Regularity of Solutions to a Lubrication problem with discountinuous separation data

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Abstract

We study the regularity of the solution to the Reynolds equation for incompressible and compressible fluids when the gap between the lubricated surfaces, "$h(x,y)$", presents a discontinuity in a two dimensional bounded domain. As in the one dimensional problem studied by Rayleight the solution P does not belong to $C^1(\Omega)$ but we obtain that $|\nabla P|$ is bounded, i.e. $P \in W^{1,\infty}(\Omega)$.

Keywords: Reynolds equation, Regularity, Elliptic equations.

1 Introduction

Lubrication is a known phenomenon since the early civilizations. Since then, lubrication has been an important part of engineering, and has helped to solve many problems of engineering, as in gears, rollers, journal bearings, thrust pads etc. One of the earlier mathematical models was presented by O. Reynolds in [12] starting a very rich literature on this subject. We assume that the distance h between the lubricated surfaces is small, that one of the surfaces is given by $z = 0$ and that it moves with a given velocity $(U, 0, 0)$ (i.e. parallel to the x-axis). The pressure of the lubricant is denoted by P. Then, the incompressible Reynolds equation has the expression

$$\left\{ \begin{array}{ll} -\mathrm{div}(h^3 \nabla P) = -6\mu U \frac{\partial}{\partial x} h & \text{in } \Omega \\ P - P_0 = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1.1}$$

where μ is the viscosity of the fluid and P_0 the pressure at the boundary. The compressible equation is given by

$$\left\{ \begin{array}{ll} -\mathrm{div}(h^3 P \nabla P) = -6\mu U \frac{\partial}{\partial x}(hP) & \text{in } \Omega \\ P - P_0 = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{1.2}$$

The classical Reynolds equation (1.2) was later modified due to many different purposes. Some of these modifications take into account the slip flow and, so, the diffusion

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coefficient h^3P is replaced by $\alpha h^2 + \beta h^3 P$ (see Burgdorfer [3]). Here, we assume the mean free path of the molecules λ is small in comparison to the film thickness h (i.e. $\lambda/h < 10^{-2}$) and so we shall work with the classical Reynolds equation (1.2). In [11] Rayleigh considered the one dimensional problem. He optimized the load carrying capacity, obtaining the optimal form of "h" given by

$$h(x) = \begin{cases} h_0 & \text{if } x \in (0, L_1) \\ h_1 & \text{if } x \in (L_1, L), \end{cases}$$

whose solution P is given by

$$P = \begin{cases} P_0 + Ax & \text{if } x \in (0, L_1) \\ P_0 + \frac{AL_1}{L-L_1}(L-x) & \text{if } x \in (L_1, L) \end{cases}$$

where $A = A(\mu, U, h_0, h_1)$. This explicit solution to the one dimensional problem belongs to the Sobolev space $W^{1,\infty}(0, L)$, i.e. $|P_x| < C$ (see Cameron [4]). Nevertheless notice that $P \notin C^1(0, L)$. During the last century, many phenomena have been studied in lubrication theory, as cavitation or deformation of the surfaces, and many of them have coupled the Reynolds equation to other equations describing the new phenomena, see Elrod [8], Burgdorfer [4], Cameron [4], Bayada [1], Rodríguez and Liñán [13], etc. In the last 30 years many authors have been studied Reynolds equation from a theoretical point of view (see e.g. Chipot and Luskin [6], Chipot [5]). Problems where lubricated surfaces present discontinuities of type (1.3) frequently appear in different engineering applications, as in "feedbox", "shaft-bearing" systems, "magnetic head recording" or in "hard or floppy disc drives" see Bhusman [2] and also Friedman and Tello [9]. In this paper we study the regularity of the solution to (1.1) and (1.2) where $\Omega = (0, L) \times (0, B)$ and h is given by the discontinuous function

$$h(x, y) = \begin{cases} h_0 & \text{if } x \in (0, \frac{L}{2}) \\ h_1 & \text{if } x \in (\frac{L}{2}, L). \end{cases} \tag{1.3}$$

Since h presents a discontinuity the source term $\frac{\partial h}{\partial x} \approx \delta_{x=L_1}$ and we can not expect that the solution P belongs to $C^2(\Omega)$. For that reason we need introduce the standard notion of week solution, given in Definition 2.1 and 3.1. The regularity $C^{0,\alpha}(\Omega)$, $\forall \alpha \in [0, 1)$, of the weak solution of (1.1) is a direct consequence of the regularity theory (see e.g. Kinderlehrer and Stampacchia [9; Th.9.2]). The $W^{1,p}(\Omega)$ regularity is a more delicate question due to the lack of continuity of h. The main goal of this paper is to present a delicate and improved version of the results advanced in Diaz and Tello [7].

Theorem 1.1 *Let P be the unique weak solution to (1.1). Then $u := P - P_0 \in W_0^{1,\infty}(\Omega)$.*

Theorem 1.2 *Let P be a weak solution of (1.2). Then $P \in W^{1,\infty}(\Omega)$.*

The proofs of the above theorems are presented in Sections 2 and 3 respectively. We consider an auxiliary problem in order to construct a test function, which helps us to obtain the needed estimates in the regularized problems(see (2.3) and (3.2)). Once obtained the estimates we take limits in the weak formulation to obtain the desired results. Through the paper, we assume

$$h_0 < h_1 \quad \text{and} \quad U < 0, \tag{1.4}$$

n denotes the outward unit normal to $\partial\Omega$ and $d\sigma$ denotes $dxdy$.

2 Incompressible case

Since $h \in L^\infty(\Omega)$, then $-6\mu U \frac{\partial h}{\partial x} \in H^{-1}(\Omega)$ and we have

$$< -6\mu U \frac{\partial h}{\partial x}, \phi >_{H^{-1}(\Omega)\times H_0^1(\Omega)} = 6\mu U \int_\Omega h \frac{\partial \phi}{\partial x} d\sigma \qquad \phi \in H_0^1(\Omega).$$

Definition 2.1 *We say that P is a weak solution of (1.1) if $P = u + P_0$, $u \in H_0^1(\Omega)$ and u satisfies*

$$\int_\Omega h^3 \nabla u \cdot \nabla \phi d\sigma \ = \int_\Omega 6\mu U h \frac{\partial \phi}{\partial x} d\sigma \qquad \forall\ \phi \in H_0^1(\Omega). \tag{2.1}$$

The existence of a unique weak solution is a direct application of Lax-Milgram Theorem. The proof of Theorem 1.1 is structured in several steps. In the first step we approximate h by some continuous functions h_ε,

$$h_\varepsilon(x,y) = \begin{cases} h_0 & \text{if } 0 \le x \le \frac{L}{2}, \\ \frac{1}{\varepsilon}(x - \frac{L}{2} + \varepsilon h_0) & \text{if } \frac{L}{2} \le x \le \frac{L}{2} + \varepsilon(h_1 - h_0), \\ h_1 & \text{if } \frac{L}{2} + \varepsilon(h_1 - h_0) \le x \le L, \end{cases} \tag{2.2}$$

and consider the approximated problem

$$\begin{cases} -\operatorname{div}(h_\varepsilon^3 \nabla u_\varepsilon) = -6\mu U \frac{\partial}{\partial x}(h_\varepsilon) & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

We have

Proposition 2.1 *The solution u_ε to (2.3) belongs to $W_0^{1,q}(\Omega)$ for any $1 \le q < \infty$.*

According Troianiello [14; Theorem 3.7 and Theorem 3.14] the proof of the above result reduces to prove that the outer normal derivative $\frac{\partial u_\varepsilon}{\partial n}$ is a bounded function. It is deduced from the following lemma.

Lemma 2.1 $\frac{\partial u_\varepsilon}{\partial n} \in L^\infty(\partial\Omega)$.

Proof. Define the operator $L_\varepsilon(\cdot) = -\operatorname{div}\ (h_\varepsilon^3 \nabla(\cdot))$. Then we have

$$L_\varepsilon(u_\varepsilon) = -6\mu U \frac{\partial}{\partial x}(h_\varepsilon) = \begin{cases} 0, & \text{if } 0 \le x \le \frac{L}{2}, \\ -\frac{6\mu U}{\varepsilon} & \text{if } \frac{L}{2} \le x \le \frac{L}{2} + \varepsilon(h_1 - h_0), \\ 0 & \text{if } \frac{L}{2} + \varepsilon(h_1 - h_0) \le x \le L. \end{cases}$$

Let $\bar{u}_\varepsilon(x,y) = Cx(x-L)y(y-B)$ with $C = \frac{48\mu|U|}{h_0^3 L^2 \varepsilon}$. A routine computation shows that

$$L_\varepsilon(\bar{u}_\varepsilon) = \begin{cases} -2Ch_0^3 y(y-B) - 2Ch_0^3 x(x-L) \ \text{ on } 0 \le x \le \frac{L}{2}, \\ -2Ch_1^3 y(y-B) - 2h_1^3 x(x-L) \ \text{ on } \frac{L}{2} + \varepsilon(h_1 - h_0) \le x \le L, \\ -Cy(y-B)[2h_\varepsilon^3 + \frac{3}{\varepsilon}(2x-L)h_\varepsilon^2] - Ch_\varepsilon^3 x(x-L) \ \text{ otherwise.} \end{cases}$$

Since $L_\varepsilon(\bar{u}_\varepsilon) > L_\varepsilon(u_\varepsilon) \ge L_\varepsilon(0) = 0$ in Ω, and $\bar{u}_\varepsilon = u_\varepsilon = 0$ on $\partial\Omega$, we get that $\bar{u}_\varepsilon$ is a supersolution and 0 is a subsolution to (2.1). Thus, by the comparison principle and the uniqueness of solutions, we obtain $\bar{u}_\varepsilon \ge u_\varepsilon \ge 0$ in Ω. In consequence

$$\frac{\partial \bar{u}_\varepsilon}{\partial n} \le \frac{\partial u_\varepsilon}{\partial n} \le 0$$

which implies that

$$-C(B^2 + L^2) \le \frac{\partial u_\varepsilon}{\partial n} \le 0 \quad \text{and so} \quad \frac{\partial u_\varepsilon}{\partial n} \in L^\infty(\partial\Omega).$$

□ In a second step, given $q > 1$, we construct a test function w_ε as solution to the auxiliary problem

$$\begin{cases} -\operatorname{div}(h_\varepsilon^3 \nabla w_\varepsilon) = -\operatorname{div}(|\nabla u_\varepsilon|^{q-2} \nabla u_\varepsilon) & \text{in } \Omega \\ h_\varepsilon^3 \frac{\partial w_\varepsilon}{\partial \vec{n}} + a_\varepsilon(x,y) w_\varepsilon = |\nabla u_\varepsilon|^{q-2} \frac{\partial u_\varepsilon}{\partial n} & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where

$$a_\varepsilon(x,y) := G_\varepsilon^{-1}(h_\varepsilon^3 \frac{\partial u_\varepsilon}{\partial n} - h_\varepsilon^3 \frac{\partial G_\varepsilon}{\partial n}),$$

$$G_\varepsilon(x) := \int_0^x \frac{6\mu U}{h_\varepsilon^2(s)} ds - k_0,$$

and

$$k_0 := -\frac{(\|\frac{\partial u_\varepsilon}{\partial n}\|_{L^\infty(\partial\Omega)}^q + |U| h_1) C^2(\Omega) h_0^3}{6\mu}.$$

In order to solve (2.4) we introduce the Hilbert space

$$V(\Omega) := \{\phi \in H^1(\Omega) \text{ such that } \int_{\partial\Omega} a_\varepsilon \phi d\sigma_2 = 0\}$$

associated to the inner product $< \phi, \psi >_V := \int_\Omega \nabla\phi \cdot \nabla\psi d\sigma + (\int_\Omega \phi d\sigma)(\int_\Omega \psi d\sigma)$.

Proposition 2.2 *Problem (2.4) has a unique weak solution $w_\varepsilon \in V(\Omega)$.*

For the proof we need a previous result

Lemma 2.2 $\int_{\partial\Omega} a_\varepsilon d\sigma_2 < 0$ *for any* $\epsilon > 0$.

Proof. Integrating in (2.3) it results

$$\int_{\partial\Omega} h_\varepsilon^3 \frac{\partial u_\varepsilon}{\partial n} - h_\varepsilon^3 \frac{\partial G_\varepsilon}{\partial n} d\sigma_2 = 0.$$

By construction of a_ε we have

$$\int_{\partial\Omega} a_\varepsilon d\sigma_2 = \int_{\partial\Omega} (G_\varepsilon^{-1} - G^{-1}(0))(h_\varepsilon^3 \frac{\partial u_\varepsilon}{\partial n} - h_\varepsilon^3 \frac{\partial G_\varepsilon}{\partial n}) d\sigma_2.$$

Since (by Lemma 2.1)

$$\frac{\partial u_\varepsilon}{\partial n} \leq 0 \text{ on } \partial\Omega, \quad \frac{\partial G_\varepsilon}{\partial n} = 0 \text{ on } y(B-y) = 0, \quad \frac{\partial G_\varepsilon}{\partial n} > 0 \text{ on } x = L,$$

and $G_\varepsilon^{-1} - G^{-1}(0) > 0$ we obtain the desired result. □ We point out that any function in $H^1(\Omega)$ can be expressed as the sum of a function in $V(\Omega)$ plus a constant.

Proof of Proposition 2.2. We define the bilinear form $A_\varepsilon : V(\Omega) \times V(\Omega) \to I\!R$ by

$$A_\varepsilon(\phi, \psi) := \int_\Omega h_\varepsilon^3 \nabla\psi \cdot \nabla\phi d\sigma + \int_{\partial\Omega} a_\varepsilon \psi\phi d\sigma_2.$$

Since

$$\int_{\partial\Omega} a_\varepsilon \psi\phi d\sigma_2 = \int_{\partial\Omega} a_\varepsilon (\psi - \frac{1}{|\Omega|}\int_\Omega \psi d\sigma)(\phi - \frac{1}{|\Omega|}\int_\Omega \phi d\sigma) d\sigma_2 - \frac{1}{|\Omega|^2}\int_{\partial\Omega} a_\varepsilon d\sigma_2 \int_\Omega \psi d\sigma \int_\Omega \phi d\sigma$$

and $\| \psi - \frac{1}{|\Omega|}\int_\Omega \psi \|_{L^2(\partial\Omega)} \leq C(\Omega) \| \nabla\psi \|_{(L^2(\Omega))^2}$ we obtain the continuity of A_ε. Coerciveness is a consequence of the fact that $\int_{\partial\Omega} a_\varepsilon < 0$. Since A_ε is a continuous, coercive and bilinear form, by Lax-Milgram Theorem, there exists a unique $w_\varepsilon \in V(\Omega)$ weak solution to (2.3). □

Lemma 2.3 *There exists a positive constant $K(\Omega, h_0, U, \mu)$ such that*

$$\| u_\varepsilon \|_{W_0^{1,q}(\Omega)} \leq K(\Omega, h_0, U, \mu) \quad \forall \quad 1 < q < \infty.$$

Proof. We take w_ε as test function in (2.3) and we get

$$\int_\Omega h_\varepsilon^3 \nabla u_\varepsilon \cdot \nabla w_\varepsilon d\sigma = \int_\Omega \nabla w_\varepsilon \cdot (6\mu U h_\varepsilon, 0) d\sigma + \int_{\partial\Omega} h_\varepsilon^3 (\frac{\partial u_\varepsilon}{\partial n} - < (6\mu U h_\varepsilon, 0), n >) w_\varepsilon d\sigma_2.$$

Taking u_ε as test function in (2.4) we obtain

$$\int_\Omega h_\varepsilon^3 \nabla u_\varepsilon \cdot \nabla w_\varepsilon d\sigma = \int_\Omega |\nabla u_\varepsilon|^q d\sigma. \tag{2.5}$$

Therefore

$$\int_\Omega |\nabla u_\varepsilon|^q d\sigma = \int_\Omega \nabla w_\varepsilon \cdot (6\mu U h_\varepsilon, 0) d\sigma + \int_{\partial\Omega} (h_\varepsilon^3 \frac{\partial u_\varepsilon}{\partial n} - < (6\mu U h_\varepsilon, 0), n >) w_\varepsilon d\sigma_2. \tag{2.6}$$

By construction of G_ε we know that $h_\varepsilon^3 \nabla G_\varepsilon = (6\mu U h_\varepsilon, 0)$. Thus

$$\int_\Omega \nabla w_\varepsilon \cdot (6\mu U h_\varepsilon, 0) d\sigma = \int_\Omega h_\varepsilon^3 \nabla G_\varepsilon \cdot \nabla w_\varepsilon d\sigma. \tag{2.7}$$

Taking G_ε as test function in (2.4) we obtain

$$\int_\Omega h_\varepsilon^3 \nabla G_\varepsilon \cdot \nabla w_\varepsilon d\sigma = \int_\Omega |\nabla u_\varepsilon|^{q-2} \nabla u_\varepsilon \cdot \nabla G_\varepsilon d\sigma$$

$$- \int_{\partial\Omega} G_\varepsilon |\nabla u_\varepsilon|^{q-2} \frac{\partial u_\varepsilon}{\partial n} d\sigma_2 + < G_\varepsilon, h_\varepsilon^3 \frac{\partial w_\varepsilon}{\partial n} >_{H^{1/2}, H^{-1/2}}$$

From (2.6), (2.7) and the above expression we get

$$\int_\Omega |\nabla u_\varepsilon|^q d\sigma = \int_\Omega |\nabla u_\varepsilon|^{q-2} \nabla u_\varepsilon \cdot \nabla G_\varepsilon d\sigma - \int_{\partial\Omega} G_\varepsilon |\nabla u_\varepsilon|^{q-2} \frac{\partial u_\varepsilon}{\partial n} d\sigma_2 +$$

$$+ < G_\varepsilon, h_\varepsilon^3 \frac{\partial w_\varepsilon}{\partial n} >_{H^{1/2}, H^{-1/2}} + \int_{\partial\Omega} (h_\varepsilon^3 \frac{\partial u_\varepsilon}{\partial n} - < (6\mu U h_\varepsilon, 0), n >) w_\varepsilon d\sigma_2.$$

Using the definition of w_ε we arrive to

$$\int_\Omega |\nabla u_\varepsilon|^q d\sigma = \int_\Omega |\nabla u_\varepsilon|^{q-2} \nabla u_\varepsilon \cdot \nabla G_\varepsilon d\sigma,$$

which implies

$$\int_\Omega |\nabla u_\varepsilon|^q d\sigma \leq \|\nabla G_\varepsilon\|_{(L^\infty(\Omega))^2} \int_\Omega |\nabla u_\varepsilon|^{q-1} d\sigma.$$

In consequence

$$\|\nabla u_\varepsilon\|^q_{(L^q)^2} \leq \frac{6\mu U}{h_0^2} \|\nabla u_\varepsilon\|^{q-1}_{(L^{q-1}(\Omega))^2} \leq C(\Omega) \frac{6\mu U}{h_0^2} \|\nabla u_\varepsilon\|^{q-1}_{(L^q(\Omega))^2},$$

i.e. $\|\nabla u_\varepsilon\|_{(L^q)^2} \leq C(\Omega) \frac{6\mu U}{h_0^2}$, and so we conclude that $\|u_\varepsilon\|_{W_0^{1,q}} \leq C(\Omega) \frac{6\mu U}{h_0^2}$ for any $q \in (1, \infty)$. $\square$ **End of the proof of Theorem 1.** Since $\| u_\varepsilon \|_{W_0^{1,q}(\Omega)} \leq C$ then there exists a subsequence u_ε which converges weakly in $W_0^{1,q}(\Omega)$, to v. Taking limits in the weak formulation of (2.3), we get that v is a solution to (1.1). From the uniqueness of solution we deduce that $u = v$ and then $u \in W_0^{1,q}(\Omega)$ for any $q < \infty$. Moreover $\| u \|_{W_0^{1,q}(\Omega)} \leq C(\Omega, h_0, \mu, U)$, which is independent of q. Since $\| u \|_{W_0^{1,\infty}} = lim_{q \to \infty} \| u \|_{W_0^{1,q}}$ taking limits when $q \to \infty$ we conclude the proof. $\square$

Lemma 2.4 *The solution P to problem (2.1) is such that $P \notin C^1(\Omega)$.*

Proof. Since $-\Delta P = 0$ outside of $x = \frac{L}{2}$, and $P \geq P_0$, P takes his maximum at some point (x_0, y_0), where $x_0 = \frac{L}{2}$ and $0 < y_0 < B$. Consider the equation

$$\begin{cases} -\Delta P_1 = 0 & \text{in } \Omega_1 \\ P_1 = P & \text{on } \partial\Omega_1, \end{cases}$$

where Ω_1 is an open subset of Ω with regular boundary such that $\Omega_1 \subset \Omega \cap \{0 < x < \frac{L}{2}\}$ and $(\frac{L}{2}, y_0)$, $(0, y_0)$ belonging to $\partial\Omega_1$. By the strong maximum principle we obtain

$$\frac{\partial}{\partial x} P_1 > 0. \tag{2.8}$$

In the same way we consider P_2, $\Omega_2 \subset \Omega \cap \{\frac{L}{2} < x < L\}$ and $(\frac{L}{2}, y_0)$, (L, y_0) belonging to $\partial\Omega_2$ and we obtain

$$\frac{\partial}{\partial x} P_2 < 0. \tag{2.9}$$

By the uniqueness of solution it results $P_1 = P$ in Ω_1 and $P_2 = P$ in Ω_2 and by (2.8) and (2.9) we get

$$\lim_{(x,y)\to(\frac{L}{2},y_0),\ x<\frac{L}{2}} \frac{\partial}{\partial x} P > 0 > \lim_{(x,y)\to(\frac{L}{2},y_0),\ x>\frac{L}{2}} \frac{\partial}{\partial x} P,$$

which proves the lemma. □

3 Compressible case

In this section we study the compressible case, modeled by the compressible Reynolds equation (1.2), where h is given by (1.3). As in Section 2 we introduce the standard notion of weak solution

Definition 3.1 *We say $P \geq 0$ is a weak solution to (1.2) if $P - P_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfies*

$$\int_\Omega h^3 P \nabla P \cdot \nabla\psi d\sigma = \int_\Omega 6\mu U h P \frac{\partial\psi}{\partial x} d\sigma \qquad \forall \;\; \psi \in H_0^1(\Omega).$$

The proof of the Theorem 1.2 consists in several steps. First we introduce the unknown function $u = \frac{1}{2}P^2$ which satisfies

$$\begin{cases} -\text{div}(h^3 \nabla u) = -6\sqrt{2} U \mu \dfrac{\partial}{\partial x}(h\sqrt{u}) & \text{in } \Omega, \\ \\ u - \frac{1}{2}P_0^2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

The existence of weak solution can be proved by several ways: using a fixed point theorem (see Chipot [5]) or as limit of the solutions of some regular approximated problems. Let us consider a continuous approximation of h (given by (2.2)) and the approximated problem

$$\begin{cases} -\text{ div } (h_\varepsilon^3 \nabla u_\varepsilon) = -6\sqrt{2} U \mu \frac{\partial}{\partial x}(h_\varepsilon \sqrt{u_\varepsilon}) & \text{in } \Omega, \\ u_\varepsilon = \frac{1}{2}P_0^2 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Then

Lemma 3.1 *There exists a weak solution to (3.2). Furthermore the weak solution u_ε belongs to $W^{1,q}(\Omega)$ for any $1 \leq q < \infty$.*

Proof. Let us consider the operator

$$L_\varepsilon(\cdot) = -\text{ div } (h_\varepsilon^3 \nabla(\cdot)) + 6\sqrt{2}\mu U \frac{\partial}{\partial x}(h_\varepsilon \sqrt{(\cdot)}).$$

Let $u_0 = \frac{1}{2}P_0^2$, then, by (1.4) we get $L_\varepsilon(u_0) \leq 0$. Let us consider $\bar{u}_\varepsilon$ defined by $\bar{u}_\varepsilon(x,y) = Cx(x-L)y(y-B) + P_0^2$, where $C = (\frac{12\mu BU_0}{\varepsilon h_0^3 L})^2$. Then we have

$$\begin{cases} L_\varepsilon(\bar{u}_\varepsilon) > L_\varepsilon(u_\varepsilon) \geq L_\varepsilon(u_0) & \text{in } \Omega, \\ \bar{u}_\varepsilon = u_\varepsilon = u_0 & \text{on } \partial\Omega, \end{cases}$$

and by comparison we obtain the existence of solution u_ε satisfying

$$\bar{u}_\varepsilon \geq u_\varepsilon \geq P_0^2 \text{ in } \Omega.$$

Then

$$\frac{\partial \bar{u}_\varepsilon}{\partial n} \leq \frac{\partial u_\varepsilon}{\partial n} \leq 0$$

and we get

$$-C(B^2 + L^2) \leq \frac{\partial u_\varepsilon}{\partial n} \leq 0. \tag{3.3}$$

□ The second step of the proof of Theorem 1.2 is to prove the regularity of $P_\varepsilon = \sqrt{2u_\varepsilon}$, where P_ε is the solution to

$$\begin{cases} -\text{ div}(h_\varepsilon^3 P_\varepsilon \nabla P_\varepsilon) = -6\mu U \frac{\partial}{\partial x}(h_\varepsilon P_\varepsilon) & \text{in } \Omega, \\ P_\varepsilon = P_0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Proposition 3.1 *The solution to (3.3) satisfies*

$$\| P_\varepsilon \|_{W^{1,q}(\Omega)} \leq K$$

for any $1 \leq q < \infty$ and $K = K(\Omega, h_0, U_0, \mu, P_0)$, i.e. independent of q and ε.

In order to prove Proposition 3.1, given $q > 1$, we introduce the auxiliary problem

$$\begin{cases} -\text{ div } (h_\varepsilon^3 P_\varepsilon \nabla w_\varepsilon) = -\text{ div } (|\nabla P_\varepsilon|^{q-2} \nabla P_\varepsilon) & \text{in } \Omega, \\ P_0 h_\varepsilon^3 \frac{\partial w_\varepsilon}{\partial \vec{n}} + a_\varepsilon(x,y) w_\varepsilon = |\nabla P_\varepsilon|^{q-2} \frac{\partial P_\varepsilon}{\partial n} & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where

$$a_\varepsilon(x,y) := P_0 G_\varepsilon^{-1}(h_\varepsilon^3 \frac{\partial P_\varepsilon}{\partial n} - h_\varepsilon^3 \frac{\partial G_\varepsilon}{\partial n}),$$

$$G_\varepsilon(x) := \int_0^x \frac{6\mu U_0}{h_\varepsilon^2(s)} ds - k_0$$

and

$$k_0 := -\frac{(\| \frac{\partial P_\varepsilon}{\partial n} \|_{L^\infty(\partial\Omega)}^q + |U_0| h_1) C^2(\Omega) P_0 h_0^3}{6\mu}.$$

Lemma 3.2 *Problem (3.4) has a unique weak solution $w_\varepsilon \in V(\Omega)$.*

Proof. As in Section 2 we consider the bilinear continuous and coercive form $A_\varepsilon : V(\Omega) \times V(\Omega) \to I\!R$ defined by

$$A_\varepsilon(\phi, \psi) := \int_\Omega h_\varepsilon^3 P_\varepsilon \nabla\psi \cdot \nabla\phi dx + \int_{\partial\Omega} a_\varepsilon \psi\phi d\sigma.$$

Since $P_0 \le P_\varepsilon \le \sqrt{2\bar{u}_\varepsilon}$ we obtain the continuity and coerciveness of A_ε (as in Proposition 2.2). Applying Lax-Milgram Theorem we conclude the proof. □ **Proof of**

Proposition 3.1. Taking w_ε as test function in (3.3) we obtain

$$\textstyle\int_\Omega P_\varepsilon h_\varepsilon^3 \nabla P_\varepsilon \cdot \nabla w_\varepsilon d\sigma = \int_\Omega P_\varepsilon \nabla w_\varepsilon \cdot (6\mu U h_\varepsilon, 0) d\sigma +$$

$$\textstyle + \int_{\partial\Omega} P_0 (h_\varepsilon^3 \frac{\partial P_\varepsilon}{\partial n} - < (6\mu U h_\varepsilon, 0), n >) w_\varepsilon d\sigma_2.$$

Taking $P_\varepsilon - P_0$ as test function in (3.4) it results

$$\int_\Omega P_\varepsilon h_\varepsilon^3 \nabla P_\varepsilon \cdot \nabla w_\varepsilon d\sigma = \int_\Omega |\nabla P_\varepsilon|^q d\sigma.$$

Then we get

$$\int_\Omega |\nabla P_\varepsilon|^q d\sigma = \int_\Omega P_\varepsilon 6\mu U h_\varepsilon \frac{\partial w_\varepsilon}{\partial x} d\sigma + \int_{\partial\Omega} P_0 (h_\varepsilon^3 \frac{\partial P_\varepsilon}{\partial n} - < (6\mu U h_\varepsilon, 0), n >) w_\varepsilon d\sigma_2. \quad (3.6)$$

Since $(6\mu U h_\varepsilon, 0) = h_\varepsilon^3 \nabla G_\varepsilon$ it results

$$\int_\Omega P_\varepsilon 6\mu U h_\varepsilon \frac{\partial w_\varepsilon}{\partial x} d\sigma = \int_\Omega P_\varepsilon h_\varepsilon^3 \nabla G_\varepsilon \cdot \nabla w_\varepsilon d\sigma.$$

Taking G_ε as test function in (3.4) and using (3.6) we obtain

$$\int_\Omega |\nabla P_\varepsilon|^q d\sigma = \int_\Omega |\nabla P_\varepsilon|^{q-2} \nabla P_\varepsilon \cdot \nabla G_\varepsilon d\sigma -$$

$$\int_{\partial\Omega} G_\varepsilon |\nabla P_\varepsilon|^{q-2} \frac{\partial P_\varepsilon}{\partial n} d\sigma_2 + < G_\varepsilon, P_0 h_\varepsilon^3 \frac{\partial w_\varepsilon}{\partial n} >_{H^{1/2}, H^{-1/2}} =$$

$$\int_{\partial\Omega} P_0 (h_\varepsilon^3 \frac{\partial P_\varepsilon}{\partial n} - < (6\mu U h_\varepsilon, 0), n >) w_\varepsilon d\sigma_2.$$

Using the definition of w_ε it results

$$\int_\Omega |\nabla P_\varepsilon|^q d\sigma = \int_\Omega |\nabla P_\varepsilon|^{q-2} \nabla P_\varepsilon \cdot \nabla G_\varepsilon d\sigma$$

and then

$$\int_\Omega |\nabla P_\varepsilon|^q dx \le \|\nabla G_\varepsilon\|_{(L^\infty(\Omega))^2} \int_\Omega |\nabla P_\varepsilon|^{q-1} d\sigma.$$

Consequently

$$\|\nabla P_\varepsilon\|^q_{(L^q(\Omega))^2} \le \frac{6\mu U}{h_0^2} \|\nabla P_\varepsilon\|^{q-1}_{(L^{q-1}(\Omega))^2} \le C(\Omega) \frac{6\mu U}{h_0^2} \|\nabla P_\varepsilon\|^{q-1}_{(L^q(\Omega))^2},$$

i.e. $\|\nabla P_\varepsilon\|_{(L^q(\Omega))^2} \leq C(\Omega)\frac{6\mu U}{h_0^2}$, and since $P_\varepsilon = P_0$ on $\partial\Omega$ we obtain the desired result. □ **Proof of Theorem 1.2.** Since $\| P_\varepsilon \|_{W^{1,q}(\Omega)} \leq K$ then there exists a subsequence $P_{i\varepsilon}$ which converges to v weakly in $W^{1,q}(\Omega)$. Since $W^{1,q}(\Omega) \subset L^\infty(\Omega)$ is a compact embedding it results $P_\varepsilon \longrightarrow v$ in $L^\infty(\Omega)$. Taking limits in the weak formulation of (3.3) we obtain

$$\int_\Omega v h^3 \nabla v \cdot \nabla \psi dx = \int_\Omega v \nabla \psi \cdot (6\mu U h, 0) dx \quad \forall\ \psi \in H_0^1(\Omega)$$

which implies that v is a weak solution to (1.2). Since $v \in W^{1,q}(\Omega)$ for any $1 \leq q < \infty$ and

$$\| v \|_{W^{1,q}(\Omega)} \leq C(\Omega, h_0, P_0, \mu, U)$$

taking limits as q goes to ∞ we obtain $v \in W^{1,\infty}(\Omega)$. □ ACKNOWLEDGMENT. The author thanks to J.I. Díaz for many helpful suggestions. The author is partially supported by project DGES (Spain) REN 2000/0766.

References

[1] Bayada, G. *Inequations variationelles elliptiques avec conditions aux limites periodiques : application a la resolution de l'equation de Reynolds.* These de 3^{eme} cycle. Univ. de Lyon, 1972.

[2] Bhushan, B. *Tribology and Mechanics of Magnetic Storage Devices,* Springer-Verlag, New York, 1990.

[3] Burgdorfer, A. "The influence of the molecular mean free path on the performance of hydrodynamics gas lubricated bearings". *J. Basic Eng. Trans. ASME* **81** (1956) 94-100.

[4] Cameron, A. *Principles of Lubrication,* Longman, London, 1966.

[5] Chipot, M. "On the Reynolds lubrication equation", *Nonlinear analysis, theory methods and applications* **12** (1988) 699-718.

[6] Chipot, M. and Luskin, M. "Existence and uniqueness of solutions to the compressible Reynolds lubrication equation", *SIAM J. Math. Anal.* **17** (1986) 1390 - 1399.

[7] Díaz, J.I. and Tello, J.I. "On a problem lacking a classical solution in lubrication theory", *Actas del XV-CEDYA, Vigo* (1997) 429-434.

[8] Elrod, H.G. "A derivation of the basic equations for hydrodynamic lubrication with a fluid having constant properties", *Quart. Appl. Math.* **17** (1959) 349-359.

[9] Friedman, A. and Tello, J.I. "Head-Media interaction in magnetic recording", *J. Dif. Eq.* **171** (2001) 443-461.

[10] Kinderlehrer, D. and Stampacchia, G. *An Introduction to Variational Inequalities and Their Applications,* Academic Press, New York, 1980.

[11] Rayleight, L. "Notes on the theory of lubrication", *Phil. Mag. J.* (1918) 1-12.

[12] Reynolds, O. "On the theory of lubrication and its application to Mr Beauchamp Tower′s experiments, including an experimental determination of the viscosity of olive oil", *Phil. Trans. R. Soc.* **177** (1886) 157-234.

[13] Rodríguez, M. and Liñán, A. "Cavitation in short Bearings", *AMSE Journal of Tribology* **107** (1985) 142-144.

[14] Tello, J.I. *Tratamiento matemático de procesos físicos con difusión y convección*, Ph.D. Thesis, U. Complutense de Madrid, 2001.

[15] Troianiello, G.M. *Elliptic differential equations and obstacle problems,* Plenum Press, 1987.