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Sequential strong measurements and the heat vision effect

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Abstract. We study the scenarios where a finite set of non-demolition von Neumann measurements is available. We note that, in some situations, the repeated application of such measurements allows estimation of an infinite number of parameters of the initial quantum state, and we illustrate the point with a physical example. We also study how the system under consideration is perturbed after several rounds of projective measurements. While in the finite dimensional case the effect of this perturbation always saturates, there are instances of infinite dimensional systems where such a perturbation is accumulative, and the act of retrieving information about the system increases its energy indefinitely (i.e. we have ‘heat vision’). We analyze this effect and discuss a specific physical system with two dichotomic von Neumann measurements where heat vision is expected to show.

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1. Introduction

In a classical system, the ability to carry out two different measurements of d outcomes each can only retrieve, at most, $2 \log_2(d)$ bits of information. Indeed, if we measure the height of an individual and then its weight, we shall not expect to gather more information by measuring its height one more time. However, in the quantum world, two measurements $+$, $-$ will not commute in general. Given three instants of time $t_0 \leq t_1 \leq t_2$, this can lead to ‘paradoxical’ situations where the output of measurement $+$ at time t_0 may differ from the output at time t_2 if there has been an intermediate measurement of $-$ at time t_1 . In that case, it is natural to wonder if this second outcome of $+$ is just random noise or, in contrast, contains meaningful information about the initial state of our quantum system, in which case sequential measurements could enhance our tomographic abilities.

Sequential measurement schemes have already proven useful in the so-called ‘weak’ measurement scenario. In this scenario, a probe is measured after weakly interacting with the system, which is followed by a standard (or ‘strong’) projective measurement—usually postselecting one of the outcomes. This measurement scenario, initially proposed by Aharonov *et al* [1], has some unexpected features that made it controversial [2]. Now well understood [3], it has led to important steps forward in both theory and experiments. In the former, it has helped in solving apparent quantum paradoxes, such as Hardy’s [4], whereas in the latter it has allowed us to observe the average trajectories of single photons [5] or to observe the spin-Hall effect of light [6]. In [7], Mitchison *et al* went beyond the case of two subsequent measurements using arbitrarily long sequences of weak measurements in order to give a characterization of counterfactual quantum computation.

However, the possible applications of many consecutive strong (i.e. invasive, not weak) non-commuting measurements for tomography seem to be absent from the scientific literature, probably because nothing especially interesting is expected to happen (in a completely different context, however, experiments involving sequential strong measurements were proposed

recently by Fritz [8] to test the validity of quantum theory). This is certainly peculiar if one takes into account that, given two non-commuting projectors P, P' , the operator space spanned by the positive operator-valued measure (POVM) elements $\{P, PP'P, P'PP', (\mathbb{I} - P)P'(\mathbb{I} - P), \dots\}$ describing consecutive measurements may well be infinite dimensional, and so the corresponding estimated probabilities can give us access to an infinite number of parameters characterizing the state of our system (figure 2).

The current experimental credo, however, does not echo this simple observation: even when non-demolition interactions are available, systems are typically discarded after a single strong measurement (e.g. in Wigner function estimation [10]). Although in many situations this classical approach works perfectly, the use of sequential strong measurements could be especially relevant for experimental situations where either (i) only a small number—say, two—of different measurement setups are available at the optical table or (ii) initializing the system is a highly costly task. In (i), instead of having access to only two parameters of the system, by using sequential strong measurements one is in principle able to obtain arbitrarily many. In (ii), for each initialization of the system, instead of gaining information about a single parameter, we add to this information relevant knowledge about many others.

We will illustrate these ideas in the first part of the paper, using as a guiding example the particle-in-a-box model, which we refer to as the ‘case study’. In particular, we will show how the statistical analysis of two *fixed* dichotomic von Neumann measurements allows, if used sequentially, estimation of the full probability density of the position of the particle.

In the second part of the paper, also using the particle-in-a-box model as an illustrative example, we analyze the evolution of systems whose dynamics is driven by *random* sequences of von Neumann measurements alone. Note that this scenario is in sharp contrast to the Aharonov–Vardi mechanism [9], where a fast *selected* sequence of projective measurements can drive a system from any initial state to any desired final one. Our main contribution there will be to show that in infinite dimensions a rare phenomenon occurs: as opposed to the finite dimensional case, where any quantum state subject to random measurements thermalizes after a time, there are instances of finitely many von Neumann measurements with finitely many outcomes whose sequential application leads to a non-convergent dynamics in infinite dimensional systems.

We show how this new phenomenon is always associated with an unbounded energy increase (hence its name ‘heat vision’) and can be observed already in scenarios with just two dichotomic von Neuman measurements, like our case study. Heat vision is a subtle purely infinite dimensional effect. Indeed, in the two-measurement/two outcome case in finite dimensions, Jordan’s lemma² ensures that there are fixed points of the evolution map with entropy ≤ 1 . Since this happens for *any* finite dimension, one cannot understand the heat vision effect as a limit of finite dimensional behaviors. Although this paper is mainly mathematical, we also show how, to a certain extent, the heat vision effect can already be observed in current ion trap experiments.

The paper is organized as follows. In section 2, we will describe the particle-in-a-box model, which will be used to exemplify the concepts developed in the paper. This model is simple enough to allow for a complete analytical study. On the other hand, it is a relevant physical model with enough complexity to exhibit the purely infinite dimensional effects we

² Jordan’s lemma lies at the basis of many results and protocols in quantum information [11]. Its breakdown in infinite dimensions, as shown here in a simple physical model, could have future surprises in the physics of infinite dimensional systems for quantum information purposes.

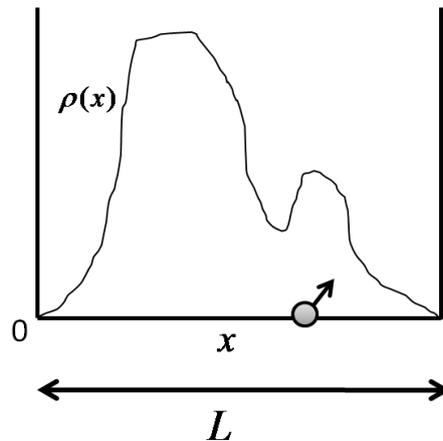


Figure 1. Our case study. A neutral spin-1/2 1D particle confined in a box of length L .

want to illustrate. Then, in section 3, we will show how to perform partial tomography of such a system by alternately measuring two different dichotomic properties. In section 4, we will study the general dynamics of a particle subject to sequential projective measurements, distinguishing between finite and infinite dimensional systems. We will arrive at the concept of heat vision and show that our case study exhibits this effect. In section 5, we will briefly suggest physical implementations of sequential measurement schemes. Finally, we will present our conclusions.

2. The case study

In this paper, we will illustrate our ideas by referring to a specific physical model. What follows is a description of such a case study; see figure 1.

Consider a spin-1/2 neutral particle in a rectangular box of size $L \times L_0 \times L_0$, with one of its vertices situated at the origin of coordinates. We will assume that $L_0 \ll L$, so this system can be regarded as a one-dimensional (1D) object subject to the potential

$$V(x) = \begin{cases} 0, & \text{for } 0 < x < L, \\ \infty, & \text{otherwise.} \end{cases} \quad (1)$$

The Hamiltonian describing the evolution of such a particle is thus

$$H = \frac{p^2}{2m} + V(x), \quad (2)$$

where m denotes the mass of the particle. Taking $\hbar = 1$, the eigenvectors of this system are $\{|n\rangle : n \in \mathbb{N}^+\}$ with $\langle x|n\rangle =: \Psi_n(x) = \sqrt{\frac{2}{L}} \sin(\frac{\pi n x}{L})$, each with the associated energy $E_n = \pi^2 n^2 / 2mL^2$. The spin degree of freedom of the particle can be modeled through a 2D Hilbert space \mathbb{C}^2 . Along these pages, the three pairs of kets $\{|0\rangle, |1\rangle\}$, $\{|+\rangle, |-\rangle\}$ and $\{|+i\rangle, |-i\rangle\}$ will represent the eigenstates of the Pauli matrices σ_z , σ_x and σ_y , respectively. It follows that the Hilbert space $\mathcal{H} \otimes \mathbb{C}^2$ where this particle lives can then be expanded in the basis $\{|n, \pm i\rangle : n \in \mathbb{N}^+\}$. As usual, quantum states in this scenario will be regarded as positive semidefinite normalized elements of $S_1 = \{A : \text{tr}(|A|) < \infty\}$, the set of trace-class operators. Not to be

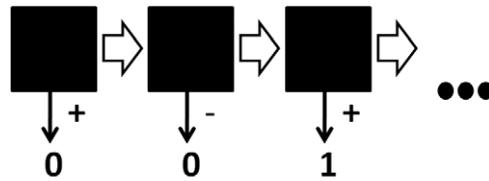


Figure 2. Tomography based on sequential measurements. Measurements + and – are sequentially applied to our physical system. Iterating the experiment we thus obtain several strings of bits, whose statistical analysis will allow us to estimate the probability density of the particle in the box. During this process, the system can accumulate an arbitrary amount of energy.

confused with the class of operators $S_2 = \{A : \text{tr}(AA^\dagger) < \infty\}$, which will also play an important role shortly.

Suppose now that our technology allows carrying out almost instantaneous von Neumann spin measurements along the \hat{u}_z -direction over such a particle without affecting its canonical degrees of freedom. Take $k \in \mathbb{R}^+$ to be a scale factor, whose role will be clear later. Defining $|\phi^\pm(x)\rangle \equiv \cos(kx)|0\rangle \pm \sin(kx)|1\rangle$, then, the two von Neumann measurements + and – associated with the projectors

$$F^\pm = \int |x\rangle\langle x| \otimes |\phi^\pm(x)\rangle\langle\phi^\pm(x)| dx \quad (3)$$

can be physically realized by applying a magnetic field $\pm\vec{B}$ along the \hat{u}_y direction with an intensity varying linearly with the x coordinate³, measuring the spin and then applying the inverse field $\mp\vec{B}$. Indeed, if

$$\vec{B}|_{y,z=0} = -(bx)\hat{u}_y$$

and the magnetic interaction is mediated through a Hamiltonian $H_s = -\mu\vec{B} \cdot \vec{\sigma}$, one can check that

$$F^\pm = e^{\mp iH_s\Delta t} (\mathbb{I} \otimes |0\rangle\langle 0|) e^{\pm iH_s\Delta t}, \quad (4)$$

provided that we switch the field \vec{B} for a time $\Delta t = \frac{k}{\mu b}$. This time will have to be very short (and so the magnetic density b will have to be very strong) if we want to neglect the evolution of the system due to the main term (2) during Δt .

Using module technicalities, we are thus able to implement two different von Neumann measurements over our system. We will call the outputs of such measurements 0 and 1 when the related projectors are F^\pm and $\mathbb{I} - F^\pm$, respectively.

3. Sequential quantum tomography

In this section, we will prove that the statistical analysis of the data obtained when measurements + and – are sequentially applied allows reconstruction of the probability density $\rho(x)$ of the particle inside the box (see figure 2). Before proceeding, however, note that we are not making

³ This can be implemented by creating a constant gradient field of the form $\vec{B} = -by\hat{u}_x - bx\hat{u}_y$ by means of a Maxwell coil [12]. Since the particle's Y coordinate is null, the effective magnetic field will be $\vec{B} = -bx\hat{u}_y$.

any assumption on the initial coupling between the spin of the particle and its canonical degree of freedom (i.e. they could be classically correlated, or even entangled).

Denoting by \tilde{F}^\pm the complementary of F^\pm ($\tilde{F}^\pm = \mathbb{I} - F^\pm$), we have that

$$\tilde{F}^\pm = \int dx |x\rangle\langle x| \otimes |\tilde{\phi}^\pm(x)\rangle\langle\tilde{\phi}^\pm(x)|, \quad (5)$$

where $|\tilde{\phi}^\pm(x)\rangle = i\sigma_y|\phi^\pm(x)\rangle$.

Now, let $\sigma \in S_1(\mathcal{H} \otimes \mathbb{C}^2)$ be the initial quantum state of the particle. The probability of obtaining outcomes $0, 0, 0, \dots$ after the sequential performance of N measurements

$$\overbrace{+, -, +, \dots}^{N \text{ times}}$$

is given by

$$P(0, 0, \dots | + - \dots) = \text{tr} \left\{ \sigma \left(\int dx |x\rangle\langle x| \otimes h(x)h(x)^\dagger \right) \right\}, \quad (6)$$

where $h(x)$ is equal to

$$\overbrace{|\phi^+(x)\rangle\langle\phi^+(x)| \cdot |\phi^-(x)\rangle\langle\phi^-(x)| \cdot |\phi^+(x)\rangle\langle\phi^+(x)| \cdot \dots}^{N \text{ times}}. \quad (7)$$

Note that, for any set of unitary vectors $|\psi_i\rangle$,

$$\left(\prod_{i=1}^N |\psi_i\rangle\langle\psi_i| \right) \left(\prod_{i=1}^N |\psi_i\rangle\langle\psi_i| \right)^\dagger = |\psi_1\rangle\langle\psi_1| \prod_{i=1}^{N-1} |\langle\psi_i|\psi_{i+1}\rangle|^2. \quad (8)$$

This implies that

$$h(x)h(x)^\dagger = |\phi^+(x)\rangle\langle\phi^+(x)| \cdot \prod_{i=1}^{N-1} |\langle\phi^+(x)|\phi^-(x)\rangle|^2 = |\phi^+(x)\rangle\langle\phi^+(x)| \cdot \cos^{2(N-1)}(2kx). \quad (9)$$

Analogously,

$$P(1, 1, \dots, | + - \dots) = \text{tr} \left\{ \sigma \left(\int dx |x\rangle\langle x| \otimes \tilde{h}(x)\tilde{h}(x)^\dagger \right) \right\}, \quad (10)$$

with

$$\tilde{h}(x)\tilde{h}(x)^\dagger = |\tilde{\phi}^+(x)\rangle\langle\tilde{\phi}^+(x)| \cdot \cos^{2(N-1)}(2kx). \quad (11)$$

Since $|\phi^+(x)\rangle\langle\phi^+(x)| + |\tilde{\phi}^+(x)\rangle\langle\tilde{\phi}^+(x)| = \mathbb{I}_2$, we arrive at the following useful identity:

$$P(0, 0, 0, \dots | + - \dots) + P(1, 1, 1, \dots | + - \dots) = \langle \cos^{2(N-1)}(2kx) \rangle. \quad (12)$$

Using the fact that $\cos^{2M}(\theta)$ is a linear combination of the functions $\{\cos(2j\theta) : j = 0, \dots, M\}$, we can therefore estimate the values $\{\langle \cos(4kjx) \rangle : j = 0, \dots, N-1\}$ from the statistical analysis of N sequential measurements. If $k \leq \frac{\pi}{4L}$, we can then extend $\rho(x)$ to an even function defined in $[-\pi/4k, \pi/4k]$ and use our statistical knowledge to infer the coefficients $\{c_j\}_j$ of the Fourier expansion

$$\rho(x) = \sum_{j=0}^{\infty} c_j \cos(4kjx), \quad \text{for } x \in [0, L]. \quad (13)$$

Note the difference with respect to traditional tomography: a conventional experiment with initial preparation $\sigma = \sigma' \otimes |0\rangle\langle 0|$ would stop right after the first measurement, thus giving us an estimate of the value of $\langle \cos^2(kx) \rangle$. In contrast, a single-shot experiment of sequential tomography provides us with estimates of arbitrarily many relevant quantities of the form $\{\langle \cos^{2j}(kx) \rangle : j = 1, 2, \dots\}$. At first glance, one could argue that, since the probabilities appearing in equation (12) decrease exponentially faster with N , it would require a large number of samples to estimate them. However, a proper calculation shows that, as long as $\rho(x)$ has a finite slope at $x = 0$, the right-hand side of equation (12) decreases as $O(1/N)$.⁴

We have proven that, as we repeat measurements $+$ and $-$, we obtain more and more useful information about our initial state. However, as we will see soon, this information comes at a price.

4. Measurement-driven dynamics

We have already discussed the benefits of subjecting our system to a sequence of $+$ and $-$ measurements. The next question to ask is: what effect could such a sequence of measurements have on the state of the particle?

4.1. The general case

Let us first address the problem in full generality: that is, picture a physical scenario where a limited (finite) number of projective measurements with a finite number of outcomes are each available. What will the state of the system be after N measurements? Obviously, the answer will depend on the implemented *measurement scheme*, the process by which we choose which measurement to apply at a given time.

For the effects of tomography, any physical system with a finite number of von Neumann measurements available can be studied by analyzing the statistics that result from randomly applying one measurement or the other. It is thus legitimate to consider the behavior of the system under *independent random measurement schemes*, where the probability $p_x > 0$ of carrying out measurement x is the same on each round. We will see that the overall effect of random measurement strategies can have a very different nature depending on whether the dimension of the underlying Hilbert space is finite or infinite.

Suppose, indeed, that we carry out with probability p_x a measurement $x \in \{1, 2, \dots, s\}$ defined by the complete set of projectors $\{F_a^x : a = 1, 2, \dots, d\} \subset B(\mathcal{H})$, for some Hilbert space \mathcal{H} . The action of the resulting map Ω over an initial state σ would then be given by

$$\Omega(\sigma) = \sum_{x,a} p_x F_a^x \sigma F_a^x. \quad (14)$$

Note that we can see $\sigma = \sum_{i,j} \sigma_{ij} |i\rangle\langle j| \in S_2$ as an element of $\mathcal{H} \otimes \mathcal{H}$ via the isomorphism⁵ $\sigma \rightarrow |\sigma\rangle = \sum_{ij} \sigma_{ij} |i\rangle|j\rangle$. Ω can then be regarded as a superoperator $\bar{\Omega} \in B(\mathcal{H} \otimes \mathcal{H})$, given by

$$\bar{\Omega} = \sum_{x,a} p_x F_a^x \otimes (F_a^x)^*. \quad (15)$$

⁴ Suppose that $\rho(x) \geq \lambda x$, for $0 \leq x \leq \epsilon < 1/k$, and note that $|\cos(kx)| \geq 1 - k^2 x^2/2$, for $x \leq \epsilon$. It follows that $\langle \cos^{2(N-1)}(kx) \rangle \geq \int_0^\epsilon \lambda x \cos(kx)^{2(N-1)} dx \geq \int_0^\epsilon \lambda x (1 - x^2/2)^{2(N-1)} dx = \frac{\lambda}{k^2} \frac{1 - (1 - k^2 \epsilon^2/2)^{2N-1}}{2N-1}$.

⁵ Indeed, note that $\text{tr}(\sigma^\dagger \sigma) = \langle \sigma | \sigma \rangle$.

Seen as an operator, $\overline{\Omega}$ is both Hermitian and positive semidefinite. Moreover, since this map is also unital, its operator norm will be upperbounded by 1 [13]. It follows that the spectrum of $\overline{\Omega}$ is in $[0, 1]$ and so the limit $\lim_{N \rightarrow \infty} \overline{\Omega}^N$ exists and is equal to Π_1 , the projector onto the space of eigenvectors of $\overline{\Omega}$ with eigenvalue 1. Note that such a limit does not depend on the initial probabilities p_x as long as all of them are strictly positive. This is so because Π_1 is just the (possibly null) projector onto the intersection of the spaces $\mathcal{H}_x = \overline{\text{span}}\{|\phi\rangle : \sum_a F_a^x \otimes (F_a^x)^* |\phi\rangle = |\phi\rangle\}$.

The previous arguments apply to both the finite and infinite dimensional cases, i.e. the limit $\lim_{N \rightarrow \infty} \overline{\Omega}^N |\sigma\rangle$ always exists in $S_2 \cong H \otimes H$. They are also valid when we replace the projectors $\{F_a^x\}$ in (14) with the slightly more general Kraus operators of the form $M_a^x \geq 0$, with $\sum_a (M_a^x)^2 = \mathbb{I}$, $\forall x$.

Since in finite dimensions S_2 and the set of trace-class operators coincide, we can conclude that repeated applications of the mapping Ω will bring any quantum state σ to a limiting state $\Omega^\infty(\sigma) \in S_1$. That is, even though the system may experience some perturbations at the beginning of the measurement process, given some time it will stabilize into a steady state.

In infinite dimensional systems, however, the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent, so the convergence of the sequence of vectors $(\overline{\Omega}^N |\sigma\rangle)$ in $\mathcal{H} \otimes \mathcal{H}$ does not necessarily imply the convergence of the sequence of states $(\Omega^N(\sigma))$ in S_1 . Actually, as we will see, in some situations $(\Omega^N(\sigma))$ will *not* converge in S_1 . We will call this phenomenon *heat vision* and, in order to grasp its physical meaning, a short digression is necessary.

The common perception among physicists is that any self-adjoint operator whose spectrum is bounded from below qualifies as a potentially physical Hamiltonian. Most interactions available in the laboratory, however, fall into the more restrictive class of *0-band Hamiltonians*. A 0-band Hamiltonian is any self-adjoint operator E with a discrete spectrum such that, for any $\bar{E} \in \mathbb{R}$, the space of eigenvectors of E with eigenvalue smaller than or equal to \bar{E} is finite dimensional. Examples of 0-band energy operators are the harmonic oscillator, the double-well potential and a particle in a box, and, more generally, the Hamiltonian of any finite number of particles subject to a potential that can be bounded from below by a harmonic trap⁶.

Now, most experiments in physics are not performed in the open air, but inside closed chambers; 0-band energy operators thus provide a very good dynamical description of those quantum systems accessible in the laboratory⁷.

Coming back to heat vision, the physical significance of this phenomenon is given by the next proposition (see appendix A).

⁶ To see why, let K denote the kinetic energy of such a set of particles, V the potential they are subject to and V' its harmonic approximation. Then, from $V - V' \geq 0$, we have that $H - H' \geq 0$, for $H = K + V$, and $H' = K + V'$. Now, for any $\bar{E} \in \mathbb{R}$, define W (W') as the vector space spanned by those eigenvectors of H (H') with eigenvalues smaller than or equal to \bar{E} . We will next show that $\dim(W) \leq \dim(W')$. Since H' is 0-band, this will imply that H is 0-banded as well. Indeed, suppose that $\dim(W) > \dim(W')$. Then, there is a normalized vector $|\psi\rangle \in W$ with $|\psi\rangle \perp W'$. But that would imply $\langle \psi | H | \psi \rangle \leq \bar{E}$ and $\langle \psi | H' | \psi \rangle > \bar{E}$, hence contradicting the initial assumption $H - H' \geq 0$.

⁷ When modelling coulomb interactions, note that atomic nuclei have a finite radius. Assuming that the positive charge of such nuclei is uniformly distributed inside a sphere, the electrostatic potential between nuclei and electrons is bounded from below. The Hamiltonian describing a measurement of the spectrum of an ensemble of molecules confined in a finite recipient is thus 0-banded.

Lemma 1. *Let $(\sigma^{(N)})$ be a sequence of normalized quantum states such that $\lim_{N \rightarrow \infty} \sigma^{(N)}$ exists in S_2 . Then, $\lim_{N \rightarrow \infty} \sigma^{(N)}$ does not exist in S_1 iff, for all 0-band energy operators E , the sequence of averages $(\text{tr}\{E\sigma^{(N)}\})$ tends to infinity.*

Consequently, if $(\Omega^N(\sigma))$ does not converge in S_1 , then the action of gathering information about σ will increase the system's energy up to arbitrarily high values. This is analogous to Superman's famous ability to induce heat with the power of his stare (see, for instance, [14]), hence the name 'heat vision'.

4.2. Specific examples

So far, we have spoken of heat vision as a hypothetical possibility. We will next prove that, remarkably, our case study gives rise to heat vision for *any* initial state $\sigma \in S_1$ we place as an input.

First, since the heat vision effect does not depend on the actual weights we assign to each measurement, we can assume w.l.o.g. that we carry out any of the two measurements with probability 1/2. The measurement channel Ω is thus equal to

$$\Omega(\sigma) = \frac{1}{2} \left\{ \sum_{j=+,-} F^j \sigma F^j + (\mathbb{I} - F^j) \sigma (\mathbb{I} - F^j) \right\}. \quad (16)$$

As shown in appendix B, for any state σ ,

$$\lim_{N \rightarrow \infty} \text{tr}\{[\Omega^N(\sigma)]^2\} = 0. \quad (17)$$

Clearly, the values $(\|\Omega^N(\sigma) - 0\|_1)$ do not tend to 0, and so the sequence of states $(\Omega^N(\sigma))$ does not converge in trace norm. Also, equation (17), together with the Rényi inequality $S(\chi) \geq -\log_2(\text{tr}\{\chi^2\})$, implies that the von Neumann entropy of $\Omega^N(\sigma)$ tends to infinity.

The fact that $(\Omega^N(\sigma))$ does not converge in S_1 implies that the energy of the system will diverge for any input state, as long as the Hamiltonian describing the system is 0-banded. Such is the case of the Hamiltonian described by equation (2), and indeed, one can check that the kinetic energy of the particle after N applications of the channel Ω obeys the formula

$$E^{(N)} = E^{(0)} + \frac{k^2}{m} N. \quad (18)$$

That is, on average, the temperature of the system will grow linearly with the number of measurements carried out.

Let us briefly recapitulate what is happening here: in order to carry out measurement \pm , first we have to apply a strong magnetic field $\pm B$ for a time. It is not surprising then that the energy of the system increases when we perform such a change. However, after the spin measurement we will apply the opposite field, $\mp B$, thus inverting the previous unitary process. Thus the only reason why the energy of the system (or for the same sake, the state of the system) changes is that we are measuring a single qubit in between! Indeed, the system is engineered in such a way that the entropy associated with such a spin measurement accumulates and accumulates in the canonical degree of freedom of our particle until the setup cannot stand any more energy. This behavior is to be compared to the finite dimensional case, where it is always

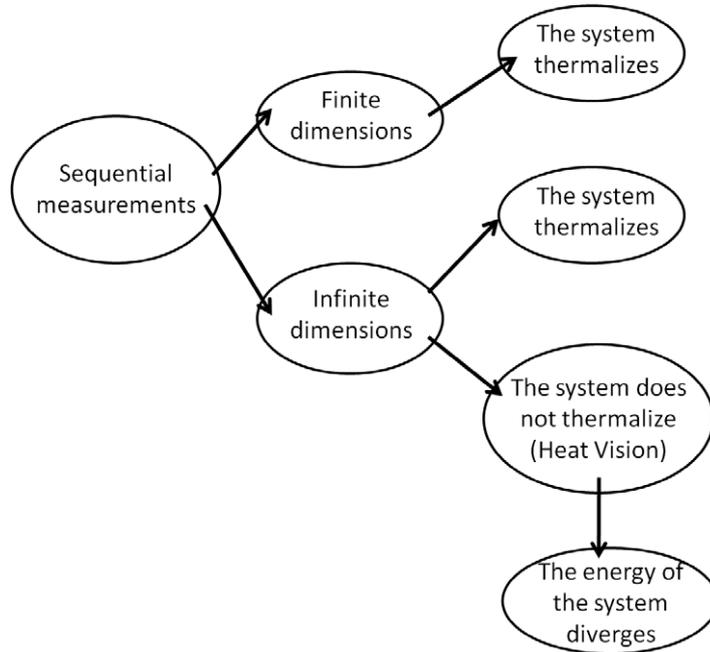


Figure 3. Dynamics of a quantum system subjected to repeated von Neumann measurements.

possible to find states of entropy ≤ 1 that are invariant under the action of any two dichotomic measurements⁸.

But the heat vision effect can manifest in even more extreme ways: in appendix C we describe systems with $s \geq 5$ dichotomic observables where the purity of any initial state σ subject to N sequential measurements is bounded by μ_s^{2N} , where $\mu_s < 1$ is independent of σ . The entropy of such states will thus increase at least linearly in N . The measurements involved, however, are quite abstract and most likely impossible to implement in any present laboratory.

By showing that the heat vision effect does emerge (at least, theoretically) in some physical scenarios, now we have a clear picture of how quantum systems evolve through sequential von Neumann measurements. The conclusions are illustrated in figure 3, where the finite and infinite dimensional cases are differentiated.

4.3. Heat vision and information

It is tempting to think that the heat vision effect appears in some infinite dimensional systems and not in finite dimensional ones just because in the former we can extract an infinite amount of information about the initial state of the system. However, this intuition is false.

Indeed, consider the following counterexample: let $\{|n\rangle : n \in \mathbb{N}\}$ be an orthonormal basis for \mathcal{H} . Then, we can define projector operators acting over $\mathcal{H} \otimes \mathbb{C}^2$ as

$$G^\pm = \sum_{n=1}^{\infty} |n\rangle\langle n| \otimes |\phi^\pm(n)\rangle\langle\phi^\pm(n)|. \quad (19)$$

⁸ In a finite dimensional system, Jordan's lemma [20] states that any pair of projectors F^+ and F^- can be simultaneously 2×2 -block-diagonalized, i.e. there exists a basis where $F^+ = \oplus_n F_n^+$, $F^- = \oplus_n F_n^-$, with F_n^+ , $F_n^- \in M_{2 \times 2}$. Any state of the form $\sigma = 0_{2 \times 2} \oplus \dots \oplus \frac{\mathbb{1}}{2} \oplus \dots \oplus 0_{2 \times 2}$ thus satisfies $F^\pm \sigma F^\pm + (\mathbb{1} - F^\pm) \sigma (\mathbb{1} - F^\pm) = \sigma$.

The analogue of equation (12) follows in a straightforward manner, and so we can combine our projective measurements to estimate the mean values $\{\langle \cos(4kjn) \rangle : j \in \mathbb{N}\}$. Choosing k irrational, the statistical analysis of repeated measurements can thus provide us with an infinite amount of information about the occupation number distribution. However, let Ω be the channel that results when we randomly apply one measurement or the other. Then, for any initial quantum state σ , it can be shown that

$$\lim_{N \rightarrow \infty} \Omega^N(\sigma) = \sum_{n=1}^{\infty} \langle n | \text{tr}_{\mathbb{C}^2}(\sigma) | n \rangle | n \rangle \langle n | \otimes \mathbb{I}_2/2, \quad (20)$$

that is, the system does not exhibit the heat vision effect in any case. Moreover, if the energy operator is diagonal in the $\{|n\rangle : n \in \mathbb{N}\}$ basis, the energy of the system does not even vary during the measurement process.

Heat vision is therefore not equivalent to the possibility of accessing an infinite amount of information, but is an independent property of the system under study.

5. Experimental implementations

So far, when exemplifying the effects and applications that follow from sequential measurements, we have referred to our idealized case study. However, we would like to point out that both tomography with strong sequential measurements and the heat vision effect can already be experienced with current technology. Indeed, consider a regular ion trap setup with just one ion. Then two internal states of such an ion could play the role of the spin in our case study, while the displacement of the ion along the trap could account for the canonical degree of freedom. ‘Spin’ measurements in this scenario can be carried out in the standard way, i.e. exciting one of the internal levels with a laser and counting the number of emitted photons. Analogously, an interaction of the form H_s can be induced by a laser beam in standing wave configuration [15]. In a usual ion trap setup, the former dispositions would effectively implement the measurements (3) over a particle subject to a harmonic potential; in order to recreate the square potential, a trap of the form [16] can be used. Due to the Brans–Dicke approximation, however, the whole experiment must be conducted in the low temperature regime (up to ~ 1 K).

6. Conclusion

In this paper, we have studied the use and effect of repeated von Neumann measurements. We have pointed out an extreme scenario where the statistical analysis of binary outcomes of sequential measurements allows estimation of the full probability density of a trapped particle. We have also shown how sometimes the action of alternating measurements can lead to a non-convergent dynamics in infinite dimensional systems. This phenomenon, the heat vision effect, always comes together with an unbounded energy increase, and can be observed experimentally in current ion trap setups. Is this the end of the story? Dreaming on, one could conceive a new architecture for quantum computing based on sequential strong measurements. In this model, the user could carry out a (small) number of non-commuting dichotomic measurements over a continuous variable system, and computations would be carried out by deciding which measurement to carry out at every step. Although still vague, we hope to explore this idea in future work.

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Appendix A. Convergence in S_1

Theorem 1. *Let $\sigma^{(N)}$ be a sequence of normalized quantum states. Then, the following conditions are equivalent:*

1. $\lim_{N \rightarrow \infty} \sigma^{(N)}$ exists in S_1 .
2. For some (and thus for all) arbitrary countable orthonormal basis $\{|n\rangle\}$ of \mathcal{H} , the limits

$$c_{n,m} = \lim_{N \rightarrow \infty} \text{tr}\{\sigma^{(N)}|n\rangle\langle m|\} \quad (\text{A.1})$$

exist and are such that

$$\sum_{n=0}^{\infty} c_{n,n} = 1. \quad (\text{A.2})$$

Proof. Suppose that (1) is true. Then, there exists an element $\hat{\sigma} \in S_1$ such that $\lim_{N \rightarrow \infty} \|\hat{\sigma} - \sigma^{(N)}\|_1 = 0$. We remind the reader that, for any self-adjoint operator A ,

$$\|A\|_1 = \sup_{\mathbb{I} \geq X \geq -\mathbb{I}} \text{tr}\{A \cdot X\}. \quad (\text{A.3})$$

Now, let $\{|n\rangle\}$ be any orthonormal basis for H . The operator $|n\rangle\langle n|$ satisfies $\mathbb{I} \geq \pm|n\rangle\langle n| \geq -\mathbb{I}$, so

$$\|\hat{\sigma} - \sigma^{(N)}\|_1 \geq |\text{tr}\{|n\rangle\langle n|\hat{\sigma}\} - \text{tr}\{|n\rangle\langle n|\sigma^{(N)}\}|. \quad (\text{A.4})$$

It follows that the limit $\lim_{N \rightarrow \infty} \text{tr}\{\sigma^{(N)}|n\rangle\langle n|\}$ exists and is equal to $\text{tr}\{\hat{\sigma}|n\rangle\langle n|\}$. Analogously, from the relations

$$\mathbb{I} \geq |n\rangle\langle m| + |m\rangle\langle n| \geq -\mathbb{I}, \quad \mathbb{I} \geq i(|n\rangle\langle m| - |m\rangle\langle n|) \geq -\mathbb{I}, \quad (\text{A.5})$$

it can be shown that $\lim_{N \rightarrow \infty} \text{tr}\{\sigma^{(N)}|n\rangle\langle m|\}$ exists as well. Finally, $\mathbb{I} \geq \pm\mathbb{I} \geq -\mathbb{I}$, which means that $\|\hat{\sigma} - \sigma^{(N)}\|_1 \geq |\text{tr}\{\hat{\sigma} - \sigma^{(N)}\}|$. Since $\text{tr}(\sigma^{(N)}) = 1, \forall N$, we have that $\text{tr}(\hat{\sigma}) = 1$, and so $\sum_n c_{n,n} = \text{tr}\{\sum_n |n\rangle\langle n|\hat{\sigma}\} = \text{tr}(\hat{\sigma}) = 1$, and (2) is proven true.

Conversely, suppose that (2) is true, and consider the operator

$$\hat{\sigma} \equiv \sum_{n,m=0}^{\infty} c_{n,m}|m\rangle\langle n|. \quad (\text{A.6})$$

This operator is bounded. Indeed, let $|v\rangle, |w\rangle \in \text{span}\{|n\rangle\}$. Then,

$$|\langle v|\hat{\sigma}|w\rangle| = \lim_{N \rightarrow \infty} |\text{tr}\{\sigma^{(N)}|w\rangle\langle v|\}| \leq \sqrt{\langle w|w\rangle\langle v|v\rangle}. \quad (\text{A.7})$$

Likewise, it can be shown that $\langle v|\hat{\sigma}|v\rangle \geq 0$ for all $|v\rangle \in \text{span}\{|n\rangle\}$, i.e. $\hat{\sigma} \geq 0$. Moreover, by equation (A.2) $\text{tr}\{\hat{\sigma}\} = 1$, so $\hat{\sigma} \in S_1$.

Let $P_K \equiv \sum_{n=0}^K |n\rangle\langle n|$. Then equation (A.2) implies that, for any $\epsilon > 0$, there exist K, M such that $\text{tr}(\sigma^{(N)} P_K) \geq 1 - \epsilon$, $\forall N \geq M$. Applying twice the relation [17]

$$\|\rho - P_K \rho P_K\|_1 \leq 2\sqrt{\text{tr}(\rho)\{\text{tr}(\rho) - \text{tr}(\rho P_K)\}}, \quad (\text{A.8})$$

valid for any $\rho \geq 0 \in S_1$, we have that

$$\|\sigma^{(N)} - \hat{\sigma}\|_1 \leq 4\sqrt{\epsilon} + \|P_K \sigma^{(N)} P_K - P_K \hat{\sigma} P_K\|_1, \quad (\text{A.9})$$

for $N > M$. Note that the last term of equation (A.9) tends to 0 as N tends to infinity (because we are evaluating the trace distance between two $K+1 \times K+1$ matrices that converge entry-wise). It follows that $\lim_{N \rightarrow \infty} \|\sigma^{(N)} - \hat{\sigma}\|_1 \leq 4\sqrt{\epsilon}$. Since ϵ was arbitrary, we conclude that $\lim_{N \rightarrow \infty} \|\sigma^{(N)} - \hat{\sigma}\|_1 = 0$, and so $(\sigma^{(N)})$ converges in S_1 . \square

In the particular case when $\sigma^{(N)} = \Omega^N(\sigma)$, for some initial state σ and some channel Ω with $\mathbb{I} \geq \bar{\Omega} \geq 0$, the existence of the limits (A.1) is automatic, since $\bar{\Omega}^N$ converges in $B(\mathcal{H} \otimes \mathcal{H})$ and $|n\rangle\langle m|^* \in \mathcal{H} \otimes \mathcal{H}$. This implies that convergence in S_1 in that case is equivalent to the existence of a basis $\{|n\rangle : n \in \mathbb{N}\}$ such that $\lim_{K \rightarrow \infty} \lim_{N \rightarrow \infty} \text{tr}(P_K \Omega^N(\sigma)) = 1$ (note the order of the limits). If the latter is the case, then $\sigma^{(N)}$ can always be described by a finite dimensional system, i.e. for a sufficiently high K , we can approximate $\sigma^{(N)}$ by the state $P_K \sigma^{(N)} P_K \in B(\mathbb{C}^{K+1})$, for all N .

In order to establish a connection between energy and convergence in S_1 in realistic scenarios, we will have to restrict the usual definition of the Hamiltonian.

Definition 1. 0-band energy operator

Let E be a self-adjoint operator acting over an infinite dimensional (separable) Hilbert space \mathcal{H} . We will say that E is a 0-band energy operator iff

1. The spectrum of E is discrete.
2. For any $\bar{E} \in \mathbb{R}$, there is only a finite number of linearly independent eigenvectors of E with eigenvalues less than or equal to \bar{E} .

The next lemma relates convergence in S_1 with energy considerations.

Lemma 1. Let $(\sigma^{(N)})$ be a sequence of normalized quantum states such that $\lim_{N \rightarrow \infty} \sigma^{(N)}$ exists in S_2 . Then, $\lim_{N \rightarrow \infty} \sigma^{(N)}$ does not exist in S_1 iff, for all 0-band energy operators E , the sequence of energy averages $(\text{tr}\{E\sigma^{(N)}\})$ tends to infinity.

Proof. Suppose that the sequence $(\sigma^{(N)})$ does not converge in S_1 , and let E be an arbitrary 0-band energy operator $E = \sum_{n=0}^{\infty} E_n |n\rangle\langle n|$, where $\{|n\rangle : n \in \mathbb{N}\}$ are a basis of eigenvectors of E , with energies $E_0 \leq E_1 \leq E_2 \leq \dots$. Let P_K be the projector $P_K \equiv \sum_{n=0}^K |n\rangle\langle n|$. We will prove that there exists $1 > \lambda > 0$ such that, for any $\bar{E} \geq E_0$, $\lim_{N \rightarrow \infty} \text{tr}\{E\sigma^{(N)}\} \geq (1 - \lambda)E_0 + \lambda\bar{E}$.

Indeed, let K be such that $E_{K+1} \geq \bar{E}$. If $\lim_{N \rightarrow \infty} \sigma^{(N)}$ does not converge in S_1 , by theorem 1 we have that $\lim_{N \rightarrow \infty} \text{tr}\{\sigma^{(N)} P_K\} \leq \lim_{n \rightarrow \infty} \lim_{N \rightarrow \infty} \text{tr}\{\sigma^{(N)} P_n\} = 1 - \lambda$, with $\lambda > 0$ independent of K . On the other hand,

$$E = \sum E_n |n\rangle\langle n| \geq E_0 P_K + \bar{E}(\mathbb{I} - P_K), \quad (\text{A.10})$$

so we arrive at

$$\lim_{N \rightarrow \infty} \text{tr}\{E\sigma^{(N)}\} \geq (1 - \lambda)E_0 + \lambda\bar{E}. \quad (\text{A.11})$$

To prove the opposite implication, suppose that $(\sigma^{(N)})$ converges to the normalized state $\hat{\sigma}$. We will show that there exists a 0-band energy operator E such that $\text{tr}\{\sigma^{(N)}E\}$ is bounded and $\lim_{N \rightarrow \infty} \text{tr}\{\sigma^{(N)}E\}$ exists. Let $\{|n\rangle : n \in \mathbb{N}\}$ be any basis for \mathcal{H} and define the probability distributions $p^{(N)}(n) \equiv \langle n|\sigma^{(N)}|n\rangle$, $\hat{p}(n) \equiv \langle n|\hat{\sigma}|n\rangle$. Also, let $K : \mathbb{N} \rightarrow \mathbb{N}$ be the mapping defined as

$$K(s) \equiv \left\{ \min K \geq 0 : \sum_{n=0}^K p^{(N)}(n) \geq 1 - \frac{1}{2^s}, \forall N \right\}. \quad (\text{A.12})$$

It is important to note that $K(s) < \infty$ for all s . Indeed, suppose that $K(s) = \infty$, for some s . This would imply that, for any number K , there exists an \tilde{N} such that $\sum_{n=0}^K p^{(\tilde{N})}(n) < 1 - \frac{1}{2^s}$. Now, for $0 < \epsilon < 1/2^{s+1}$, choose M such that $\|\hat{\sigma} - \sigma^{(N)}\|_1 < \epsilon$, for all $N > M$, and choose K such that

$$\begin{aligned} \sum_{n=0}^K p^{(N)}(n) &> 1 - \frac{1}{2^s} + \epsilon, \quad \forall N \leq M, \\ \sum_{n=0}^K \hat{p}(n) &> 1 - \frac{1}{2^s} + 2\epsilon. \end{aligned} \quad (\text{A.13})$$

Then, for $N > M$, $|\sum_{n=0}^K p^{(N)}(n) - \hat{p}(n)| \leq \|\sigma^{(N)} - \hat{\sigma}\|_1 < \epsilon$. It follows that the expression $\sum_{n=0}^K p^{(N)}(n) > 1 - \frac{1}{2^s} + \epsilon$ holds for all N , thus contradicting our initial claim.

We will differentiate two cases depending on the existence of the limit $\lim_{s \rightarrow \infty} K(s)$.

If $\lim_{s \rightarrow \infty} K(s) = \hat{K} < \infty$, then $\sum_{n=0}^{\hat{K}} p^{(N)}(n) = 1$ for all N . We can thus simply define the 0-band energy operator $E = \sum_{n=\hat{K}+1}^{\infty} n|n\rangle\langle n|$ and we would have that $\text{tr}\{\sigma^{(N)}E\} = 0 < \infty$.

Suppose, in contrast, that $\lim_{s \rightarrow \infty} K(s) = \infty$, and define the sets of natural numbers $I_0 = [0, K(1)]$ and

$$I_s \equiv \begin{cases} [K(s) + 1, K(s + 1)], & \text{if } K(s) + 1 \leq K(s + 1), \\ \emptyset, & \text{otherwise,} \end{cases} \quad (\text{A.14})$$

for $s \geq 1$. These sets are finite and disjoint, and satisfy $\cup_{s=0}^{\infty} I_s = \mathbb{N}$. Denoting the projector $\sum_{n \in I_s} |n\rangle\langle n|$ as P_s , we thus have that the positive operator

$$E = \sum_{s=0}^{\infty} \sqrt{2^s} P_s \quad (\text{A.15})$$

is 0-banded.

Finally, note that

$$\sum_{n \in I_s} p^{(N)}(n) \leq \sum_{n=K(s)+1}^{\infty} p^{(N)}(n) \leq \frac{1}{2^s}, \quad (\text{A.16})$$

for $s \geq 1$. This implies that, for any N ,

$$E^{(N)} \equiv \text{tr}(\sigma^{(N)} E) \leq 1 + \sum_{s=1}^{\infty} \left(\frac{\sqrt{2}}{2}\right)^s = 1 + \frac{1}{\sqrt{2}-1}, \quad (\text{A.17})$$

i.e. the sequence $(E^{(N)})$ is bounded.

Now, define $Q_t = \sum_{s=0}^t P_s$. From all the above, it is clear that

$$0 \leq E^{(N)} - \text{tr}(Q_t \sigma^{(N)} Q_t E) \leq \sum_{s=t+1}^{\infty} \left(\frac{\sqrt{2}}{2}\right)^s. \quad (\text{A.18})$$

Likewise,

$$0 \leq \hat{E} - \text{tr}(Q_t \hat{\sigma} Q_t E) \leq \sum_{s=t+1}^{\infty} \left(\frac{\sqrt{2}}{2}\right)^s, \quad (\text{A.19})$$

where $\hat{E} \equiv \text{tr}(\hat{\sigma} E)$. It follows that

$$|\hat{E} - E^{(N)}| \leq |\text{tr}(Q_t [\hat{\sigma} - \sigma^{(N)}] Q_t E)| + \frac{\sqrt{2}-1}{\sqrt{2}^t}. \quad (\text{A.20})$$

Taking the limit $N \rightarrow \infty$ and then $t \rightarrow \infty$, the right-hand side of equation (A.20) vanishes, and thus $\lim_{N \rightarrow \infty} \text{tr}(\sigma^{(N)} E) = \text{tr}(\hat{\sigma} E)$. \square

Let us make a final remark.

Lemma 2. Suppose that the channel $\mathbb{I} \geq \bar{\Omega} \geq 0$ has the property that, for any quantum state σ , the sequence $(\Omega^N(\sigma))$ does not converge in S_1 . Then, $\lim_{N \rightarrow \infty} \Omega^N(\sigma) = 0$ in S_2 , i.e. $\lim_{N \rightarrow \infty} \text{tr}\{\sigma^{(N)} |\phi\rangle\langle\psi|\} = 0$ for any pair of states $|\phi\rangle, |\psi\rangle$.

Proof. Suppose that there exists a basis $\{|n\rangle : n \in \mathbb{N}\}$ for the Hilbert space such that the coefficients $c_{n,m} = \lim_{N \rightarrow \infty} \text{tr}(\Omega^N(\sigma) |n\rangle\langle m|)$ do not satisfy condition (A.2). Then, following the proof of theorem 1, one could build an operator $\hat{\sigma} \geq 0 \in S_1$ such that $\text{tr}(\hat{\sigma}) \leq 1$. Now, if $\text{tr}(\hat{\sigma}) \neq 0$, then $\hat{\sigma}' \equiv \hat{\sigma} / \text{tr}(\hat{\sigma})$ would be a quantum state such that $\Omega(\hat{\sigma}') = \hat{\sigma}'$, contradicting the main assumption. We thus have that $\text{tr}(\hat{\sigma}) = 0$, which, together with $\hat{\sigma} \geq 0$, implies that $\hat{\sigma} = 0$ (and so, $\langle\phi|\hat{\sigma}|\psi\rangle = \lim_{N \rightarrow \infty} \text{tr}\{\sigma^{(N)} |\psi\rangle\langle\phi|\} = 0$, for all $|\psi\rangle, |\phi\rangle$). \square

Appendix B. Derivation of equation (17)

Viewed as a superoperator, the channel (16) can be seen as equal to

$$\bar{\Omega} = \frac{1}{2}(\mathbb{I} \otimes U) \int dx dy |x\rangle\langle x| \otimes |y\rangle\langle y| \otimes = \otimes (M(x-y) \oplus M(x+y)) (\mathbb{I} \otimes U^\dagger), \quad (\text{B.1})$$

where

$$M(x) = \begin{pmatrix} 1 & \cos[2kx] \\ \cos[2kx] & 1 \end{pmatrix} = \cos^2[kx] |+\rangle\langle+| + \sin^2[kx] |-\rangle\langle-| \quad (\text{B.2})$$

and

$$U = |+i, -i\rangle\langle 0, 0| + |-i, +i\rangle\langle 0, 1| + |+i, +i\rangle\langle 1, 0| + |-i, -i\rangle\langle 1, 1|. \quad (\text{B.3})$$

It thus follows that

$$\overline{\Omega}^N = \frac{1}{2}(\mathbb{I} \otimes U) \int dx dy |x\rangle\langle x| \otimes |y\rangle\langle y| \otimes \otimes (M(x-y)^N \oplus M(x+y)^N) (\mathbb{I} \otimes U^\dagger), \quad (\text{B.4})$$

and one can then check that

$$\begin{aligned} \text{tr} \left\{ [\Omega^N(|n, \pm i\rangle\langle n, \pm i|)]^2 \right\} &= \langle n, \pm i, n, \mp i | \overline{\Omega}^{2N} |n, \pm i, n, \mp i\rangle \\ &= \frac{1}{2} \int_0^L \int_0^L dx dy \Psi_n(x)^2 \Psi_n(y)^2 (\cos^{4N}[k(x-y)] + \sin^{4N}[k(x-y)]). \end{aligned} \quad (\text{B.5})$$

Taking into account that $\Psi_n(x)^2 \Psi_n(y)^2 \leq 4/L^2$ and changing to variables $x' = (x-y)/L$, $y' = (x+y)/L$, we have that

$$\text{tr} \left\{ [\Omega^N(|n, \pm i\rangle\langle n, \pm i|)]^2 \right\} \leq \varphi(N)^2 := 2 \int_{-1}^1 [1 - |x|] \cdot [\cos^{4N}(kLx) + \sin^{4N}(kLx)] dx. \quad (\text{B.6})$$

For any initial state σ , applying the Schwartz inequality, one arrives at

$$\begin{aligned} \text{tr}\{\Omega^N(\sigma)|n, \pm i\rangle\langle n, \pm i|\} &= \text{tr}\{\sigma \Omega^N(|n, \pm i\rangle\langle n, \pm i|)\} \\ &\leq \sqrt{\text{tr}(\sigma^2)} \sqrt{\text{tr} \left\{ [\Omega^N(|n, \pm i\rangle\langle n, \pm i|)]^2 \right\}} \leq \varphi(N), \end{aligned} \quad (\text{B.7})$$

with $\lim_{N \rightarrow \infty} \varphi(N) = 0$. Note that this expression is independent of n : the occupation of each of the states $\{|n, \pm i\rangle\}$ tends uniformly to zero. In other words, the energy density distribution of a quantum state subject to sequential measurements + and – neither converges nor displaces, but flattens.

Now, theorem 1 in appendix A states that any sequence $(\Omega^N(\sigma))$ does not converge in S_1 iff there exists some orthonormal basis $\{|\psi_n\rangle : n \in \mathbb{N}\}$ such that $\sum_n \lim_{N \rightarrow \infty} \langle \psi_n | \Omega^N(\sigma) | \psi_n \rangle < 1$. Moreover, by lemma 2 in the same appendix, if such is the case for any initial state σ , then $\lim_{N \rightarrow \infty} \Omega^N(\sigma)$ tends to 0 in the $\|\cdot\|_2$ norm. It thus follows that

$$\lim_{N \rightarrow \infty} \text{tr}\{[\Omega^N(\sigma)]^2\} = 0, \quad (\text{B.8})$$

for all initial states σ .

Appendix C. Extreme cases of heat vision

An extreme case of Heat Vision can be found in the following system. Consider the group G that results from the free product [18] of \mathbb{Z}_2 with itself s times, i.e.

$$G = \overbrace{\mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2}^{s \text{ times}},$$

and take $l_2(G)$ to be our Hilbert space.

The left regular representation of an element $g \in G$ is defined as the unitary operator $\lambda(g) : l_2(G) \rightarrow l_2(G)$ such that $\lambda(g)|g'\rangle = |gg'\rangle$, for any $g' \in G$ [19]. The left regular representation of the generators $\lambda(g_i)$ thus satisfies $\lambda(g_i)^2 = 1$, $\lambda(g_i) = \lambda(g_i)^\dagger$. This implies that, for each generator g_i , there exists an associated projector $(\lambda(g_i) + \mathbb{I})/2$, and so each g_i defines a

quantum dichotomic measurement. The channel Ω that results when we carry out one of the s measurements with probability $1/s$ can be written as

$$\bar{\Omega} = \frac{1}{2} \left(\frac{1}{s} \sum_{i=1}^s \lambda(g_i) \otimes \lambda(g_i)^* + \mathbb{I} \right). \quad (\text{C.1})$$

Define the operator Π_i as the projection onto the subspace of $l_2(G)$ spanned by all the elements of G that start with the symbol g_i , and note that $\lambda(g_i) = x_i + y_i$, with $x_i \equiv \lambda(g_i)\Pi_i$, $y_i \equiv \Pi_i\lambda(g_i)$. We have that

$$\left\| \sum_{i=1}^s \lambda(g_i) \otimes \lambda(g_i)^* \right\| = \left\| \sum_{i=1}^s \lambda(g_i) \right\| \leq \left\| \sum_{i=1}^s x_i \right\| + \left\| \sum_{i=1}^s y_i \right\|, \quad (\text{C.2})$$

where in the first equality we have made use of Fell's absorption principle [19]. On the other hand, for any two sets of operators $\{A_i\}\{B_i\}$,

$$\left\| \sum_i A_i B_i \right\| \leq \left\| \sum_i A_i A_i^\dagger \right\|^{1/2} \left\| \sum_i B_i^\dagger B_i \right\|^{1/2}. \quad (\text{C.3})$$

Taking $(A_i = \mathbb{I}, B_i = x_i)$ and $(A_i = y_i, B_i = \mathbb{I})$, we have that the last term of equation (C.2) is upperbounded by

$$\sqrt{s} \left(\left\| \sum_i \Pi_i \right\|^{1/2} + \left\| \sum_i \Pi_i \right\|^{1/2} \right), \quad (\text{C.4})$$

which, in turn, is upperbounded by $2\sqrt{s}$, since $\sum_i \Pi_i \leq \mathbb{I}$.

It follows that

$$\|\bar{\Omega}\| \leq 1/2 + 1/\sqrt{s}. \quad (\text{C.5})$$

The norm of $\bar{\Omega}$ as an operator in $l_2(G) \otimes l_2(G)$ is thus smaller than 1 whenever the number of measurements is greater than 4. This phenomenon can only occur in infinite dimensional systems, since for any finite dimensional unital map $\bar{\omega}$ on $\mathbb{C}^d \otimes \mathbb{C}^d$, $\bar{\omega}|\mathbb{I}_d\rangle = |\mathbb{I}_d\rangle$, for $|\mathbb{I}_d\rangle \equiv \sum_{i=1}^d |i, i\rangle$, and so $\|\bar{\omega}\| = 1$.

Suppose then that $s \geq 5$ and so $\|\bar{\Omega}\| = \mu < 1$, and let $\sigma \in S_1(l_2(G))$ be any arbitrary normalized quantum state, with $\sigma^{(N)} \equiv \Omega^N(\sigma)$. Our previous discussion implies that

$$\text{tr}\{\sigma^{(N)}\}^2 \leq \mu^{2N} \text{tr}\{\sigma\}^2 \leq \mu^{2N}. \quad (\text{C.6})$$

That is, the purity of any initial state decreases exponentially with the number of applications of the channel and so the system exhibits heat vision for any input state.

References

- [1] Aharonov Y, Albert D Z and Vaidman L 1988 *Phys. Rev. Lett.* **60** 1351
- [2] Aharonov Y and Vaidman L 2002 The two-state vector formalism of quantum mechanics: an updated review *Time in Quantum Mechanics*, ed J G Muga, R S Mayato and I L Egusquiza (Heidelberg: Springer) pp 399–448
- [3] Dressel J, Agarwal S and Jordan A N 2010 *Phys. Rev. Lett.* **104** 240401
- [4] Hardy L 1992 *Phys. Rev. Lett.* **68** 2981
- [5] Kocsis S *et al* 2011 *Science* **332** 1170–3

- [6] Hosten O and Kwiat P 2008 *Science* **319** 787 (arXiv:10.1126/science.1152697)
- [7] Mitchison G, Jozsa R and Popescu S 2007 *Phys. Rev. A* **76** 062105
- [8] Fritz T 2010 *J. Math. Phys.* **51** 052103
- [9] Aharonov Y and Vardi M 1980 *Phys. Rev. D* **21** 2235–40
- [10] Lutterbach L G and Davidovich L 1997 *Phys. Rev. Lett.* **78** 2547
Nogues G, Rauschenbeutel A, Osnaghi S, Bertet P, Brune M, Raimond J M, Haroche S, Lutterbach L G and Davidovich L 2000 *Phys. Rev. A* **62** 054101
- [11] Schwarz M, Temme K and Verstraete F 2011 arXiv:1104.1410
Temme K, Osborne T J, Vollbrecht K G, Poulin D and Verstraete F 2011 *Nature* **471** 87
Wolf M M, Perez-Garcia D and Fernandez C 2009 *Phys. Rev. Lett.* **103** 230402
Masanes LI 2006 *Phys. Rev. Lett.* **97** 050503
- [12] Clerk-Maxwell J 1873 *Treatise on Electricity and Magnetism* (Oxford: Clarendon) p 319
- [13] Perez-Garcia D, Wolf M M, Petz D and Ruskai M B 2006 *J. Math. Phys.* **47** 083506
- [14] Verheiden M, Benes E, Lei A, Lea R and Benes M 2005 Burn! *Superman* vol 218 (New York: DC Comics)
- [15] James D F V 1998 *Appl. Phys. B* **66** 181
- [16] Staantum P, Kraft S D, Lange J, Wester R and Weidemller M 2006 *Phys. Rev. Lett.* **96** 023201
- [17] Renner R 2005 Security of quantum key distribution *PhD Thesis* Swiss Federal Institute of Technology (ETH), Zurich arXiv:quant-ph/0512258
- [18] Voiculescu D V, Dykema K J and Nica A 1992 *Free Random Variables: a Noncommutative Probability Approach to Free Products with Applications to Random Matrices, Operator Algebras and Harmonic Analysis on Free Groups (CRM Monograph Series vol 1)* (Providence, RI: American Mathematical Society)
- [19] Pisier G 2003 *An Introduction to Operator Spaces, London Math. Soc. (Lecture Notes Series vol 294)* (Cambridge: Cambridge University Press)
- [20] Jordan C 1875 *Bulletin de la S. M. F.* **3** 103