ORTHOGONALLY ADDITIVE POLYNOMIALS ON C*-ALGEBRAS

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Abstract. Let A be a C*-algebra which has no quotient isomorphic to $M_2(\mathbb{C})$. We show that for every orthogonally additive scalar n-homogeneous polynomials P on A such that P is Strong* continuous on the closed unit ball of A, there exists φ in A^* satisfying that $P(x) = \varphi(x^n)$, for each element x in A. The vector valued analogue follows as a corollary.

1. Introduction

Let A be a C*-algebra. Two elements x, y in A are said to be *orthogonal* if $xy = yx = 0$. A mapping Φ from A to a Banach space is said to be *orthogonally additive* if for every pair of mutually orthogonal elements x, y in A we have $\Phi(x + y) = \Phi(x) + \Phi(y)$.

In the case of a Banach lattice L, a function Φ from L to a Banach space is said to be *orthogonally additive* if $\Phi(x + y) = \Phi(x) + \Phi(y)$, whenever x and y are orthogonal.

By an *X-valued n-homogeneous polynomial* on a Banach space E we mean a *continuous* X-valued mapping P on E for which there exists a continuous n-linear operator $T : E \times \cdots \times E \longrightarrow X$ satisfying $P(x) = T(x, \ldots, x)$, for every x in X.

Orthogonally additive polynomials on Banach lattices have been studied and described by several authors (see for instance [2, 7, 8, 9, 16, 20]).

When, by the Gelfand-Naimark theorem, the $C(K)$ spaces are regarded as unital abelian C*-algebras, the main result in [16] asserts that for each orthogonally additive n-homogeneous polynomial $P : C(K) \to \mathbb{C}$, there exists a Borel regular measure μ on K such that $P(f) = \int_K f^n d\mu$, $(f \in C(K))$. In other words, for each P as above, there exists ϕ in $C(K)^*$ satisfying that $P(f) = \phi(f^n)$, $(f \in C(K))$.

It seems natural to ask whether the above description remains true for orthogonally additive homogeneous polynomials on a general C*-algebra. This is the main goal treated in this note. It is not hard to see that for each ϕ in the dual space of a C*-algebra A, the law $x \mapsto P_\phi(x) := \phi(x^n)$

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defines an n-homogeneous polynomial P_ϕ on A. Since, for each pair of mutually orthogonal elements $x, y \in A$, we have $(x+y)^n = x^n + y^n$, it follows that P_ϕ is orthogonally additive. We shall check in §3, that the above polynomial P_ϕ also has the following topological property: P_ϕ is $S^*(A, A^*)$-to-norm continuous on the closed unit ball of A (see definitions bellow). Section two provides a study of those multilinear operators on the cartesian product of C*-algebras and Banach spaces which are jointly Strong*-to-norm continuous on the closed unit balls.

Section three deals with the study of orthogonally additive homogeneous polynomials on a C*-algebra. When considering (non necessarily unital) abelian C*-algebras, the description provided in [16] can be adapted word by word, to show that for every $M \in \mathbb{R}^+$, every locally compact space $L \subset (0, M]$, such that $L \cup \{0\}$ is compact, and every orthogonally additive homogeneous polynomials P on $C_0(L)$, there exists a functional $\phi \in C_0(L)^*$ satisfying that $P(f) = \phi(f^n)$ (Proposition 12). This result guarantees every orthogonally additive homogeneous polynomials on a C*-algebra can be "locally" good-described on the subalgebra generated by a self-adjoint element. The result that allows us to extend this "local" description to a global description will follow from the solution of the Mackey-Gleason problem given by L. J. Bunce and J. D. M. Wright in [4, 5] and [6]. More concretely, Theorem 3.1 in [6] gives us the key tool used in Proposition 13 to prove linearity for quasi-linear mappings on a C*-algebra.

Finally, in our main result (Theorem 16) we establish the following description: Let A be a C*-algebra which has no quotient isomorphic to $M_2(\mathbb{C})$, and let P be an orthogonally additive n-homogeneous polynomials P on A which is Strong* continuous on the closed unit ball of A, then there exists φ in A^* satisfying that $P(x) = \varphi(x^n)$, for each element x in A. The vector valued case follows easily.

Given Banach spaces X and Y, $L(X, Y)$ will denote the space of all bounded linear mappings from X to Y. Throughout the paper the word "operator" will always mean bounded linear operator. We shall write $L(X)$ instead of $L(X, X)$. The symbol $\mathcal{P}^n(X, Y)$ will stand for the Banach space of all n-homogeneous polynomials from X to Y. When $Y = \mathbb{K}$ we will omit it. The closed unit ball of a Banach space X is denoted by B_X while X^* will stand for its dual space.

Let $X_1, \ldots, X_n$, and X be Banach spaces and $T : X_1 \times \cdots \times X_n \to X$ be a n-linear operator. Let $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation. There is a unique n-linear extension $AB(T)_\pi : X_1^{**} \times \cdots \times X_n^{**} \to X^{**}$ verifying, for every $z_i \in X_i^{**}$ and every net $(x^i_{\alpha_i} \in X_i$ $(1 \leq i \leq n)$

$$AB(T)_\pi(z_1, \ldots, z_k) = \text{weak}^* - \lim_{\alpha_{\pi(1)}} \cdots \text{weak}^* - \lim_{\alpha_{\pi(k)}} T(x^1_{\alpha_1}, \ldots, x^n_{\alpha_n}).$$

Moreover, $AB(T)_\pi$ is bounded and has the same norm as T. The extensions $AB(T)_\pi$ are the same as the extensions constructed by Aron and

Berner for polynomials in [1], and we call them the *Aron-Berner extensions* of T (see also [15, Proposition 3.1]).

Let W be a von Neumann algebra and let ϕ be a positive functional in W_*, the law $z \mapsto \phi(zz^*)$ defines a seminorm on W. The Strong topology (respectively, the Strong* topology) on W is the topology generated by all the seminorms $\phi(zz^*)$ (respectively, $\phi(\frac{z^*z+zz^*}{2}))$) where ϕ runs on all positive functionals in W_*. The Strong* topology on W is usually denoted by $S^*(W, W_*)$. Given a C*-algebra A, the Strong* topology on A ($S^*(A, A^*)$) is the topology $S^*(A^{**}, A^*)|_A$. In [15, §2] the reader can find several interesting characterizations of the Strong* topology on Banach spaces.

2. Joint Strong*-Continuity of multilinear maps

The following characterization of the joint Right-to-norm continuity for T is borrowed from [15, Proposition 3.20]: Let $X_1, \ldots, X_n$ and X be Banach spaces and let $T : X_1 \times \cdots \times X_n \to X$ be a multilinear operator. Consider the following statements:

a) T is jointly Right-to-norm continuous.
b) There exists a positive constant M (depending only on T), reflexive Banach spaces $R_1, \ldots, R_n$ and bounded linear operators $T_i : X_i \to R_i$ satisfying, for each x_i in X_i,

$$\|T(x_1, \ldots, x_n)\| \leq M \ |||x_1|||_{T_1} \ \cdots \ |||x_n|||_{T_n}.$$

c) T is jointly Right-to-norm continuous on bounded subsets.

Then the following implications hold a) $\Rightarrow$ b) $\Rightarrow$ c).

We recall that for each bounded operator T between two Banach spaces X and Y, the symbol $|||.|||_T$ will denote the seminorm on X defined by $x \mapsto |||x|||_T := \|T(x)\|$.

The above result remains true when we replace the Right topology with the $S^*(X_i, X_i^*)$ topology and the reflexive spaces are assumed to be Hilbert spaces (see [15, §3]).

Another interesting question is whether the joint Strong*-to-norm continuity on the cartesian product of the closed unit balls could be also characterised. In the linear case, [13, Theorem 2.4] establishes the following:

Proposition 1. [13] *Let X and Y be Banach spaces and $T : X \to Y$ a linear operator. The following are equivalent:*

a) *$T|_{B_X} : B_X \to Y$ is Right$|_{B_X}$-to-norm continuous;*
b) *There exist a bounded linear operator S from X into a reflexive Banach space and a mapping $N : (0, +\infty) \to (0, +\infty)$ satisfying that*

$$\|T(x)\| \leq N(\varepsilon) \ |||x|||_S + \varepsilon \ \|x\|,$$

for all $x \in X$, $\varepsilon > 0$.

In [14, 13], it was shown that conditions *a)* and *b)* in the above results are also equivalent to:

a)' T is weakly compact;
b)' T is Right-to-norm continuous.

Using the polarization formula for symmetric multilinear operators [11, p. 6] we easily get the following:

Lemma 2. *Let X and Y be a Banach spaces, $P \in \mathcal{P}^n(X,Y)$ an n-homogeneous polynomial and $T : X \times \ldots \times X \to Y$ the symmetric multilinear operator associated to it. Then $T|_{B_X \times \ldots \times B_X}$ is jointly $S^*(X,X^*)$-to-norm continuous if and only if $P|_{B_X}$ is $S^*(X,X^*)$-to-norm continuous.* □

Lemma 3. *Let $X_1, \ldots, X_n$ and X be Banach spaces and let*

$$S : X_1 \times \cdots \times X_n \to X$$

be a multilinear operator. Suppose that

$$S|_{B_{X_1} \times \ldots \times B_{X_n}} : B_{X_1} \times \ldots \times B_{X_n} \to X$$

is jointly Strong-to-norm continuous, then there are Hilbert spaces H_1, $H_2, \ldots, H_n$ and bounded linear operators $S_i : X_i \to H_i$ such that*

$$\|S(x_1, \ldots, x_n)\| \leq \prod_{i=1}^{n} \left(|||x_i|||_{S_i} + \|x_i\| \right),$$

for every $x_i \in X_i$.

Proof. According to our hypothesis, the set

$$\mathcal{O} := \{(x_1, \ldots, x_n) \in B_{X_1} \times \ldots \times B_{X_n} : \|T(x_1, \ldots, x_n)\| \leq 1\}$$

is a Strong*-neighborhood of 0 in $B_{X_1} \times \ldots \times B_{X_n}$. Thus, by the definition of the Strong* topology, for each $i = 1, \ldots n$, there exist a positive constant δ, Hilbert spaces $H_1^i, \ldots, H_{p_i}^i$ and bounded linear operators $T_j^i : X_i \to H_j^i$ $(1 \leq j \leq p_i)$, such that

$$\mathcal{O} \supseteq \mathcal{O}' := \{(x_1, \ldots, x_n) \in B_{X_1} \times \ldots \times B_{X_n} : |||x_i|||_{T_j^i} \leq \delta, \ \forall\, 1 \leq j \leq p_i\}.$$

We denote

$$H_i := \bigoplus_{1 \leq j \leq p_i}^{\ell_2} H_j^i$$

and $S_i : X_i \to H_i$ the bounded linear operator given by $S_i(x_i) := (\delta^{-1}\, T_j^i(x))$. Clearly, for each i, H_i is a Hilbert space.

For each $(x_1, \ldots, x_n) \in X_1, \ldots, X_n$, with $x_i \neq 0$ $(1 \leq i \leq n)$, the element

$$\left(\frac{1}{\|S_1(x_1)\| + \|x_1\|}\ x_1, \ldots, \frac{1}{\|S_n(x_n)\| + \|x_n\|}\ x_n \right)$$

belongs to $\mathcal{O}' \subseteq \mathcal{O}$, and hence

$$\left\| T \left(\frac{1}{\|S_1(x_1)\| + \|x_1\|}\ x_1, \ldots, \frac{1}{\|S_n(x_n)\| + \|x_n\|}\ x_n \right) \right\| \leq 1,$$

which implies that

$$\|T(x_1,\ldots,x_n)\| \leq \prod_{i=1}^{n} \left(\||x_i\||_{S_i} + \|x_i\|\right).$$

When any $x_i = 0$, the above inequality is trivial. □

We can now prove:

Proposition 4. *Let $X_1,\ldots,X_n$ and X be Banach spaces and let*

$$T : X_1 \times \cdots \times X_n \to X$$

be a multilinear operator. Then the following are equivalent:

a) $T|_{B_{X_1}\times\ldots\times B_{X_n}} : B_{X_1} \times \ldots \times B_{X_n} \to X$ *is jointly Strong*-to-norm continuous.*

b) *There exist mappings $N_i : (0,+\infty) \to (0,+\infty)$ (depending only on T), Hilbert spaces $H_1,\ldots,H_n$ and bounded linear operators $T_i : X_i \to R_i$ satisfying that*

$$\|T(x_1,\ldots,x_n)\| \leq \prod_{i=1}^{n} \left(N_i(\varepsilon)\ \||x_i\||_{T_i} + \varepsilon\ \|x_i\|\right),$$

for each x_i in X_i and $\varepsilon > 0$.

Proof. a) ⇒ b) For each natural m, the mapping mT is is jointly Strong*-to-norm continuous on $B_{X_1} \times \ldots \times B_{X_n}$. Thus, by Lemma 3, there are Hilbert spaces H^i_m $(1 \leq i \leq n)$ and bounded linear operators $S_{i,m} : X_i \to H^i_n$ satisfying that

$$\|mT(x_1,\ldots,x_n)\| \leq \prod_{i=1}^{n} \left(\||x_i\||_{S_{i,m}} + \|x_i\|\right),$$

for all $(x_1,\ldots,x_n)$. We may assume $S_{i,m} \neq 0$, for all $m \in \mathbb{N}$, $1 \leq i \leq n$. Define

$$H_i := \bigoplus_{m\in\mathbb{N}}^{\ell_2} H^i_m,$$

and

$$S_i : X_i \to H_i$$

the bounded linear operator given by

$$S_i(x_i) := \left(\frac{1}{m\ \|S_{i,m}\|} S_{i,m}(x_i)\right),$$

and $N_i : (0,+\infty) \to (0,+\infty)$ by $N_i(\varepsilon) := \|S_{i,m(\varepsilon)}\|$, where

$$m(\varepsilon) = \inf\{m \in \mathbb{N} : \frac{1}{\sqrt[n]{m}} < \varepsilon\}.$$

Finally, given $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$ we have

$$\|m(\varepsilon)T(x_1, \ldots, x_n)\| \leq \prod_{i=1}^{n} \left(\|S_{i,m(\varepsilon)}(x_i)\| + \|x_i\|\right);$$

$$\|T(x_1, \ldots, x_n)\| \leq \prod_{i=1}^{n} \left(\frac{1}{\sqrt[n]{m(\varepsilon)}}\|S_{i,m(\varepsilon)}(x_i)\| + \frac{1}{\sqrt[n]{m(\varepsilon)}}\|x_i\|\right)$$

$$\|T(x_1, \ldots, x_n)\| \leq \prod_{i=1}^{n} \left(\|S_{i,m(\varepsilon)}\| \ \|S_i(x_i)\| + \varepsilon \ \|x_i\|\right)$$

$$\|T(x_1, \ldots, x_n)\| \leq \prod_{i=1}^{n} \left(N_i(\varepsilon) \ \|S_i(x_i)\| + \varepsilon \ \|x_i\|\right).$$

b) $\Rightarrow$ *a)* We proceed by mathematical induction on n. When $n = 1$, the statement follows from Proposition 1. Let us assume that our statement is true for all $k \leq n-1$. Suppose that $T : X_1 \times \cdots \times X_n \to X$ is a multilinear operator satisfying b). It can be easily seen, from *b*), that $T(x_{\lambda_1}, \ldots, x_{\lambda_n})$ tends to zero in norm, whenever (x_{λ_i}) is a bounded $S^*(X_i, X_i^*)$-null net in X_i.

For each $1 \leq i \leq n$, let (x_{μ_i}) be a net in B_{X_i}, converging, in the $S^*(X_i, X_i^*)$-topology to an element $x_0^i \in B_{X_i}$. Since, for each $1 \leq i \leq n$, $x_{\mu_i} - x_0^i$ is a bounded $S^*(X_i, X_i^*)$-null net in X_i, it follows that

$$\|T(x_{\mu_1} - x_0^1, \ldots, x_{\mu_n} - x_0^n)\| \to 0.$$

Then we deduce that

$$\begin{aligned}
(1)\ T(x_{\mu_1}, \ldots, x_{\mu_n}) - T(x_{\mu_1} - x_0^1, \ldots, x_{\mu_n} - x_0^n) &= +T(x_0^1, x_{\mu_2}, \ldots, x_{\mu_n}) \\
+T(x_{\mu_1}, x_0^2, x_{\mu_3}, \ldots, x_{\mu_n}) + \ldots &+ T(x_{\mu_1}, \ldots, x_{\mu_{n-2}}, x_0^{n-1}, x_{\mu_n}) \\
-T(x_0^1, x_0^2, x_{\mu_3}, \ldots, x_{\mu_n}) - \ldots &- T(x_{\mu_1}, \ldots, x_{\mu_{n-2}}, x_0^{n-1}, x_0^n) - \ldots \\
-(-1)^{n-1}T(x_{\mu_1}, x_0^2, \ldots, x_0^n) - \ldots &- (-1)^{n-1}T(x_0^1, \ldots, x_{\mu_{n-1}}, x_0^n) \\
-(-1)^{n-1}T(x_0^1, \ldots, x_0^{n-1}, x_{\mu_n}) &- (-1)^n T(x_0^1, \ldots, x_0^n)
\end{aligned}$$

tends to zero in norm.

Let $1 \leq j \leq n-1$. We observe that whenever, in the operator T we fix j-variables $x_0^{i_1}, \ldots, x_0^{i_j}$, then we get an $(n-j)$-multilinear operator which also satisfies statement *b*). So, by the induction hypothesis, the latter is an $(n-j)$-multilinear operator which is jointly Strong*-to-norm continuous on the cartesian product of the corresponding closed unit balls. This implies that every summand in the right hand side of equality (1) tends to $T(x_0^1, \ldots, x_0^n)$ in norm, which implies that $T(x_{\mu_1}, \ldots, x_{\mu_n})$ tends to $T(x_0^1, \ldots, x_0^n)$ in norm. Thus T is jointly Strong*-to-norm continuous on the cartesian product of the closed unit balls. $\square$

The next corollary follows now from the above result together with Grothendieck's inequality for C*-algebras [10, 17]. We recall that for each positive functional ϕ in the dual space of a C*-algebra A, $\|.\|_\phi$ denotes the seminorm on A defined by $\|x\|_\phi^2 = 2^{-1}\phi(xx^* + x^*x)$.

Corollary 5. *Let $A_1, \ldots, A_n$ be C*-algebras, X a Banach space and*

$$T : A_1 \times \cdots \times A_n \to X$$

a multilinear operator. Then the following are equivalent:

a) *$T|_{B_{A_1}\times\ldots\times B_{A_n}} : B_{A_1} \times \ldots \times B_{A_n} \to X$ is jointly Strong*-to-norm continuous.*
b) *There exist mappings $N_i : (0, +\infty) \to (0, +\infty)$ (depending only on T) and positive norm-one functionals $\phi_1 \in A_1^*, \ldots, \phi_n \in A_n^*$ satisfying that*

$$\|T(x_1, \ldots, x_n)\| \leq \prod_{i=1}^{n} \left(N_i(\varepsilon)\ \|x_i\|_{\phi_i} + \varepsilon\ \|x_i\|\right),$$

for each x_i in A_i and $\varepsilon > 0$. □

Corollary 6. *Let A be C*-algebra and P an element in $\mathcal{P}^n(A)$. Then the following are equivalent:*

a) *$P|_{B_A}$ is $S^*(A, A^*)$-to-norm continuous.*
b) *There exists a mapping $N : (0, +\infty) \to (0, +\infty)$, (depending only on P) and positive norm-one functionals $\phi_1, \ldots, \phi_n \in A^*$ satisfying that*

$$|P(x)| \leq \prod_{i=1}^{n} \left(N(\varepsilon)\ \|x\|_{\phi_i} + \varepsilon\right),$$

for each x in B_A and $\varepsilon > 0$. □

Corollary 7. *Let A be C*-algebra and P an element in $\mathcal{P}^n(A)$. Then the following are equivalent:*

a) *$P|_{B_{A^{**}}}$ is $S^*(A^{**}, A^*)$-to-norm continuous.*
b) *There exists a mapping $N : (0, +\infty) \to (0, +\infty)$, (depending only on P) and positive norm-one functionals $\phi_1, \ldots, \phi_n \in A^*$ satisfying that*

$$|P^{**}(x)| \leq \prod_{i=1}^{n} \left(N(\varepsilon)\ \|x\|_{\phi_i} + \varepsilon\right),$$

for each x in $B_{A^{**}}$ and $\varepsilon > 0$. □

Corollary 8. *Let $P \in \mathcal{P}^n(A)$ be an n-homogeneous polynomial whose restriction to the closed unit ball of A is $S^*(A, A^*)$-to-norm continuous. Then the Aron-Berner extension P^{**} is $S^*(A^{**}, A^*)$-to-norm continuous on $B_{A^{**}}$.*

Proof. Let $T : A\times\ldots\times A \to \mathbb{C}$ be the symmetric n-linear form satisfying that $P(x) = T(x, \ldots, x)$. By hypothesis, Lemma 2 and Corollary 5, there exists a mapping $N : (0, +\infty) \to (0, +\infty)$, (depending only on P) and positive norm-one functionals $\phi_1, \ldots, \phi_n \in A^*$ satisfying that

$$|T(x_1, \ldots, x_n)| \leq \prod_{i=1}^{n} \left(N(\varepsilon)\ \|x_i\|_{\phi_i} + \varepsilon\ \|x_i\|\right), \tag{2}$$

for each x_i in A and $\varepsilon > 0$. Let $z \in A^{**}$, $\|z\| \leq 1$. Since the $S^*(A^{**}, A^*)$ topology is compatible with the duality (A^{**}, A^*), by Alaoglu's Theorem, there exists a net (x_λ) in B_A converging in the $S^*(A^{**}, A^*)$ topology to z.

Since T^{**} is separately weak*-continuous and x_λ tends to z in the weak* topology, we deduce that

$$\lim |T^{**}(x_\lambda, x_2, \ldots, x_n)| = |T^{**}(z, x_2, \ldots, x_n)|,$$

for every $x_2, \ldots, x_n$. Now inequality (2) gives that

$$|T^{**}(z, x_2, \ldots, x_n)| \leq \left(N(\varepsilon)\ \|z\|_{\phi_i} + \varepsilon\right) \prod_{i=2}^{n} \left(N(\varepsilon)\ \|x_i\|_{\phi_i} + \varepsilon\right),$$

for every $z \in B_{A^{**}}$, $x_1, \ldots, x_n \in B_A$, $\varepsilon > 0$.

Similarly, we have

$$|P(z)| \leq \prod_{i=1}^{n} \left(N(\varepsilon)\ \|z\|_{\phi_i} + \varepsilon\right),$$

for every $\varepsilon > 0$ and $z \in B_{A^{**}}$, which, by Corollary 7 shows that P^{**} is $S^*(A^{**}, A^*)$-to-norm continuous on $B_{A^{**}}$. □

3. Main Result

Let A be a C*-algebra and let X be a Banach space. Let $\mathcal{P}^n_o(A, X)$ denote the space of all orthogonally additive elements in $\mathcal{P}^n(A, X)$. When X is the scalar field we just write $\mathcal{P}^n_o(A)$

A natural example of an orthogonally additive n-homogeneous scalar polynomial on a C*-algebra A can be built as follows: Let ϕ be a functional in A^*, the dual space of A. The law $x \mapsto P_\phi(x) := \phi(x^n)$ defines an n-homogeneous polynomial P_ϕ on A. Since, for each pair of mutually orthogonal elements $x, y \in A$, we have $(x + y)^n = x^n + y^n$, it follows that P_ϕ is orthogonally additive.

For each ϕ in A^*, the polynomial P_ϕ defined as above has another interesting properties that should be also considered. P_ϕ admits a natural extension to A^{**} that will be denoted by $P^{**}_\phi : A^{**} \to \mathbb{C}$, $P^{**}_\phi(z) = \phi(z^n)$. The functional ϕ is in the predual of A^{**} and hence ϕ is Strong*-to-norm continuous (see [19, Corollary 1.8.10]). Since the product of A^{**} is jointly $S^*(A^{**}, A^*)$ continuous on bounded sets (compare [19, Proposition 1.8.12]), it follows that

$$P^{**}_\phi|_{B_{A^{**}}} : B_{A^{**}} \to \mathbb{C}$$

is $S^*(A^{**}, A^*)|_{B_{A^{**}}}$-to-norm continuous. Thus, $P_\phi|_{B_A} : B_A \to \mathbb{C}$ is $S^*(A, A^*)$-to-norm continuous.

For each $\phi \in A^*$, let T_ϕ be the multilinear operator defined as follows:

$$T_\phi : A \times \ldots \times A \to \mathbb{C}$$

$$T_\phi(x_1, \ldots, x_n) = \phi(x_1 x_2 \ldots x_n).$$

Similar arguments to those given in the above paragraph show that T_ϕ is $S^*(A, A^*)$-to-norm continuous when restricted to the cartesian product of the closed unit ball of A.

Let $\mathcal{P}^n_{o,S^*}(A)$ be the space of all elements in $\mathcal{P}^n_o(A)$ whose restrictions to the closed unit ball of A are $S^*(A, A^*)$-to-norm continuous.

Lemma 9. *Let A be a C*-algebra and $n \in \mathbb{N}$. The mapping*

$$\Phi : A^* \to \mathcal{P}^n_{o,S^*}(A)$$

defined by $\Phi(\phi) = P_\phi$ is an injective linear operator.

Proof. The mapping is clearly linear and bounded, with $\|\Phi\| = 1$. Suppose that $P_\phi = P_\varphi$ for suitable ϕ, φ in A^*. Let a be a positive element in A. Then there exists a positive element x in A such that $x^n = a$. Therefore $\phi(a) = P_\phi(x) = P_\varphi(x) = \varphi(a)$. Since every element b in A can be written as a linear combination of positive elements in A, we have $\phi = \varphi$. □

Remark 10. The notion of orthogonality can also be defined in a natural way in a Banach lattice. For $C(K)$ spaces both notions of orthogonality (in the C^*-algebra sense and in the lattice sense) coincide. The results in our paper are inspired on the results in [16], which deal with the $C(K)$ case. The answer for $C(K)$ spaces was also proved independently in [2], where the authors also study the problem in the setting of lattices.

Let A be a C*-algebra. Following [12], we say that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is *strongly continuous* if for each $S^*(A, A^*)$-convergent net (x_λ) in A_{sa} with limit x the net $(f(x_\lambda))$ is $S^*(A, A^*)$-convergent to f(x).

Remark 11. Let A be a C*-algebra. Throughout the paper A_{sa} will stand for the self-adjoint part of A. For each element $x \in A_{sa}$ nd each $q > 0$, we define $x^{[q]} = (x^+)^q - (x^-)^q$, where x^+ and x^- are the positive and negative part of x, respectively (compare [12, 1.1.10]).

For each $n \in \mathbb{N}$ and $\psi \in A^*_{sa}$ the restriction of $P_{[\psi]}(x)(:= \psi(x^{[n]}))$ to the closed unit ball of A_{sa} is Strong*-to-norm continuous (although $P_{[\psi]}$ is not a polynomial if n is even). Let (x_λ) be a $S^*(A, A^*)$ convergent net in $B_{A_{sa}}$ with limit x.

Let us denote $g, h : \mathbb{R} \to \mathbb{R}$ the continuous functions defined by

$$g(t) = \begin{cases} t, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}, \text{ and } h(t) = \begin{cases} t, & \text{if } t \leq 0 \\ 0, & \text{if } t > 0 \end{cases}$$

It follows from [12, Proposition 2.3.2] that g and h are strongly continuous. Therefore $x^+_\lambda = g(x_\lambda)$ converges in the $S^*(A, A^*)$ topology to $x^+ = g(x)$ and hence, by the joint Strong*-continuity of the product on B_A, we deduce that $(x_\lambda^+)^n \to (x^+)^n$, with respect to the $S^*(A, A^*)$ topology. Similarly, $(x_\lambda^-)^n = (h(x_\lambda)^n)$ converges to $(x^-)^n$ in the $S^*(A, A^*)$ topology. Thus $(x_\lambda^{[n]})$ converges to $x^{[n]}$ in the $S^*(A, A^*)$ topology.

Now, by the Strong*-continuity of ψ, we deduce that $P_{[\psi]}(x_\lambda) = \psi(x_\lambda^{[n]})$ tends to $\psi(x^{[n]}) = P_{[\psi]}(x)$ in norm.

Proposition 12. *Let M be a positive number and let L be a locally compact subset of $(0, M]$ such that $L \cup \{0\}$ is compact. Let $C_0(L)$ denote the C*-algebra of all complex continuous functions on $L \cup \{0\}$ vanishing at 0. The law*

$$\phi \mapsto P_\phi$$

defines a linear bijection from $C_0(L)^*$ onto $\mathcal{P}_o^n(C_0(L))$. A posteriori, we get that $\mathcal{P}_o^n(C_0(L)) = \mathcal{P}_{o,S^*}^n(C_0(L))$

Proof. The proof is the same, with the obvious modifications, as in the $C(K)$ case ([16]). □

Let A be a C*-algebra. A mapping $\nu : A_{sa} \to \mathbb{R}$ is said to be a *local quasi-linear functional* if ν is bounded on the closed unit ball of A_{sa} and for each $x \in A_{sa}$, ν is linear on the smallest norm closed subalgebra of A containing x.

The solution of the Mackey-Gleason problem for von Neumann algebras obtained by L. Bunce and J. D. M. Wright in [4, 5] is one of the fundamental contributions to the theory of von Neumann algebras. In the just quoted papers, the authors show that for every von Neumann algebra A having no direct summand of Type I_2, and for every bounded and orthogonally additive function m defined on the lattice of projections of A, m has a unique extension to a bounded linear functional on A. As a consequence, every quasi-linear functional on a von Neumann algebra A, where A has no direct summands of Type I_2, is linear.

Let A be a C*-algebra. We recall that a scalar function $\mu : A_{sa} \to \mathbb{R}$ is said to be *uniformly continuous on $B_{A_{sa}}$* with respect to the $S^*(A, A^*)$-topology if for every $\varepsilon > 0$ there exists an $S^*(A, A^*)$-open symmetric neighborhood of 0, V, satisfying that for each $x, y \in B_{A_{sa}}$ with $x - y \in V$, we have $|\mu(x) - \mu(y)| < \varepsilon$.

The following result derives from [6, Theorems 3.1 and 3.3].

Proposition 13. *Let A be a C*-algebra which has no quotient isomorphic to $M_2(\mathbb{C})$. Let $\nu : A_{sa} \to \mathbb{R}$ be a local quasi-linear mapping, which is uniformly continuous on $B_{A_{sa}}$, when the latter is equipped with the $S^*(A, A^*) \equiv S^*(A^{**}, A^*)|_A$ topology. Then ν is linear.*

Proof. By [6, Lemma 2.2], A^{**} has no direct summand of type I_2. Let $\pi : A \to L(H)$ be the universal representation of A (see [19, Definition 1.16.5]).

By [12, Theorem 3.7.7] and [19, Proposition 1.16.2], π extends to a weak*-continuous representation $\widetilde{\pi} : A^{**} \to L(H)$ satisfying that $\widetilde{\pi}(A^{**})$ is weak*-closed in $L(H)$ and $\widetilde{\pi} : A^{**} \to \widetilde{\pi}(A^{**})$ is a weak*-continuous isomorphism. In particular $\widetilde{\pi}$ is $S^*(A^{**}, A^*)$-to-$S^*(\widetilde{\pi}(A^{**}), \widetilde{\pi}(A^{**})_*)$ continuous (see [15, Remark 2.11] or [18, Corollary 3 and Remark 3]).

By [3, Corollary] and [18, Remark 3], it follows that

$$S^*(L(H), L(H)_*)|_{\widetilde{\pi}(A^{**})} = S^*(\widetilde{\pi}(A^{**}), \widetilde{\pi}(A^{**})_*).$$

Since we also have $S^*(A, A^*) = S^*(A^{**}, A^*)|_A$ we deduce that $[\nu] : \pi(A_{sa}) \to \mathbb{R}$, $[\nu](\pi(x)) = x$, $(x \in A_{sa})$ is a local quasi-linear functional which is uniformly continuous on the closed unit ball of $\pi(A_{sa})$. Finally, Theorem 3.1 in [6] implies that $[\nu]$ (and hence ν) is linear. □

In the above result the hypothesis of ν being uniformly continuous on $B_{A_{sa}}$ is cannot always easily checked. In [6, Theorem 3.1], this hypothesis is needed to guarantee the existence of an extension of ν to A_{sa}^{**}. In our particular setting of orthogonally additive polynomials, the existence of this extension will be assured by the Aron-Berner transpose. Similar ideas to those presented in [6, Theorem 3.1] are also enough to get the statement of following lemma. The proof is included here for completeness reasons.

Lemma 14. *Let A be a C^*-algebra which has no quotient isomorphic to $M_2(\mathbb{C})$. Let $\nu : A_{sa} \to \mathbb{R}$ be a local quasi-linear mapping satisfying the following:*

a) *ν is $S^*(A, A^*)$-to-norm continuous on $B_{A_{sa}}$*
b) *ν admits an extension $\overline{\nu} : A_{sa}^{**} \to \mathbb{R}$ which is $S^*(A^{**}, A^*)$-to-norm continuous on $B_{A_{sa}^{**}}$.*
c) $\sup\{|\overline{\nu}(p)| :$ *p is a projection in* $A^{**}\} < \infty$.

Then $\overline{\nu}$ is local quasi-linear and ν and $\overline{\nu}$ are linear.

Proof. Let us observe that every norm convergent sequence is bounded and converges also in the Strong* topology. Thus ν and $\overline{\nu}$ are clearly norm continuous.

We shall now show that $\overline{\nu}$ is local quasi-linear. To this end, let z be a non-zero element in A_{sa}^{**}. We may assume that $\|z\| = 1$. Since the $S^*(A^{**}, A^*)$ topology is compatible with the duality (A^{**}, A^*), by Alaoglu's Theorem, there exists a net (x_λ) in B_A converging in the $S^*(A^{**}, A^*)$ topology to z.

Since $\overline{\nu}$ is an extension of ν and $\overline{\nu}$ is $S^*(A^{**}, A^*)$-to-norm continuous on $B_{A_{sa}^{**}}$, we have $\nu(x_\lambda) = \overline{\nu}(x_\lambda) \to \overline{\nu}(z)$. Moreover, for each $k \in \mathbb{N}$, $\nu(x_\lambda^k) = \overline{\nu}(x_\lambda^k) \to \overline{\nu}(z^k)$. This implies that whenever Υ_1 and Υ_2 are real polynomials with zero constant term, we have

$$\nu(\Upsilon_1(a_\lambda)) + \nu(\Upsilon_2(a_\lambda)) = \nu(\Upsilon_1(a_\lambda) + \Upsilon_2(a_\lambda)),$$

by the local quasi-linearity of ν. And, by the joint $S^*(A^{**}, A^*)$-continuity of the product on bounded sets we deduce $\Upsilon_i(a_\lambda) \to \Upsilon_i(z)$ in the $S^*(A^{**}, A^*)$-topology ($i = 1, 2$). The corresponding continuity of $\overline{\nu}$ gives

$$\overline{\nu}(\Upsilon_1(z)) + \overline{\nu}(\Upsilon_2(z)) = \overline{\nu}(\Upsilon_1(z) + \Upsilon_2(z)).$$

Standard norm-continuity arguments allow us to conclude that $\overline{\nu}$ is linear in the norm closed subalgebra of A^{**}_{sa} generated by z. This shows that $\overline{\nu}$ is local quasi-linear.

Let $p_1, p_2, \ldots, p_k$ be orthogonal projections in A^{**}_{sa} and denote $z = p_1 + \frac{1}{2}p_2 + \ldots + \frac{1}{2^{k-1}}p_k + \frac{1}{2^k}(1 - p_1 - p_2 - \ldots - p_k)$. The sequence $(z^m)_m$ converges in norm to p_1. This implies that p_1 lies is the norm closed subalgebra of A^{**}_{sa} generated by z. The sequence $((2z - 2p_1)^m)_m$ converges in norm to p_2. This argument can be repeated to show that $p_1, p_2, \ldots, p_k$ all lie in the norm closed subalgebra generated by z. The local quasi-linearity of $\overline{\nu}$ gives that $\overline{\nu}$ is a finitely additive measure on the projections of A^{**}.

Let us observe that, by [6, Lemma 2.2], A^{**} has no direct summands of type $M_2(\mathbb{C})$. Now the Bunce-Wright-Mackey-Gleason Theorem ([4, Theorem B] or [5, Theorem B]) guarantees that $\overline{\nu}$ extends to a bounded linear functional on A^{**}_{sa}. Finally, by local quasi-linearity and spectral theory we conclude that ν and $\overline{\nu}$ are linear. $\square$

We will need the following technical lemma.

Lemma 15. *Let A be a C^*-algebra, $n \in \mathbb{N}$ and $P, Q \in \mathcal{P}^n(A)$. Suppose that $P(a) = Q(a)$, for every $a \in A_{sa}$. Then $P(x) = Q(x)$, for every $x \in A$.*

Proof. We shall proceed by mathematical induction on n. When $n = 1$ the statement follows immediately. Let us then assume that our thesis holds for $n - 1$. Assume $P, Q \in \mathcal{P}^n(A)$ are such that $P(a) = Q(a)$, for every $a \in A_{sa}$. Let $T, S : A \times \ldots \times A \to \mathbb{C}$ be the symmetric n-linear forms satisfying that $P(x) = T(x, \ldots, x)$ and $Q(x) = S(x, \ldots, x)$ for every $x \in A$. By hypothesis we know that $T(a, \ldots, a) = S(a, \ldots, a)$, for every $a \in A_{sa}$, and hence, by the polarization formula [11, p.6], $T(a_1, \ldots, a_n) = S(a_1, \ldots, a_n)$ for every $a_1, \ldots, a_n \in A_{sa}$.

Fixed $a \in A_{sa}$, we define the symmetric $n - 1$ linear forms

$$T_a, S_a : A \times \ldots \times A \to \mathbb{C}$$

as

$$T_a(x_1, \ldots, x_{n-1}) := T(a, x_1, \ldots, x_{n-1})$$

and

$$S_a(x_1, \ldots, x_{n-1}) := S(a, x_1, \ldots, x_{n-1}).$$

We take their associated $(n-1)$-homogeneous polynomials $P_a, Q_a : A \to \mathbb{C}$ given by $P_a(x) = T_a(x, \ldots, x)$ and $Q_a(x) = S_a(x, \ldots, x)$, for every $x \in A$. We have seen above that, for each $b \in A_{sa}$, $P_a(b) = Q_a(b)$ so, by the induction hypothesis, we conclude that $P_a(x) = Q_a(x)$ for every $x \in A$. Then we have $T(a, x \ldots, x) = S(a, x \ldots, x)$, for every $a \in A_{sa}$ and $x \in A$.

Now, using the linearity of T and S it follows that $P(x) = T(x, \ldots, x) = S(x, \ldots, x) = Q(x)$, for every $x \in A$. □

We can now prove our main result:

Theorem 16. *Let A be a C*-algebra which has no quotient isomorphic to $M_2(\mathbb{C})$. For each P in $\mathcal{P}^n_{o,S^*}(A)$ there exists φ in A^* satisfying $P(x) = \varphi(x^n)$, for each element x in A.*

Proof. Suppose that $P \in \mathcal{P}^n_{o,S^*}(A)$. Let $T : A \times \ldots \times A \to \mathbb{C}$ be the symmetric n-linear form satisfying that $P(x) = T(x, \ldots, x)$ and let $T^{**} : A^{**} \times \ldots \times A^{**} \to \mathbb{C}$ denote the corresponding Aron-Berner extension of T. P^{**} will stand for the polynomial on A^{**} defined by $P^{**}(z) = T^{**}(z, \ldots, z)$, ($z$ in A^{**}).

Let us define two mappings $\mu : A \to \mathbb{C}$ and $\overline{\mu} : A^{**} \to \mathbb{C}$ given by the laws

$$\mu(a) = \Re e P(a^{1/n}), \text{ for each } a \in A^+,$$
$$\mu(b) = \mu(b^+) - \mu(b^-), \text{ for each } b \in A_{sa},$$
$$\mu(b + ic) = \mu(b) + i\mu(c), \ (\forall b, c \in A_{sa}),$$

and

$$\overline{\mu}(a) = \Re e P^{**}(a^{1/n}), \text{ for each } a \in (A^{**})^+,$$
$$\overline{\mu}(b) = \overline{\mu}(b^+) - \overline{\mu}(b^-), \text{ for each } b \in A^{**}_{sa},$$
$$\overline{\mu}(b + ic) = \overline{\mu}(b) + i\overline{\mu}(c), \ (\forall b, c \in A^{**}_{sa}).$$

It is clear that $\sup\{|\overline{\mu}(p)| : p \text{ is a projection in } A^{**}\} \leq \|P^{**}\| < \infty$.

Since P is $S^*(A, A^*)$-to-norm continuous on the closed unit ball of A, we deduce from Corollary 8 and Lemma 2 that T (respectively, T^{**}) is jointly $S^*(A, A^*)$-continuous on $B_A \times \ldots \times B_A$ (respectively, jointly $S^*(A^{**}, A^*)$-continuous on $B_{A^{**}} \times \ldots \times B_{A^{**}}$).

As we have seen in Remark 11, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} \sqrt[n]{t}, & \text{if } t \geq 0 \\ -\sqrt[n]{-t}, & \text{if } t < 0 \end{cases},$$

is strongly continuous (compare [12, Proposition 2.3.2]). This fact together with the above paragraph imply that μ (respectively, $\overline{\mu}$) is $S^*(A, A^*)$-to-norm continuous on B_A (respectively, $S^*(A^{**}, A^*)$-to-norm continuous on $B_{A^{**}}$). We also have $\mu(a) = \overline{\mu}(a)$, for each $a \in A$.

In order to simplify notation, we write ν for the restriction of μ to A_{sa} and $\overline{\nu}$ for the restriction of $\overline{\nu}$ to A^{**}_{sa}.

We claim that ν is local quasi linear. Indeed, for each $x \in A_{sa}$ let C_x be the smallest norm closed subalgebra of A_{sa} containing x. It is known that C_x is isomorphically isometric to $C_0(L, \mathbb{R})$ for a locally compact Hausdorff space $L \subset (0, \|x\|]$ with $L \cup \{0\}$ compact.

Since $P \in \mathcal{P}^n_o(A)$ (and thus, $\Re e P \in \mathcal{P}^n_o(A_{sa})$) we have that Q, the restriction of $\Re e P$ to C_x, also is orthogonally additive. Now, by Proposition 12, we know that there exists a functional $\varphi_x \in (C_x)^*$ such that $Q(z) = \varphi_x(z^n)$ for every $z \in C_x$. Now, we take a positive element $a \in C_x$ and a positive element $y \in C_x$ with $y^n = a$. Then $\nu(a) = \nu(y^n) = Q(y) = \varphi_x(y^n) = \varphi(a)$. This implies, immediately, that $\nu = \varphi$ on C_x. Therefore $\nu|_{C_x} : C_x \to \mathbb{R}$ is linear, which implies that ν is local quasi-linear.

Now, Lemma 14 gives that $\nu : A_{sa} \to \mathbb{R}$ is linear. As a consequence

$$\mu(x^n) = \nu(x^n) = \nu((x^+)^n + (-1)^n (x^-)^n) = \nu((x^+)^n) + (-1)^n \nu((x^-)^n)$$

$$\text{(by definition of } \nu) = \Re\text{e}P(x^+) + (-1)^n \Re\text{e}P(x^-) = \Re\text{e}P(x^+) + \Re\text{e}P(-x^-)$$

$$\text{(by orthogonality)} = \Re\text{e}P(x^+ - x^-) = \Re\text{e}P(x),$$

for every $x \in A_{sa}$. Similarly, we can get a linear functional $\psi : A_{sa} \to \mathbb{R}$ such that $\psi(x^n) =: \Im\text{m}P(x)$, for every $x \in A_{sa}$. We then have a linear form $\varphi(a + ib) := \nu(a) + i\psi(b)$ in A^* satisfying that $P(a) = \varphi(a^n)$, for every $a \in A_{sa}$.

Finally, let $O \in \mathcal{P}^n_{o,S^*}(A)$, be the polynomial defined by $O(x) := \varphi(x^n)$. Since, for each $a \in A_{sa}$, $O(a) = \varphi(a^n) = P(a)$, Lemma 15 gives the statement. □

Corollary 17. *Let A be a C*-algebra which has no quotient isomorphic to $M_2(\mathbb{C})$ and let $n \in \mathbb{N}$. The operator*

$$\Phi : A^* \to \mathcal{P}^n_{o,S^*}(A)$$

$$\Phi(\varphi) = P_\varphi$$

is a bicontinuous linear bijection from A^* onto $\mathcal{P}^n_{o,S^*}(A)$. Moreover, for each $\phi \in A^*_{sa}$, $\|\phi\| = \|P_\phi\|$ while $\|\phi\| \geq \|P_\phi\| \geq 2\|\phi\|$, for all $\phi \in A^*$. □

Let A be a C*-algebra. By the bilinear Grothendieck's inequality for C*-algebras (see [17, 10]), we clearly have $\mathcal{P}^2_{S^*}(A) = \mathcal{P}^2(A)$. The failure of a trilinear Grothendieck's inequality implies that $\mathcal{P}^n_{S^*}(A) \neq \mathcal{P}^n(A)$, for $n \geq 3$. It seem natural to ask whether $\mathcal{P}^n_{o,S^*}(A) = \mathcal{P}^n_o(A)$.

Problem 18. *Let A be a C*-algebra. Let P be an orthogonally additive n-homogeneous polynomial on A. Is P automatically $S^*(A, A^*)$-to-norm continuous on bounded sets?*

4. Vector valued polynomials

We consider now the space $\mathcal{P}^n_{o,S^*}(A, X)$ be the space of all elements in $\mathcal{P}^n_o(A, X)$ whose restrictions to the closed unit ball of A are $S^*(A, A^*)$-to-norm continuous.

As in the scalar case, it is easy to see that, for every $m \in \mathcal{L}(A, X)$ we can define a polynomial $P_m \in \mathcal{P}^n_{o,S^*}(A, X)$ by

$$P_m(a) = m(a^n).$$

We have

Corollary 19. *Let A be a C^*-algebra which has no quotient isomorphic to $M_2(\mathbb{C})$. For each P in $\mathcal{P}^n_{o,S^*}(A,X)$ there exists $m \in \mathcal{L}(A,X)$ satisfying $P(a) = m(a^n)$, for each element a in A. Therefore, the operator*

$$\Phi : \mathcal{L}(A,X) \longrightarrow \mathcal{P}^n_{o,S^*}(A,X)$$

$$\Phi(m) = P_m$$

is a bicontinuous linear bijection from $\mathcal{L}(A,X)$ onto $\mathcal{P}^n_{o,S^*}(A,X)$.

Proof. Everything is either immediate or similar to the scalar case except (maybe) for the surjectivity. We prove this. Let P be as in the hypothesis. In that case, for every $x^* \in X^*$, $x^* \circ P \in \mathcal{P}^n_{o,S^*}(A)$. Hence, Theorem 16 guarantees the existence of an element $m_{x^*} \in A^*$ such that, for every $a \in A$,

$$x^* \circ P(a) = m_{x^*}(a^n).$$

So, we define a mapping

$$m : A \longrightarrow X^{**}$$

by

$$\langle m(a), x^* \rangle = m_{x^*}(a).$$

We need to check that

- $m(a)$ is indeed in X^{**}. First we check that $m(a) : X^* \longrightarrow \mathbb{K}$ is linear. We first consider the case when $a \geq 0$. Then

$$\langle m(a), \alpha x^* + \beta y^* \rangle = m_{\alpha x^* + \beta y^*}(a) = ((\alpha x^* + \beta y^*) \circ P)(a^{\frac{1}{n}}) =$$
$$= \alpha (x^* \circ P)(a^{\frac{1}{n}}) + \beta (y^* \circ P)(a^{\frac{1}{n}}) = \alpha m_{x^*}(a) + \beta m_{y^*}(a) =$$
$$= \alpha \langle m(a), x^* \rangle + \beta \langle m(a), y^* \rangle$$

  Using this we can prove that $m(a) : X^* \longrightarrow \mathbb{K}$ is linear for a general $a \in A$ using the decomposition of a as linear combination of positive elements and standard reasonings

  We now have to check that $m(a) : X^* \longrightarrow \mathbb{K}$ is bounded, but this is easier.
- m is linear: for every $x^* \in X^*$,

$$\langle m(\alpha a + \beta b), x^* \rangle = m_{x^*}(\alpha a + \beta b) =$$
$$= \alpha m_{x^*}(a) + \beta m_{x^*}(b) = \langle \alpha m(a) + \beta m(b), x^* \rangle$$

- m is bounded: Let $a \in B_A$. We first assume that a is positive. Then

$$\|m(a)\| = \sup_{x^* \in B_{X^*}} |\langle m(a), x^* \rangle| = \sup_{x^* \in B_{X^*}} |x^* \circ P(a^{\frac{1}{n}})| = \|P(a^{\frac{1}{n}})\| \leq \|P\|.$$

  Again, we can remove now the condition that a be positive.
- For every $a \in A$, $m(a^n) = P(a)$: For every $x^* \in X^*$ we have

$$\langle m(a^n), x^* \rangle = m_{x^*}(a^n) = x^* \circ P(a)$$

  and the desired inequality follows. A posteriori, we have that for every positive $a \in A$, $m(a) \in X$, and it follows that m is actually X-valued.

□

References

[1] Aron, R. M. and Berner P. D., A Hahn-Banach extension theorem for analytic mappings, *Bull. Soc. Math. France* **106**, 3-24 (1978).

[2] Benyamini, Y., Lassalle and Llavona, J. G., Homogeneous orhtogonally additive polynomials on Banach lattices, *Bull. London Math. Soc.* **38**, 459-469 (2006).

[3] Bunce, L. J., Norm preserving extensions in JBW*-triple preduals, *Quart. J. Math. Oxford* **52**, no. 2, 133-136 (2001).

[4] Bunce, L. J. and Wright, J. D. M., The Mackey-Gleason problem, *Bull. Amer. Math. Soc.* **26**, no. 2, 288-293 (1992).

[5] Bunce, L. J. and Wright, J. D. M., The Mackey-Gleason problem for vector measures on projections in von Neumann algebras, *J. London Math. Soc.* (2) **49**, no. 1, 133-149 (1994).

[6] Bunce, L. and Wright, J. D. M., The quasi-linear problem for C*-algebras, *Pacific J. Math.* **172**, No. 1, 41-47 (1996).

[7] Drewnowski, L. and Orlicz, W., Continuity and representation of orthogonally additive functionals, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **17**, 647û-653 (1969).

[8] Friedman, N. and Katz M., Additive functionals of Lp spaces, *Canad. J. Math.* **18**, 1264û1271 (1966).

[9] Friedman, N. and Katz M., A representation theorem for additive functionals, *Arch. Rational Mech. Anal.* **21** 49û-57 (1966).

[10] Haagerup, U., The Grothendieck inequality for bilinear forms on C*-algebras, *Avd. Math.* **56**, 93-116 (1985).

[11] J. Mújica, *Complex Analysis in Banach spaces*, Math. Studies No. 120. North-Holland, 1986.

[12] Pedersen, G. K., *C*-algebras and their automorphism groups*, Academic Press, 1979.

[13] Peralta, A. M., Topological characterisation of weakly compact operators revisited, preprint 2006.

[14] Peralta, A. M., Villanueva, I., Wright, J.D.M. and Ylinen, K., Topological characterisation of weakly compact operators, to appear in J. Math. Anal. Appl.

[15] Peralta, A. M., Villanueva, I., Wright, J.D.M. and Ylinen, K., The Strong*-topology and quasi completely continuous operators on Banach spaces, preprint 2006.

[16] Pérez, D. and Villanueva, I., Orthogonally additive polynomials on spaces of continuous functions, *J. Math. Anal. Appl.* **306**, 97–105, (2005)

[17] Pisier, G., Grothendieck's theorem for non commutative C*-algebras with an appendix on Grothendieck's constant, *J. Funct. Anal.* **29**, 397-415 (1978).

[18] Rodríguez Palacios, Á., On the strong* topology of a JBW*-triple, *Quart. J. Math. Oxford* (2) **42**, 99-103 (1991).

[19] Sakai, S., *C*- and W*-algebras*, Springer-Verlag, Berlin-New York, 1971.

[20] Sundaresam, K., Geometry of spaces of polynomials on Banach lattices, in: *Applied Geometry and Discrete Mathematics, in: DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, vol. 4, Amer. Math. Soc., Providence, RI, 1991, pp. 571û586.

[21] M. Takesaki, *Theory of operator algebras I*, Springer Verlag, New York-Heidelberg, 1979.

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