdot, z_n) : X \to Z$ which is weakly compact by hypothesis. Continue. □

The equivalence between (a) and (b) above was already known for polynomials ([33]).

Next, we consider the Dunford-Pettis property (DPP for short), introduced by Grothendieck in [37]. Let us recall that a Banach space X has the DPP if every weakly compact defined on X is completely continuous (that is, it sends weakly convergent sequences into norm convergent sequences. Completely continuous multilinear operators are defined in the obvious way). As before, the range space can be taken as c_0. Typical examples of spaces enjoying the DPP are $L_1(\mu)$ and $C_0(\Omega)$-spaces.

Theorem 11 ([38]). *Let X_i be Banach spaces for $1 \le i \le n$. The following are equivalent:*

(a) *Each X_i has the DPP.*
(b) *For any Banach space Z (or merely c_0), every n-linear operator $T \in \mathcal{L}^n(X_1, \ldots, X_n; Z)$ with Z-valued Aron-Berner extension is completely continuous.*

Proof. We write the proof for two Banach spaces X and Y. Suppose X lacks DPP. Then there is a Banach space Z and a weakly compact operator $L : X \to Z$ which is not completely continuous. Take a nonzero functional $y' \in Y'$ and define a bilinear map $T : X \times Y \to Z$ taking $T(x,y) = y'(y)L(x)$. Clearly, T is not completely continuous and $\alpha\beta(T)$ is Z-valued.

As for the converse, let X and Y Banach spaces with DPP and suppose $T : X \times Y \to c_0$ has c_0-valued Aron-Berner extension. Take x_n and y_n, weakly null sequences in X and Y respectively. One has to show that $T(x_n, y_n)$ is norm convergent to zero in c_0. Let us define an operator $(T(x_n, \cdot)) : Y \to c_0(c_0)$ by $(T(x_n, \cdot))(y) = (T(x_n, y))_n$. The definition makes sense because

$(T(x_n, y))_n = (T(\cdot, y)(x_n))_n$ and the operator $T(\cdot, y) : X \to c_0$, being weakly compact, is completely continuous, by the DPP of X.

The proof will be complete if we show that $(T(x_n, \cdot))$ is completely continuous. By the DPP of Y one only has to see that $(T(x_n, \cdot))$ is weakly compact. But this follows from the obvious fact that each operator $T(x_n, \cdot) : Y \to c_0$ is weakly compact, that for every $y'' \in Y''$ the sequence $T(x_n, \cdot)''(y'') = \alpha\beta(T)(x_n, y'')$ converges to zero and the following observation of Ryan whose easy proof is left to the reader. □

Lemma 13 (Ryan [52]). *Let $S : Y \to c_0(Z)$ be a linear operator, with $S(y) = (S_n(y))_n$. Then S is weakly compact if and only if*

(a) *S_n is weakly compact for all n.*
(b) *for every $y'' \in Y''$, the sequence $\left(S_n''(y'')\right)_n$ converges to 0.*

Other classical properties related to weak compactness are the reciprocal DPP and Pełczyński's property (V). A Banach space has the RDPP if every completely continuous operator on it is weakly compact and has property (V) if every unconditionally converging operator is weakly compact. Recall that a (linear or multilinear) operator T is unconditionally converging if for every weakly unconditionally Cauchy series $\sum_n x_n$ the sequence $(T(\sum_{n=1}^k x_n))_k$ is norm convergent. In the multilinear case this notion, introduced by M. Fernández Unzueta in [27], is slightly different from the "usual" one (see [32, 32]). One has,

Theorem 12 ([38]). *The following assertions are equivalent:*

(a) *The spaces X_i have property (V) (respectively, RDPP).*
(b) *For all Z and each unconditionally converging (respectively, completely continuous) $T \in \mathcal{L}^n(X_1, \ldots, X_n; Z)$, every Aron-Berner extension of T is Z-valued.*
(c) *For all Z, every unconditionally converging (respectively, completely continuous) $T \in \mathcal{L}^n(X_1, \ldots, X_n; Z)$, has an Z-valued Aron-Berner extension.*

We close this Section with some remarks about the "polynomial" versions of the preceding results. There are polynomial versions of Theorem 12 ([38]). Strangely enough, we do not know if there is a polynomial version of Theorem 11. There is a polynomial version of this last result for regular spaces:

Proposition 5. *For a regular Banach space X, the following assertions are equivalent:*

(a) *X has the DPP.*
(b) *For all Banach spaces Z, every homogeneous polynomial $P : X \to Z$ whose Aron-Berner extension is Z-valued is completely continuous.*

Proof. Let us recall here that the Aron-Berner extension of a homogeneous polynomial $P : X \to Z$ is given by

$$\alpha\beta(P)(x'') = \alpha\beta_s(T)(x'', \ldots, x''),$$

where T is the symmetric operator associated to P. That (b) implies (a) is obvious. For the converse, suppose X has DPP and $\alpha\beta(P)$ is Z-valued. If X is regular, then $\alpha\beta(T) = \alpha\beta_s(T)$ and one can recover $\alpha\beta(T)$ from $\alpha\beta(P)$ via polarization, from where it follows that $\alpha\beta(T)$ takes values in Z. Now apply Theorem 11 to conclude that T (and, a fortiori, P) is completely continuous. □

The most natural way to eliminate the hypothesis of regularity in the preceding Proposition would be to solve in the affirmative the following

Question 2. *Let $P : X \to Z$ be a homogeneous polynomial and T its associated symmetric operator. Suppose $\alpha\beta(P)$ takes values in Z. Does this imply that $\alpha\beta(T)$ is Z-valued?*

Remark 8. With the notations above, if the Aron-Berner extension of P is Z-valued, then T is separately weakly compact.

To see this, notice that if $S : X'' \times \cdots \times X'' \to Z''$ is the symmetric operator associated with $\alpha\beta(P)$ and $\alpha\beta(P)$ takes values in Z, then so does S. Even if S and $\alpha\beta(T)$ are different extensions, they agree on points having the form $(x_1, \ldots, x_{i-1}, x'', x_{i+1}, \ldots, x_n), x_i \in X, x'' \in X''$. Hence

$$\begin{aligned}(T(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n))''(x'') &= \alpha\beta(T)(x_1, \ldots, x_{i-1}, x'', x_{i+1}, \ldots, x_n)\\ &= S(x_1, \ldots, x_{i-1}, x'', x_{i+1}, \ldots, x_n)\end{aligned}$$

which belongs to Z, and so T is separately weakly compact.

Thus, a possible counterexample to Question 2 must start out being a separately weakly compact, not weakly compact symmetric multilinear operator defined on a not regular Banach space.

5. An application: multilinear operators on $C(K)$ spaces

In this section we shall show an application of the Aron-Berner extensions to the study of multilinear forms and operators in spaces of continuous functions. First, let us recall some well-known facts from the linear theory.

Let K be a compact Hausdorff space and let Σ be the Borel σ-algebra of K. We denote by $B(K)$ the Banach space of the functions which are uniform limit of (Borel) simple functions, endowed with the supremum norm. $C(K)$ is easily seen to be isometrically isomorphic to a subspace of $B(K)$. According to Riesz representation theorem, the space of linear forms on $C(K)$ is isometrically isomorphic to the space $M(K)$ of all (Borel) regular measures on K, endowed with the variation norm, via the pairing

$$\langle \mu, f \rangle = \int_K f d\mu \quad (f \in C(K), \mu \in M(K)).$$

It follows easily that $B(K)$ is isometrically isomorphic to a subspace of $C(K)''$. Let now Z be a Banach space and consider an operator $T : C(K) \to Z$, it can be proved (see [20]) that there exists one only (vector) measure $\mu : \Sigma \to Z''$ such that

(a) μ has bounded semivariation and $\|T\| = \|\mu\|$.
(b) μ is weakly* regular, that is, for every $z' \in Z'$ the scalar measure $z' \circ \mu$ is regular.
(c) The application $z' \in Z' \mapsto z' \circ \mu \in (C(K))'$ is weak* to weak* continuous.
(d) For every $f \in C(K)$, one has $T(f) = \int_K f d\mu$.

The easiest way to obtain such measure is to consider the bitranspose $T'' : C(K)'' \to Z''$ and to define μ on the Borel sets of K by

$$\mu(A) = T''(\chi_A)$$

where χ_A is the characteristic function of A.

The representation of linear operators *via* measures has been fruitfully used to, among other things, characterize classes of operators with conditions on their representing measure, and to prove the coincidence of different ideals of operators on $C(K)$ spaces. Thus, one has the following

Theorem 13. *Let $T : C(K) \to Z$ be a linear operator and $\mu : \Sigma \to Z''$ be its representing measure. Then the following assertions are equivalent:*

(a) *T is weakly compact.*
(b) *T'' is Z-valued.*
(c) *T is completely continuous.*
(d) *T is unconditionally converging.*
(e) *μ is regular.*
(f) *μ is countably additive.*
(g) *μ is Z-valued.*

Let us see now what can be done in the multilinear case. We first need some definitions.

Definition 4 (Dobrakov [23])**.** *A function $\gamma : \Sigma_1 \times \cdots \times \Sigma_n \to Z$ is a (countably additive) polymeasure if it is separately (countably) additive. We say that γ is regular if it is separately regular.*

Definition 5 (Dobrakov [23])**.** *Given a polymeasure $\gamma : \Sigma_1 \times \cdots \times \Sigma_n \to Z$, its semivariation $\|\gamma\| : \Sigma_1 \times \cdots \times \Sigma_n \to [0, +\infty]$ is given by*

$$\|\gamma\|(A_1, \ldots, A_n) = \sup \left\{ \left\| \sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} a_1^{j_1} \ldots a_n^{j_n} \gamma(A_1^{j_1}, \ldots, A_n^{j_n}) \right\| \right\}$$

where the supremum is taken over all finite Σ_i-partitions $(A_i^{j_i})_{j_i=1}^{k_i}$ of A_i $(1 \le i \le n)$, and all numbers $a_i^{j_i}$ in the unit ball of the scalar field.

If we denote by $S(K_i)$ the normed space of (Borel) simple functions on K_i with the supremum norm and $s_i = \sum_{j_i=1}^{k_i} a_{i,j_i} \chi_{A_{i,j_i}} \in S(K_i)$, for every

Z-valued polymeasure γ the formula

$$T_\gamma(s_1, \dots, s_n) = \sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} a_{1,j_1}, \dots a_{n,j_n} \gamma(A_{1,j_1}, \dots A_{n,j_n})$$

defines a multilinear map $T_\gamma : S(K_1) \times \cdots \times S(K_n) \to Z$ such that $\|T_\gamma\| = \|\gamma\|(K_1, \dots, K_n) \leq \infty$. From now, $\|\gamma\|$ will denote the total semivariation of the polymeasure γ, that is, $\|\gamma\| = \|\gamma\|(K_1, \dots, K_n)$.

So, if γ has finite semivariation (that is, $\|\gamma\| < \infty$), the map T_γ can be uniquely extended (with the same norm) to $B(K_1) \times \cdots \times B(K_n)$. We will still denote this extension by T_γ and we shall write also

$$T_\gamma(g_1, \dots, g_n) = \int (g_1, \dots, g_n) d\gamma.$$

It is easily seen that the correspondence $\gamma \mapsto T_\gamma$ is an isometric isomorphism between the space of all Z-valued polymeasures of finite semivariation on $\Sigma_1 \times \cdots \times \Sigma_n$ and $\mathcal{L}^n(B(K_1) \dots B(K_n); Z)$ (see [24] and the references there included for a quite exhaustive study of integrals respect to polymeasures).

Let now $T : C(K_1) \times \cdots \times C(K_n) \to \mathbb{K}$ be a continuous multilinear form, and let

$$\alpha\beta(T) : C(K_1)'' \times \cdots \times C(K_n)'' \longrightarrow \mathbb{K}$$

be its extension defined as in Section 2 (note that it follows from Lemma 10 and Corollary 5 that $\alpha\beta(T)$ is uniquely defined).

Then we can define the set function $\gamma : \Sigma_1 \times \cdots \times \Sigma_n \to \mathbb{K}$ by

$$\gamma(A_1, \dots, A_n) = \alpha\beta(T)(\chi_{A_1}, \dots, \chi_{A_n}).$$

It is easy to check that γ is a regular polymeasure with bounded semivariation. It is also not difficult to see that, for every $(f_1, \dots, f_n) \in C(K_1) \times \cdots \times C(K_n)$,

$$T(f_1, \dots, f_n) = \int_{K_1 \times \cdots \times K_n} (f_1, \dots, f_n) d\gamma.$$

Looking at the linear model and with just a little of care it can be proved that $(C(K_1) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_n))'$ is isometrically isomorphic to the space of regular scalar polymeasures defined on $\Sigma_1 \times \cdots \times \Sigma_n$ endowed with the semivariation norm (see [25] and [10]).

Let us now consider the case of a multilinear operator $T : C(K_1) \times \cdots \times C(K_n) \to Z$. We can again consider its Aron-Berner extension

$$\alpha\beta(T) : C(K_1)'' \times \cdots \times C(K_n)'' \longrightarrow Z'',$$

which again is unique.

We can use it to define a (vector) polymeasure $\gamma : \Sigma_1 \times \cdots \times \Sigma_n \to Z''$ by

$$\gamma(A_1, \dots, A_k) = \alpha\beta(T)(\chi_{A_1}, \dots, \chi_{A_n}).$$

Then we obtain the following theorem and corollary, whose proof can be seen in [10].

Theorem 14. *Let $T : C(K_1) \times \cdots \times C(K_n) \to Z$ be a multilinear operator and let γ be defined as above. Then γ is a polymeasure of bounded semivariation that satisfies*

(a) $\|T\| = \|\gamma\|$.
(b) $T(f_1, \ldots, f_n) = \int (f_1, \ldots, f_n) d\gamma$ *for all* $f_i \in C(K_i)$
(c) *For every* $z' \in Z'$, $z' \circ \gamma$ *is a regular polymeasure and the map* $z' \mapsto z' \circ \gamma$ *from* Z' *into* $(C(K_1) \hat{\otimes} \cdots \hat{\otimes} C(K_n))'$ *is weak*-to-weak* continuous.*

Conversely, if $\gamma : \Sigma_1 \times \cdots \times \Sigma_n \to Z''$ *is a polymeasure satisfying* (c)*, then it has finite semivariation and formula* (b) *defines a multilinear operator from* $C(K_1) \times \cdots \times C(K_n)$ *into* Z *for which* (a) *holds.*

Therefore the correspondence $T \leftrightarrow \gamma$ *is an isometric isomorphism.*

Corollary 7. *Let* $(g_i^k)_k$ *be sequences in* $B(K_i)$ *converging to* $g_i \in B(K_i)$ *with respect to the* $\sigma(B(K_i), M(K_i))$ *topology for* $1 \leq i \leq n$*. If* $\alpha\beta(T)$ *is* Z*-valued, then*

$$\lim_{k \to \infty} \alpha\beta(T)(g_1^k, \ldots, g_n^k) = \alpha\beta(T)(g_1, \ldots, g_n)$$

in the norm topology.

We present further below a multilinear version of Theorem 13. We still need some technicalities before the main result of this section. The following Lemma can be seen in [9].

Lemma 14. *For a multilinear operator* $T : X_1 \times \cdots \times X_n \to Z$*, the following assertions are equivalent:*

(a) T *is unconditionally converging.*
(b) *Given weakly unconditionally Cauchy series* $\sum_{k=1}^{\infty} x_j^k$ *in* X_j *such that, for some* $1 \leq i \leq n$, $\sum_{k=1}^{\infty} x_i^k$ *weakly null, then*

$$\lim_{m \to \infty} \|T(s_1^m, \ldots, s_n^m)\| = 0 \ , \text{ where } \quad s_j^m = \sum_{k=1}^{m} x_j^k \quad .$$

Lemma 15. *Let* $T : C(K_1) \times \cdots \times C(K_n) \to Z$ *be an unconditionally converging multilinear operator. For every* $1 \leq j \leq n$*, let* $\sum_k f_j^k$ *be a weakly unconditionally Cauchy series in* $C(K_j)$ *and let* $s_j^m = \sum_{k=1}^{m} f_j^k$*. Pick* $1 \leq i \leq n$ *and, for each* $m \geq 1$*, define a regular measure* $\gamma_m : \Sigma_i \to Z$ *by*

$$\gamma_m(A_i) = \alpha\beta(T)(s_1^m, \ldots, s_{i-1}^m, \chi_{A_i}, s_{i+1}^m, \ldots, s_n^m).$$

Then, the measures $\{\gamma_m\}_{m=1}^{\infty}$ *are uniformly countably additive.*

Proof. For each m, let us define $T_m : C(K_i) \to Z$ by

$$T_m(f_i) = T(s_1^m, \ldots, s_{i-1}^m, f_i, s_{i+1}^m, \ldots, s_n^m).$$

It is clear that T_m is unconditionally converging, hence weakly compact ($C(K)$ spaces have property (V)). Therefore its representing measure, which clearly coincides with γ_m, is countably additive. If the measures $\{\gamma_m\}_{m=1}^{\infty}$ are not uniformly countably additive, then there exist $\epsilon > 0$ and a sequence $(A_i^p)_{p\in\mathbb{N}} \subset \Sigma_i$ of open disjoint sets such that, for every $p \in \mathbb{N}$,

$$\sup_m \|\gamma_m(A_i^p)\| > \epsilon \quad .$$

Then there exists a increasing sequence of indexes $(m(l))_{l\in\mathbb{N}}$ with $m(0) = 0$, and sets $A_{p(l)}$ such that $\|\gamma_{m(l)}(A_i^{p(l)})\| > \epsilon$. Since each γ_m is regular we have that for every $l \in \mathbb{N}$ there exists a norm-one function $f_i^{p(l)} \in C(K_i)$ with support in $A_i^{p(l)}$ so that

$$\left\|\int f_i^{p(l)} d\gamma_{m(l)}\right\| = \left\|T\left(s_1^{m(l)}, \dots, s_{i-1}^{m(l)}, f_i^{p(l)}, s_{i+1}^{m(l)}, \dots, s_n^{m(l)}\right)\right\| > \epsilon \quad .$$

Let now

$$y_j^q = \sum_{k=m(q-1)+1}^{m(q)} f_j^k \quad \text{for every } q \geq 1 \text{ and } j \neq i \text{ and}$$

$$y_i^1 = f_i^{p(1)}, \quad y_i^q = f_i^{p(q)} - f_i^{p(q-1)} \quad \text{for every } q \geq 2.$$

All of these series are easily seen to be weakly unconditionally Cauchy. Now, for every $l \in \mathbb{N}$,

$$\left\|T\left(\sum_{k=1}^{l} y_1^k, \dots, \sum_{k=1}^{l} y_n^k\right)\right\|$$
$$= \left\|T\left(\sum_{k=1}^{m(l)} f_1^k, \dots, \sum_{k=1}^{m(l)} f_{i-1}^k, f_i^{p(l)}, \sum_{k=1}^{m(l)} f_{i+1}^k, \dots, \sum_{k=1}^{m(l)} f_n^k\right)\right\| > \epsilon$$

which is contradiction with Lemma 14, since $\sum_q y_i^q$ is weakly null. □

The lemma states that an unconditionally converging multilinear operator is "uniformly unconditionally converging" when we fix in $n-1$ of the variables the partial sums of weakly unconditionally Cauchy series. We mention that the same idea holds true when $T : X_1 \times \cdots \times X_n \to Z$ is an unconditionally converging multilinear operator acting on general Banach spaces ([38]). Similarly, it follows with the essentially the same proof that if $T : X_1 \times \cdots \times X_n \to Z$ is a completely continuous multilinear operator then T is "uniformly completely continuous" when we fix in $n-1$ of the variables the k-th terms of weakly Cauchy sequences [38].

Finally we can prove the main result of the Section. We note that in the proof we use "backwards" a good idea of Ryan ([52])

Theorem 15. *Let $T : C(K_1) \times \cdots \times C(K_n) \to Z$ be a multilinear operator and $\gamma : \Sigma_1 \times \cdots \times \Sigma_n \to Z''$ its representing polymeasure. Then the following assertions are equivalent:*

(a) *T is completely continuous.*

(b) *T is unconditionally converging.*
(c) *$\alpha\beta(T)$ is Z-valued.*
(d) *γ is Z-valued.*
(e) *γ is separately countably additive.*
(f) *γ is separately regular.*

Proof. That (a) implies (b) is always true. Let us see that (b) implies (c) by induction on n. If $n = 1$ the result is well known. Let us suppose it true for $n-1$. If we fix $(f_1, \ldots, f_{n-1}) \in C(K_1) \times \cdots \times C(K_{n-1})$ then the operator $T(f_1, \ldots, f_{n-1}, \cdot) : C(K_n) \to Z$ is unconditionally converging, hence weakly compact. Let us now fix $g \in C(K_n)''$. The operator $T_g : C(K_1) \times \cdots \times C(K_{n-1}) \to Z$ defined by

$$T_g(f_1, \ldots, f_{n-1}) = \alpha\beta(T)(f_1, \ldots, f_{n-1}, g)$$

is indeed Z-valued. Let us see that it is unconditionally converging. For $1 \leq j \leq n-1$, let $\sum_k f_j^k$ be weakly unconditionally Cauchy series in $C(K_j)$ and suppose one of them converges weakly to 0. We can assume without loss of generality that $\sum_{k=1}^{\infty} f_1^k = 0$ in the weak topology. If we fix $f_n \in C(K_n)$ it is easy to see, using Lemma 14, that

$$\lim_{m\to\infty} \left\| T\left(s_1^m, \ldots, s_{n-1}^m, f_n\right) \right\| = 0,$$

where $s_i^m = \sum_{k=1}^m f_i^k$. So, we can define the operator $S : C(K_n) \to c_0(Z)$ by

$$S(f_n) = (T(s_1^m, \ldots, s_{n-1}^m, f_n))_m = (T_m(f_n))_m.$$

Let us now see that S is unconditionally converging. Using Lemma 14 it suffices to check that $S(s_n^m) \to 0$ when $s_n^m = \sum_{k=1}^m f_n^k$ and $\sum_k f_n^k$ is a weakly unconditionally Cauchy series in $C(K_n)$ that converges weakly to 0. We can suppose without loss of generality that $\|s_n^m\| \leq 1$ for every m. Lemma 15 states that the measures $\{\gamma_m\}$ are uniformly countably additive. Let $\lambda : \Sigma_n \to [0, +\infty]$ be a countably additive measure such that the measures $\{\gamma_m\}$ are uniformly λ-continuous. For every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{n\in\mathbb{N}} \|\gamma_n(A)\| < \epsilon/2 \quad \text{when} \quad \lambda(A) < \delta \quad .$$

Since s_n^m converges weakly to 0 (as $m \to \infty$) we have that, for every $t \in K_n$, $s_n^m(t)$ converges to 0. Hence, we can use the Egoroff theorem to produce a compact set $K' \subset K_n$ such that $s_n^m \to 0$ uniformly in K' and so that $\lambda(K_n \setminus K') < \delta$. Let m_0 be such that, for every $m > m_0$,

$$\|s_n^m\|_{K'} \leq \frac{\epsilon}{2\,\|\gamma\|}, \text{ where } \|f\|_{K'} = \sup_{t\in K'} |f(t)| \quad .$$

Then, for every $m > m_0$ and every $p \in \mathbb{N}$, one has

$$\begin{aligned}\left\|T(s_1^p, \ldots, s_{n-1}^p, s_n^m)\right\| &= \left\|\int_{K_n} s_n^m d\gamma_p\right\| \\ &\leq \left\|\int_{K'} s_n^m d\gamma_p\right\| + \left\|\int_{K_n \setminus K'} s_n^m d\gamma_p\right\| \\ &\leq \|s_n^m\|_{K'} \|\gamma_p\| (K') + \|\gamma_p\| (K_n \setminus K') \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Hence,

$$\lim_{m\to\infty} \sup_{p\in\mathbb{N}} \left\|T(s_1^p, \ldots, s_{n-1}^p, s_n^m)\right\| = 0,$$

so, S is unconditionally converging.

Therefore, S is weakly compact. Lemma 13 proves that, for every $g \in C(K_n)''$,

$$\lim_{m\to\infty} \|(T_m)''(g)\| = 0.$$

Since $(T_m)''(g) = \alpha\beta(T)(s_1^m, \ldots, s_{n-1}^m, g) = T_g(s_1^m, \ldots, s_{n-1}^m)$, it follows that T_g is unconditionally converging. Now, the induction hypothesis tells us that $\alpha\beta(T_g)$ is Z-valued. Since this happens for every $g \in C(K_n)''$, it follows that $\alpha\beta(T)$ is Z-valued.

That (c) implies (a) follows immediately from Corollary 7.

The equivalence between (c), (d), (e) and (f) follows from standard measure theory, and it can be seen in [57]. □

References

[1] R. Arens, Operations induced in fucntion classes, *Monatsh. Math.* **55** (1951), 1–19.
[2] R. Arens, The adjoint of a bilinear operation, *Proc. Amer. Math. Soc.* **2** (1951), 839–848.
[3] R. Aron and P. Berner, A Hahn-Banach extension theorem for analytic mappings, *Bull. Soc. Math. France* **106** (1978), 3–24.
[4] R. Aron, B. Cole, and T. Gamelin, Spectra of algebras of analytic functions on a Banach space, *J. Reine Angew. Math.* **415** (1991), 51–93.
[5] R. Aron and P. Galindo, Weakly compact multilinear mappings, *Proc. of the Edinburgh Math. Soc.* **40** (1997), 181–192.
[6] R. Aron, C. Hervés, and M. Valdivia, Weakly continuous mappings on Banach spaces, *J. Funct. Anal.* **52** (1983), 189–204.
[7] F. Blasco, Complementation in spaces of symmetric tensor products and polynomials, *Studia Math.* **123** (1997), 165-175.
[8] C. Boy, S. Dineen, and P. Mellon, Dublin's list problems (1994).
[9] F. Bombal, M. Fernández, and I. Villanueva, Unconditionally converging multilinear operators, to appear in *Math. Nach.*
[10] F. Bombal and I. Villanueva, Multilinear operators on spaces of continuous functions, *Funct. Approx. Comment. Math.* **XXVI** (1998), 117–126.
[11] F. Cabello Sánchez, On Ring Homomorphisms, *Expo. Math.* **17** (1999), 91–96.
[12] F. Cabello Sánchez, J. M. F. Castillo, and R. García, Polynomials on dual-isomorphic spaces, *Ark. Mat.* **38** (2000), 37–44 .

[13] F. Cabello Sánchez, J.M.F. Castillo, and D. Yost, Sobczyk's theorem from A to B, *Extracta Math.* (this issue).

[14] F. Cabello Sánchez, R. García, and I. Villanueva, On a question of Pełczyński about multilinear operators, to appear in *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.*

[15] D. Carando, Extendible polynomials on Banach spaces, *J. Math. Anal. Appl.* **233** (1999), 359–372.

[16] P. Civin and B. Yood, The second conjugate space of a Banach algebra as an algebra, *Pacific J. Math.* **11** (1961), 847–870.

[17] A.M. Davie and T.W. Gamelin, A theorem on polynomial-star appoximation, *Proc. Amer. Math. Soc.* **106** (2) (1989), 351–356.

[18] J.C. Díaz and S. Dineen, Polynomials on stable spaces, *Ark. Mat.* **36** (1998), 87–96.

[19] J. Diestel, Sequences and Series in Banach Spaces, *Graduate Texts in Math.* **92**, Springer, Berlin 1984.

[20] J. Diestel and J. J. Uhl, Vector Measures, *Mathematical Surveys* **15**, *American Mathematical Society, Providence, R.I.*, 1977.

[21] S. Dineen, Complex Analysis in Locally Convex Spaces, *Math. Studies* ***57**, North-Holland,* Amsterdam 1981.

[22] S. Dineen, Complex analysis on infinite dimensional spaces, *Springer Monographs in Math.* London, Berlin, Heidelberg, 1999.

[23] I. Dobrakov, On integration in Banach spaces, VIII (polymeasures), *Czech. Math. J.* **37 (112)** (1987), 487–506.

[24] I. Dobrakov, Representation of Multilinear Operators on $C_0(T_i, X_i)$, I, *Atti Sem. Mat. Fis. Univ. Modena* **XXXIX** (1991), 131–138.

[25] I. Dobrakov, Representation of multilinear operators on $C_0(T_i)$, *Czech. Math. J.* **39 (114)** (1989), 288–302.

[26] J. Duncan and S.A.R. Hosseiniun, The second dual of a Banach algebra, *Proc. Royal Soc. Edinburgh* **84A** (1979) 309–325.

[27] M. Fernández Unzueta, Unconditionally convergent polynomials in Banach spaces and related properties, *Extracta Math.* **12** (1997), 305–307.

[28] P. Galindo, D. García, M. Maestre, and J. Mujica, Extension of multilinear mappings on Banach spaces, *Studia Math.* **108** (1984), 55–77.

[29] G. Godefroy and B. Iochum, Arens regularity of Banach algebras and the geometry of Banach spaces, *J. Funct. Anal.* **80** (1988), 47–59.

[30] M. González and J. M. Gutiérrez, When every polynomial is unconditionally converging, *Arch. Math.* **63** (1994), 145–151.

[31] M. González and J. M. Gutiérrez, Unconditionally converging polynomials on Banach spaces, *Math. Proc. Cambridge Philos. Soc.* **117** (1995), 321–331.

[32] M. González and J. M. Gutiérrez, Injective factorization of holomorphic mappings, *Proc. Amer. Math. Soc.* **127** (1999), 1715–1721.

[33] M. González and J. M. Gutiérrez, Polynomials Grothendiek Properties, *Glasgow Math. J.* **37** (1995), 211–219.

[34] F. Gourdeau, Amenability and the second dual of a Banach algebra, *Studia Math.* **125** (1997), 75–81.

[35] A. Grothendieck, Critéres de compacité dans les espaces fonctionels généraux, *Amer. J. Math.* **74** (1952), 168–186.

[36] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, *Mem. Amer. Math. Soc.* **16** (1955).

[37] A. Grothendieck, Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$, *Canad. J. Math.* **5** (1953), 129–173.

[38] J. Gutiérrez and I. Villanueva, Extensions of multilinear operators and Banach space properties, Preprint.

[39] A. Ya. Helemskii, Banach and locally convex algebras, Oxford Science Publications, 1993.

[40] J. Heinrich, Ultraproducts in Banach space theory, *J. Reine Angew. Math.* **313** (1980), 72–104.

[41] B.E. Johnson, Cohomology of Banach algebras, Memoirs AMS **127**, 1972.

[42] N.J. Kalton, Locally complemented subspaces and $\mathcal{L}_p$-spaces for $0 < p < 1$, *Math. Nachr.* **115** (1984), 71–97.

[43] N. J. Kalton and N. T. Peck, Twisted sums of sequence spaces and the three space problem, *Proc. Trans. Amer. Math. Soc.* **255** (1979), 1-30.

[44] J. L. Kelley and I. Namioka, Linear topological spaces, *Van Nostrand*, Princeton, N.J., 1962.

[45] S. Lasalle and I. Zalduendo, To what extent does the dual Banach space E' determine the polynomials over?, *Preprint*, 1998.

[46] M. Lindström and R. A. Ryan, Applications of Ultraproducts to infinite dimensional holomorphy, *Math. Scand.* **71** (**2**) (1992), 229–242.

[47] D. H. Leung**,** Some remarks on regular Banach spaces, *Glasgow Math. J.* **38** (1996), 243–248.

[48] J. Lindenstrauss and H.P. Rosenthal, The $\mathcal{L}_p$-spaces, *Israel J. Math.* **7** (1969), 325–349.

[49] J. Mujica, Complex Analysis in Banach Spaces, *Math. Studies* **120**, North-Holland, Amsterdam 1986.

[50] O. Nicodemi, Homomorphisms of algebras of germs of holomorphic funtions, Functional Analysis, Holomorphy and Approximation theory, Proceedings, Rio de Janeiro, 1978 (S. Machao, Ed.). Lecture Notes in Math. Springer 843 (1981), 534–546.

[51] A. Pełczyński, Banach spaces on which every unconditionally converging operator is weakly compact, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* **10** (1962), 641–648.

[52] R. A. Ryan, Dunford-Pettis properties, *Bull. Acad. Polon. Sci. Sér. Sci. Math.* **27** (1979), 373–379.

[53] S. Sherman, The second dual of a C*-algebra, Proc. Int. Cong. Math. Cambridge, Vol 1 (1950), 470.

[54] B. Sims, "Ultra"-techniques in Banach space theory, *Queen's papers in Pure Appl. Math.* **60**, Queen's University, Kingston, Ontario, Canada 1982.

[55] Z. Takeda, Conjugate spaces of operator algebras, *Proc. Japan Acad.* **30** (1954), 90–95.

[56] A. Ülger, Weakly compact bilinear forms and Arens regularity, *Proc. Amer. Math. Soc.* **101** (1987), 697–704.

[57] I. Villanueva, Completely continuous multilinear operators on $C(K)$ spaces, to appear in *Proc. Amer. Math. Soc.*

[58] I. Zalduendo, A canonical extension for analytic functions on Banach spaces, *Trans. Amer. Math. Soc.* **320** (**2**) (1990), 747–763.

[59] I. Zalduendo**,** Extending polynomials- a survey. *Publ. Dep. Análisis Mat.*, Universidad Complutense de Madrid, (1998).

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