ON AN EVOLUTION PROBLEM ASSOCIATED TO THE MODELLING OF INCERTITUDE INTO THE ENVIRONMENT

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Abstract. We consider a mathematical model, posed by J.E. Scheinkman, simulating that an industrial project take place into the environment without destroy it. We introduce a change of variable leading the formulation to a nonlinear evolution problem which we study by means of L^∞-accretive operators techniques. We prove that under suitable conditions there is extinction in finite time, which corresponds to some special behaviour of the solution of the original stochastic control problem.

1. Introduction

In this note we consider a mathematical model simulating that an industrial project take place into the environment without destroy it. Following Scheinkman models [6], we assume that the benefit of the environment at the instant t changes according to a positive diffusion process such

$$\mathrm{d}\mathcal{X}(t) = \mu_1(\mathcal{X}(t))\mathrm{d}t + \sqrt{2}\sigma_1(\mathcal{X}(t))\mathrm{d}\mathcal{B}_1(t). \tag{1.1}$$

The alternative project also changes according to a positive diffusion process

$$\mathrm{d}\mathcal{Y}(t) = \mu_2(\mathcal{Y}(t))\mathrm{d}t + \sqrt{2}\sigma_2(\mathcal{Y}(t))\mathrm{d}\mathcal{B}_2(t), \tag{1.2}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are standard Brownian motions defined in a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and with a coefficient of correlation $\varrho \in (-1, 1)$. Let us assume that, for $i = 1, 2$, the functions μ_i and σ_i are Lipschitz continuous and vanishing at the origin, *i.e.*

$$|\mu_i(z_1) - \mu_i(z_2)| + |\sigma_i(z_1) - \sigma_i(z_2)| \leq K|z_1 - z_2|, \tag{1.3}$$

$$\mu_i(0) = \sigma_i(0) = 0, \quad i = 1, 2. \tag{1.4}$$

By $\theta(t)$ we denote the fraction of the environment transforming to the industrial project, then we will assume that the utility flow is as

$$U\left((1 - \theta(t))\,\mathcal{X}(t), \theta(t)\mathcal{Y}(t)\right), \tag{1.5}$$

where U is a continuous concave function and non decreasing in their arguments and $U(0, 0) = 0$. Here the initial data are given by $\mathcal{X}(0) = x$, $\mathcal{Y}(0) = y$ and the environment transformation fraction by $\theta(0) = \theta$. We define the accumulate utility fuctional

$$J(x, y, \theta) = \mathbb{E}\left[\int_0^\infty e^{-\alpha s} U\left((1 - \theta(s))\,\mathcal{X}(s), \theta(s)\mathcal{Y}(s)\right)\mathrm{d}s\right] \tag{1.6}$$

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where $\mathcal{X}(t)$ and $\mathcal{Y}(t)$ are the solutions of (1.1) and (1.2) respectively and

$$1 \geq \theta(t) = \theta + M(t). \tag{1.7}$$

Here $\alpha > 0$ is a discount coefficient and $M(t)$ an increasing process describing the irreversible development. Finally, $\mathcal{A}_\theta$ is the domain of process $M(t)$ where (1.7) holds. We point out that by hypothesis the cost of development is zero.

The main goal of this paper is to study the qualitative behavior of the optimal value function v given by

$$v(x, y, \theta) = \sup_{\mathcal{A}_\theta} J(x, y, \theta). \tag{1.8}$$

We introduce the variable $t = 1 - \theta$ and study the associated evolution problem by means of L^∞-accretive operators. We prove that under suitable conditions there is extinction in finite time, which corresponds to some special behavior of the solution of the original stochastic control problem.

2. Modelling of the evolution problem

First of all, we are going to show that the optimal function v satisfies, in some sense to be detailed, the obstacle problem

$$\min\{-\mathcal{L}v + \alpha v - U((1-\theta x, \theta y), -v_\theta\} = 0 \quad \text{in} \quad \Omega, \tag{2.1}$$

where $\mathcal{L}$ is the differential operator associated to the diffusion $(\mathcal{X}(t), \mathcal{Y}(t))$, thus,

$$\mathcal{L}v \doteq \sigma_1^2(x)v_{xx} + \sigma_2^2(y)v_{yy} + 2\sigma_1(x)\sigma_2(y)\varrho v_{xy} + \mu_1(x)v_x + \mu_2(y)v_y.$$

In order to simplify the exposition, we assume for the moment that v is smooth enough. At each instant t there are two possibilities: to protect the environment or to develop the industrial project. If the decision is to protect the environment in the period of time $[0, h]$, then $\theta(t) = \theta$. Therefore it follows

(2.2)
$$v(x, y, \theta) \geq \mathbb{E}\left[\int_0^h e^{-\alpha s} U\left((1-\theta)\mathcal{X}(s), \theta\mathcal{Y}(s)\right) \mathrm{d}s + e^{-\alpha h} v\left(\mathcal{X}(h), \mathcal{Y}(h), \theta(h)\right)\right],$$

with equality if the optimal decision is $\theta(t) = \theta$ in $[0, h]$.

Applying the Ito's rule to $e^{-\alpha h} v\left(\mathcal{X}(h), \mathcal{Y}(h), \theta(h)\right)$ in (2.2), classical arguments in passing to the limit as $h \to 0$ prove that

$$-\mathcal{L}v + \alpha v - U((1-\theta x, \theta y) \geq 0, \tag{2.3}$$

with equality if the optimal decision in the instant $t = 0$ is to protect the environment.

If the decision is to develop the environment into the private project, and $\triangle\theta$ represent the instantaneous increasing of the fraction of develop, then $\theta(0^+) = \theta + \triangle\theta$. Applying the Fleming and Soner theory [4] to our problem we obtain: if τ is stopping time then the principle of dynamic programming:

$$v(x, y, \theta) = \sup_{\mathcal{A}_\theta} \mathbb{E}\left[\int_0^\tau e^{-\alpha s} U\left((1-\theta(s))\mathcal{X}(s), \theta(s)\mathcal{Y}(s)\right) \mathrm{d}s + e^{-\alpha\tau} v\left(\mathcal{X}(\tau), \mathcal{Y}(\tau), \theta(\tau)\right)\right].$$

Therefore,

$$v(x, y, \theta) \geq v\left(x, y, \theta(0^+)\right)$$

consequently

$$v_\theta \leq 0, \tag{2.4}$$

with equality if $\triangle\theta > 0$, in other words, with equality if the optimal decision at the instant $t = 0$ is develop the environment into the industrial project.
Writing (2.3) and (2.4) under the two possible options we obtain the Hamilton-Jacobi-Bellman equation:

$$\min\left\{-\mathcal{L}v + \alpha v - U\left((1-\theta)x, \theta y\right), -v_\theta\right\} = 0 \quad \text{in} \quad \Omega. \tag{2.5}$$

The correct formulation of (2.5) must be formulated by means the viscosity solution theory (see [3]), but here we omit it.

In order to get some qualitative properties, we reformulate the problem in a different way. For $\Omega = \mathbb{R}_+^2$ we define the function

$$\widehat{U}(x, y, \theta) \doteq U((1-\theta)x, \theta y).$$

Then, the problem under consideration can be stated as follows: Find $v : \Omega \times [0,1] \longrightarrow \mathbb{R}$ such that:

$$\begin{cases} \min\left\{-v_\theta, -\mathcal{L}v + \alpha v - \widehat{U}(x,y,\theta)\right\} = 0 & \text{in} \quad \Omega \times (0,1), \\ \mathcal{A}\nabla v \cdot \boldsymbol{\nu} = 0 & \text{on} \quad \partial\Omega \times (0,1), \\ v(x,y,1) = \mathbb{E}\left[\int_0^\infty e^{-\alpha s} U\left(0, \mathcal{Y}(s)\right) \mathrm{d}s\right] & \text{in} \quad \Omega, \end{cases} \tag{2.6}$$

where $\boldsymbol{\nu}$ is the unitary out-normal vector and $\mathcal{A}$ is the matrix

$$\mathcal{A} = \begin{pmatrix} \sigma_1^2(x) & \sigma_1(x)\sigma_2(y)\varrho \\ \sigma_1(x)\sigma_2(y)\varrho & \sigma_2^2(y) \end{pmatrix}.$$

We introduce the change of variable $t = 1 - \theta$ and define the function

$$\overline{U}(x, y, t) \doteq \widehat{U}(tx, y, 1-t) = U(tx, (1-t)y).$$

for which we introduce the problem

$$\begin{cases} -\mathcal{L}f + \alpha f = \overline{U}(x,y,t) & \text{in} \quad \Omega \times (0,1), \\ \mathcal{A}\nabla f \cdot \boldsymbol{\nu} = 0 & \text{on} \quad \partial\Omega \times (0,1). \end{cases} \tag{2.7}$$

Due to the Lipschitz condition assumed on of the functions μ_i and σ_i one proves that the solution of (2.7) verifies $f \in H^2_{\text{loc}}(\mathcal{A}, \Omega)$ (see [5]). Finally, for the function

$$u(x, y, t) \doteq v(x, y, 1-t) - f(x, y, t)$$

problem (2.6) becomes

$$\begin{cases} \min\left\{u_t + f_t, -\mathcal{L}u + \alpha u\right\} = 0 & \text{in} \quad \Omega \times (0,1), \\ \mathcal{A}\nabla u \cdot \boldsymbol{\nu} = 0 & \text{on} \quad \partial\Omega \times (0,1), \\ u(x,y,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.8}$$

Other general formulation on the evolution problem (2.8) can be obtained by the multivalued operator theory (see [2]). It is know that we may write the evolution problem as

$$\begin{cases} \dfrac{\partial u}{\partial t} + \gamma(-\mathcal{L}u + \alpha u) \ni g(x,y,t) & \text{in} \quad \Omega \times (0,1), \\ \mathcal{A}\nabla u \cdot \boldsymbol{\nu} = 0 & \text{on} \quad \partial\Omega \times (0,1), \\ u(x,y,0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.9}$$

where $g(x,y,t) = -\dfrac{\partial f}{\partial t}(x,y,t)$ and γ is the maximal monotone graph given by

$$\gamma(u) = \begin{cases} \emptyset & \text{if} \quad u < 0, \\ (-\infty, 0] & \text{if} \quad u = 0, \\ 0 & \text{if} \quad u > 0. \end{cases}$$

3. Existence of solution of the problem (2.9).

Here we introduce the Banach space

$$L^\infty_w(\Omega) = \{v : wv \in L^\infty(\Omega)\},$$

equipped with the norm

$$\|u\|_{L^\infty_w(\Omega)} = \|wu\|_{L^\infty(\Omega)}.$$

Analogously one introduces the space $L^2_w(\Omega)$, $H^1_w(\mathcal{A},\Omega)$ and $H^2_{\text{loc}}(\Omega)$ (see [5]).

Next we define the multivalued operator $\mathcal{C} : \quad D(\mathcal{C}) \longrightarrow \mathcal{P}(L^\infty(\Omega))$ by

$$\begin{cases} D(\mathcal{C}) = \left\{v \in L^\infty(\Omega) \cap H^1(\mathcal{A},\Omega) : \quad -\mathcal{L}v + \alpha v \in L^\infty(\Omega), \quad \gamma(-\mathcal{L}v + \alpha v) \in L^\infty(\Omega)\right\}, \\ \mathcal{C}u = \gamma(-\mathcal{L}v + \alpha v) \quad \text{if} \quad u \in D(\mathcal{C}). \end{cases}$$

Lemma 3.1. *The multivalued operator $\mathcal{C}$ is m-T-accretive in $L^\infty(\Omega)$.*

Proof. First of all, we note that the operator $\mathcal{C}$ is T-accretive in $L^\infty(\Omega)$ if and only if for all (u_1, v_1), $(u_2, v_2) \in \mathcal{C}$ the inequality

$$\tau^+(u_1 - u_2, v_1 - v_2) \geq 0$$

holds, where τ^+ is given by

$$\tau^+(f,g) = \max\left\{\lim_{\lambda \downarrow 0} \operatorname{ess\,sup}\left[\alpha(x)g(x) : \ x \in \Omega(f,\lambda)\right], \ \alpha \in L^\infty(\Omega), \quad \alpha(x) \in \operatorname{sign}^+ f(x) \ a.e.\right\}$$

with $\Omega(f,\lambda) = \{x \in \Omega : \quad |f(x)| \leq |f|_{L^\infty} - \lambda\}$ and

$$\operatorname{sign}^+(v) = \begin{cases} 1 & \text{if} \quad v > 0, \\ [0,1] & \text{if} \quad v = 0, \\ 0 & \text{if} \quad v < 0. \end{cases}$$

We will use the auxiliary function

$$\gamma_\epsilon(x) = \epsilon \inf\{0, x\} = \epsilon x^-$$

verifying $\gamma_\epsilon \to \gamma$, as $\epsilon \to 0$. So that, for (u_1, v_1), $(u_2, v_2) \in \mathcal{C}$ we claim

$$\tau^+\left(u_1 - u_2, \gamma_\epsilon(-\mathcal{L}u_1 + \alpha u_1) - \gamma_\epsilon(-\mathcal{L}u_2 + \alpha u_2)\right) \geq 0 \quad \text{for all} \quad \epsilon > 0. \tag{3.1}$$

Indeed, if there exists $\lambda > 0$ with

$$\gamma_\epsilon(-\mathcal{L}u_1 + \alpha u_1) - \gamma_\epsilon(-\mathcal{L}u_2 + \alpha u_2) < 0 \quad \text{c.p.t.} \quad \Omega\left((u_1 - u_2)^+, \lambda\right)$$

the monotonicity of the operator γ_ϵ leads to

$$-\mathcal{L}(u_1 - u_2) + \alpha(u_1 - u_2) \leq 0 \quad \text{in} \quad \Omega((u_1 - u_2)^+, \lambda,).$$

As on the boundary of $\Omega((u_1 - u_2)^+, \lambda,)$ we have $u_1 - u_2 = |(u_1 - u_2)^+|_{L^\infty} - \lambda$ the maximum principle derives a contradiction. So, taking limit in (3.1) as $\epsilon \downarrow 0$ we obtain

$$\tau^+(u_1 - u_2, v_1 - v_2) \geq 0.$$

On the other hand, in proving the m–accretivity, we note that the problem

$$\begin{cases} \gamma(-\mathcal{L}u + \alpha u) + \lambda u \ni F & \text{in} \quad \Omega, \\ \mathcal{A}\nabla u \cdot \nu = 0 & \text{on} \quad \partial\Omega \end{cases}$$

has one and only one solution if $F \in L^\infty$ and $\lambda > 0$. Indeed, this problem is equivalent to

$$\begin{cases} -\mathcal{L}u + \alpha u + \gamma(\lambda u - F) \ni 0 & \text{in} \quad \Omega, \\ \mathcal{A}\nabla u \cdot \nu = 0 & \text{on} \quad \partial\Omega \end{cases}$$

studied in [5].

4. The Environment preservation domain.

In this section we shall obtain an estimate of the vanishing domain of the operator $-\mathcal{L}u + \alpha u$ representing the domain where the optimal control problem is to protect the environment.

Theorem 4.1. *Let $f(x, y, t)$ be the unique solution of the problem (2.7) and assume that*

$$\text{(4.1)} \qquad \begin{cases} \exists t_0 \in [0,1] \quad \textit{that} \quad \rho(t) \geq 0 \quad \forall t \in [t_0, 1] \quad \textit{for} \\ \rho(t) \doteq \min\limits_{(x,y)\in\overline{\Omega}} \left(\dfrac{\partial f}{\partial t}(x, y, t) \right) = - \max\limits_{(x,y)\in\overline{\Omega}} (g(x, y, t)) \end{cases}$$

and

$$\text{(4.2)} \qquad \exists t_1 \in [t_0, 1] \quad \textit{that} \int_0^{t_0} \|g(s)\|_{L^\infty(\Omega)} \mathrm{d}s \leq \int_{t_0}^{t_1} - \max[g(x, y, \tau)]\mathrm{d}\tau.$$

Then

$$\text{(4.3)} \qquad v(x, y, \theta) = f(x, y, 1 - \theta) \quad \forall \theta \in [0, 1 - t_1] \quad \textit{and} \quad \textit{a.e.}\ (x, y) \in \Omega.$$

Remark 4.2. The result says that v can be determined in terms of U in the interval $[0, 1 - t_1]$. □

Remark 4.3. In the case of $\sigma_i(z) = \sigma_i z$, and $\mu_i(z) = \mu_i z$ the hypothesis (4.1) and (4.2) are verified, for instance, for $U(x, y, \theta) = \log((1 - \theta)x) - \log(\theta y) + \sigma_1^2 - \sigma_2^2 - \mu_1 + \mu_2$. □

Proof. We will suppose that $u(\cdot, \cdot, t) \neq 0$ for all $t \in (0, 1)$, because if there exists t^* such that $u(\cdot, \cdot, t^*) = 0$ then the function

$$\widehat{u}(\cdot, \cdot, t) = \begin{cases} u(\cdot, \cdot, t) & \text{if} \quad t \in (0, t^*), \\ 0 & \text{if} \quad t \in (t^*, 1), \end{cases}$$

is also a solution of problem (2.9) and then the uniqueness of solutions yields the conclusion. We note that by assumption (4.1) one has

$$\text{(4.4)} \qquad B_{\rho(t)}(g(\cdot, \cdot, t)) \subset \mathcal{C}(0) = \{w \in L^\infty(\Omega) : \quad w \leq 0 \quad \textit{in} \quad \Omega\}.$$

By the accretivity of the operator $\mathcal{C}$ in $L^\infty(\Omega)$ we obtain that

$$\text{(4.5)} \quad |u(t) - \widetilde{v}(t)| \leq |u(r) - \widetilde{v}(r)| + \int_r^t \tau\left(u(s) - v(s), g(s) - h(s)\right) \mathrm{d}s \quad \forall r \in [0, 1],$$

for any $\widetilde{v}$ solution of $\dfrac{\mathrm{d}\widetilde{v}}{\mathrm{d}t}(t) + \mathcal{C}\widetilde{v}(t) \ni h(t)$ (see Benilan[1]). Choosing $\widetilde{v} = 0$ and $h(s) = g(s) + \rho(s)\varepsilon(s)$ with $|\varepsilon(s)|_{L^\infty(\Omega)} \leq 1$ and $r = t_0$ we obtain

$$|u(t)| \leq |u(t_0)| + \int_{t_0}^t \tau\left(u(s), -\rho(s)\varepsilon(s)\right) \mathrm{d}s.$$

Now from

$$\tau(x, \lambda y) = \lambda \tau(x, y) \quad \forall \lambda \in \mathbb{R} \quad \text{and} \quad \forall x, y \in L^\infty(\Omega),$$

we obtain

$$|u(t)| \leq |u(t_0)| + \int_{t_0}^{t} -\rho(s)\tau\left(u(s),\varepsilon(s)\right)\mathrm{d}s.$$

So, taking $\varepsilon(s) = \dfrac{u(s)}{|u(s)|}$ the property $\tau(x,x) = |x|,\ \forall x \in L^\infty(\Omega)$, implies

$$|u(t)| \leq |u(t_0)| - \int_{t_0}^{t} \rho(s)\mathrm{d}s.$$

On the other hand, by choosing $\widetilde{v} = 0$ and $h = 0$ in (4.5) and using that $\tau(x,y) \leq \|y\|_{L^\infty(\Omega)}$ we deduce

$$|u(t_0)| \leq \int_0^{t_0} \|g(s)\|_{L^\infty(\Omega)}\mathrm{d}s.$$

Therefore

$$|u(t)| \leq \int_0^{t_0} \|g(s)\|_{L^\infty(\Omega)}\mathrm{d}s - \int_{t_0}^{t} \rho(s)\mathrm{d}s$$

whence assumption (4.2) leaves to the contradiction $|u(t_1)| = 0$. □.

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