WHERE DO HOMOGENEOUS POLYNOMIALS ON ℓ_1^n ATTAIN THEIR NORM?

DAVID PÉREZ-GARCÍA AND IGNACIO VILLANUEVA

Abstract. Using a 'reasonable' measure in $\mathcal{P}(^2\ell_1^n)$, the space of 2-homogeneous polynomials on ℓ_1^n, we show the existence of a set of positive (and independent of n) measure of polynomials which do not attain their norm in the vertices of the unit ball of ℓ_1^n. Next we prove that, when n grows, the measure of the set of polynomials which attain their norm in a face of 'high' dimension of the unit ball tends to 0.

1. Introduction, Notation and Definitions

In the past few years there has been an increasing interest, within the theory of polynomials in Banach spaces, in the study of the geometry of the spaces of polynomials (see, for instance, [1], [3], [4], [5], [6]).

In this direction, in the conference Function Theory on Infinite Dimensional Spaces VII, held in Madrid in 2001, professor I. Zalduendo asked the question of 'How many' homogeneous polynomials will attain their norm in the vertices of the unit ball of ℓ_∞^n when n tends to infinity. He conjectured that 'almost everyone'. In this direction, he and D. Carando published recently a paper giving qualitative general results (see [2]). As they say in the introduction, the question is to study how likely it is for a polynomial $P : E \longrightarrow \mathbb{R}$ to attain its norm at a given subset A of the unit ball B_E. In our paper we give quantitative results referring to 2-homogeneous poynomials on ℓ_1^n, as an example of the results that can be expected in more general cases. We use normalized Lebesgue's measure μ_n on the unit ball of the space $\mathcal{L}_s(^2\ell_1^n)$ of symmetric bilinear forms to count 'how many' polynomials attain their norm wherever. The reason for using this measure, instead of normalized Lebesgue's measure on the polynomial unit ball is that it is (by far) easier to deal with. On the other hand, it is also a reasonable measure since, by the polarization formula, for every 2−homogeneous polynomial P on ℓ_1^n, we have that $\|P\| \leq \|A\|_s \leq 2\|P\|$, where $\|\cdot\|_s$ is the norm given by the associated symmetric bilinear form A.

The first result we have is that Zalduendo's conjecture fails in this setting (see Theorem 2.3). This is not so surprising since the number of vertices in the unit ball of l_1^n is just $2n$, whereas in the unit ball of ℓ_∞^n there are

1991 *Mathematics Subject Classification.* ??
Key words and phrases. polynomials, extreme points, ??
Both authors were partially supported by DGICYT grant BMF2001-1284.

2^n vertices. The main result (Theorem 2.4), however, shows that even in this case Zalduendo's conjecture is not far from the truth, in the sense that, asymptotically, almost every polynomial attains its norm in a face of 'low' dimension.

The notation will be the usual in this context. E will denote a finite dimensional Banach space. Associated to it, we are going to consider its unit ball B_E, the space of real-valued 2-homogeneous polynomials $\mathcal{P}(^2E)$, and the space of real-valued symmetric bilinear forms $\mathcal{L}_s(^2E)$. Given a polynomial P, we are going to write A for its associated bilinear form. We are going to consider only polynomials P such that $A \in B_{\mathcal{L}_s(^2E)}$. ℓ_1^n will be the Banach space $(\mathbb{R}^n, \|\cdot\|_1)$, ℓ_∞^n will be $(\mathbb{R}^n, \|\cdot\|_\infty)$ and $\{e_i\}_{i=1}^n$ will denote the canonical basis of $\mathbb{R}^n$. In $B_{\ell_\infty^{\frac{n(n+1)}{2}}}$, we are going to consider the probability measure $\mu_n = \frac{1}{2^{\frac{n(n+1)}{2}}}\lambda_{\frac{n(n+1)}{2}}$, where $\lambda_{\frac{n(n+1)}{2}}$ is the Lebesgue measure in $\mathbb{R}^{\frac{n(n+1)}{2}}$.

Via the mapping $P \mapsto (A(e_i, e_j) = a_{ij})_{1\le i\le j}^n$ we are going to identify the set of polynomials $P \in \mathcal{P}(^2\ell_1^n)$ such that $A \in B_{\mathcal{L}_s(^2E)}$ with the unit ball of $\ell_\infty^{\frac{n(n+1)}{2}}$, and μ_n, via the previous identification, is just the probability measure we are going to use to see how likely it is for a 2-homogeneous polynomial to attain its norm at a given subset of $B_{\ell_1^n}$.

For a general definition of a vertex and a m-dimensional face of a convex polytope, we refer the reader to [7]. Here all we are going to use is that ([7, pages 55-56]) in $B_{\ell_1^n}$, the vertices are just $\pm e_i, i = 1, \ldots, n$, and a $(m-1)$-dimensional face (or $(m-1)$-face) is just the convex hull of m linearly independent vertices. The interior of a m-face C is the set of points of C that are not in any k-face, for $k < m$.

Though we are not going to say it from now on, it is not difficult to show that all the sets we are going to consider are measurable.

2. The results

Lemma 2.1. *Let E be a normed vector space, let P be a 2-homogeneous scalar polynomial defined on E and let $T \in \mathcal{L}(^2E)$ be its associated symmetric bilinear form. Suppose $x, y \in E$ and suppose $|P(x)| \ge |P(y)|$. If $|P(x)| > |T(x,y)|$ then for every $0 < \lambda < 1$, $|P(\lambda x + (1-\lambda)y)| < |P(x)|$. Conversely, if $|T(x,y)| > |P(x)|$ and $P(x)$ and $T(x,y)$ have the same sign, then there exists $\lambda \in (0,1)$ such that $|P(\lambda x + (1-\lambda)y)| > |P(x)|$.*

Proof. Let us suppose first that $|P(x)| > |T(x,y)|$. Then, for very $\lambda \in (0,1)$,

$$\begin{aligned}|P(\lambda x + (1-\lambda)y)| &= |\lambda^2 P(x) + (1-\lambda)^2 P(y) + 2\lambda(1-\lambda)T(x,y)| \\ &\le |\lambda^2 P(x)| + |(1-\lambda)^2 P(y)| + |2\lambda(1-\lambda)T(x,y)| \\ &< |P(x)|,\end{aligned}$$

because $\lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) = 1$.

Conversely, suppose that $T(x, y) > P(x) \geq 0$ (the other case is similar). Let

$$f(\lambda) = P(\lambda x + (1-\lambda)y) = \lambda^2 P(x) + (1-\lambda)^2 P(y) + 2\lambda(1-\lambda)T(x, y).$$

Then

$$f'(\lambda) = 2(\lambda P(x) + (\lambda - 1)P(y) + (1 - 2\lambda)T(x, y)),$$

and $f'(\lambda) = 0$ only when

$$\lambda = \lambda_0 = \frac{P(y) - T(x, y)}{P(x) + P(y) - 2T(x, y)}.$$

Clearly $0 < \lambda_0 < 1$ and, since $f''(\lambda) = 2(P(x) + P(y) - 2T(x, y)) < 0$, we get that $f(\lambda_0) = P(\lambda_0 x + (1-\lambda_0)y) > P(x)$. Moreover, we get that

$$f(\lambda_0) = \frac{P(y)P(x) - T(x, y)^2}{P(x) + P(y) - 2T(x, y)}.$$

□

As a first application of the lemma, we have the following

Proposition 2.2. *Let $P \in \mathcal{P}(^2\ell_1^n)$, let $A \in \mathcal{L}^2(\ell_1^n)$ be its associated symmetric bilinear form. Let $i \in \{1, \ldots, n\}$ be such that $|P(e_i)| \geq |P(e_j)|$ for every $j \in \{1, \ldots, n\}$. Suppose that, for every $j \in \{1, \ldots, n\}$, $|P(e_i)| \geq |A(e_i, e_j)|$. Then P attains its norm either at e_i or at one of the $(n-2)$-dimensional faces not adjacent to e_i or $-e_i$.*

Proof. Let us suppose without loss of generality that $i = 1$. A point y in one of the not adjacent $(n-2)$-dimensional faces can always be written in the form $y = \sum_{j=2}^n \alpha_j e_j$, where $\sum_{j=2}^n |\alpha_j| = 1$. Let us note that

$$|A(e_1, y)| = |\sum_{j=2}^{n} \alpha_j A(e_1, e_j)| \leq \sum_{j=2}^{n} |\alpha_j||A(e_1, e_j)| \leq |P(e_1)|.$$

So, consider any point z in the unit ball of ℓ_1^n. There exists y in one of the $(n-2)$-dimensional faces not adjacent to e_1 or $-e_1$ and $\lambda \in [0, 1]$ such that $z = \lambda e_1 + (1-\lambda)y$. If $|P(e_1)| \geq |P(y)|$, we can use Lemma 2.1 to prove that $|P(z)| \leq |P(x)|$. If $|P(e_1)| \leq |P(y)|$ the same lemma can be used to prove that $|P(z)| \leq |P(y)|$. □

We can now use the previous results to prove the next theorem.

Theorem 2.3 (Failure of Zalduendo's conjecture for ℓ_n^1)**.** *For any $n \geq 2$, if we note $C = \{P \in \mathcal{P}(^2\ell_1^n)$ such that $A \in B_{\mathcal{L}(^2\ell_1^n)}$ and P does not attain its norm in a vertex$\}$, then $\mu_n(C) \geq \frac{1}{6}$.*

Proof. We define the following sets

$$B := \left\{ P \text{ such that there exists } i_0, j_0 \text{ with } \left\{ \begin{array}{c} \max_i |P(e_i)| = |P(e_{i_0})| \\ |P(e_{i_0})| < |A(e_{i_0}, q_{j_0})| \end{array} \right\} \right\}$$

$$\hat{B} := \left\{ P \text{ s. t. there exists } i_0, j_0 \text{ with } \left\{ \begin{array}{c} \max_i |P(e_i)| = |P(e_{i_0})| \\ |P(e_{i_0})| < |A(e_{i_0}, q_{j_0})| \\ \operatorname{sign} P(e_{i_0}) \neq \operatorname{sign} A(e_{i_0}, e_{j_0}) \end{array} \right\} \right\}$$

$$\tilde{B} := \left\{ P \text{ s. t. there exists } i_0, j_0 \text{ with } \left\{ \begin{array}{c} \max_i |P(e_i)| = |P(e_{i_0})| \\ |P(e_{i_0})| < |A(e_{i_0}, q_{j_0})| \\ \operatorname{sign} P(e_{i_0}) = \operatorname{sign} A(e_{i_0}, e_{j_0}) \end{array} \right\} \right\}.$$

Let us consider the linear isometry $\ell_\infty^{\frac{n(n+1)}{2}} \longrightarrow \ell_\infty^{\frac{n(n+1)}{2}}$ given by $(a_{ij})_{j \geq i} \mapsto (\tilde{a}_{i_j})_{j \geq i}$, where $\tilde{a}_{ij} = -a_{ij}$ if $j > i$ and $\tilde{a}_{ii} = a_{ii}$. Clearly the image of $\tilde{B}$ is just $\hat{B}$. Using the theorem of the change of variables, we obtain that $\mu_n(\tilde{B}) = \mu_n(\hat{B})$. Besides, $B = \tilde{B} \cup \hat{B}$ and, by Lemma 2.1, $\tilde{B} \subset C$. Therefore

$$\mu_n(C) \geq \frac{\mu_n(B)}{2}.$$

Now, using the usual identification $P \leftrightarrow (A(e_i, e_j) = a_{ij})_{j > i}$, we have that

$$B^c \subset \bigcup_k \{|a_{kk}| = \max_i |a_{ii}| \text{ and } |a_{kk}| = \max_j |a_{kj}|\},$$

where we take $a_{kj} = a_{jk}$ if $k > j$.

For each $k = 1, \ldots, n$, the measure of the set $\{|a_{kk}| = \max_i |a_{ii}|$ and $|a_{kk}| = \max_j |a_{kj}|\}$ can be calculated easily by integration to be $\frac{1}{2n-1}$, therefore we have that

$$\mu_n(C) \geq \frac{\mu_n(B)}{2} \geq \frac{n-1}{4n-2} \geq \frac{1}{6},$$

for every $n \geq 2$. □

This result shows the existence of a set of positive measure of polynomials which do not attain their norm in the vertices. We are reasonably sure of the existence of another set of positive measure of polynomials which do attain their norm in the vertices, but we have not been able to prove this yet.

Indeed, it seems to be the case that 'most' of the polynomials $P \in \mathcal{P}(^2\ell_1^n)$ attain their norm in the low-dimensional faces. This is the content of our next theorem, which probably can be substantially refined.

Let us define A_n^m to be the set of polynomials $P \in \mathcal{P}(^2\ell_1^n)$ such that $\|A\|_s \leq 1$ and P attains its norm in the interior of an $(m-1)$-face. Our main theorem states the following

Theorem 2.4. *With the previous notation,*

$$\lim_{n\to\infty} \mu_n \left(\bigcup_{m>16\sqrt{n}}^{n} A_n^m \right) = 0. \tag{1}$$

The idea behind the proof of Theorem 2.4 is to find sets B_n^m such that $A_n^m \subset B_n^m$, each B_n^m is 'easy' to measure, and condition (1) still holds for B_n^m. To do this we need some previous results.

Proposition 2.5. *If P is a polynomial that attains its maximum in the interior of the $(m-1)$-face given by the vertices $v_1, \ldots, v_m$, and if $P(v_1) \leq P(v_2) \leq \cdots \leq P(v_m)$, then*

$$\begin{aligned} P(v_1) &\leq A(v_1, v_j) && \forall j > 1 \\ P(v_2) &\leq A(v_2, v_j) && \forall j > 2 \\ &\cdots \\ P(v_{m-1}) &\leq A(v_{m-1}, v_m). \end{aligned}$$

Proof. The interior of C is given by

$$int(C) = \{\lambda_1 v_1 + \cdots \lambda_{m-1} v_{m-1} + (1 - \lambda_1 - \cdots \lambda_{m-1}) v_m, \text{ where}$$
$$\lambda_i \in (0,1) \quad (1 \leq i \leq m-1) \text{ and } \sum_{i=1}^{m-1} \lambda_i < 1\}.$$

We call $D = \{(\lambda_1, \cdots, \lambda_{m-1}) \in (0,1)^{m-1} : \quad \sum_{i=1}^{m-1} \lambda_i < 1\}$ and we define $f : D \longrightarrow \mathbb{R}$ by

$$f(\lambda_1, \cdots, \lambda_{m-1}) = P(\lambda_1 v_1 + \cdots + \lambda_{m-1} v_{m-1} + (1 - \lambda_1 - \cdots - \lambda_{m-1}) v_m).$$

We have that f is the polynomial of degree 2 given by

$$\begin{aligned} &f(\lambda_1, \ldots, \lambda_{m-1}) = \\ &\sum_{i=1}^{m-1} \lambda_i^2 P(v_i) + \left(1 + \sum_{i=1}^{m-1} \lambda_i^2 + 2 \sum_{1=i<j}^{m-1} \lambda_i \lambda_j - 2 \sum_{i=1}^{m-1} \lambda_i \right) P(v_m) + \\ &+ 2 \sum_{1=i<j}^{m-1} \lambda_i \lambda_j A(v_i, v_j) + 2 \sum_{i=1}^{m-1} \lambda_i A(v_i, v_m) - 2 \sum_{i=1}^{m-1} \lambda_i^2 A(v_i, v_m) - \\ &- 2 \sum_{i \neq j}^{m-1} \lambda_i \lambda_j A(v_i, v_m). \end{aligned}$$

As f attains its maximum in D, we have that the hessian matrix $H = (H_{ij})_{i,j=1}^{m-1}$ of f, which is constant, is semidefinite negative. Then, considering

$u_{ij} = e_i - e_j$ for $i < j$, we have that

$$\frac{1}{2}(H_{ii} + H_{jj} - 2H_{ij}) = \frac{1}{2}u_{ij}^t H u_{ij} \leq 0$$

Now,

$$\frac{1}{2}H_{ii} = P(v_i) + P(v_m) - 2A(v_i, v_m)$$
$$\frac{1}{2}H_{jj} = P(v_j) + P(v_m) - 2A(v_j, v_m)$$
$$\frac{1}{2}H_{ij} = P(v_m) + A(v_i, v_j) - A(v_i v_m) - A(v_j, v_m),$$

and so

(2) $$P(v_i) + P(v_j) \leq 2A(v_i, v_j)$$

holds for $1 \leq i < j \leq m-1$.

As, in addition, $P(v_i) + P(v_m) - 2A(v_i, v_m) = \frac{1}{2}H_{ii} \leq 0$ for $1 \leq i \leq m-1$, we have that (2) holds for $1 \leq i < j \leq m$. Using the condition $P(v_1) \leq \cdots \leq P(v_m)$ it is straightforward to conclude the result. □

The following two lemmata can be easily proved by induction.

Lemma 2.6. *If $n \geq 2$, we have that*

$$\int_{x_n}^1 \cdots \int_{x_1}^1 (1-x_1)\cdots(1-x_{n-1})^{n-1} dx_0 \cdots dx_{n-1} = \frac{(1-x_n)^{\frac{n(n+1)}{2}}}{\prod_{k=2}^n \frac{k(k+1)}{2}}.$$

Lemma 2.7.

$$\prod_{k=2}^m \frac{k(k+1)}{2} = \frac{(m+1)!^2}{2^m(m+1)}.$$

We can use now Proposition 2.5 and the previous lemmata to prove

Proposition 2.8.

(3) $$\mu_n(A_n^m) \leq \frac{\binom{n}{m} 2^{2m+1}}{(m+1)!}.$$

Proof. Given a $(m-1)$-face C, we will call A_C the set of polynomials $P \in \mathcal{P}(^2\ell_1^n)$ with $\|A\|_s \leq 1$ such that P attain its maximum in the interior of C (we can do the same with the minimum).

Let us call C_0 the $(m-1)$-face given by $e_1, \ldots, e_m$. It is not difficult to see that, given any other $(m-1)$-face, say C, there exists a linear isometry $T : \ell_\infty^{\frac{n(n+1)}{2}} \longrightarrow \ell_\infty^{\frac{n(n+1)}{2}}$ (with $|\det(T)| = 1$) that maps A_C onto A_{C_0}. Using the theorem of change of variables, it follows that

$$\mu_n(A_C) = \mu_n(A_{C_0}).$$

We also know [7, page 56] that there are $\binom{n}{m}2^m$ different $(m-1)$-faces in $B_{\ell_1^n}$. Therefore, we have that

$$\mu_n(A_n^m) \leq 2\binom{n}{m}2^m\mu_n(A_{C_0}) = \binom{n}{m}2^{m+1}\mu_n(A_{C_0}), \tag{4}$$

where the additional 2 comes from considering both the polynomials that attain its maximum or its minimum in the interior of C_0.

Now, if we make the convention $a_{ij} = a_{ji}$ if $i > j$ and define, for each permutation $\sigma : \{1, \dots, m\} \longrightarrow \{1, \dots, m\}$, the set B_σ by

$$B_\sigma = \Big\{A = (a_{ij}) \subset B_{\mathcal{L}_s(^2E)} \text{ such that } a_{\sigma(1),\sigma(1)} \leq \cdots \leq a_{\sigma(m),\sigma(m)} \text{ and}$$

$$\left\{ \begin{array}{c} a_{\sigma(1),\sigma(1)} \leq a_{\sigma(1),\sigma(2)}, \dots, a_{\sigma(1),\sigma(m)} \\ a_{\sigma(2),\sigma(2)} \leq a_{\sigma(2),\sigma(3)}, \dots, a_{\sigma(2),\sigma(m)} \\ \cdots \\ a_{\sigma(m-1),\sigma(m-1)} \leq a_{\sigma(m-1),\sigma(m)} \end{array} \right\}\Bigg\},$$

we get, using Proposition 2.5, that

$$A_{C_0} \subset \bigcup_\sigma B_\sigma.$$

But we have as above that $\mu_n(B_\sigma) = \mu_n(B_{id})$ for every σ. Moreover, $2^{\frac{m(m+1)}{2}}\mu_n(B_{id})$ is just

$$\int_{-1}^{1}\int_{a_{22}=a_{11}}^{1} \cdots \int_{a_{mm}=a_{m-1,m-1}}^{1} \prod_{j=1}^{m-1}(1-a_{jj})^{m-j}da_{mm}\cdots da_{22}da_{11}. \tag{5}$$

Now, by Lemma 2.6

$$(5) = \int_{-1}^{1}(1-x)^{\frac{m(m-1)}{2}+m-1} = 2^{\frac{m(m+1)}{2}}\frac{1}{\prod_{k=2}^{m}\frac{k(k+1)}{2}},$$

and by Lemma 2.7,

$$\mu_n(B_{id}) = \frac{2^m(m+1)}{(m+1)!^2}.$$

So

$$\mu_n(A_{C_0}) \leq m!\mu_n(B_{id}) = \frac{2^m}{(m+1)!},$$

and an appeal to (4) finishes the proof.

□

Finally we need a technical result

Proposition 2.9. *There exists a natural number n_0 such that for every $n \geq n_0$, we have*

$$\sum_{m=8n}^{n^2}\frac{\binom{n^2}{m}2^{2m}}{m!} \leq \frac{1}{n}.$$

Proof. The proof lies in the following two claims:

Claim 1: There exists a natural number n_0 such that for every $n \geq n_0$, we have that

$$\frac{\binom{n^2}{8n} 2^{16n}}{(8n)!} \leq \frac{1}{n^3}.$$

Claim2: If $8n \leq m \leq n^2 - 1$ and we call

$$x_m = \frac{\binom{n^2}{m} 2^{2m}}{m!},$$

we have that $x_m \geq x_{m+1}$.

With this two claims, if $n \geq n_0$ then

$$\sum_{m=8n}^{n^2} \frac{\binom{n^2}{m} 2^{2m}}{m!} \leq \sum_{m=8n}^{n^2} \frac{\binom{n^2}{8n} 2^{16n}}{(8n)!} \leq \frac{n^2}{n^3} = \frac{1}{n},$$

and we are done.

In order to prove the first claim we call

$$y_n = \frac{\binom{n^2}{8n} 2^{16n} n^3}{(8n)!}.$$

We will see that $\lim_{n \to \infty} y_n = 0$.
We have

$$\frac{y_{n+1}}{y_n} =$$

$$= 2^{16} (1 + \frac{1}{n})^3 \frac{(n^2 + 1 + 2n) \cdots (n^2 + 1)}{(8n+8)^2 \cdots (8n+1)^2 (n^2 + 1 - 6n - 8) \cdots (n^2 - 8n + 1)}$$

$$= \frac{(1 + 1/n)^3}{2^{32}} \frac{(n^2 + 1 + 2n) \cdots (n^2 + 2n - 6)}{\left(\frac{8n+8}{8}\right)^2 \cdots \left(\frac{8n+1}{8}\right)^2} \frac{(n^2 + 2n - 7) \cdots (n^2 + 1)}{(n^2 + 1 - 6n - 8) \cdots (n^2 - 8n + 1)}$$

$$\leq \frac{(1 + \frac{1}{n})^3}{2^{32}} \frac{(n^2 + 1 + 2n) \cdots (n^2 + 2n - 6)}{\left(\frac{8n+8}{8}\right)^2 \cdots \left(\frac{8n+1}{8}\right)^2} \left(\frac{n^2 + 2n - 7}{n^2 - 8n + 1} \right)^{2n-8} = B_n.$$

It is easy to see that

$$\lim_{n \to \infty} B_n = \frac{e^{20}}{2^{32}} < \frac{1}{2}.$$

Therefore, by the quotient criterium, $\lim_{n \to \infty} y_n = 0$.

To see the second claim, we are going to prove that $\frac{x_{m+1}}{x_m} \leq 1$.
We have that

$$\frac{x_{m+1}}{x_m} = \frac{4(n^2 - m)}{(m+1)^2},$$

and

$$\frac{4(n^2-m)}{(m+1)^2} \leq 1 \Leftrightarrow m \geq 1+\sqrt{4n^2-1}.$$

As $8n \geq 1+\sqrt{4n^2-1}$, we can conclude the result. $\square$

Proof of Theorem 2.4. We have that

$$\mu_n\left(\bigcup_{m>16\sqrt{n}}^{n} A_n^m\right) \leq \sum_{m>16\sqrt{n}}^{n} \mu_n(A_n^m) \leq \sum_{m>16\sqrt{n}}^{n} \frac{\binom{n}{m}2^{2m+1}}{(m+1)!}$$

$$\leq \sum_{m=16[\sqrt{n}]}^{n} \frac{\binom{n}{m}2^{2m}}{m!} \leq \sum_{m=8([\sqrt{n}]+1)}^{([\sqrt{n}]+1)^2} \frac{\binom{([\sqrt{n}]+1)^2}{m}2^{2m}}{m!},$$

where $[\cdot]$ notes the integer part.

Therefore, by Proposition 2.9, there exists a natural number n_0 such that

$$\mu_n\left(\bigcup_{m>16\sqrt{n}}^{n} A_n^m\right) \leq \frac{1}{[\sqrt{n}]+1}.$$

for every $n \geq n_0$, and we are done. $\square$

References

1. R. Aron, B. Beauzamy, and P. Enflo, *Polynomials in many variables: Real vs complex norms.*, J. Approx. Theory **74** (1993), 181–198.
2. D. Carando and I. Zalduendo, *Where do homogeneous polynomials attain their norm?*, To appear in Publicationes Mathematicae Debrecen.
3. Y.S. Choi and S.G. Kim, *Extreme polynomials and multilinear forms on ℓ_1*, J. Math. Anal. Appl. **228** (1998), 467–482.
4. ———, *The unit ball of* $\surd(^2\ell_2^2)$, Arch. Math. **71** (1998), 472–480.
5. B. Grecu, *Smooth 2-homogeneous polynomials on hilbert spaces*, ??, ??
6. ———, *Geometry of 3-homogeneous polynomials on real hilbert spaces*, J. Math. Anal. Appl. **246** (2000), 271–289.
7. B. Grünbaum, *Convex polytopes*, UMI, Books on Demand, 2000.

E-mail address: `David_Perez@mat.ucm.es, Ignacio_Villanueva@mat.ucm.es`

Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad Complutense de Madrid, Madrid 28040