DIFFEOMORPHISMS BETWEEN SPHERES AND HYPERPLANES IN INFINITE-DIMENSIONAL BANACH SPACES

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Abstract. We prove that for every infinite-dimensional Banach space X with a Fréchet differentiable norm, the sphere S_X is diffeomorphic to each closed hyperplane in X. We also prove that every infinite-dimensional Banach space Y having a (not necessarily equivalent) C^p norm (with $p \in \mathbb{N} \cup \{\infty\}$) is C^p diffeomorphic to $Y \setminus \{0\}$.

In 1966 C. Bessaga [1] proved that every infinite-dimensional Hilbert space H is C^∞ diffeomorphic to its unit sphere. The key to prove this astonishing result was the construction of a diffeomorphism between H and $H \setminus \{0\}$ being the identity outside a ball, and this construction was possible thanks to the existence of a C^∞ non-complete norm in H. T. Dobrowolski [5] developed Bessaga's non-complete norm technique and proved that every infinite-dimensional Banach space X which is linearly injectable into some $c_0(\Gamma)$ is C^∞ diffeomorphic to $X \setminus \{0\}$. More generally he proved that every infinite-dimensional Banach space X having a C^p non-complete norm is C^p diffeomorphic to $X \setminus \{0\}$. If in addition X has an equivalent C^p smooth norm $\|.\|$ then one can deduce that the sphere $S = \{x \in X : \|x\| = 1\}$ is C^p diffeomorphic to any of the hyperplanes in X. So, regarding the generalization of Bessaga and Dobrowolski's results to every infinite-dimensional Banach space having a differentiable norm (resp. C^p smooth norm, with $p \in \mathbb{N} \cup \{\infty\}$), the following problem naturally arises: does every infinite-dimensional Banach space with a C^p smooth equivalent norm have a C^p smooth non-complete norm? Surprisingly enough, this seems to be a difficult question which still remains unsolved. Without proving the existence of smooth non-complete norms we show that every infinite-dimensional Banach space X with a Fréchet differentiable (resp. C^p smooth) norm $\|.\|$ is diffeomorphic (resp. C^p diffeomorphic) to $X \setminus \{0\}$, and we deduce that the sphere $S_X = \{x \in X : \|x\| = 1\}$ is (C^p) diffeomorphic to any of the closed hyperplanes H in X. We also prove that every infinite-dimensional Banach space Y having a (not necessarily equivalent) C^p smooth norm is C^p diffeomorphic to $Y \setminus \{0\}$. Our method of defining deleting diffeomorphisms can be viewed, in a sense, as an analytical adaptation of Klee's geometrical approach in [14], which was rediscovered and simplified in [10],

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where a recipe for a construction of homeomorphisms removing convex bodies from non-reflexive Banach spaces is given.

Let us formally state our main result. Recall that a norm in a Banach space X is said to be Fréchet differentiable (resp. C^p smooth) if it is so on $X \setminus \{0\}$.

Theorem 1. *Let $(X, \|.\|)$ be an infinite-dimensional Banach space with a C^p smooth norm $\|.\|$, and let S_X be its unit sphere. Then, for every closed hyperplane H in X, there exists a C^p diffeomorphism between S_X and H.*

The argument in the proof of this result is a modification of that in [1], changing the non-complete norm and the use of Banach's contraction principle for a different kind of *non-complete convex function* and the following *fixed point lemma*

Lemma 2. *Let $F : (0, \infty) \longrightarrow [0, \infty)$ be a continuous function such that, for every $\beta \geq \alpha > 0$,*

$$F(\beta) - F(\alpha) \leq \frac{1}{2}(\beta - \alpha) \, and \limsup_{t \to 0^+} F(t) > 0.$$

Then there exists a unique $\alpha > 0$ such that $F(\alpha) = \alpha$.

Proof. Note that $\lim_{\beta\to\infty}[F(\beta) - \beta] \leq \lim_{\beta\to\infty}[F(1) + \frac{1}{2}(\beta - 1) - \beta] = -\infty$, while $\limsup_{\beta\to 0^+}[F(\beta) - \beta] > 0$. Then, from Bolzano's theorem we get an $\alpha > 0$ such that $F(\alpha) = \alpha$. Moreover, the first condition in the statement implies that the function defined by $\beta \longrightarrow F(\beta) - \beta$ is strictly decreasing, which yields the uniqueness of this α.

The key to the proof of theorem 1 is the following

Proposition 3. *Let $(X, \|.\|)$ be a non-reflexive infinite-dimensional Banach space with a C^p smooth norm $\|.\|$. Then there exists a C^p diffeomorphism φ between X and $X \setminus \{0\}$ such that $\varphi(x) = x$ whenever $\|x\| \geq 1$.*

Proof. Since X is not reflexive, according to James's theorem [13], there exists a continuous linear functional $T : X \longrightarrow \mathbb{R}$ such that T does not attain its norm. We may assume $\|T\| = 1$, so that $T(x) < \|x\|$ for every $x \neq 0$, and there exists a sequence (y_k) of vectors such that $\|y_k\| = 1$ and

$$\|y_k\| - T(y_k) = 1 - T(y_k) \leq \frac{1}{4^{k+1}}$$

for every $k \in \mathbb{N}$. Let us define $\omega : X \longrightarrow \mathbb{R}$ by $\omega(x) = \|x\| - T(x)$. Note that $\omega(x) = 0$ if and only if $x = 0$, $\omega(x + y) \leq \omega(x) + \omega(y)$ and $\omega(rx) = r\omega(x)$ for each $r > 0$, although ω is not a norm in X because $\omega(x) \neq \omega(-x)$ in general. Now, let $\gamma : [0, \infty) \longrightarrow [0, 1]$ be a non-increasing C^∞ function such that $\gamma = 1$ in $[0, 1/2]$, $\gamma = 0$ in $[1, \infty)$ and $\sup\{|\gamma'(t)| : t \in [0, \infty)\} \leq 4$, and let us define the following deleting path $p : (0, \infty) \longrightarrow X$,

$$p(t) = \sum_{k=1}^{\infty} \gamma(2^{k-1}t) y_k.$$

It is quite clear that p is a well defined C^∞ path such that $p(t) = 0$ for $t \geq 1$. Let y be an arbitrary vector in X and let $F : (0, \infty) \longrightarrow [0, \infty)$ be defined by $F(\alpha) = \omega(y - p(\alpha))$ for $\alpha > 0$. Let us see that $F(\alpha)$ satisfies the conditions of lemma 2. If $\beta \geq \alpha$ then $\gamma(2^{k-1}\alpha) - \gamma(2^{k-1}\beta) \geq 0$ because γ is non-increasing, and also $\gamma(2^{k-1}\alpha) - \gamma(2^{k-1}\beta) \leq 4|2^{k-1}\alpha - 2^{k-1}\beta|$ because $\sup\{|\gamma'(t)| : t \in [0, \infty)\} \leq 4$. Let us also note that the property $\omega(z + y) \leq \omega(z) + \omega(y)$ implies that $\omega(x) - \omega(y) \leq \omega(x - y)$, as well as $\omega(\sum_{k=1}^\infty z_k) \leq \sum_{k=1}^\infty \omega(z_k)$ for every convergent series $\sum_{k=1}^\infty z_k$. Taking this into account and recalling the positive homogeneity of ω we may deduce

$$\begin{aligned}
&F(\beta) - F(\alpha) = \omega(y - p(\beta)) - \omega(y - p(\alpha))\\
&\leq \omega\big((y - p(\beta)) - (y - p(\alpha))\big) = \omega(p(\alpha) - p(\beta))\\
&= \omega(\sum_{k=1}^\infty (\gamma(2^{k-1}\alpha) - \gamma(2^{k-1}\beta))y_k) \leq \sum_{k=1}^\infty \omega((\gamma(2^{k-1}\alpha) - \gamma(2^{k-1}\beta))y_k)\\
&= \sum_{k=1}^\infty (\gamma(2^{k-1}\alpha) - \gamma(2^{k-1}\beta))\omega(y_k) \leq \sum_{k=1}^\infty 4|2^{k-1}\alpha - 2^{k-1}\beta|\omega(y_k)\\
&= \sum_{k=1}^\infty 2^{k+1}\omega(y_k)|\beta - \alpha| \leq \sum_{k=1}^\infty 2^{k+1}\frac{1}{4^{k+1}}|\beta - \alpha| = \frac{1}{2}(\beta - \alpha)
\end{aligned}$$

for every $\beta \geq \alpha$, so that the first condition in lemma 2 is satisfied. Let us check that F also satisfies the second condition. Let $M > 0$ and choose $k_0 \in \mathbb{N}$ such that $\sum_{j=1}^{k_0} T(y_j) > M + T(y)$ (this is clearly possible, as $T(y_k) \to 1$ when $k \to \infty$). Then, if $0 < \alpha < 1/2^{k_0}$, $\gamma(2^{j-1}\alpha) = 1$ for $j = 1, 2, ..., k_0$, which implies

$$\begin{aligned}
&F(\alpha) = \omega(y - p(\alpha)) = \|y - p(\alpha)\| - T(y) + T(p(\alpha))\\
&\geq -T(y) + T(p(\alpha)) = -T(y) + \sum_{k=1}^\infty \gamma(2^{k-1}\alpha)T(y_k)\\
&\geq -T(y) + \sum_{j=1}^{k_0} \gamma(2^{j-1}\alpha)T(y_j) = -T(y) + \sum_{j=1}^{k_0} T(y_j)\\
&> -T(y) + M + T(y) = M
\end{aligned}$$

for every $\alpha > 0$ such that $\alpha < 1/2^{k_0}$. This proves that

$$\lim_{t \to 0^+} F(t) = +\infty.$$

So, according to lemma 2, the equation $F(\alpha) = \alpha$ has a unique solution. This means that for any $y \in X$, a number $\alpha(y) > 0$ with the property

$$\omega(y - p(\alpha(y))) = \alpha(y),$$

is uniquely determined. This implies that the mapping

$$\psi(x) = x + p(\omega(x))$$

is one-to-one from $X \setminus \{0\}$ onto X, with

$$\psi^{-1}(y) = y - p(\alpha(y)).$$

As ω and p are C^p, so is ψ. Let $\Phi(y, \alpha) = \alpha - \omega(y - p(\alpha))$. Since for any $y \in X$ we have $y - p(\alpha(y)) \neq 0$, the mapping Φ is differentiable on a neighbourhood of any point $(y_0, \alpha(y_0))$ in $X \times (0, \infty)$. On the other hand, since $F(\beta) - F(\alpha) \leq \frac{1}{2}(\beta - \alpha)$ for $\beta \geq \alpha > 0$, it is clear that $F'(\alpha) \leq \frac{1}{2}$ for every α in a neighbourhood of $\alpha(y)$, and so

$$\frac{\partial\Phi(y, \alpha)}{\partial\alpha} = 1 - F'(\alpha) \geq 1 - 1/2 > 0.$$

Thus, using the implicit function theorem we obtain that the map $y \rightarrow \alpha(y)$ is of class C^p and therefore $\psi : X \setminus \{0\} \longrightarrow X$ is a C^p diffeomorphism. Let $h : X \longrightarrow X \setminus \{0\}$ be the inverse of ψ. It should be noted that $h(x) = x$ whenever $\omega(x) = \|x\| - T(x) \geq 1$. In order to conclude the proof we only need to compose h with a C^p diffeomorphism $g : X \longrightarrow X$ transforming the set $\{x \in X : \|x\| \leq 1\}$ onto $\{x \in X : \omega(x) \leq 1\}$. The existence of such a diffeomorphism is ensured by the following lemma, which is a restatement of a result in [7]. So define $\varphi = g^{-1} \circ h \circ g$. It is clear that φ is a C^p diffeomorphism from X onto $X \setminus \{0\}$ such that φ is the identity outside the unit ball of X.

Lemma 4. *Let X be a Banach space, and let U_1, U_2 be C^p smooth closed convex bodies containing no ray emanating from the origin, and such that the origin is an interior point of both U_1 and U_2. Then there exists a C^p diffeomorphism $g : X \longrightarrow X$ such that $g(0) = 0$, $g(U_1) = U_2$, and $g(\partial U_1) = \partial U_2$, where ∂U_j stands for the boundary of U_j. Moreover, $g(x) = \lambda(x)x$, where $\lambda : X \longrightarrow [0, \infty)$, and hence g takes each of the rays emanating from the origin onto itself.*

The proof of this lemma, which we will also use later on, can be found in [7], lemma 2; see also [2].

In the case when X is a reflexive infinite-dimensional Banach space the problem was solved quite a long time ago. We can recall the results of T. Dobrowolski [5] to state the following

Proposition 5. *Let $(X, \|.\|)$ be a reflexive infinite dimensional Banach space having a C^p smooth norm $\|.\|$. Then there exists a C^p diffeomorphism φ between X and $X \setminus \{0\}$ such that $\varphi(x) = x$ whenever $\|x\| \geq 1$.*

Proof. Since X is reflexive, X can be linearly injected into some $c_0(\Gamma)$ and, according to proposition 5.1 of [5], X admits a C^∞ non-complete norm ω (which may be assumed to satisfy $\omega(x) \leq \|x\|$). Then, using proposition 3.1

of [5], we get a C^∞ diffeomorphism $h : X \longrightarrow X \setminus \{0\}$ such that $h(x) = x$ if $\omega(x) \geq 1$. An application of lemma 4 as at the end of proposition 3 gives us the desired diffeomorphism φ.

Combining propositions 3 and 5 we get the following

Theorem 6. *Let $(X, \|.\|)$ be an infinite-dimensional Banach space with a C^p smooth norm $\|.\|$. Then there exists a C^p diffeomorphism φ between X and $X \setminus \{0\}$ such that $\varphi(x) = x$ whenever $\|x\| \geq 1$.*

In fact this result can be viewed as a corollary of the following more general result. Recall that a (not necessarily equivalent) norm ϱ in a Banach space $(X, \|.\|)$ is said to be C^p smooth if it is so with respect to $\|.\|$, which in principle does not imply the differentiability of ϱ with respect to itself.

Theorem 7. *Let $(X, \|.\|)$ be an infinite-dimensional Banach space having a (not necessarily complete) C^p smooth norm ϱ. Then there exists a C^p diffeomorphism φ between X and $X \setminus \{0\}$ such that $\varphi(x) = x$ whenever $\varrho(x) \geq 1$. If in addition the extension of ϱ to the completion of the normed space (X, ϱ) is C^p differentiable (with respect to itself), then there exists a bijection φ between X and $X \setminus \{0\}$ which is a C^p diffeomorphism in each of the norms $\|.\|$ and ϱ and such that $\varphi(x) = x$ whenever $\varrho(x) \geq 1$.*

Proof. If ϱ is complete then it is an equivalent C^p smooth norm on X, and we can deduce that X and $X \setminus \{0\}$ are C^p diffeomorphic from propositions 3 and 5. If, on the contrary, ϱ is not complete, we can use proposition 3.1 of [5] to conclude that X and $X \setminus \{0\}$ are C^p diffeomorphic.

Let us complete the proof of theorem 1. We will do nothing but adapt the ideas of Bessaga [1] to the more general setting of a differentiable C^p norm $\|.\|$ ($p \in \mathbb{N} \cup \{\infty\}$) whose sphere might contain segments and consequently the usual stereographic projection might not be well defined for the whole sphere.

Let us choose a point $x_0 \in S_X$ and see first that $S_X \setminus \{x_0\}$ is diffeomorphic to any hyperplane H in X. Put $x^* = d\|.\|(x_0)$, $Z = ker x^*$, and consider the decomposition $X = [x_0] \oplus Z = \mathbb{R} \times Z$. Take a C^∞ convex body U on the plane $\mathbb{R}^2$ such that the set $\{(t,s) : t^2 + s^2 = 1, t \geq 0\} \cup \{(-1, s) : |s| \leq 1/2\}$ is contained in ∂U, the boundary of U. Consider the Minkowski functional of U, $q_U(t,s) = \inf\{\lambda > 0 : (t,s) \in \lambda U\}$, which is C^∞ smooth away from $(0,0)$. Define $Q(t,z) = q_U(t, \|z\|)$ for every $(t,z) \in \mathbb{R} \times Z$. It is quite clear that Q is a C^p function away from the ray $\{\lambda x_0 : \lambda > 0\}$ (and Q is C^1 smooth on $X \setminus \{0\}$). Now consider the convex body $V = \{(t,z) \in X : Q(t,z) \leq 1\}$ and its boundary ∂V. The proof of lemma 4 (see [2] or [7]) shows that the sets $\partial V \setminus \{x_0\}$ and $S_X \setminus \{x_0\}$ are C^p diffeomorphic (whereas ∂V and S_X are C^1 diffeomorphic). Note that for every $z \in Z$ the ray joining z to x_0 intersects the set ∂V at a unique point. This means that the stereographic projection $\pi : \partial V \setminus \{x_0\} \longrightarrow Z_{-1}$ (where $Z_{-1} = \{x \in X : x^*(x) = -1\}$ is the tangent hyperplane to ∂V at $-x_0$), defined by means of the rays emanating from x_0, is a well defined one-to-one mapping from $\partial V \setminus \{x_0\}$ onto Z_{-1}, and it is easy to check that π is a C^p diffeomorphism between $\partial V \setminus \{x_0\}$ and Z_{-1}. Since any two closed hyperplanes in X are isomorphic this proves that $\partial V \setminus \{x_0\}$ is C^p diffeomorphic to each hyperplane H in X, and hence so is $S_X \setminus \{x_0\}$.

Thus, to complete the proof of theorem 1 it only remains to show that $S_X \setminus \{x_0\}$ and S_X are C^p diffeomorphic, which we can do by choosing a suitable atlas for S_X and using theorem 6. Let us recall that $x^* = d\|.\|(x_0)$ and $Z = kerx^*$. Define $D_1 = \{x \in S_X : x^*(x) > -1/2\}$ and $D_2 = \{x \in S_X : x^*(x) < 1/2\}$, and let $\pi_1 : D_1 \longrightarrow Z$ be the stereographic projection defined by means of the rays coming from $-x_0$, and $\pi_2 : D_2 \longrightarrow Z$ the stereographic projection defined by means of the rays emanating from x_0. Note that, although the sphere S_X might contain segments, these stereographic projections are well defined because they have been restricted to D_1 and D_2, sets which cannot contain a segment passing through $-x_0$ and x_0 respectively. Let $G_1 = \{x \in D_1 : x^*(x) > 1/2\}$ and consider $\pi_1(G_1) \subseteq Z$. Since $\pi_1(G_1)$ is an open set in Z containing 0, there exists $\varepsilon > 0$ such that $\{z \in Z : \|z\| \le \varepsilon\} \subseteq \pi_1(G_1)$. Now, from theorem 6 we get a diffeomorphism $\varphi : Z \longrightarrow Z \setminus \{0\}$ such that $\varphi(z) = z$ whenever $\|z\| \ge 1$. Let $h(z) = \varepsilon\varphi(\frac{1}{\varepsilon}z)$ for each $z \in Z$. It is clear that h is a C^p diffeomorphism between Z and $Z \setminus \{0\}$ such that $h(z) = z$ whenever $\|z\| \ge \varepsilon$. Finally, define $g : S_X \longrightarrow S_X \setminus \{x_0\}$ by

$$g(x) = \begin{cases} x \quad \text{if } x \in D_2 \\ \pi_1^{-1}(h(\pi_1(x))) \quad \text{if } x \in D_1 \end{cases}$$

It is easy to check that g is a C^p diffeomorphism from S_X onto $S_X \setminus \{x_0\}$. This concludes the proof of theorem 1.

FINAL REMARKS

1. It is worth noting that theorem 6 above enlarges the class of spaces for which some results of B. M. Garay [8, 9] concerning the existence of solutions to ordinary differential equations and cross-sections of solution funnels in infinite-dimensional Banach spaces are valid.

2. Let $(X, \|.\|)$ be an infinite-dimensional Banach space having a (not necessarily complete) Fréchet differentiable norm ϱ. It is natural to consider the unit sphere $S_\varrho = \{x \in X : \varrho(x) = 1\}$ and ask whether S_ϱ is diffeomorphic to each closed hyperplane H in X. One can show that this is the case, using theorems 6 or 7 as in the proof of theorem 1.

3. Let X be the reflexive Banach space constructed by W. T. Gowers and B. Maurey in [12] which is not isomorphic and therefore is not diffeomorphic to its closed hyperplanes. Being reflexive, X has an equivalent Fréchet differentiable norm $\|.\|$ (see, e.g., [15] or [4]). By theorem 1, the unit sphere S_X is diffeomorphic to a hyperplane of X and hence S_X is not diffeomorphic to the whole of X.

4. The following problem concerning smooth negligibility of points remains unsolved: let X be an infinite-dimensional Banach space having a C^p smooth bump function. Is there a C^p diffeomorphism φ between X and $X \setminus \{0\}$ such that $\varphi(x) = x$ whenever $\|x\| \geq 1$?

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