

SMOOTH LIPSCHITZ RETRACTIONS OF STARLIKE BODIES ONTO THEIR BOUNDARIES IN INFINITE-DIMENSIONAL BANACH SPACES

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ABSTRACT. Let X be an infinite-dimensional Banach space and let A be a C^p Lipschitz bounded starlike body (for instance the unit ball of a smooth norm). We prove that

- (1) The boundary ∂A is C^p Lipschitz contractible.
- (2) There is a C^p Lipschitz retraction from A onto ∂A .
- (3) There is a C^p Lipschitz map $T : A \rightarrow A$ with no approximate fixed points.

1. INTRODUCTION AND MAIN RESULTS

The well known Brouwer's fixed point theorem states that every continuous self-map of the unit ball of a finite-dimensional Banach space admits a fixed point. This is equivalent to saying that there is no continuous retraction from the unit ball onto the unit sphere, or that the unit sphere is not contractible (the identity map on the sphere is not homotopic to a constant map). This result is no longer true in infinite dimensions (see [8]). In [14] B. Nowak showed that for several infinite-dimensional Banach spaces Brouwer's theorem fails even for Lipschitz mappings, and in [6] Y. Benyamini and Y. Sternfeld generalized Nowak's result for all infinite-dimensional normed spaces, establishing that for every infinite-dimensional space $(X, \|\cdot\|)$ there exists a Lipschitz retraction from the unit ball $B_X = \{x \in X : \|x\| \leq 1\}$ onto the sphere $S_X = \{x \in X : \|x\| = 1\}$, and that S_X is Lipschitz contractible.

In recent years a lot of work has been done on smoothness and Lipschitz properties in Banach spaces (see [11, 5]). Following this trend it is natural to ask whether Nowak-Benyamini-Sternfeld's results can be sharpened so as to get C^p smooth Lipschitz retractions of the unit ball onto the sphere of every infinite-dimensional Banach space whose norm is C^p smooth. In this note we will show that this is indeed possible. In fact we generalize those results in two ways. Not only do they hold for the smooth category but also for a wider class of objects than balls and spheres: we show that for every infinite-dimensional Banach space with a C^p Lipschitz bounded starlike body A (where $p = 0, 1, 2, \dots, \infty$), there is a C^p Lipschitz retraction of A onto its boundary ∂A , and ∂A is also C^p Lipschitz contractible.

At this point we need to introduce some terminology. A closed subset A of a Banach space X is said to be a starlike body provided A has a non-empty interior and there exists a point $x_0 \in \text{int}A$ such that each ray emanating from x_0 meets the boundary of A at most once. In this case we will say that A is *starlike with respect to* x_0 . When dealing with starlike bodies, we can always assume that they are starlike

with respect to the origin (up to a suitable translation), and we will do so unless otherwise stated. For a starlike body A , we define the Minkowski functional of A as

$$q_A(x) = \inf\{\lambda > 0 \mid \frac{1}{\lambda}x \in A\}$$

for all $x \in X$. It is easily seen that for every starlike body A its Minkowski functional q_A is a continuous function which satisfies $q_A(rx) = rq_A(x)$ for every $r \geq 0$. Moreover, $A = \{x \in X \mid q_A(x) \leq 1\}$, and $\partial A = \{x \in X \mid q_A(x) = 1\}$, where ∂A stands for the boundary of A . Conversely, if $\psi : X \rightarrow [0, \infty)$ is continuous and satisfies $\psi(\lambda x) = \lambda\psi(x)$ for all $\lambda \geq 0$, then $A_\psi = \{x \in X \mid \psi(x) \leq 1\}$ is a starlike body. Convex bodies (that is, closed convex sets with nonempty interior) are an important kind of starlike bodies. We will say that A is a C^p smooth (Lipschitz) starlike body provided its Minkowski functional q_A is C^p smooth (and Lipschitz) on the set $X \setminus q_A^{-1}(0)$.

Smooth starlike bodies are interesting because they are strongly related to bump functions and to n -homogeneous polynomials in Banach spaces (see [2] and [3]), therefore their geometrical properties are worth studying. It is worth noting that every Banach space having a C^p smooth (Lipschitz) bump function has a C^p smooth (Lipschitz) bounded starlike body too (and the converse is also true).

Before stating our main result we need a few topological definitions. Let M, N be closed subsets of a Banach space X . We will say that two maps $f, g : M \rightarrow N$ are C^p Lipschitz homotopic provided there exist an open subset U of X containing M , an $\varepsilon > 0$, and a C^p smooth mapping $H : (-\varepsilon, 1 + \varepsilon) \times U \rightarrow X$ such that the restriction of H to $[0, 1] \times M$ is a Lipschitz homotopy joining f to g , that is, $H : [0, 1] \times M \rightarrow N$ is Lipschitz continuous and satisfies $H(0, x) = f(x)$ and $H(1, x) = g(x)$ for all $x \in M$. Moreover we will demand that $H(t, x) = f(x)$ for $t \leq 0$, $x \in M$, and $H(t, x) = g(x)$ for $t \geq 1$, $x \in M$.

It is not difficult to see that, with this definition, ‘being C^p Lipschitz homotopic’ endows the set of C^p Lipschitz mappings from M into N with an equivalence relationship (one can join C^p smooth homotopies without losing smoothness or Lipschitzness).

A closed subset M of X is said to be C^p Lipschitz contractible if the identity map on M is C^p Lipschitz homotopic to a constant map on M . For instance, it is easy to check that every C^p Lipschitz starlike body A is C^p Lipschitz contractible. It is also easy to see that every two maps on a (C^p Lipschitz) contractible set are always (C^p Lipschitz) homotopic (they are both homotopic to a constant).

Finally, we will say that $r : A \rightarrow \partial A$ is a C^p smooth Lipschitz retraction from the starlike body A onto its boundary provided there exist an open subset U of X containing A and a C^p smooth mapping $R : U \rightarrow X$ such that R fixes all the points of ∂A , and the restriction of R to A is Lipschitz continuous and coincides with r .

Our main result is the following

Theorem 1.1. *Let X be an infinite-dimensional Banach space and let A be a C^p Lipschitz bounded starlike body. Then:*

- (1) *The boundary ∂A is C^p Lipschitz contractible.*
- (2) *There is a C^p Lipschitz retraction from A onto ∂A .*

- (3) *There is a C^p Lipschitz map $T : A \longrightarrow A$ with no approximate fixed points, that is, $\inf\{\|x - T(x)\| : x \in A\} > 0$.*

As a corollary we obtain the following generalization of Benyamini-Sternfeld's theorem:

Corollary 1.2. *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with an equivalent norm $\|\cdot\|$ which is C^p smooth, and let B_X and S_X be its unit ball and unit sphere respectively. Then*

- (1) *S_X is C^p Lipschitz contractible.*
- (2) *There is a C^p Lipschitz retraction of B_X onto S_X .*
- (3) *There is a C^p Lipschitz map $T : B_X \longrightarrow B_X$ with no approximate fixed points.*

If one is not interested in the Lipschitz property, it is a trivial consequence of the main result in [1] (see also [4]) that the sphere S_X is C^p contractible and there are C^p smooth retractions from B_X onto S_X . Unfortunately, the deleting diffeomorphisms obtained in [1, 4] are not Lipschitz, and corollary 1.2 cannot be deduced using those results. As a matter of fact, corollary 1.2 provides a new result even in the case $X = \ell_2$ with the usual hilbertian norm.

2. THE PROOFS

The proof of the main result is rather technical and will be split into three propositions and several lemmas. The general scheme of the proof follows that of [6], which in turn is a generalization with some modifications of Nowak's approach [14]. The proofs in [6, 14] are already involved in themselves and here they will be complicated with the difficulties peculiar to smooth maps and starlike bodies.

First of all it should be noted that parts (2) and (3) of theorem 1.1 are straightforward consequences of (1). Indeed, assume that ∂A is C^p Lipschitz contractible. Then there are an open subset U of X containing ∂A and a C^p smooth map $H : (-\varepsilon, 1 + \varepsilon) \times U \longrightarrow X$ such that the restriction of H to $[0, 1] \times \partial A$ is a Lipschitz homotopy joining the identity to a constant x_0 in ∂A . Without loss of generality we may assume that H is defined on $(-\infty, +\infty) \times U$ and has the property that $H(t, x) = x_0$ for all $t \leq 0$, $x \in \partial A$, and $H(t, x) = x$ for all $t \geq 1$, $x \in \partial A$. Then the formula

$$R(x) = H(2\psi(x) - 1, \frac{x}{\psi(x)}),$$

where ψ is the Minkowski functional of A , defines a C^p smooth map on $X \setminus \{0\}$ with the property that $R(x) = x/\psi(x)$ whenever $\psi(x) \geq 1$ and $R(x) = x_0$ if $\psi(x) \leq \frac{1}{2}$. Then one can obviously extend R (by putting $R(0) = x_0$) to a C^p smooth map $R : X \longrightarrow X$ such that $R(x) = x$ whenever $\psi(x) = 1$ and $R(x) = x_0$ for $\psi(x) \leq \frac{1}{2}$. The restriction of R to the set $A = \{x \in X : \psi(x) \leq 1\}$ gives us a C^p smooth retraction r from A onto its boundary. By using the fact that $H : [0, 1] \times \partial A \longrightarrow \partial A$ is Lipschitz, it is easily seen that $r : A \longrightarrow \partial A$ is Lipschitz as well. This shows that part (1) of the theorem implies (2). On the other hand, once we have such a C^p Lipschitz retraction r one can easily get a C^p Lipschitz map $T : A \longrightarrow A$ with no approximate fixed points: it is enough to take $T(x) = -r(x)$.

Let us now start the proof of part (1) of 1.1. The following lemma tells us that for every two C^p Lipschitz bounded starlike bodies A_1, A_2 the pair $(A_1, \partial A_1)$ is C^p Lipschitz equivalent to the pair $(A_2, \partial A_2)$. We omit the proof of this result since it is an easy adaptation of that of Proposition 3 in [2].

Lemma 2.1. *Let X be a Banach space, and let A_1, A_2 be C^p Lipschitz bounded starlike bodies. Then there exist a C^p bi-Lipschitz diffeomorphism $g : X \rightarrow X$ such that $g(A_1) = A_2$, $g(\partial A_1) = \partial A_2$, and $g(0) = 0$. Moreover, $g(x) = \mu(x)x$, where $\mu : X \rightarrow [0, \infty)$, and hence g preserves the rays emanating from the origin.*

Therefore, any C^p Lipschitz property of a bounded starlike body or its boundary is shared with all the bounded starlike bodies and their boundaries. In particular the main theorem and all the auxiliary results which we will introduce in this section can be proved for any particular C^p Lipschitz bounded starlike body in a Banach space X and then, by using this lemma, extended for the rest of C^p Lipschitz bounded starlike bodies, which are all equivalent. We will use this fact later on without further notice.

We will also need the following technical definition.

Definition 2.2. *Let X be a Banach space with a C^p smooth Lipschitz bounded starlike body A , and let ψ be its Minkowski functional. Let M be a closed subset of X , $y_0 \in M$, and $\varepsilon > 0$. For every $y \in M$, $\delta > 0$, define the pseudoball $B_M^\psi(y, \delta) = \{x \in M : \psi(x - y) \leq \delta\}$.*

A point y_0 is said to be an ε -escaping point for ψ in M provided there exists a C^p smooth Lipschitz mapping $T : M \rightarrow M$ satisfying:

- (1) *T is Lipschitz homotopic to the identity on M .*
- (2) *$\inf\{\psi(T^n y_0 - T^m y_0) : n > m \geq 0\} \geq 10\varepsilon$.*
- (3) *For all $n \geq 0$, T maps $B_M^\psi(T^n y_0, 2\varepsilon)$ isometrically onto $B_M^\psi(T^{n+1} y_0, 2\varepsilon)$ and, moreover, T is merely a translation when restricted to these sets.*
- (4) *For all $n \geq 0$, $T^{-1}(B_M^\psi(T^{n+1} y_0, 2\varepsilon)) = B_M^\psi(T^n y_0, 2\varepsilon)$.*

Now we state the three auxiliary propositions that we will use in the proof of the main theorem.

Proposition 2.3. *Let M, N be closed subsets of a Banach space X which has a C^p Lipschitz bounded starlike body A with Minkowski functional ψ . Suppose there is an ε -escaping point y_0 in M . Let $g : [-1, 1] \times M \rightarrow N$ be a C^p Lipschitz map which constantly attains the value $z_0 \in N$ outside the set $[\frac{1}{4}, \frac{3}{4}] \times B_M^\psi(y_0, \varepsilon)$. Assume moreover that there exists an open subset U of X containing M and an extension $\bar{g} : (-1 - \varepsilon, 1 + \varepsilon) \times U \rightarrow X$ of g such that \bar{g} is C^p smooth and satisfies $\bar{g}(t, x) = z_0$ for all $t \in (-1 - \varepsilon, 1 + \varepsilon)$ and $x \notin B_U^\psi(y_0, \varepsilon)$. Then g is C^p Lipschitz homotopic to the constant function z_0 in $[-1, 1] \times M$ by means of a C^p Lipschitz homotopy $H_\tau(t, x)$ ($0 \leq \tau \leq 1$, $(t, x) \in [-1, 1] \times M$) for which $H_\tau(t, x) = z_0$ whenever $|t| \geq \frac{3}{4}$.*

Proposition 2.4. *Let X be an infinite-dimensional Banach space with a C^p Lipschitz bounded starlike body. Then there exist $\varepsilon > 0$ and another C^p Lipschitz bounded symmetric starlike body W such that its boundary ∂W has an ε -escaping point with respect to $\psi = q_W$, the Minkowski functional of W .*

Proposition 2.5. *Let X be a Banach space and let A be a C^p Lipschitz starlike body which is bounded and symmetric, $x_0 \in \partial A$, $\varepsilon > 0$. Then the identity map on ∂A is C^p Lipschitz homotopic to a map $f : \partial A \rightarrow \partial A$ which constantly attains the value $-x_0$ outside the set $\{x \in \partial A : \psi(x - x_0) < \varepsilon\}$ (where ψ is the Minkowski functional of A). Moreover, f can be assumed to have a C^p smooth extension $\bar{f} : U \rightarrow X$ (where U is an open subset of X containing ∂A) such that $\bar{f}(x) = -x_0$ whenever $\psi(x - x_0) \geq \varepsilon$, $x \in U$.*

Proof of the theorem.

Let Y be a closed hyperplane of X . By Proposition 2.4 there is a C^p Lipschitz bounded symmetric starlike body W on Y such that its boundary ∂W admits an ε -escaping point y_0 , for some $\varepsilon > 0$ that can be assumed to satisfy $0 < \varepsilon < \frac{1}{4}$. Let q_W be the Minkowski functional of this starlike body. We may write $X = \mathbb{R} \times Y$. Now, let V be a C^∞ smooth Lipschitz bounded symmetric convex body of the plane \mathbb{R}^2 such that its boundary ∂V contains the set

$$\{(t, s) \in \mathbb{R}^2 : |t| \leq 1, |s| = 1\},$$

and consider the Minkowski functional q_V of V , which is a C^∞ smooth equivalent norm on \mathbb{R}^2 . Define now

$$\psi(t, y) = q_V(t, q_W(y))$$

for every $(t, y) \in \mathbb{R} \times Y = X$. It is clear that ψ is a C^p Lipschitz function on $X \setminus \{0\}$ which is symmetric and positive homogeneous. Then

$$U = \{(t, y) \in X : \psi(t, y) \leq 1\}$$

is a C^p Lipschitz bounded symmetric starlike body with the property that its boundary ∂U contains the band $[-1, 1] \times \partial W$. Without loss of generality we can assume that $U = A$ (see Lemma 2.1 and the preceding remarks), and it suffices to prove the theorem for this particular starlike body.

Next put $x_0 = (\frac{1}{2}, y_0) \in \partial A$ and $z_0 = -x_0$. By Proposition 2.5 there exists a C^p Lipschitz map $f : \partial A \rightarrow \partial A$ which is C^p Lipschitz homotopic to the identity on ∂A , and which has a C^p smooth extension $\bar{f} : U \rightarrow X$ such that $\bar{f}(x) = -x_0$ whenever $\psi(x - x_0) \geq \varepsilon$, $x \in U$. Note that if $x = (t, y) \in \partial A$ satisfies $\psi(x - x_0) < \varepsilon$ then, since $\varepsilon < \frac{1}{4}$, and taking into account the particular shape of ∂A , we have that $(t, y) \in [\frac{1}{4}, \frac{3}{4}] \times B_{\partial W}^\psi(y_0, \varepsilon) \subset [-1, 1] \times \partial W$. Then it is clear that $g = f|_{[-1, 1] \times \partial W}$ satisfies the conditions of Proposition 2.3 with $M = \partial W$ (bear in mind that $\bar{g}(t, y) = \bar{f}(t, y) = z_0$ whenever $\psi(0, y - y_0) \geq \varepsilon$ because the pseudoball $\{x \in X : \psi(x - x_0) < \varepsilon\}$ is contained in the cylinder $\{(t, y) \in X : q_W(y - y_0) < \varepsilon\}$). Since y_0 is an ε -escaping point in ∂W , it follows that g is C^p Lipschitz homotopic, as a map from $[-1, 1] \times \partial W$ into ∂A , to the constant $z_0 = -x_0 \in \partial A$, by a C^p Lipschitz homotopy $H_\tau(t, y)$ satisfying $H_\tau(t, y) = z_0$ whenever $|t| \geq \frac{3}{4}$.

Now, from the particular construction of ∂A , it is clear that one can extend H_τ to a C^p Lipschitz homotopy F_τ by defining $F_\tau(x) = z_0$ for $x \in \partial A \setminus ([-1, 1] \times \partial W)$, and it is easily checked that F_τ is a C^p Lipschitz homotopy joining f to the constant z_0 in ∂A . Since f is itself C^p Lipschitz homotopic to the identity on ∂A , we can conclude that ∂A is C^p Lipschitz contractible to a point.

Now we will give the proofs of Propositions 2.3, 2.4 and 2.5.

Proof of Proposition 2.3.

Let T be the map associated to ψ and the ε -escaping point y_0 in Definition 2.2. Let $\theta : \mathbb{R} \rightarrow [0, \infty)$ be a C^p Lipschitz mapping such that θ is strictly increasing in $(0, \infty)$, $\theta(-t) = \theta(t)$, $\theta(0) = 0$, and $\theta(t) = |t|$ for $|t| \geq \frac{1}{8}$. Pick another non-decreasing C^p map $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ such that $\zeta(t) = 0$ for $t \leq \frac{1}{4}$ and $\zeta(t) = 1$ for $t \geq \frac{3}{4}$. Now let us define two maps $f_0, f_1 : [-1, 1] \times M \rightarrow N$ by

$$f_0(t, x) = \begin{cases} g(\theta(t), T^{-n}(x)) & \text{whenever } t \geq 0, x \in B_M^\psi(T^n(y_0), \varepsilon), \text{ and } n \geq 0; \\ g(\theta(t), T^{-n}(x)) & \text{whenever } t \leq 0, x \in B_M^\psi(T^n(y_0), \varepsilon), \text{ and } n \geq 1; \\ z_0 & \text{otherwise;} \end{cases}$$

and

$$f_1(t, x) = \begin{cases} g(\theta(t), T^{-n}(x)) & \text{whenever } t \geq 0, x \in B_M^\psi(T^n(y_0), \varepsilon), \text{ and } n \geq 0; \\ z_0 & \text{otherwise.} \end{cases}$$

Note that on each ‘‘rectangle’’ $[\frac{1}{4}, \frac{3}{4}] \times B_M^\psi(T^n(y_0), \varepsilon)$ or $[-\frac{3}{4}, -\frac{1}{4}] \times B_M^\psi(T^n(y_0), \varepsilon)$, $n \geq 0$, the mappings f_0 and f_1 are defined by the corresponding value (with respect to T^{-n}) of g in the rectangle $[\frac{1}{4}, \frac{3}{4}] \times B_M^\psi(y_0, \varepsilon)$. All these rectangles are disjoint, by the definition of ε -escaping point. Since T^{-n} is merely an affine traslation of $B_M^\psi(T^n(y_0), 2\varepsilon)$ onto $B_M^\psi(y_0, 2\varepsilon)$ and g is C^p smooth and Lipschitz, it is clear that the maps f_0, f_1 are C^p and Lipschitz as well.

By assumption, T is C^p Lipschitz homotopic to the identity; let $G_\tau, \tau \in [0, 1]$, be a C^p Lipschitz homotopy joining the identity to T in M . Then

$$F_\tau(t, x) = \begin{cases} f_0(t, x) & \text{for } t \geq 0; \\ f_0(t, G_\tau(x)) & \text{for } t \leq 0, \end{cases}$$

is a C^p Lipschitz homotopy joining f_0 to f_1 in $[-1, 1] \times M$. Now, the map $F_\tau^1(t, x) = f_1(\theta(t)(1-\zeta(\tau)) + \zeta(\tau), x)$ defines a C^p Lipschitz homotopy joining f_1 to the constant z_0 , and it is not difficult to see that the map

$$F_\tau^0(t, x) = \begin{cases} f_0(\theta(t)\zeta(\tau) + (1-\zeta(\tau)), x) & \text{for } x \notin B_M^\psi(y_0, \varepsilon); \\ g(t, x) & \text{for } x \in B_M^\psi(y_0, \varepsilon) \end{cases}$$

is a C^p Lipschitz homotopy joining g to f_0 (here we use the fact that $\bar{g}(t, x) = z_0$ whenever $\psi(x - x_0) \geq \varepsilon$, $x \in U$). We can then obtain the desired homotopy H_τ by applying successively F_τ^0 , F_τ and F_τ^1 . Since all of these homotopies have the constant value z_0 for $|t| \geq \frac{3}{4}$, the same is true of H_τ .

In order to prove Proposition 2.4 a number of rather technical lemmas and facts will be required. Let us fix some standard notation used throughout these statements. If K is a subset of X and $x \in X$, we denote by $d_\psi(x, K) := \inf\{\psi(x - y) : y \in K\}$. Also, the closed starlike body $B_X^\psi(x, r) = \{y \in X : \psi(y - x) \leq r\}$ will be simply written as $B^\psi(x, r)$.

The first technical tool we need is somehow a smooth version of *Uryshon's lemma*.

Lemma 2.6. *Let X be a Banach space, and let A be a C^p Lipschitz bounded symmetric starlike body with Minkowski functional ψ and K be a compact subset of X . Then, for every $r > 0$ there exists a C^p Lipschitz function $f = f_{\psi,r,K} : X \rightarrow [0, 1]$ such that*

- (1) $f(x) = 1$ whenever $d_\psi(x, K) \leq r/2$, and
- (2) $f(x) = 0$ whenever $d_\psi(x, K) \geq r$.

Proof. Let L_ψ be the Lipschitz constant of ψ (i.e., $\psi(x) - \psi(y) \leq L_\psi \|x - y\|$, for all $x, y \in X$). Since K is compact there exist $x_1, \dots, x_l \in K$ such that

$$K \subset \bigcup_{j=1}^l B^{\|\cdot\|}\left(x_j, \frac{r}{4L_\psi}\right). \quad (*)$$

Then pick a non-decreasing C^∞ function $g : \mathbb{R} \rightarrow [0, 1]$ such that $g^{-1}(0) = (-\infty, \frac{3}{4}r]$ and $g^{-1}(1) = [\frac{7}{8}r, \infty)$. Put

$$h(x) = \prod_{j=1}^l g(\psi(x - x_j))$$

for all $x \in X$. Since the functions $x \mapsto g(\psi(x - x_j))$ are all bounded, Lipschitz and C^p , the function h , being a finite product of such functions, is also C^p smooth and Lipschitz. Moreover, note that the Lipschitz constant of h only depends on ψ , r and the number of elements of the covering $(*)$.

By the construction of h it is quite clear that $h(x) = 1$ if $x \notin \bigcup_{j=1}^l B^\psi(x_j, \frac{7}{8}r)$, and therefore $h(x) = 1$ whenever $d_\psi(x, K) \geq r$. Moreover, is easy to see that $h(x) = 0$ if $x \in G := \bigcup_{j=1}^l B^\psi(x_j, \frac{3}{4}r)$. Let us check that $G \supseteq \{x \in X : d_\psi(x, K) \leq r/2\}$. In fact, if $x \in X$ is such that $d_\psi(x, K) = r/2$, take $y \in K$ in such a way that $\psi(x - y) = r/2$ and x_j so that $y \in B^{\|\cdot\|}(x_j, \frac{r}{4L_\psi})$. Then it follows

$$\psi(x - x_j) \leq \psi(x - y) + L_\psi \|y - x_j\| \leq \frac{3}{4}r.$$

In order to conclude the proof it suffices to take $f(x) = 1 - h(x)$. □

Fact 2.7. *Let X be a Banach space which has a C^p Lipschitz bounded symmetric starlike body A with Minkowski functional ψ and $r > 0$. Then for some $M > 0$ one has that for every a, b with $\|a\| = \|b\| = \frac{1}{4}$ there exists a C^p Lipschitz function $f_{a,b} : X \rightarrow [0, 1]$ whose Lipschitz constant is less than or equal to M , and which satisfies that*

- (1) $f_{a,b}(x) = 1$ whenever $d_\psi(x, [a, b]) \leq r/2$, and
- (2) $f_{a,b}(x) = 0$ whenever $d_\psi(x, [a, b]) \geq r$.

Proof. Fix $r > 0$. For every two arbitrary points a, b of X satisfying $\|a\| = \|b\| = \frac{1}{4}$, consider the compact set $K = [a, b]$. From Lemma 2.6, there exists a function $f_{a,b}$ that verifies conditions (1) and (2). We only have to ensure that the function $f_{a,b}$ constructed in the proof of Lemma 2.6 can be chosen with a Lipschitz constant that does not depend on the segment $[a, b]$. As we remarked before, the Lipschitz constant of $f_{a,b}$ only depends on the number of elements of the finite covering chosen in $(*)$. But, since the diameter of any segment $[a, b]$ is uniformly bounded, for every

pair a and b it is easy to find an appropriate covering of $[a, b]$ with a fixed number of elements. \square

Lemma 2.8. *Let X be a Banach space which has a C^p Lipschitz bounded symmetric starlike body A with Minkowski functional ψ . Then for every $r > 0$ there exists a constant $L > 0$ so that for every $a, b \in X$ with $\|a\| = \|b\| = \frac{1}{4}$ there is a map $F = F_{a,b} : A \rightarrow A$ satisfying*

- (1) F is C^p Lipschitz, and the Lipschitz constant of F is less than or equal to L (and therefore only depends on ψ and r , but not on a, b).
- (2) F maps $B^\psi(a, r/2)$ isometrically onto $B^\psi(b, r/2)$; in fact F is merely a translation when restricted to these sets, and $F(a) = b$.
- (3) $F^{-1}(B^\psi(b, r/2)) = B^\psi(a, r/2)$.
- (4) $F(x) = x$ whenever $d_\psi(x, [a, b]) \geq r$.
- (5) F maps lines parallel to the segment $[a, b]$ into themselves.

Proof. For every such a, b let us define $F = F_{a,b} : X \rightarrow X$ by

$$F(x) = x + f_{a,b}(x)(b - a)$$

for all $x \in X$, where $f_{a,b}$ is the corresponding function obtained from fact 2.7. It is clear that F is C^p smooth and Lipschitz on X , with a Lipschitz constant not greater than $L = M + 1$. Therefore F satisfies condition (1) of the lemma.

It is evident from the definitions of F and $f_{a,b}$ that F satisfies properties (2), (4) and (5) as well. Let us see that F satisfies property (3). If $F(x) \in B^\psi(b, r/2)$ then

$$\frac{r}{2} \geq \psi(b - F(x)) = \psi(f_{a,b}(x)a + (1 - f_{a,b}(x))b - x).$$

Since $0 \leq f_{a,b}(x) \leq 1$ we have $f_{a,b}(x)a + (1 - f_{a,b}(x))b \in [a, b]$ and, therefore, it follows that $d_\psi(x, [a, b]) \leq r/2$ and $f_{a,b}(x) = 1$. Henceforth, we have

$$\frac{r}{2} \geq \psi(b - F(x)) = \psi(b - (x + (b - a))) = \psi(a - x),$$

which means that $x \in B^\psi(a, r/2)$. This shows that $F^{-1}(B^\psi(b, r/2)) = B^\psi(a, r/2)$. \square

Lemma 2.9. *Let X be an infinite-dimensional Banach space which has a C^p Lipschitz bounded symmetric starlike body A with Minkowski functional ψ . Then there exist some $\varepsilon > 0$ and a point x_0 in the interior of A which is an ε -escaping point in A with respect to a map $T : A \rightarrow A$ which in addition to properties (1)–(4) of Definition 2.2 satisfies $T(x) = x$ whenever $\psi(x) \geq \frac{3}{4}$.*

Proof. Without loss of generality we can assume that $B_X \subseteq A$. Since A is a bounded starlike body we know that there exists some $\alpha > 0$ such that $\alpha\|x\| \leq \psi(x) \leq \|x\|$ for all $x \in X$. Note that no matter how T is defined, T will be C^p Lipschitz homotopic to the identity on A because A is starlike and hence C^p Lipschitz contractible (so that both T and the identity are homotopic to a constant on A).

Let $(w_n)_{n \in \mathbb{N}}$ be a normalized basic sequence in X with biorthogonal functionals $(w_n^*)_{n \in \mathbb{N}} \subset X^*$ satisfying $\|w_n^*\| \leq 4$ (one can always take such a sequence, see [10], p. 93), and put $z_n = \frac{1}{4}w_n$ for all $n \in \mathbb{N}$. Let us denote by $L_{n,k}$ the straight line $\{tz_n + (1-t)z_k : t \in \mathbb{R}\}$ passing through z_n and z_k (for $n \neq k$). It is easy to see

that, if $\{n, k\} \cap \{m, l\} = \emptyset$ then $\|x - y\| \geq \frac{1}{32}$ for all $x \in L_{n,k}, y \in L_{m,l}$. This implies that $\psi(x - y) \geq \frac{\alpha}{32}$ for all $x \in L_{n,k}, y \in L_{m,l}$; that is, $d_\psi(L_{n,k}, L_{m,l}) \geq \frac{\alpha}{32}$.

Now take $r = \frac{\alpha}{320}$, and for every $n, k \in \mathbb{N}$, $n \neq k$, pick a function $F_{n,k} : A \rightarrow A$ satisfying the conditions of Lemma 2.8 for $a = z_n$ and $b = z_k$, and put $\varepsilon = r/4$. For this choice of ε and r we have

$$d_\psi(L_{n,k}, L_{m,l}) \geq \frac{\alpha}{32} \geq 10r > 20\varepsilon. \quad (**)$$

Note that, by this inequality and the construction of F , if $\{n, k\} \cap \{m, l\} = \emptyset$ then $F_{n,k}(x) = x$ whenever $d_\psi(x, L_{m,l}) \leq r = 4\varepsilon$, and in particular whenever $F_{m,l}(x) \neq x$ or $x = F_{m,l}(y)$ for some $y \neq x$. Then the infinite composition

$$V_1(x) = (\cdots \circ F_{2n-1,2n} \circ \cdots \circ F_{3,4} \circ F_{1,2})(x)$$

is well defined and satisfies

- (1) V_1 is Lipschitz. Indeed, take into account that all the maps $F_{n,k}$ involved in the definition of V_1 have a Lipschitz constant which is less than or equal to a fixed constant L , and the infinite composition defining V_1 is uniformly locally finite. In fact for every $x \in A$ there exists a neighbourhood of x in A such that V_1 coincides with one of the $F_{n,k}$ when restricted to this neighbourhood. From these properties and from the facts that A contains the unit ball B_X , which is a convex set, and V_1 obviously restricts to the identity outside B_X , one can easily deduce that V_1 is Lipschitz (with a Lipschitz constant less than or equal to L) on A .
- (2) V_1 is C^p smooth (this is again a consequence of the fact that the infinite composition defining V_1 is locally finite and all the functions $F_{n,k}$ are C^p smooth).
- (3) V_1 maps $B^\psi(z_{2n-1}, 2\varepsilon)$ isometrically (in fact it is merely a translation when restricted to this set) onto $B^\psi(z_{2n}, 2\varepsilon)$.
- (4) $V_1^{-1}(B^\psi(z_{2n}, 2\varepsilon)) = B^\psi(z_{2n-1}, 2\varepsilon)$.
- (5) $V_1(x) = x$ whenever $\psi(x) \geq \frac{3}{4}$.

Let us define as well

$$V_2(x) = (\cdots \circ F_{2n,2n+1} \circ \cdots \circ F_{4,5} \circ F_{2,3})(x).$$

Then V_2 is also a C^p Lipschitz map that satisfies

- (3') V_2 maps $B^\psi(z_{2n}, 2\varepsilon)$ isometrically (in fact it is a translation when restricted to this set) onto $B^\psi(z_{2n+1}, 2\varepsilon)$.
- (4') $V_2^{-1}(B^\psi(z_{2n+1}, 2\varepsilon)) = B^\psi(z_{2n}, 2\varepsilon)$.
- (5') $V_2(x) = x$ whenever $\psi(x) \geq \frac{3}{4}$.

Now let us define $T = V_2 \circ V_1$. It is clear that T is a C^p Lipschitz map. It only remains to check that z_1 is an ε -escaping point for T . Indeed, as said above, condition (1) of Definition 2.2 is trivially satisfied. It is also clear that $T^n z_1 = z_{2n+1}$, and condition (2) of 2.2 follows from $(**)$ above. Finally, conditions (3) and (4) of 2.2 follow respectively from (3, 3') and (4, 4') above. \square

Proof of Proposition 2.4.

Let Y be a closed hyperplane of X and identify $X = Y \times \mathbb{R}$. Let A be a C^p Lipschitz bounded symmetric starlike body in Y with Minkowski functional q_A . By Lemma 2.9 there exist $\varepsilon > 0$ and $y_0 \in A$ with $q_A(y_0) < \frac{3}{4}$ such that y_0 is an ε -escaping point with respect to a map $V : A \rightarrow A$ which satisfies $V(x) = x$ for $q_A(x) \geq \frac{3}{4}$. Take a C^∞ Lipschitz convex body U on the plane \mathbb{R}^2 such that its boundary ∂U contains the set

$$\{(s, t) : |s| = 1, |t| \leq \frac{3}{4}\} \cup \{(s, t) : |s| \leq \frac{3}{4}, |t| = 1\},$$

and consider the Minkowski functional of U , q_U , which is C^p and Lipschitz. Define now

$$\psi(y, t) = q_U(q_A(y), t)$$

for every $(y, t) \in Y \times \mathbb{R} = X$. It is clear that ψ is a C^p Lipschitz function on $X \setminus \{0\}$ which is symmetric and positive homogeneous. Then $W = \{(y, t) \in X : \psi(y, t) \leq 1\}$ is a C^p Lipschitz bounded symmetric starlike body with the property that its boundary ∂W contains the set

$$\{(y, t) \in X : q_A(y) = 1, |t| \leq \frac{3}{4}\} \cup \{(y, t) \in X : q_A(y) \leq \frac{3}{4}, |t| = 1\}.$$

Now define $T : \partial W \rightarrow \partial W$ by

$$T(y, t) = \begin{cases} (V(y), 1) & \text{if } t = 1; \\ (y, t) & \text{if } t \neq 1 \end{cases}$$

It is clear that T is a well-defined Lipschitz map. Moreover T is C^p smooth on ∂W because T can be extended to a map $\bar{T} : X \rightarrow X$ which is C^p smooth (indeed, $\bar{T}(y, t) = (\frac{1}{2}(t+1)V(y) + \frac{1}{2}(1-t)y, t)$ does the job; note that if $(y, t) \in \partial W$ and $|t| \neq 1$ then $q_A(y) \geq \frac{3}{4}$ and therefore $V(y) = y$; this implies that $\bar{T}|_{\partial W} = T$).

Let us check that $(y_0, 1)$ is an ε -escaping point of ∂W associated with this map T . Conditions (2), (3) and (4) of Definition 2.2 follow directly from the corresponding properties of V (take into account that $\psi(y, 0) = q_A(y)$ for all $y \in Y$). It only remains to prove that T is C^p Lipschitz homotopic to the identity on ∂W . Take a C^∞ smooth function $\theta : \mathbb{R} \rightarrow \mathbb{R}$ such that $\theta(t) = 0$ for $t \leq \frac{1}{4}$ and $\theta(t) = 1$ for $t \geq \frac{3}{4}$, and define $H : [0, 1] \times \partial W \rightarrow \partial W$ by

$$H_\tau(x) = H_\tau(y, t) = \frac{\theta(\tau)T(x) + (1 - \theta(\tau))x}{\psi(\theta(\tau)T(x) + (1 - \theta(\tau))x)}.$$

Note that, since $T(x) \neq x$ only for points $x = (y, t)$ with $t = 1$ and $q_A(y) \leq \frac{3}{4}$, we have $\psi(\tau T(x) + (1 - \tau)x) \geq 1$ for all x, τ , so that H is clearly well defined and Lipschitz on $[0, 1] \times \partial W$. It is also clear that H is a homotopy joining T to the identity on ∂W . Moreover, H is C^p smooth because it can be extended to a C^p defined on an open neighbourhood of $[0, 1] \times \partial W$ (indeed, it is enough to consider

$$H_\tau(x) = H_\tau(y, t) = \frac{\theta(\tau)(\frac{1}{2}(t+1)V(y) + \frac{1}{2}(1-t)y, t) + (1 - \theta(\tau))(y, t)}{\psi(\theta(\tau)(\frac{1}{2}(t+1)V(y) + \frac{1}{2}(1-t)y, t) + (1 - \theta(\tau))(y, t))},$$

which is C^p smooth on the subset of points of $\mathbb{R} \times X$ for which the denominator of this quotient is not zero, an open neighbourhood of $[0, 1] \times \partial W$).

Proof of Proposition 2.5.

Let $\theta : \mathbb{R} \rightarrow [0, 1]$ be a non-increasing C^∞ smooth function such that $\theta(t) = 1$ for $t \leq \varepsilon/2$ and $\theta(t) = 0$ for $t \geq \varepsilon$. Let us define

$$\bar{f}(x) = \frac{\theta(\psi(x - x_0))x - (1 - \theta(\psi(x - x_0)))x_0}{\psi(\theta(\psi(x - x_0))x - (1 - \theta(\psi(x - x_0)))x_0)}$$

for every x such that the above denominator is not zero. It is clear that \bar{f} is a C^p smooth map with the property that $\bar{f}(x) = -x_0$ whenever $\psi(x - x_0) \geq \varepsilon$. The restriction of \bar{f} to ∂A gives us a C^p smooth Lipschitz function $f : \partial A \rightarrow \partial A$ which constantly attains the value $-x_0$ outside $\{x \in \partial A : \psi(x - x_0) < \varepsilon\}$. To see that f is Lipschitz it is enough to take into account that there is some $\delta > 0$ such that

$$\|\theta(\psi(x - x_0))x - (1 - \theta(\psi(x - x_0)))x_0\| \geq \delta$$

for all $x \in \partial A$, and the maps θ, ψ are Lipschitz. Finally, define $H : [0, 1] \times \partial A \rightarrow \partial A$ by

$$H(\tau, x) = \frac{(1 - \theta(\tau))x + \theta(\tau)f(x)}{\psi((1 - \theta(\tau))x + \theta(\tau)f(x))}.$$

It is not difficult to see that H is a C^p Lipschitz homotopy joining the identity to the function f on ∂A .

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