

INTEGRAL OPERATORS ON THE PRODUCT OF $C(K)$ SPACES

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ABSTRACT. We study and characterize the integral multilinear operators on a product of $C(K)$ spaces in terms of the representing polymeasure of the operator. Some applications are given. In particular, we characterize the Borel polymeasures that can be extended to a measure in the product σ -algebra, generalizing previous results for bimeasures. We also give necessary conditions for the weak compactness of the extension of an integral multilinear operator on a product of $C(K)$ spaces.

1. INTRODUCTION

The modern theory of Banach spaces is greatly indebted to the work of A. Grothendieck. In his papers [11] and [12] he introduced the most important classes of *operator ideals*, whose study and characterization in different concrete classes of Banach spaces has been a permanent subject of interest since then. One of the classes defined in [11] and now intensively studied, is the class of *integral operators* (see below), whose definition establishes a first connection between the linear and the multilinear (bilinear, in fact) theory.

Grothendieck himself started the study of several classes of operators on $C(K)$ spaces in [12]. As a consequence of the Riesz representation Theorem, every continuous linear map T from $C(K)$ into another Banach space X has a *representing measure*, i.e., a finitely additive measure m of bounded semi variation defined on the Borel σ -field of K , with values in X^{**} (the bidual of X), in such a way that

$$T(f) := \int f dm, \text{ for each } f \in C(K).$$

(see, e.g. [5] or [6]). The study of the relationships between T and its representing measure plays a central role in this research.

When T is a continuous k -linear map from a product $C(K_1) \times \cdots \times C(K_k)$ (where K_i are compact Hausdorff spaces) into a Banach space X , there exists also an integral representation theorem with respect to the representing *polymeasure* of T (see below for the definitions). If $k = 1$, the integral operators (G-integral in our notation; see Definition 2.3 below) are precisely those whose representing measure has bounded variation (see e.g. [16, p. 477] and [5, Th. VI.3.3]). The aim of this paper is to study and characterize the multilinear vector valued integral operators on a product of $C(K)$ spaces in terms of the corresponding representing polymeasure. As an application we obtain an intrinsic characterization of the Borel polymeasures than can be extended to measures in the product Borel σ -algebra, extending some previous results for the case of bimeasures. We also study the relationship between the weak compactness of an integral multilinear map on a

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product $C(K_1) \times \cdots \times C(K_k)$ and that of its linear extension to $C(K_1 \times \cdots \times K_k)$. Some other applications are given.

2. DEFINITIONS AND PRELIMINARIES

The notation and terminology used throughout the paper will be the standard in Banach space theory, as for instance in [5]. However, before going any further, we shall establish some terminology: $\mathcal{L}^k(E_1 \dots, E_k; X)$ will be the Banach space of all the continuous k -linear mappings from $E_1 \times \cdots \times E_k$ into X and $\mathcal{L}_{wc}^k(E_1 \dots, E_k; X)$ will be the closed subspace of it formed by the weakly compact multilinear operators. When $X = \mathbb{K}$ or $k = 1$, we will omit them. If $T \in \mathcal{L}^k(E_1 \dots, E_k; X)$ we shall denote by $\hat{T} : E_1 \otimes \cdots \otimes E_k \rightarrow X$ its linearization. As usual, $E_1 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon E_k$ will stand for the (complete) injective tensor product of the Banach spaces E_1, \dots, E_k .

We shall use the convention $.[i]$ to mean that the i -th coordinate is not involved.

If $T \in \mathcal{L}^k(E_1, \dots, E_k; X)$ we denote by T_i ($1 \leq i \leq k$) the operator $T_i \in \mathcal{L}(E_i; \mathcal{L}^{k-1}(E_1, \dots, E_k, X))$ defined by

$$T_i(x_i)(x_1, \dots, x_k) := T(x_1, \dots, x_k),$$

Let now Σ_i ($1 \leq i \leq k$) be σ -algebras (or simply algebras) of subsets on some non void sets Ω_i . A function $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \rightarrow X$ or $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \rightarrow [0, +\infty]$ is a (countably additive) k -polymeasure if it is separately (countably) additive. ([8, Definition 1]). A countably additive polymeasure γ is *uniform in the i th variable* if the measures $\{\gamma(A_1, \dots, A_{i-1}, \cdot, A_{i+1}, \dots, A_k) : A_j \in \Sigma_j (j \neq i)\}$ are uniformly countably additive. As in the case $k = 1$ we can define the *variation* of a polymeasure $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \rightarrow X$, as the set function

$$v(\gamma) : \Sigma_1 \times \cdots \times \Sigma_k \rightarrow [0, +\infty]$$

given by

$$v(\gamma)(A_1, \dots, A_k) = \sup \left\{ \sum_{j_1=1}^{n_1} \cdots \sum_{j_k=1}^{n_k} \left\| \gamma(A_1^{j_1}, \dots, A_k^{j_k}) \right\| \right\}$$

where the supremum is taken over all the finite Σ_i -partitions $(A_i^{j_i})_{j_i=1}^{n_i}$ of A_i ($1 \leq i \leq k$). We will call $bvpm(\Sigma_1, \dots, \Sigma_k; X)$ the Banach space of the polymeasures with bounded variation defined on $\Sigma_1 \times \cdots \times \Sigma_k$ with values in X , endowed with the variation norm.

We can define also its *semivariation*

$$\|\gamma\| : \Sigma_1 \times \cdots \times \Sigma_k \rightarrow [0, +\infty]$$

by

$$\|\gamma\|(A_1, \dots, A_k) = \sup \left\{ \left\| \sum_{j_1=1}^{n_1} \cdots \sum_{j_k=1}^{n_k} a_1^{j_1} \cdots a_k^{j_k} \gamma(A_1^{j_1}, \dots, A_k^{j_k}) \right\| \right\}$$

where the supremum is taken over all the finite Σ_i -partitions $(A_i^{j_i})_{j_i=1}^{n_i}$ of A_i ($1 \leq i \leq k$), and all the collections $(a_i^{j_i})_{j_i=1}^{n_i}$ contained in the unit ball of the scalar field. We will call $bpm(\Sigma_1, \dots, \Sigma_k; X)$ the Banach space of the polymeasures with bounded semivariation defined on $\Sigma_1 \times \cdots \times \Sigma_k$ with values in X , endowed with the semivariation norm.

If γ has finite semivariation, an elementary integral $\int (f_1, f_2, \dots, f_k) d\gamma$ can be defined, where f_i are bounded, Σ_i -measurable scalar functions, just taking the limit of the integrals of k -uples of simple functions (with the obvious definition) uniformly converging to the f_i 's (see [8]).

If K_1, \dots, K_k are compact Hausdorff spaces, then every multilinear operator $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k); X)$ has a unique *representing polymeasure* $\gamma : \text{Bo}(K_1) \times \dots \times \text{Bo}(K_k) \rightarrow X^{**}$ (where $\text{Bo}(K)$ denotes the Borel σ -algebra of K) with finite semivariation, in such a way that

$$T(f_1, \dots, f_k) = \int (f_1, \dots, f_k) d\gamma \quad \text{for } f_i \in C(K_i),$$

and such that for every $x^* \in X^*$, $x^* \circ \gamma \in \text{rcapm}(\text{Bo}(K_1), \dots, \text{Bo}(K_k))$, the set of all regular, countably additive scalar polymeasures on $\text{Bo}(K_1) \times \dots \times \text{Bo}(K_k)$. (Cfr. [2]).

Given a polymeasure γ we can consider the set function γ_m defined on the semi-ring of all measurable rectangles $A_1 \times \dots \times A_k$ ($A_i \in \Sigma_i$) by

$$\gamma_m(A_1 \times \dots \times A_k) := \gamma(A_1, \dots, A_k)$$

It follows, f. i. from [7, Prop. 1.2] that γ_m is finitely additive and then it can be uniquely extended to a finitely additive measure on the algebra $a(\Sigma_1 \times \dots \times \Sigma_k)$ generated by the measurable rectangles. In general, this finitely additive measure cannot be extended to the σ -algebra $\Sigma_1 \otimes \dots \otimes \Sigma_k$ generated by $\Sigma_1 \times \dots \times \Sigma_k$. But if there is a countably additive measure μ of bounded variation on $\Sigma_1 \otimes \dots \otimes \Sigma_k$ that extends γ_m , then by standard measure theory (see e.g. [6, Th. I.5.3] and [7]) we have

$$v(\gamma_m)(A_1 \times \dots \times A_k) = v(\gamma)(A_1, \dots, A_k) = v(\mu)(A_1 \times \dots \times A_k), \quad \text{for } A_i \in \Sigma_i. \quad (*)$$

The next definition extends Grothendieck's notion of multilinear integral forms to the multilinear integral operators:

Definition 2.1. A multilinear operator $T \in \mathcal{L}^k(E_1, \dots, E_k; X)$ is *integral* if \hat{T} (i.e., its linearization) is continuous for the injective (ϵ) topology on $E_1 \otimes \dots \otimes E_k$. Its norm (as an element of $\mathcal{L}(E_1 \hat{\otimes}_\epsilon \dots \hat{\otimes}_\epsilon E_k; X)$) is the *integral norm* of T , $\|T\|_{int} := \|\hat{T}\|_\epsilon$.

Proposition 2.2. $T \in \mathcal{L}^k(E_1, \dots, E_k; X)$ is integral if and only if $x^* \circ T$ is integral for every $x^* \in X^*$.

Proof. For the non-trivial part, let us consider the map $X^* \ni x^* \mapsto x^* \circ \hat{T} \in (E_1 \hat{\otimes}_\epsilon \dots \hat{\otimes}_\epsilon E_k)^*$, well defined by hypothesis. A simple application of the closed graph theorem proves that this linear map is continuous. Hence,

$$\sup_{\|x^*\| \leq 1} \|x^* \circ \hat{T}\|_\epsilon = M < \infty.$$

But it is easy to see that $\|T\|_{int} := \sup_{\|u\|_\epsilon \leq 1} \|\hat{T}(u)\| = M$. □

The next definition is well known.

Definition 2.3. An operator $S \in \mathcal{L}(E; X)$ is *G-integral* (the "G" comes from "Grothendieck") if the associated bilinear form

$$\begin{aligned} B_S : E \times X^* &\longrightarrow \mathbb{K} \\ (x, y) &\longmapsto y(T(x)) \end{aligned}$$

is integral. In that case the integral norm of S , $\|S\|_{int} := \|B_S\|_{int}$.

Let us recall that a bilinear form $T \in \mathcal{L}^2(E_1, E_2)$ is integral if and only if any of the two associated linear operators $T_1 \in \mathcal{L}(E_1; E_2^*)$ and $T_2 \in \mathcal{L}(E_2; E_1^*)$ is G-integral in the above sense (cfr., e.g. [5, Ch- VI]).

Proposition 2.4. *Let $k \geq 2$, E_1, \dots, E_k be Banach spaces and $T \in \mathcal{L}^k(E_1, \dots, E_k)$. Then T is integral if and only if there exists i , $1 \leq i \leq k$, such that*

- a) *For every $x_i \in E_i$, $T_i(x_i)$ is integral.*
- b) *The mapping*

$$\tilde{T}_i : E_i \rightarrow (E_1 \otimes_\epsilon \cdots \otimes_\epsilon E_k)^*$$

defined by

$$\tilde{T}_i(x_i) := \widehat{T_i(x_i)}$$

is a G-integral operator.

If (a) and (b) are satisfied for some i , then the same happens for any other index j , $1 \leq j \leq k$. Moreover, in this case, $\|T\|_{int} = \|\tilde{T}_i\|_{int}$.

Proof. If (a) and (b) are satisfied and we put $F_i := E_1 \otimes_\epsilon \cdots \otimes_\epsilon E_k$, the bilinear map $B_{T_i} : E_i \times F_i \rightarrow \mathbb{K}$ is integral and $\|B_{T_i}\|_{int} = \|\tilde{T}_i\|$ ([5, Corollary VIII.2.12]). From the associativity, the commutativity of the ϵ -tensor product and the definitions, it follows that T is integral and the norms are equal.

Conversely, suppose that T is integral. We shall prove that (a) and (b) hold for $i = 1$: From the hypothesis and the associativity of the injective tensor product, it follows that the bilinear map

$$\begin{aligned} B_T : E_1 \times (E_2 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon E_k) &\longrightarrow \mathbb{K} \\ (x, u) &\longmapsto \hat{T}(x \otimes u) \end{aligned}$$

is integral. By [5, Corollary VIII.2.12], the associated linear operator from E_1 into $(E_2 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon E_k)^*$ is G-integral. Clearly, this operator coincides with \tilde{T}_1 , and this proves (a) and (b) for $i = 1$. \square

3. INTEGRAL FORMS ON $C(K)$ SPACES

Let now K, K_1, \dots, K_k be compact Hausdorff spaces.

Recall that, for every Banach space X , $C(K, X)$, the Banach space of all the X -valued continuous functions on K endowed with the sup norm, is canonically isometric to $C(K) \hat{\otimes}_\epsilon X$ ([5, Example VIII.1.6]). Moreover, if $X = C(S)$ (S a compact Hausdorff space) then $C(K, C(S))$ is canonically isometric to $C(K \times S)$. Thus, we have the following identifications

$$C(K_1) \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon C(K_k) \approx C(K_i, C(K_1) \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon C(K_k)) \approx C(K_1 \times \cdots \times K_k).$$

Suppose that $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k))$ with representing polymasure γ . If there exists a regular measure μ on the Borel σ -algebra of $K_1 \times \cdots \times K_k$ that extends γ_m , then, by the Riesz representation theorem, μ is the representing measure of some continuous linear form \hat{T} on $C(K_1 \times \cdots \times K_k) \approx C(K_1) \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon C(K_k)$ and clearly

$$\begin{aligned} (1) \quad \hat{T}(f_1 \otimes \cdots \otimes f_k) &= \int_{K_1 \times \cdots \times K_k} f_1 \cdots f_k d\mu = \\ &= T(f_1, \dots, f_k) = \int_{K_1 \times \cdots \times K_k} (f_1, \dots, f_k) d\gamma. \end{aligned}$$

Consequently T is integral. Note also that, as follows from the introduction, $\|\hat{T}\| = v(\mu) = v(\gamma)$.

Conversely, if T is such that its linearization \hat{T} on $C(K_1) \otimes \cdots \otimes C(K_k)$ is continuous for the ϵ -topology (i.e., T is integral), another application of the Riesz representation theorem yields a measure μ on $Bo(K_1 \times \cdots \times K_k)$ such that (1) holds. By the uniqueness of the representation theorem for k -linear maps, we have

$$\mu(A_1 \times \cdots \times A_k) = \gamma(A_1, \dots, A_k) \quad (\text{for every } A_i \in \Sigma_i)$$

and so μ extends γ . Summarizing, we have proved

Proposition 3.1. *Let $k \geq 2$ and $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k))$ with representing polymeasure γ . Then T is integral if and only if γ can be extended to a regular measure μ on $Bo(K_1 \times \cdots \times K_k)$ in such a way that*

$$\mu(A_1 \times \cdots \times A_k) = \gamma(A_1, \dots, A_k) \quad \text{for every } A_i \in \Sigma_i, (1 \leq i \leq k).$$

In this case, $\|\hat{T}\| = \|T\|_{int} = v(\gamma) = v(\mu)$.

Consequently, $(C(K_1) \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon C(K_k))^*$ can be isometrically identified with a subspace of the space of all regular polymeasures on $Bo(K_1) \times \cdots \times Bo(K_k)$ with finite variation, endowed with the variation norm.

Now we are going to obtain an intrinsic characterization of the extendible Radon polymeasures which will allow us to see that the previous isometry is onto.

If $\Sigma_1, \dots, \Sigma_k$ are σ -algebras, X is a Banach space and $\gamma \in bpm(\Sigma_1, \dots, \Sigma_k; X)$, then we can define a measure

$$\varphi_1 : \Sigma_1 \longrightarrow bpm(\Sigma_2, \dots, \Sigma_k; X)$$

by

$$\varphi_1(A_1)(A_2, \dots, A_k) = \gamma(A_1, A_2, \dots, A_k).$$

It is known that $\|\varphi_1\| = \|\gamma\|$ (see [3]). Related to this we have the following lemma, whose easy proof we include for completeness:

Lemma 3.2. *Let X be a Banach space, $\Omega_1, \dots, \Omega_k$ sets and $\Sigma_1, \dots, \Sigma_k$ σ -algebras defined on them. Let now $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \longrightarrow X$ be a polymeasure. Then $v(\gamma) < \infty$ if and only if φ_1 takes values in $bvpm(\Sigma_2, \dots, \Sigma_k; X)$ and $v(\varphi_1) < \infty$ when we consider the variation norm in the image space. In that case, $v(\varphi_1)(A_1) = v(\gamma)(A_1, \Omega_2, \dots, \Omega_k)$ and $v(\varphi_1(A_1))(A_2, \dots, A_k) \leq v(\gamma)(A_1, A_2, \dots, A_k)$. Of course the role played by the first variable could be played by any of the other variables.*

Proof. Let us first assume that $v(\gamma) < \infty$. In the following we will adopt the convention that \sup_{j_2, \dots, j_k} means the supremum over all the finite Σ_i -partitions $(A_i^{j_i})_{j_i=1}^{n_i}$ of A_i ($2 \leq i \leq k$). Then, with this notation,

$$\begin{aligned} v(\varphi_1(A_1)) &= v(\varphi_1(A_1))(\Omega_2, \dots, \Omega_k) = \sup_{j_2, \dots, j_k} \sum_{j_2} \cdots \sum_{j_k} \|\varphi_1(A_1)(A_2^{j_2}, \dots, A_k^{j_k})\| = \\ (2) \quad &= \sup_{j_2, \dots, j_k} \sum_{j_2} \cdots \sum_{j_k} \|\gamma(A_1, A_2^{j_2}, \dots, A_k^{j_k})\| \leq v(\gamma)(A_1, \Omega_2, \dots, \Omega_k) \leq \\ &\leq v(\gamma)(\Omega_1, \Omega_2, \dots, \Omega_k) = v(\gamma). \end{aligned}$$

Therefore, φ_1 is $bvpm(\Sigma_2, \dots, \Sigma_k; X)$ -valued. Let us now see that it has bounded variation when we consider the variation norm in the image space:

$$\begin{aligned}
(3) \quad v(\varphi_1) &= v(\varphi_1)(\Omega_1) = \sup_{j_1} \sum_{j_1} \|\varphi_1(A_1^{j_1})\| = \sup_{j_1} \sum_{j_1} v(\varphi_1(A_1^{j_1})) \leq \\
&\leq \sup_{j_1} \sum_{j_1} v(\gamma)(A_1^{j_1}, \Omega_2, \dots, \Omega_k) \leq v(\gamma)(\Omega_1, \Omega_2, \dots, \Omega_k) = v(\gamma) < \infty.
\end{aligned}$$

In the next to last inequality we have used that the variation of a polymasure is itself separately countably additive ([8, Theorem 3]).

Conversely, if φ_1 is $bvpm(\Sigma_2, \dots, \Sigma_k; X)$ -valued and with bounded variation when we consider the variation norm in the image space, then

$$\begin{aligned}
(4) \quad v(\gamma) &= v(\gamma)(\Omega_1, \dots, \Omega_k) = \sup_{j_1, j_2, \dots, j_k} \sum_{j_1} \sum_{j_2} \cdots \sum_{j_k} \|\gamma(A_1^{j_1}, A_2^{j_2}, \dots, A_k^{j_k})\| \leq \\
&\leq \sup_{j_1} \sum_{j_1} \sup_{j_2, \dots, j_k} \sum_{j_2} \cdots \sum_{j_k} \|\varphi_1(A_1^{j_1})(A_2^{j_2}, \dots, A_k^{j_k})\| = \sup_{j_1} \sum_{j_1} \|\varphi_1(A_1^{j_1})\| = \\
&= v(\varphi_1)(\Omega_1) = v(\varphi_1) < \infty.
\end{aligned}$$

Putting together both inequalities we get that

$$v(\gamma) = v(\varphi_1).$$

To prove the first of the last two statements of the lemma we replace Ω_1 by A_1 in (3), (4). To prove the last statement we replace $(\Omega_2, \dots, \Omega_k)$ by (A_2, \dots, A_k) in (2). \square

Let $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k))$ with representing polymasure γ . Let us consider $T_1 : C(K_1) \longrightarrow \mathcal{L}^{k-1}(C(K_2), \dots, C(K_k))$ and let

$$\varphi_1 : Bo(K_1) \longrightarrow rcapm(Bo(K_2), \dots, Bo(K_k))$$

be defined as above.

It is known that φ_1 is countably additive if and only if γ is uniform ([3, Lemma 2.2]) and in this case φ_1 is the representing measure of T_1 ([3, Theorem 2.4]). From the definitions, it is easy to check that every polymasure with finite variation is uniform.

Now we can prove the first of our main results.

Theorem 3.3. *Let $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k))$ with representing polymasure γ . Then the following are equivalent:*

- a) $v(\gamma) < \infty$.
- b) T is integral.
- c) γ can be extended to a regular measure μ on $Bo(K_1 \times \dots \times K_k)$.
- d) γ can be extended to a countably additive (not necessarily regular) measure μ_2 on $Bo(K_1) \otimes \dots \otimes Bo(K_k)$.

Proof. The equivalence between (b) and (c) is just Proposition 3.1. If (c) holds, defining $\mu_2 := \mu|_{Bo(K_1) \otimes \dots \otimes Bo(K_k)}$ proves (d). Since a countably additive scalar measure has bounded variation, from (*) in the previous Section, we get that (d) implies (a). Finally let us prove that (a) implies (b): By Proposition 2.4 we have to show that

$$\text{i) } \widehat{T_1(f_1)} \in (C(K_2) \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon C(K_k))^* := F^*$$

and

$$\text{ii) } \tilde{T}_1 : C(K_1) \longrightarrow F^* \text{ is G-integral.}$$

We shall proceed by induction on k . For $k = 1$ there is nothing to prove.

Let $k = 2$. In this case we only have to prove (ii). By the discussion following Lemma 3.2, the representing measure of T_1 is $\varphi_1 : Bo(K_1) \rightarrow C(K_2)^*$ and $v(\varphi_1) = v(\gamma) < \infty$. Since every dual space is 1-complemented in its bidual, by Corollary VIII.2.10 and Theorems VI.3.3 and VI.3.12 of [5], T_1 is integral and

$$\|T_1\|_{int} = \|\hat{T}\| = v(\varphi) = v(\gamma)$$

by Lemma 3.2.

Let us now suppose the result true for $k - 1$. Let $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k))$.

i) For $f_1 \in C(K_1)$, the representing polymeasure of $T_1(f_1)$ is γ_{f_1} , defined by

$$\gamma_{f_1}(A_2, \dots, A_k) = \int (f_1, \chi_{A_2}, \dots, \chi_{A_k}) d\gamma,$$

as can be easily checked. Since $|\gamma_{f_1}(A_2, \dots, A_k)| \leq \|f_1\|_\infty v(\gamma)(K_1, A_2, \dots, A_k)$, it follows that γ_{f_1} has finite variation and $v(\gamma_{f_1}) \leq v(\gamma)\|f_1\|_\infty$. Hence, by the induction hypothesis, $T_1(f_1)$ is integral.

ii) As before, we have to prove that the representing measure of

$$\tilde{T}_1 : C(K_1) \rightarrow (C(K_2) \hat{\otimes}_\epsilon \dots \hat{\otimes}_\epsilon C(K_k))^*$$

has finite variation. The representing measure of \tilde{T}_1 is $\tilde{\varphi}_1$, where

$$\tilde{\varphi}_1(A_1)(f_2 \otimes \dots \otimes f_k) = \int (\chi_{A_1}, f_2, \dots, f_k) d\gamma$$

and, clearly, $\tilde{\varphi}_1$ is just φ_1 of Lemma 3.2 considering the integral (equivalently variation) norm in the image space. Therefore, Lemma 3.2 proves that $v(\tilde{\varphi}_1) = v(\gamma) < \infty$, and so \tilde{T}_1 is G-integral. \square

REMARK 3.4. The equivalence (a) \Leftrightarrow (d) was proved for bimeasures in [14, Corollary 2.9]. The techniques used in that paper, essentially different from ours, do not seem to extend easily to the case of k -polymeasures when $k \geq 3$.

To the best of our knowledge, it was unknown when a polymeasure could be decomposed as the sum of a positive and a negative polymeasure. It is clear now that, for the polymeasures in $rcapm(Bo(K_1), \dots, Bo(K_k))$, this happens only in the most trivial case, that is, when γ can be extended to a measure, and then decomposed as such.

Corollary 3.5. *Given $\gamma \in rcapm(Bo(K_1), \dots, Bo(K_k))$, γ can be decomposed as the sum of a positive and a negative polymeasure if and only if $v(\gamma) < \infty$.*

Proof. If $v(\gamma) < \infty$, then γ can be extended to μ as in Theorem 3.3. Let us now decompose this measure μ as the sum of a positive and a negative measure $\mu = \mu_p + \mu_n$. Clearly now $\gamma = \mu_p + \mu_n$, considering μ_p and μ_n as polymeasures. Conversely, if $\gamma = \gamma_p + \gamma_n$, where γ_p (resp. γ_n) is a positive (resp. negative) polymeasure, then

$$v(\gamma) = v(\gamma)(K_1, \dots, K_k) \leq \gamma_p(K_1, \dots, K_k) - \gamma_n(K_1, \dots, K_k) < \infty.$$

\square

4. VECTOR-VALUED INTEGRAL MAPS ON $C(K)$ SPACES

We will use now the results of the preceding section to characterize the vector valued integral operators. First we will need a new definition: Let $\Omega_1, \dots, \Omega_k$ be non-empty sets and $\Sigma_1, \dots, \Sigma_k$ be σ -algebras defined on them. If $\gamma : \Sigma_1 \times \dots \times \Sigma_k \longrightarrow X$ is a Banach space valued polymeasure, we can define its *quasivariation*

$$\|\gamma\|^+ : \Sigma_1 \times \dots \times \Sigma_k \longrightarrow [0, +\infty]$$

by

$$\|\gamma\|^+(A_1, \dots, A_k) = \sup \left\{ \left\| \sum_{j_1=1}^{n_1} \dots \sum_{j_k=1}^{n_k} a_{j_1, \dots, j_k} \gamma(A_{1, j_1}, \dots, A_{k, j_k}) \right\| \right\}$$

where $(A_{i, j_i})_{j_i=1}^{n_i}$ is a Σ_i -partition of A_i ($1 \leq i \leq k$) and $|a_{j_1, \dots, j_k}| \leq 1$ for all (j_1, \dots, j_k) .

It is not difficult to see that the quasivariation is separately monotone and sub-additive and that, for every $(A_1, \dots, A_k) \in \Sigma_1 \times \dots \times \Sigma_k$,

$$\|\gamma\|(A_1, \dots, A_k) \leq \|\gamma\|^+(A_1, \dots, A_k) \leq v(\gamma)(A_1, \dots, A_k).$$

It can also be checked that $\|\gamma\|^+ = \sup\{v(x^* \circ \gamma); x^* \in B_{X^*}\}$.

We can consider the space of polymeasures $\gamma \in bpm(\Sigma_1, \dots, \Sigma_k; X)$ such that $\|\gamma\|^+ < \infty$. Standard calculations show that $\|\cdot\|^+$ is a Banach space norm in this space. This space has been recently considered in [7], where the authors develop a theory of integration for these polymeasures. We will prove in this section that, for the polymeasures representing multilinear operators on $C(K_1) \times \dots \times C(K_k)$, the ones with finite quasivariation are precisely those which can be extended to a measure on $Bo(K_1 \times \dots \times K_k)$, and thus the previously mentioned integration theory can be dispensed with.

Theorem 4.1. *Let $k \geq 2$, $T : C(K_1) \times \dots \times C(K_k) \longrightarrow X$ be a multilinear operator and let $\gamma : Bo(K_1) \times \dots \times Bo(K_k) \longrightarrow X^{**}$ be its associated polymeasure. Then the following are equivalent:*

- a) $\|\gamma\|^+ < \infty$.
- b) T is integral.
- c) γ can be extended to a bounded ω^* -regular measure $\mu : Bo(K_1 \times \dots \times K_k) \longrightarrow X^{**}$ (in such a way that

$$\mu(A_1 \times \dots \times A_k) = \gamma(A_1, \dots, A_k) \quad \text{for every } A_i \in \Sigma_i, (1 \leq i \leq k) .$$

- d) γ can be extended to a bounded ω^* -countably additive (not necessarily regular) measure $\mu_2 : Bo(K_1) \otimes \dots \otimes Bo(K_k) \longrightarrow X^{**}$.

Proof. Let us first prove that (a) implies (b): If $\|\gamma\|^+ < \infty$, then, for every $x^* \in X^*$, $v(x^* \circ \gamma) < \infty$. Since $x^* \circ \gamma$ is clearly the representing polymeasure of $x^* \circ T$, using Theorem 3.3, we obtain that $x^* \circ T$ is integral for every $x^* \in X^*$ and now we can apply Proposition 2.2 to finish the proof.

Let us now prove that (b) implies (c): Let $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k); X)$ be integral, and call $\bar{T} \in \mathcal{L}(C(K_1 \times \dots \times K_k); X)$ its extension. If

$$\mu : Bo(K_1 \times \dots \times K_k) \longrightarrow X^{**}$$

is the representing measure of \bar{T} , it is clear that μ satisfies (c).

Clearly (c) implies (d). Now, if (d) is true, then $\|\mu_2\| = \sup_{x^* \in B_{X^*}} v(x^* \circ \mu) = \sup_{x^* \in B_{X^*}} v(x^* \circ \gamma) = \|\gamma\|^+$, and (a) holds. \square

Corollary 4.2. *Let $T : C(K_1) \times \cdots \times C(K_k) \longrightarrow X$ be a multilinear operator with representing polymasure γ . Then the following statements are equivalent:*

- a) $v(\gamma) < \infty$.
- b) T is integral and its extension $\bar{T} : C(K_1 \times \cdots \times K_k) \longrightarrow X$ is G -integral.

Proof. If T is integral and μ is the representing measure of its extension \bar{T} then, as we saw in Section 2, $v(\gamma) = v(\mu)$. Hence, the equivalence between (a) and (b) follows from $\|\gamma\|^+ \leq v(\gamma)$ and the fact (already used) that a linear operator on a $C(K)$ -space is G -integral if and only if its representing measure has finite variation. \square

The fact that every integral multilinear map $T : C(K_1) \times \cdots \times C(K_k) \longrightarrow X$ can be extended to a continuous linear map $\bar{T} : C(K_1) \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon C(K_k) \approx C(K_1 \times \cdots \times K_k) \longrightarrow X$ has some immediate consequences:

Proposition 4.3. *Let $T : C(K_1) \times \cdots \times C(K_k) \longrightarrow X$ be an integral multilinear operator and for each i , $1 \leq i \leq k$, let $(f_i^n) \subset C(K_i)$ be bounded sequences.*

a) *If at least one of the sequences (f_i^n) is weakly null and $(x_n^*) \subset X^*$ is a weakly Cauchy sequence, then*

$$\lim_{n \rightarrow \infty} \langle T(f_1^n, \dots, f_k^n), x_n^* \rangle = 0 \quad (\dagger)$$

b) *If all the sequences (f_i^n) are weakly Cauchy and (x_n^*) is a weakly null sequence in X^* , then (\dagger) holds.*

Proof. From the well known characterization of the weak topology in $C(K)$ -spaces, it follows that the sequence $(f_1^n \otimes \cdots \otimes f_k^n) \subset C(K_1) \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon C(K_k) \approx C(K_1 \times \cdots \times K_k)$ is, respectively, weakly null (under (a)) or weakly Cauchy (under (b)), and

$$\langle T(f_1^n, \dots, f_k^n), x_n^* \rangle = \langle f_1^n \otimes \cdots \otimes f_k^n, \bar{T}^*(x_n^*) \rangle$$

The result follows from the Dunford-Pettis property of $C(K_1 \times \cdots \times K_k)$. \square

If $k = 2$, it can be proved that the sequence $(f_1^n \otimes f_2^n)$ is also weakly null or weakly Cauchy, respectively, in the projective tensor product $C(K_1) \hat{\otimes}_\pi C(K_2)$ ([4, Lemma 2.1]). Hence, when X has the Dunford-Pettis Property the above result is true for any continuous bilinear map. Nevertheless, Proposition 4.3 gives a necessary condition for a multilinear map to be integral, and so it provides an easy way to see when a multilinear map is *not* integral.

Example 4.4. Let (r_n) be a bounded, orthonormal sequence (with respect to the usual scalar product) in $C([0, 1])$ and let $T : C([0, 1]) \times C([0, 1]) \longrightarrow \ell_2$ be defined by

$$T(f, g) = \left(\left(\int_0^1 f r_n \right) \left(g \left(\frac{1}{n} \right) \right) \right)_{n=1}^\infty$$

Then T (clearly weakly compact) is not integral. In fact, if g_n denotes the function which is equal to 1 at $\frac{1}{n}$, 0 in $[0, \frac{1}{2n}]$ and $[\frac{3}{2n}, 1]$ and linear elsewhere, the sequence (g_n) converges pointwise to 0 and so is weakly null in $C([0, 1])$. But $T(r_n, g_n) = e_n$, the usual ℓ_2 -basis, and so $\langle T(r_n, g_n), e_n \rangle = 1$ for every n .

Let us state a definition: given Banach spaces E_1, \dots, E_k, X , a multilinear operator $T \in \mathcal{L}^k(E_1, \dots, E_k; X)$ is called *regular* if every one if the linear operators $T_i \in \mathcal{L}(E_i; \mathcal{L}^{k-1}(E_1, \dots, E_k; X))$ associated to it are weakly compact (see [1] and [10] for some properties of these operators). In case $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k); X)$,

it follows from [3] that its representing polymeasure γ is uniform if and only if T is regular.

Proposition 4.5. *Let $T : C(K_1) \times \cdots \times C(K_k) \rightarrow X$ be an integral multilinear operator with representing polymeasure γ , and suppose that its extension $\overline{T} : C(K_1 \times \cdots \times K_k) \rightarrow X$ is weakly compact. Then γ is uniform, and therefore T is regular.*

Proof. Let us prove, for instance, that γ is uniform in the first variable. According to [3, Theorem 2.4], it suffices to prove that the corresponding operator $T_1 : C(K_1) \rightarrow \mathcal{L}^{k-1}(C(K_2), \dots, C(K_k); X)$ is weakly compact or, equivalently, it maps weakly null sequences into norm null sequences ([5, Corollary VI.2.17]). Let $(f_1^n) \subset C(K_1)$ be a weakly null sequence. We have to prove that $\|T_1(f_1^n)\| \rightarrow 0$ when $n \rightarrow \infty$. If not, there would be an $\epsilon > 0$ and a subsequence (denoted in the same way) such that $\|T_1(f_1^n)\| > \epsilon$ for every n . Then we could produce $f_j^n \in C(K_j)$ ($2 \leq j \leq k$), $\|f_j^n\| \leq 1$, such that $\|T_1(f_1^n)(f_2^n, \dots, f_k^n)\| = \|T(f_1^n, \dots, f_k^n)\| > \epsilon$ for every n . But $(f_1^n \cdot f_2^n \cdots f_k^n) \subset C(K_1 \times \cdots \times K_k)$ converges weakly to 0. Hence, from the aforementioned property of weakly compact operators on $C(K)$ -spaces, $\|\overline{T}(f_1^n \cdots f_k^n)\| = \|T(f_1^n, \dots, f_k^n)\|$ tends to 0 as n tends to ∞ , which is a contradiction. \square

REMARK 4.6. Note that if X is reflexive, in particular if $X = \mathbb{K}$, then it follows from the above result that integral operators are regular.

We do not know if the converse of the Proposition 4.5 is true. In any case, if $T : C(K_1) \times \cdots \times C(K_k) \rightarrow X$ is integral and regular and, for instance, we denote by $T_1 : C(K_1) \rightarrow \mathcal{L}^{k-1}(C(K_2), \dots, C(K_k); X)$ the associated linear map, it is easily checked that $T(\varphi_1)$ is integral for any $\varphi_1 \in C(K_1)$, and its representing measure takes also values in the space of integral $(k-1)$ -linear operators. Thus, it can be considered as a measure $m_1 : \Sigma_1 \rightarrow \mathcal{L}(C(K_2 \times \cdots \times K_k); X)$. Reasoning in a similar way as in [3, Theorem 2.4], we can prove that m_1 coincides with the representing measure of the operator $\hat{T} : C(K_1, C(K_2 \times \cdots \times K_k)) \rightarrow X$ given by the Dinculeanu-Singer Theorem (see, e.g. [5, p. 182]), and the weak compactness of \overline{T} is clearly equivalent to that of \hat{T} .

In the linear case, an operator $T : C(K) \rightarrow X$ is weakly compact if and only if its representing measure μ takes values in X , if and only if μ is countably additive. This is not longer true in the multilinear case, where the role of weakly compact operators seems to be played by the so called *completely continuous* multilinear maps (see [17]). In the case of integral multilinear maps one could conjecture that the weak compactness and the behaviour of the representing polymeasure of T should be analogous to that of the extended linear operator. This is not true, as the following example shows:

Example 4.7. Let us consider $\ell_\infty = C(\beta\mathbb{N})$. Let $q : \ell_\infty \rightarrow \ell_2$ be a linear, continuous and onto map ([15, Remark 2.f.12]), and let us take a bounded sequence $(a_n) \subset \ell_\infty$ such that $q(a_n) = e_n$ (the canonical basis of ℓ_2) for any n . Suppose $\|a_n\| \leq C$. Then $(e_n \otimes a_n)$ is a basic sequence in $\ell_\infty \hat{\otimes}_\epsilon \ell_\infty \approx C(\beta\mathbb{N}, \ell_\infty)$ ([13, Proposition 3.15]), equivalent to the canonical basis of c_0 , since

$$\left\| \sum_{i=1}^n \lambda_i e_i \otimes a_i \right\|_\epsilon = \sup_{\|x^*\| \leq 1} \left\| \sum_{i=1}^n \lambda_i x^*(a_i) e_i \right\|_\infty \leq C \|(\lambda_i)\|_\infty.$$

Moreover, if $\varphi_n := e_n^* \otimes q^*(e_n^*) \in \ell_\infty^* \otimes \ell_\infty^* \subset (\ell_\infty \hat{\otimes}_\epsilon \ell_\infty)^*$ we have $\|\varphi_n\| \leq \|q\|$ for all n and so, as $\varphi_n(a \otimes b) \rightarrow 0$ when n tends to ∞ , it turns out that (φ_n) is a weak* null sequence. Hence $P(u) := \sum_{n=1}^{\infty} \varphi_n(u) e_n \otimes a_n$ is a continuous projection from $\ell_\infty \hat{\otimes}_\epsilon \ell_\infty$ onto the closed subspace (isomorphic to c_0) spanned by $\{e_n \otimes a_n : n \in \mathbb{N}\}$. Consequently,

$$\hat{T} : \ell_\infty \hat{\otimes}_\epsilon \ell_\infty \longrightarrow c_0$$

defined as $\hat{T}(u) := (\varphi_n(u)) \in c_0$, is linear, continuous and *onto*. In particular, \hat{T} is *not* weakly compact. By construction, the corresponding bilinear map $T : \ell_\infty \times \ell_\infty \longrightarrow c_0$ is integral. Also, since $(\varphi_n(x \otimes y))_n^\infty \in \ell_2$ for $x, y \in \ell_\infty$ and $\|T(x, y)\|_2 \leq \|q\| \|x\|_\infty \|y\|_\infty$, it follows that T factors continuously through ℓ_2 and consequently it is weakly compact. In particular, the representing bimeasure γ of T takes values in c_0 , ([2, Corollary 2.2]), but the extended measure μ that represents $\bar{T} : C(\beta\mathbb{N}, \ell_\infty) \rightarrow c_0$ does not.

The next proposition characterizes when \bar{T} is weakly compact in terms of the representing polymeasure of T :

Proposition 4.8. *Let $T : C(K_1) \times \cdots \times C(K_k) \longrightarrow X$ be an integral multilinear operator with representing polymeasure γ , and let μ be the representing measure of its extension $\bar{T} : C(K_1 \times \cdots \times K_k) \longrightarrow X$. The following assertions are equivalent:*

- a) \bar{T} is weakly compact.
- b) μ takes values in X .
- c) μ is countably additive.
- d) γ_m (see Section 2) takes values in X and is strongly additive.

Proof. The equivalences between (a), (b) and (c) are well known (see [5, Theorem VI.2.5]), and obviously they imply (d). Finally, since μ is a w^* -countably additive extension of γ_m , (d) implies that γ_m is weakly countably additive (and strongly additive). The Hahn-Kluvanek extension Theorem ([5, Theorem I.5.2]) provides an (unique) X -valued countably additive extension of γ_m to $\Sigma := \text{Bo}(K_1) \otimes \cdots \otimes \text{Bo}(K_k)$, which clearly coincides with μ . Obviously, every function in $C(K_1) \otimes \cdots \otimes C(K_k)$ is Σ -measurable. Hence, by density, every continuous function on $K_1 \times \cdots \times K_k$ is Σ -measurable. Urysohn's lemma proves that every closed, F_σ set belongs to Σ , and so is sent by μ to X . A well known result of Grothendieck ([12, Théorème 6]) proves that μ sends any Borel subset of $K_1 \times \cdots \times K_k$ to X . \square

REFERENCES

- [1] R. M. Aron and P. Galindo, Weakly compact multilinear mappings, *Proc. of the Edinburgh Math. Soc.* **40** (1997) 181-192.
- [2] F. Bombal and I. Villanueva, Multilinear operators on spaces of continuous functions. *Funct. Approx. Comment. Math.* **XXVI** (1998), 117–126.
- [3] F. Bombal and I. Villanueva, Regular multilinear operators in spaces of continuous functions. *Bull. Austral. Math. Soc.*, vol. **60** (1999), 11-20.
- [4] F. Bombal and I. Villanueva, On the Dunford-Pettis property of the tensor product of $C(K)$ spaces. To appear in *Proc. Amer. Math. Soc.*
- [5] J. Diestel and J. J. Uhl, *Vector Measures*. Mathematical Surveys, No. 15. American Math. Soc., Providence, R.I., 1977.
- [6] N. Dinculeanu, *Vector Measures*. Pergamon Press, 1967.
- [7] N. Dinculeanu and M. Muthiah, Bimeasures in Banach spaces. Preprint.
- [8] I. Dobrakov, On integration in Banach spaces, VIII (polymeasures). *Czech. Math. J.* **37** (112) (1987), 487–506.
- [9] I. Dobrakov, Representation of multilinear operators on $\times C_0(T_i)$. *Czech. Math. J.* **39** (114) (1989), 288–302.

- [10] M. González and J. M. Gutiérrez, Injective factorization of holomorphic mappings, *Proc. Amer. Math. Soc.* **127** (1999), 1715–1721.
- [11] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires. *Mem. Amer. Math. Soc.* **16** (1955).
- [12] A. Grothendieck, Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$. *Canad. J. Math.* **5** (1963), 129–173.
- [13] J. R. Holub, Tensor product bases and tensor diagonals, *Trans. Amer. Math. Soc.* **151** (1970), 563–579.
- [14] S. Karni and E. Merzbach, On the extension of Bimeasures. *Journal d'Analyse Math.* **55** (1990), 1–16.
- [15] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, I* Springer, Berlin, 1977.
- [16] J. R. Retherford and C. Stegall, Fully Nuclear and Completely Nuclear Operators with applications to \mathcal{L}_1 and \mathcal{L}_∞ spaces. *Trans. Amer. Math. Soc.* **163** (1972), 457–492.
- [17] I. Villanueva, Completely continuous multilinear operators on $C(K)$ spaces, *Proc. of the Amer. Math. Soc.* **128** (1999), 793–801.

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