

Classification of blow-up with nonlinear diffusion and localized reaction

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Abstract

We study the behaviour of nonnegative solutions of the reaction-diffusion equation

$$\begin{cases} u_t = (u^m)_{xx} + a(x)u^p & \text{in } \mathbb{R} \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}. \end{cases}$$

The model contains a porous medium diffusion term with exponent $m > 1$, and a localized reaction $a(x)u^p$ where $p > 0$ and $a(x) \geq 0$ is a compactly supported function. We investigate the existence and behaviour of the solutions of this problem in dependence of the exponents m and p . We prove that the critical exponent for global existence is $p_0 = (m + 1)/2$, while the Fujita exponent is $p_c = m + 1$: if $0 < p \leq p_0$ every solution is global in time, if $p_0 < p \leq p_c$ all solutions blow up and if $p > p_c$ both global in time solutions and blowing up solutions exist. In the case of blow-up, we find the blow-up rates, the blow-up sets and the blow-up profiles; we also show that reaction happens as in the case of reaction extended to the whole line if $p > m$, while it concentrates to a point in the form of a nonlinear flux if $p < m$. If $p = m$ the asymptotic behaviour is given by a self-similar solution of the original problem.

1 Introduction and main results

This paper is motivated by the wish to understand the blow-up properties of reaction diffusion equations which combine a localized reaction term with nonlinear diffusion. In

order to fix ideas, the present study concentrates the nonlinear reaction-diffusion equation

$$\begin{cases} u_t = (u^m)_{xx} + a(x)u^p, & (x, t) \in \mathbb{R} \times (0, T), \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \end{cases} \quad (1.1)$$

Nonnegative solutions $u \geq 0$ are considered. We take exponents $m > 1$ and $p > 0$ and the coefficient $a(x) \geq 0$ is a compactly supported function; this means that the reaction term acts only *locally*, and this is the main difference with existing studies of blow-up for similar reaction-diffusion equations. Thus, the problem may be used to describe a chemical reaction-diffusion process in which, due to the effect of the catalyst, the reaction takes place only at some local sites, [BOR]. We remark that the name localized has received also some other interpretation in the literature of blow-up: a reaction depending only on the value of the unknown u in some local set, for instance a point $R(u(x, t)) = u^p(x_0, t)$, but acting throughout the whole domain of interest, see the survey [S]. In our case, we have $R(u(x, t)) = 0$ outside the support of $a(x)$. As to the corresponding n -dimensional model $u_t = \Delta u^m + a(x)u^p$, there exist several interesting possible choices of the localized reaction, see [CY]. Some of the results of this paper extend to those situations, this is the subject of a future work. Finally, the initial value u_0 is assumed to be continuous and nonnegative. More precise assumptions are made below.

Since $m > 1$, we have slow diffusion: if for instance u_0 has compact support then the function u is in general a solution only in a weak sense, i.e., u and $(u^m)_x$ are absolutely continuous functions and the equation is understood in the weak sense; u is C^∞ in its positivity set but not globally. Local in time existence, as well as a comparison principle, can easily be obtained, but the solution may only exist for $t \in [0, T)$ and become unbounded as $t \rightarrow T$ for some $T < \infty$. In other words, the solution may *blow up* in finite time, and this is our main concern.

Let us examine what is known in some standard case before presenting our results. It is well known that blow-up happens for the problem with *homogeneous reaction*, i.e., $a \equiv 1$:

$$\begin{cases} u_t = (u^m)_{xx} + u^p, & (x, t) \in \mathbb{R} \times (0, T), \\ u(x, 0) = u_0(x), & x \in \mathbb{R}. \end{cases} \quad (1.2)$$

All the solutions to this problem blow up if $1 < p \leq m + 2$, while for $p > m + 2$ they blow up provided that the initial data u_0 are large enough. In this case, the numbers $p_0 = 1$ and $p_c = m + 2$ are called the *global existence exponent* and the *Fujita exponent*, respectively. This analysis can be extended to equation (1.1) with $a(x) \geq \delta > 0$. Our investigation will show that the exponents of the problem with localized reaction are not the same.

On the other hand, there is a close connection of problem (1.1) with the problem of diffusion with nonlinear boundary flux conditions. Thus, if we take a sequence of reaction coefficients converging to a Dirac delta at the origin (i.e., if $a_n(x) \rightarrow \delta_0(x)$), it is clear from the weak formulation of the problem, at least formally, that the corresponding solutions

u_n should converge to a solution of the problem

$$\begin{cases} u_t = (u^m)_{xx}, & (x, t) \in (0, \infty) \times (0, T), \\ -(u^m)_x(0, t) = u^p(0, t), & t \in (0, T), \\ u(x, 0) = u_0(x), & x \in (0, \infty). \end{cases} \quad (1.3)$$

For this boundary reaction problem, it is known that the phenomenon of blow-up occurs in a similar way as for problem (1.2), but the critical exponents are in this case: $p_0 = (m + 1)/2$ and $p_c = m + 1$, see [GL, FPQR]. There are a number of coincidences and differences between the two problems that we will investigate below.

Main results

Our first objective is to identify the critical exponents for problem (1.1). In this respect we show that the critical exponents for problem (1.1) are the same as for problem (1.3), instead of problem (1.2). This happens even if the support of a is very large. We have the following result.

Theorem 1.1 (i) *If $0 < p \leq (m + 1)/2$ all the solutions to problem (1.1) are globally defined;*

(ii) *if $(m + 1)/2 < p \leq m + 1$ all the solutions blow up in finite time;*

(iii) *if $p > m + 1$ the solutions may blow up in finite time or not depending on the size of initial data.*

In case of linear diffusion $m = 1$ the above exponents, $p_0 = 1$, $p_c = 2$, were obtained by Pinsky in [P].

Actually, we are going to show that problem (1.1) can be considered in some way as an *intermediate problem* between problems (1.2) and (1.3). As for the critical exponents, the above theorem says that it behaves like problem (1.3). But in the way the solutions blow up, the asymptotic behaviour for $t \sim T$ depends on the reaction exponent p . This is reflected in the speed at which the solutions blow up, the *blow-up rate*, and the final shape of the solution (properly rescaled) near the blow-up time, the *blow-up profile*. Note that critical exponents have been studied for a number of related problems, see for instance [SGKM, L, DL, GL2, GV, PQR], but rarely for localized reaction, [P].

From now on, u stands for a solution to problem (1.1) which blows up at $t = T$. In order to study its asymptotic behaviour we need to make two further assumptions on the data:

(H1) u_0 is symmetric and nonincreasing for $x > 0$;

(H2) u_0 satisfies $(u_0^m)'' + au_0^p \geq 0$ in $\mathcal{D}'(\mathbb{R})$.

As a consequence of (H1), the same two properties hold for the solution $u(t) = u(\cdot, t)$ at every $0 < t < T$; on the other hand, (H2) implies that the solution is nondecreasing in t (see [BC]). This last hypothesis is useful in order to get the blow-up rates, but it can be avoided in some cases. With these assumptions, we have

Theorem 1.2 *Let $p > (m+1)/2$ and let u be a blow-up solution with blow-up time $T > 0$. If u_0 satisfies hypotheses (H1) and (H2), then for $t \sim T$,*

$$\|u(\cdot, t)\|_\infty \sim (T - t)^{-\alpha}, \quad (1.4)$$

where α is defined by

$$\alpha = \max\left\{\frac{1}{p-1}, \frac{1}{2p-m-1}\right\}. \quad (1.5)$$

	$\frac{m+1}{2}$	m	$m+1$	p
Global Solutions	All solutions blow up	Blow up Solutions and Global Solutions		
	$\alpha = \frac{1}{p-1}$	$\alpha = \frac{1}{2p-m-1}$		

Figure 1: Critical exponents and blow-up rates

In other words, if $p \geq m$ we have $\|u(\cdot, t)\|_\infty \sim (T - t)^{-1/(p-1)}$, the same rate obtained for the homogeneous reaction problem (1.2), cf. [SGKM], while for $p < m$ we have $\|u(\cdot, t)\|_\infty \sim (T - t)^{-1/(2p-m-1)}$, the blow-up rate of the boundary reaction problem (1.3), see [QR]. This is connected to the fact that we may have single-point blow-up or global blow-up. Remark: hypothesis (H2) can be eliminated in the case $p \geq m$.

Once we know the blow-up rates, we rescale the solution accordingly. We define the exponent

$$\beta = \begin{cases} \frac{p-m}{2(p-1)} & \text{if } p \geq m \\ \frac{p-m}{2p-m-1} & \text{if } p \leq m, \end{cases} \quad (1.6)$$

and introduce the renormalized function and variables:

$$f(\xi, \tau) = (T - t)^\alpha u(x, t), \quad \xi = x(T - t)^{-\beta}, \quad \tau = -\log(1 - t/T), \quad (1.7)$$

with α given by (1.5). In similarity variables, we arrive at the following equation for f

$$f_\tau = (f^m)_{\xi\xi} - \beta\xi f_\xi - \alpha f + b(\xi, \tau)f^p, \quad (\xi, \tau) \in \mathbb{R} \times [0, \infty), \quad (1.8)$$

where the reaction term takes the form

$$b(\xi, \tau) = \begin{cases} a(e^{-\beta\tau}\xi) & \text{if } p \geq m \\ e^{-\beta\tau}a(e^{-\beta\tau}\xi) & \text{if } p \leq m, \end{cases} \quad (1.9)$$

Observe the different behaviour of the reaction coefficient b as $\tau \rightarrow \infty$ depending on p :

$$\lim_{\tau \rightarrow \infty} b(\xi, \tau) = \begin{cases} 1 & \text{if } p > m \\ a(\xi) & \text{if } p = m \\ 2L\delta_0(\xi) & \text{if } p < m. \end{cases} \quad (1.10)$$

This difference is crucial in what follows. We need another definition: the ω -limit set of any initial function f_0 is the set of all possible limits of the solution f to equation (1.8) with $f(\cdot, 0) = f_0$, i.e.,

$$\omega(f_0) = \left\{ F \in C(\mathbb{R}), F \geq 0 \mid \exists \tau_j \rightarrow \infty \text{ such that } \lim_{\tau_j \rightarrow \infty} f(\cdot, \tau_j) = F \text{ uniformly in compact sets of } \mathbb{R} \right\}.$$

After these preparations, we have

Theorem 1.3 *Assume $p > (m + 1)/2$ and let u a solution to problem (1.1) which blows up at time T and satisfies the rates (1.4). Let α and β be defined in (1.5), (1.6), put $f_0(\xi) = T^\alpha u_0(\xi T^\beta)$ and let $\omega(f_0)$ be its ω -limit. Then,*

(i) *If $p > m$, then $\omega(f_0)$ is contained in the set of nontrivial solutions of*

$$(F^m)'' - \beta\xi F' - \alpha F + F^p = 0, \quad \xi \in \mathbb{R}. \quad (1.11)$$

(ii) *If $p < m$, then $\omega(f_0)$ is obtained by reflection from the unique positive bounded solution of*

$$\begin{cases} (F^m)'' - \beta\xi F' - \alpha F = 0 & \xi \in \mathbb{R}_+, \\ -(F^m)'(0) = 2LF^p(0). \end{cases} \quad (1.12)$$

(iii) *If $p = m$, then $\omega(f_0)$ is the unique nontrivial symmetric solution of*

$$(F^m)'' - \alpha F + a(\xi)F^p = 0, \quad \xi \in \mathbb{R}. \quad (1.13)$$

Remark: Uniqueness of the limit profile in the case (i) is still an open problem, see [SGKM]. In the other cases, the uniqueness implies that

$$\lim_{t \nearrow T} (T - t)^\alpha u(\xi(T - t)^\beta, t) = F(\xi),$$

uniformly in compact sets of $[0, \infty)$.

With the asymptotic behaviour, we are also able to describe the blow-up set,

$$B(u) = \{x \in \mathbb{R} : \exists x_k \rightarrow x, t_k \rightarrow T^-, u(x_k, t_k) \rightarrow \infty\}. \quad (1.14)$$

From hypothesis (H1) there are only three possibilities for the blow-up set: global blow-up, $B(u) = \mathbb{R}$; regional blow-up, $B(u)$ is a bounded interval; or single-point blow-up, $B(u) = \{0\}$.

Theorem 1.4 *Under the above hypotheses, we have*

- (i) *global blow-up if $p < m$;*
- (ii) *regional blow-up if $p = m$;*
- (iii) *single-point blow-up if $p > m$.*

The proof of these results is organized in steps in the different sections. In Sections 2 and 3 we characterize the global existence and Fujita exponents, Theorem 1.1; in Section 4 we obtain the blow-up rates, Theorem 1.2; finally Section 5 is devoted to establish the asymptotic behaviour, Theorem 1.3, and the blow-up sets, Theorem 1.4.

2 Blow-up versus global existence

In this section we perform the proof of part of the classification scheme, and characterize when solutions with finite time blow-up can exist. Without loss of generality we assume in this section that the coefficient $a(x)$ is a characteristic function, and by rescaling we put $a(x) = \chi_{[-L,L]}$.

Theorem 2.1 *There exist blowing up solutions if and only if $p > (m + 1)/2$.*

Let us start by the lower range, $p \leq (m+1)/2$; we first prove that there are no bounded solutions (which is called *grow-up* when it is not blow-up), and then that there are no blow-up solutions. Moreover, we prove that if $p < m$ the grow-up or blow-up properties hold globally in space.

Lemma 2.1 *Let $0 < p < m$. We have:*

- (i) *if u does not blow up, then $\lim_{t \rightarrow \infty} u(x, t) = \infty$ uniformly on compact sets of \mathbb{R} ;*
- (ii) *if u does blow up, then $B(u) = \mathbb{R}$.*

Proof. Fix $R > L$ and consider the following Dirichlet problem

$$\begin{cases} w_t = (w^m)_{xx} + a(x)w^p, & (x, t) \in (-R, R) \times (0, \infty), \\ w(\pm R, t) = 0, & t \in (0, \infty), \\ w(x, 0) = w_0(x), & x \in (-R, R). \end{cases} \quad (2.1)$$

We first observe that there always exists an stationary solution W to this problem: it is obtained by shooting from $W(0) = A$, $W'(0) = 0$, for different values of $A > 0$. Actually W is decreasing, concave and, since $0 < p < m$, we have that A increases with R . Moreover, it is easy to prove that W is an attractor for problem (2.1). Now take any point $x_0 \in \mathbb{R}$ and any number $M > 0$. It is clear that there exists $R > 0$ such that the corresponding stationary solution satisfies $W(x_0) > 2M$.

(i) If our solution u to problem (1.1) is global in time, since it is clearly a supersolution to the above Dirichlet problem when choosing $w_0 \leq u_0$, we have,

$$u(x, t) \geq w(x, t) \rightarrow W(x), \quad \text{as } t \rightarrow \infty,$$

and thus $u(x_0, t) \geq M$ for t large. We prove in this way global grow-up.

(ii) Assume now that u blows up in a finite time $T > 0$, and suppose that there exists $x_1 > 0$ such that $u(x_1, t) \leq K$ for every $0 < t < T$. Comparison with the porous medium equation for $x > x_1$ implies that the interface of u is bounded up to $t = T$. Therefore taking R large enough we can get $\text{supp}(u(\cdot, t)) \subset [-R, R]$, and also $u_0 \leq W$. Then u is a subsolution to problem (2.1) and, again by comparison, it is bounded by W , a contradiction that implies that the blow-up is global. \square

Lemma 2.2 *If $0 < p \leq (m + 1)/2$ every solution to problem (1.1) is global.*

Proof. By means of a comparison we may assume that u_0 is symmetric, and nonincreasing for $x > 0$, so that the maximum of $u(\cdot, t)$ is achieved at $x = 0$. The proof uses the following integral identity

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u^2 = -m \int_{\mathbb{R}} u^{m-1} (u_x)^2 + \int_{-L}^L u^{p+1}$$

which is obtained by multiplying the equation by u . Since $p \leq (m + 1)/2$ and by the previous lemma we may assume $u(0, t) \geq 1$, the last integral satisfies

$$\begin{aligned} \int_{-L}^L u^{p+1} &\leq 2Lu^{(m+3)/2}(0, t) = C \int_0^\infty u^{(m+1)/2} |u_x| \\ &\leq C \left(\int_{\mathbb{R}} u^{m-1} (u_x)^2 \right)^{1/2} \left(\int_{\mathbb{R}} u^2 \right)^{1/2} \\ &\leq m \int_{\mathbb{R}} u^{m-1} (u_x)^2 + C \int_{\mathbb{R}} u^2. \end{aligned}$$

Therefore,

$$\frac{d}{dt} \int_{\mathbb{R}} u^2 \leq C \int_{\mathbb{R}} u^2,$$

which by Gronwall's Lemma gives

$$\int_{\mathbb{R}} u^2 \leq Ce^t.$$

Then using again Lemma 2.1 we get a contradiction if u blows up. \square

We now examine the existence of blow-up solutions in the range $p > (m + 1)/2$. The existence in the case $p > m$ is a direct consequence of the existence of such solutions for problem (1.2) in bounded domains. On the other hand, when $(m + 1)/2 < p \leq m$ the proof uses comparison with a blow-up solution or a blow-up subsolution which we construct specifically for our problem.

Lemma 2.3 *If $p > m$ problem (1.1) has blowing up solutions.*

Proof. We compare with a blow-up solution to the Dirichlet problem

$$\begin{cases} u_t = (u^m)_{xx} + u^p, & (x, t) \in [-L, L] \times (0, T), \\ u(\pm L, t) = 0, & t \in (0, T), \\ u(x, 0) = u_0(x), & x \in [-L, L], \end{cases}$$

see [SGKM], which is a subsolution to our problem. \square

Lemma 2.4 *If $p = m$ then problem (1.1) admits, for any length $L > 0$, a unique symmetric blow-up solution in the self-similar form*

$$U(x, t) = (T - t)^{-1/(m-1)} F(x). \quad (2.2)$$

Proof. Substituting U into the equation for u , we have that the profile F satisfies the equation

$$(F^m)'' + a(x)F^m - \frac{1}{m-1}F = 0, \quad x > 0,$$

plus the boundary condition $F'(0) = 0$. First of all we observe that if $L \geq L_S = \pi m / (m - 1)$, the above equation admits the explicit profile,

$$F(x) = F_S(x) = \begin{cases} A_S (\cos(\pi x / 2L_S))^{2/(m-1)}, & 0 \leq x \leq L_S, \\ 0, & x \geq L_S, \end{cases} \quad (2.3)$$

where $A_S^{m-1} = 2m / (m^2 - 1)$, see [SGKM]. For $L < L_S$ we construct F by putting together in a C^1 form two pieces, one for $0 \leq x \leq L$ and another for $x \geq L$. For $x \geq L$, since $a(x) = 0$, we have the uniparametric family of explicit profiles,

$$F(x) = C(L_0 - x)_+^{2/(m-1)}, \quad (2.4)$$

for any $L_0 > L$, where $C^{m-1} = (m - 1) / (2m(m + 1))$. We will fix L_0 later on. For the piece of the profile in the inner interval $0 \leq x \leq L$ we consider the function $g = F^m$, which satisfies the problem

$$\begin{cases} g'' + g - \frac{1}{m-1}g^{1/m} = 0, & x \in (0, L), \\ g'(0) = 0, \\ g'(L) = -\sqrt{2m / (m^2 - 1)} g^{(m+1)/2m}(L). \end{cases}$$

The last condition guarantees $F \in C^1$. Shooting from $x = 0$ with $g(0) = A > 0$, we look for the point $\ell > 0$ at which g satisfies

$$g'(\ell) = -\sqrt{2m / (m^2 - 1)} g^{(m+1)/2m}(\ell).$$

and characterize the length ℓ in terms of A . Our purpose is to determine if for every length $0 < L < L_S$ there exists $A > 0$ such that $\ell(A) = L$. To this end we observe that multiplying the equation for g by g' and putting $H(s) = s^2 - \frac{2m}{m^2-1}s^{(m+1)/m}$, we have the following conservation

$$E(x) = \frac{1}{2}(g')^2 + \frac{1}{2}H(g) \equiv \text{const.}$$

Evaluating this constant at $x = 0$ we obtain $E(x) \equiv E(0) = \frac{1}{2}H(A)$. From this it is easy to see that for $A > A_S > 0$ (where $H(A_S) = 0$), the profile is decreasing and is given implicitly by

$$\int_{g(x)}^A \frac{ds}{\sqrt{H(A) - H(s)}} = x.$$

Observe that $A_S = F_S^m(0)$, where F_S is given in (2.3). Putting also $x = \ell$ we get $E(x) \equiv E(\ell) = \frac{1}{2}g^2(\ell)$, i.e. $g(\ell) = \sqrt{H(A)}$. Therefore,

$$\ell(A) = \int_{\sqrt{H(A)}}^A \frac{ds}{\sqrt{H(A) - H(s)}}.$$

On the other hand, one can check that $\ell(A)$ is monotone decreasing for $A > A_S$ and tends to zero as $A \rightarrow \infty$. Of course $\ell(A_S) = L_S$. From this the existence of the piece of the profile corresponding to $0 \leq x \leq L$ is obtained, for any $0 < L < L_S$. Uniqueness comes from the monotonicity of $\ell(A)$. The free parameter $L_0 > 0$ must now be fixed to satisfy $g(\ell) = \sqrt{H(A)}$.

Hence, we have constructed the unique blow-up self-similar solution of problem (1.1). Its support is $[-L_0, L_0]$, where $L_0 = L_S$ if $L \geq L_S$ or $L_0 = L + cH(A)^{(m-1)/4m}$ if $0 < L < L_S$. \square

Lemma 2.5 *If $(m+1)/2 < p < m$ there exist solutions to problem (1.1) that blow up in finite time.*

Proof. We construct here a blow-up subsolution. The function obtained in this case is not of self-similar form but the matching of a self-similar function with a blowing-up parabola. We fix a point $0 < x_0 \leq L$ and consider the even function obtained by reflection of

$$\underline{u}(x, t) = \begin{cases} (A(t) - B(t)x^2)^{1/m} & 0 \leq x \leq x_0, \\ V(x - x_0, t) & x \geq x_0, \end{cases} \quad (2.5)$$

where $A(t)$ and $B(t)$ are taken in order to have a C^1 function, and V is a self-similar solution of the problem,

$$\begin{cases} V_t = (V^m)_{xx} & (x, t) \in \mathbb{R}^+ \times (0, T), \\ -(V^m)_x(0, t) = V^q(0, t) & t \in (0, T). \end{cases} \quad (2.6)$$

It is well known that this problem admits blowing-up self-similar solutions if $q > (m+1)/2$, and they have the form $V(x, t) = (T - t)^{-\gamma} f(x(T - t)^{(m-q)\gamma})$, with $\gamma = 1/(2q - m - 1)$. Moreover, f has compact support $[0, \xi_0]$ if $q \leq m$. See [GL].

The above implies that

$$A(t) = V^m(0, t) + \frac{x_0}{2} V^q(0, t), \quad B(t) = \frac{1}{2x_0} V^q(0, t).$$

Therefore, for $0 \leq x \leq x_0$, the function \underline{u} is given by

$$\underline{u}(x, t) = (T - t)^{-\gamma} I^{1/m}(x, t), \quad I(x, t) = f^m(0) + \frac{1}{2x_0} f^q(0)(x_0^2 - x^2)(T - t)^{(m-q)\gamma}.$$

Notice that $I(x, t) \sim \text{const.}$ In order to see that \underline{u} is a subsolution to problem (1.1), we only have to look at the interval $(0, x_0)$. We calculate

$$\begin{aligned} \underline{u}_t &= (T - t)^{-\gamma-1} (\gamma I - \frac{m-q}{2x_0 m} I^{1-m}(x_0^2 - x^2)(T - t)^{(m-q)\gamma}), \\ (\underline{u}^m)_{xx} &= -\frac{1}{x_0} f^q(0)(T - t)^{-q\gamma}, \\ \underline{u}^p &= f^p(0) I^p (T - t)^{-p\gamma}. \end{aligned} \tag{2.7}$$

Hence the condition for \underline{u} to be a subsolution when T is small reduces to the inequality

$$c_1 T^{-\gamma-1} \leq -c_2 T^{-q\gamma} + c_3 T^{-p\gamma}.$$

This inequality can be achieved if we take $q < \min\{p, (p+m)/2\}$. Since $(m+1)/2 < p < m$ we can choose $(m+1)/2 < q < p$ in order to get the desired blow-up subsolution. Its support is given by $[-x_0 - \xi_0(T - t)^{-(m-q)\gamma}, x_0 + \xi_0(T - t)^{-(m-q)\gamma}]$. \square

3 Fujita exponent

We complete in this section the proof of Theorem 1.1 by characterizing the Fujita exponent. We divide the proof into several lemmas. The cases $(m+1)/2 < p < m$ and $p = m$ come as corollaries of the results proved in the previous section. The case $m < p \leq m+1$ uses the energy method of Levine. Finally, the existence of small global solutions for $p > m+1$ is proved by means of comparison with a self-similar global supersolution.

Lemma 3.1 *If $(m+1)/2 < p \leq m$ then every solution to problem (1.1) blows up.*

Proof. We only have to check that the self-similar solution constructed in Lemma 2.4 when $p = m$ or the subsolution constructed in Lemma 2.5 for $p < m$ can be put below any solution if we let pass enough time. If $p < m$ this holds by means of Lemma 2.1.

Thus $\underline{u}(x, 0) \leq u(x, t_0)$ for some $t_0 > 0$. On the other hand, when $p = m$ the self-similar solution has small initial value if $T > 0$ is large, but its support is not small, since the length L_0 is not small. We then must use the penetration property of the solutions of the porous medium equation (the equation in (1.1) with $a(x) \equiv 0$), which also holds trivially for our solutions by comparison, to guarantee that there exists $t_0 > 0$ such that the support of $u(\cdot, t_0)$ contains the interval $[-L_0, L_0]$. Therefore, taking T large enough, we have $u(x, t_0) \geq U(x, 0)$. By comparison u must blow up in finite time. \square

Lemma 3.2 *If $m < p < m + 1$ every solution to problem (1.1) blows up.*

Proof. The proof follows the argument of [LS] in three steps.

i) We consider the energy functional

$$E_u(t) = \frac{1}{2} \int_{-\infty}^{\infty} |(u^m)_x(x, t)|^2 dx - \frac{m}{p+m} \int_{-\infty}^{\infty} a(x) u^{p+m}(x, t) dx \quad (3.1)$$

and check that a condition to guarantee blow-up if $p > m$ is that there exist some $t_0 \geq 0$ for which $E_u(t_0) < 0$. This is done in [LS] in a more general case (the concavity argument).

ii) We now observe that the Barenblatt function

$$B(x, t; D) = t^{-1/(m+1)} (D - kx^2 t^{-2/(m+1)})_+^{1/(m-1)}, \quad (3.2)$$

$k = \frac{m-1}{2m(m+1)}$, $D > 0$, (which is a subsolution to our equation), satisfies the above requirement provided t is large. In fact,

$$\begin{aligned} E_B(t) &= c_1 t^{-(2m+1)/(m+1)} - c_2 t^{-(p+m-1)/(m+1)} \int_0^{cLt^{-1/(m+1)}} (1-z^2)^{(p+m)/(m-1)} dz \\ &\leq c_1 t^{-(2m+1)/(m+1)} - c_3 t^{-(p+m)/(m+1)}, \end{aligned}$$

and we get $E_B(t) < 0$ for t large whenever $p < m + 1$. Notice the effect of L being finite in the last integral.

iii) We finally choose $D > 0$ small in order to have $B(x, 1; D) \leq u_0(x)$. Therefore $B(x, t+1; D) \leq u(x, t)$. Since the solution to problem (1.1) with initial value $B(\cdot, t_1; D)$ with t_1 large blows up in finite time, so does u . \square

Lemma 3.3 *If $p = m + 1$ every solution to problem (1.1) blows up.*

Proof. We use the method introduced in [G] to prove the blow-up in the case $a(x) = 1$ for the critical exponent (see also [GL]). Assuming by contradiction that our solution is global in time, this method consists in the construction, by means of a rescaling and a limit procedure, of a solution to some problem which has no solution.

Given a global solution u to problem (1.1), we perform the change of variables

$$v(\xi, \tau) = (t+1)^{1/(m+1)}u(x, t), \quad \xi = x(t+1)^{-1/(m+1)}, \quad \tau = \log(t+1).$$

We have that v solves, for every $\tau > 0$, the equation

$$v_\tau = (v^m)_{\xi\xi} + \frac{1}{m+1}(\xi v)_\xi + \rho(\xi)e^{\tau/(m+1)}v^{m+1}, \quad (3.3)$$

where $\rho(\xi) = \chi_{\{|\xi| \leq Le^{-\tau/(m+1)}\}}$. We want to pass to the limit v for $\tau \rightarrow \infty$, but we are not allowed to do that, we don't know if this limit exists. Instead of this we consider another solution, precisely the solution g of equation (3.3) with initial datum

$$g(\xi, 0) = \left(D - k\xi^2\right)_+^{1/(m-1)} = B(\xi, 1; D),$$

(see (3.2)). We observe that the function $B(\cdot, 1; D)$ has negative energy if D is large. Indeed

$$E_B(1) \leq D^{(3m+1)/(2(m-1))}(c_1 - c_2D^{m/(m-1)}) < 0$$

for $D > D_*$. Therefore, if $u_0(x) \geq B(x, 1; D_*)$ we are done. In the general case we take $D > 0$ small in order to have $v_0(x) = u_0(x) \geq B(x, 1; D)$. This implies $g \leq v$, and therefore g is also a global solution.

For the special form of the initial value, it is easy to see that g is increasing in τ , symmetric and decreasing in $\xi > 0$, see [G]. Therefore there do exist the limit

$$f(\xi) = \lim_{\tau \rightarrow \infty} g(\xi, \tau), \quad (3.4)$$

finite or infinite. A first step in order to pass to the limit also in the equation is to show that f is bounded outside the origin. And this must be true for if not g would be large enough to blow up in finite time. Assume then by contradiction that there exists $\xi_0 > 0$ such that $f(\xi_0) = \infty$. Then by monotonicity we have that given $M > 0$ there exists τ_M large such that $g(\xi, \tau_M) > M$ for every $|\xi| \leq \xi_0$. Consider g in the original variables, i.e., define $w(x, t) = e^{-\tau/(m+1)}g(\xi, \tau)$. We want to see that taking M large then w blows up in finite time, contradicting the fact that g is global. To this end we consider a function in the form $W(x) = B(x, t; D)$ in such a way that

- W has negative energy;
- $\text{supp}(W) \subset (-\xi_0 e^{\tau_M/(m+1)}, \xi_0 e^{\tau_M/(m+1)})$;
- $W(x) \leq M e^{-\tau_M/(m+1)}$.

The first condition is achieved if $D > D_*$. The last two requirements are possible for some $t = t_*$ provided M is chosen large enough, depending on ξ_0 and τ_M . This implies that the solution to equation (3.3) with initial value W blows up in finite time, and it lies below w , which is the desired contradiction.

In summary, the function f defined in (3.4) is finite for every $\xi \neq 0$.

Now, exactly as in [G], by means of a Lyapunov functional, we obtain that f is a weak symmetric solution to the equation

$$0 = (f^m)'' + \frac{1}{m+1}(\xi f)', \quad (3.5)$$

for every $\xi \neq 0$. But the only solutions to that equation are the above profiles $f(\xi) = B(\xi, 1; D)$, $D > 0$. In particular $f(0)$ is finite, so we can pass to the limit in the reaction term, in the weak formulation of equation (3.3), to get the boundary condition at $\xi = 0^+$

$$-\frac{1}{2L} \lim_{\xi \rightarrow 0^+} (f^m)'(\xi) = f^{m+1}(0).$$

We end with the observation that the profiles obtained do not satisfy this condition. \square

Lemma 3.4 *If $p > m + 1$ then there exist global solutions to problem (1.1).*

Proof. This result is a consequence of the work [Q], where a global supersolution for the equation

$$u_t = (u^m)_{xx} + |x|^{-\sigma} u^p,$$

is constructed for $p > m + 2 - \sigma$ and every $\sigma < 1$. These supersolutions decay to zero and have the self-similar form

$$\bar{u}(x, t) = t^{-\mu} \psi(xt^{-\nu}), \quad \mu = \frac{2 - \sigma}{2p - (m - 1)\sigma - 2}, \quad \nu = \frac{(p - m)\mu}{2 - \sigma}.$$

Obviously the reaction coefficient is bigger than $a(x)$, multiplied by a constant if necessary, and the above functions are also supersolutions to our problem (1.1). \square

4 Blow-up Rates

Here we calculate the speed at which the blow-up solutions tend to infinity as t approaches the blow-up time T , i.e., we prove Theorem 1.2. The result and the techniques are different depending on the sign of $p - m$.

Lemma 4.1 *Let $p > m$ and set $\alpha = 1/(p - 1)$. If u_0 satisfies (H1) then we have $u(0, t) \sim (T - t)^{-\alpha}$.*

Proof. The result follows by intersection theory. We want to compare u with a self-similar solution to problem (1.2) with the same blow up time T . Such self-similar functions have the form

$$U(x, t) = (T - t)^{-\alpha} F(\xi), \quad \xi = |x|(T - t)^{-\beta},$$

where $\alpha = 1/(p - 1)$ and $\beta = \alpha(p - m)/2$. The lower estimate is obtained easily by considering the constant profile $\bar{F}(\xi) = \alpha^\alpha$. The corresponding function \bar{U} is a supersolution to our problem and, by the maximum principle, it must necessarily intersect u if they blow up at the same time.

As to the upper estimate, it is well known that for every $R > 0$ there exists a profile F_R which is positive in $[0, R)$ and vanishes at $\xi = R$, cf. [SGKM]. Notice that as $\beta > 0$ this profile gives a subsolution to problem (1.1) when extended by zero for $\xi > R$ if $RT^\beta < L$.

On the other hand, since R goes to zero as $F_R(0)$ goes to infinity, it is clear that we can find $0 < R < LT^{-\beta}$ such that $U(\cdot, 0)$ and u_0 have exactly two intersections, and also that $u_0(0) < U(0, 0)$. This implies $u(0, t) < U(0, t) = F(0)(T - t)^{-\alpha}$ for every $0 < t < T$, giving the desired estimate. Indeed, the number of intersections can not increase, see for instance [SGKM], and this number cannot be zero, as before by the maximum principle; the symmetry of the solutions does the rest: the number of intersections is always two for every $0 < t < T$.

□

Remark 4.1 *The lower estimate of the rate holds for every $p > (m + 1)/2$, though it is not sharp when $p < m$. As to the upper estimate in the case $p = m$ the above profiles exist only for $R > \pi/2$, and we can repeat the same argument only if $L > \pi/2$. The proof for general $L > 0$ is included in the following lemma.*

Lemma 4.2 *Let $p \leq m$ and set $\alpha = 1/(2p - m - 1)$. If u_0 satisfies (H1) and (H2) then we have $u(0, t) \sim (T - t)^{-\alpha}$.*

Proof. We use a rescaling technique inspired in the work [GK]. The difference lies in the final step: we do not pass to the limit, but instead we estimate the blow-up time of the rescaled function which is translated into a blow-up rate for the original solution.

Fix $t \in (0, T)$, put $\lambda = u(0, t)$ and consider the function

$$v_\lambda(x, s) = \lambda^{-1}u(\lambda^{m-p}x, \lambda^{m+1-2p}s + t), \quad (4.6)$$

which satisfies

$$(v_\lambda)_s = (v_\lambda^m)_{xx} + \rho_\lambda(x)v_\lambda^p, \quad (x, s) \in \mathbb{R} \times (-\lambda^{2p-m-1}t, \lambda^{2p-m-1}(T - t)), \quad (4.7)$$

where $\rho_\lambda(x) = \lambda^{m-p}a(\lambda^{m-p}x)$. Notice that hypotheses (H1) and (H2) imply

$$0 \leq v_\lambda(x, 0) \leq 1, \quad v_\lambda(0, 0) = 1, \quad (v_\lambda)_s \geq 0. \quad (4.8)$$

Our purpose is, by means of comparison, to show that v_λ blows up in a time $c_1 \leq S \leq c_2$ with c_i independent of λ . Since $S = \lambda^{2p-m-1}(T - t)$ and $\lambda = u(0, t)$, we conclude $u(0, t) \sim c(T - t)^{-1/(2p-m-1)}$. By symmetry the comparison is made only for $x > 0$.

We assume first $p < m$, and also we begin with the upper estimate. To this end we consider the function constructed in the proof of Lemma 2.5, with $x_0 = x_0(\lambda) = \lambda^{p-m}L$, namely

$$\underline{u}(x, s) = \begin{cases} (S^* - s)^{-\gamma} I^{1/m}(x, s) & 0 \leq x \leq x_0, \\ (S^* - s)^{-\gamma} f(x(S^* - s)^{(m-q)\gamma}) & x \geq x_0, \end{cases}$$

where $\gamma = 1/(2q - m - 1)$, f is the self-similar profile corresponding to problem (2.6) with $q \in ((m + 1)/2, p)$ to be chosen, and

$$I(x, s) = f^m(0) + \frac{1}{2x_0} f^q(0)(x_0^2 - x^2)(S^* - s)^{(m-q)\gamma}.$$

Denoting $A = f(0)$, we take $S^* = (4A)^{1/\gamma}$. Observe that for $\lambda \geq \lambda_0$ we have $A^m \leq I(x, s) \leq 2A^m$. We obtain in this way $\underline{u}(0, 0) \leq 1/2$.

To see that \underline{u} is a subsolution to the equation (4.7) we only have to check that (see (2.7)),

$$\gamma(2A)^{1/m}(S^* - s)^{-\gamma-1} \leq \lambda^{m-p} A^p (S^* - s)^{-\gamma p} \left(A^{p/m} - \frac{1}{L} A^{q-p} (S^* - s)^{\gamma(p-q)} \right),$$

or which is the same

$$\gamma(2A)^{1/m}(S^* - s)^{\gamma(p-1)-1} \leq \lambda^{m-p} A^p \left(A^{p/m} - \frac{4^{p-q}}{L} \right).$$

It is easy to see that as q tends to $(m + 1)/2$, we have that both γ and A tend to infinity. Take then q close to $(m + 1)/2$ in order to get $A^{p/m} - 4^{p-q}/L \geq 1$, and also $\gamma(p - 1) - 1 > 0$. This fixes also S^* (independent of λ). Hence taking λ large enough, we obtain the required condition and \underline{u} is a subsolution to (4.7). We now have to compare the initial values. Though we know that v_λ is large everywhere for s close to S since u has global blow-up, there is no measurement of this growth. To get then v_λ large enough to put it above $\underline{u}(\cdot, 0)$ we consider the following problem

$$\begin{cases} z_s = (z^m)_{xx} & x > 0, s > 0, \\ z(0, s) = 1 & s > 0, \\ z(x, 0) = 0 & x > 0. \end{cases}$$

Its solution is the so-called Polubarinova-Kochina self-similar function $z(x, s) = H(x/\sqrt{s})$, see [P-K]. By (4.8) we have $v_\lambda \geq z$ for $x \geq 0$, $0 \leq s < S$. On the other hand, since $\underline{u}(0, 0) \leq 1/2$, and the support of $\underline{u}(\cdot, 0)$ is bounded independently of λ , it is clear that there exist a time s_0 , independent of λ , such that $z(x, s_0) \geq \underline{u}(x, 0)$. Summing up, we have

$$v_\lambda(x, s_0) \geq z(x, s_0) \geq \underline{u}(x, 0).$$

Therefore, by comparison we get $v_\lambda(x, s + s_0) \geq \underline{u}(x, s)$ for λ large. This implies $S \leq S^* + s_0$, and the proof of the upper estimate for t close to T is completed.

In order to obtain the lower estimate, we consider the following problem

$$\begin{cases} \bar{v}_s = (\bar{v}^m)_{xx} + K\rho_\lambda(x) & x > 0, s > 0, \\ \bar{v}(x, 0) = 1 & x > 0. \end{cases} \quad (4.9)$$

If we define $w(x, s) = \int_0^x \bar{v}(y, s) dy$, the problem satisfied by w is

$$\begin{cases} w_s = ((w_x)^m)_x + K \int_0^x \rho_\lambda(y) dy & x > 0, s > 0, \\ w(x, 0) = x & x > 0, \\ w(0, s) = 0 & s > 0. \end{cases}$$

Notice that, since $\rho_\lambda \rightarrow 2L\delta_0$, the function $\bar{w}(x, s) = x(1 + LKs)$ is a supersolution to this problem. Therefore, as $w(0, s) = \bar{w}(0, s)$, we have $\bar{v}(0, s) = w_x(0, s) \leq \bar{w}_x(0, s) = 1 + LKs$.

On the other hand, whenever v satisfies $v \leq K^{1/p}$, it is a subsolution to problem (4.9). We then obtain, for $0 < s < 1/K$ with K large independent of λ , the inequality

$$v(0, s) \leq \bar{v}(0, s) \leq 1 + L < K^{1/p}.$$

This implies that the blow-up time for v satisfies $S \geq 1/K$, and the proof is concluded in the case $p < m$.

When $p = m$ the lower estimate of the rate was obtained in the previous lemma. In order to obtain the upper estimate we perform the same proof as for $p < m$ but with the self-similar solution constructed in Lemma 2.4 instead of \underline{u} . \square

5 Asymptotic behaviour

In this section we prove the stabilization result, Theorem 1.3 and, as a consequence, we complete the study of the blow-up sets, Theorem 1.4.

Proof of Theorem 1.3 For $p \neq m$ we use the approach introduced in [GV] to deal with perturbed problems. In fact we consider equation (1.8) as a small perturbation of the equation with $b(\xi)$ replaced by its limit (1.10). Thus, if $p > m$, using [G2] we are in the hypotheses of Theorem 3 in [GV], and therefore we obtain that $\omega(f_0)$ is contained in the set of stationary solutions to

$$f_\tau = (f^m)_{\xi\xi} - \beta\xi f_\xi - \alpha f + f^p, \quad (\xi, \tau) \in \mathbb{R} \times [0, \infty).$$

If $p < m$ we obtain the same result with the limit problem

$$\begin{cases} f_\tau = (f^m)_{\xi\xi} - \beta\xi f_\xi - \alpha f, & \xi > 0, \tau > 0 \\ -(f^m)_\xi(0, \tau) = 2Lf^p(0, \tau) & \tau > 0. \end{cases}$$

In the case $p = m$ the result follows easily from the existence of a Lyapunov functional. In fact this functional is explicit,

$$L_f(\tau) = \frac{1}{2} \int_{\mathbb{R}} |(f^m)_\xi|^2 + \frac{m}{m^2 - 1} \int_{\mathbb{R}} f^{m+1} - \frac{m}{p + m} \int_{-L}^L f^{p+m}.$$

It is bounded and nonincreasing along the orbits. Therefore by standard theory the limit is a stationary solution to equation (1.8) and thus it is the unique self-similar profile constructed in Lemma 2.4. \square

Proof of Theorem 1.4 The case $p < m$ has been proved in Lemma 2.1. As for the case $p > m$ we follow, step by step, the proof given in [G2, FM] in which the case $a \equiv 1$ is considered. We obtain that there exists $\delta > 0$ small enough such that

$$J = (u^m)_x + f(x)u^p < 0 \quad \text{in } (0, s(t) - \delta) \times (t_0, T),$$

where here we choose $f(x) = \lambda \int_0^x a(s) ds$, $\lambda > 0$ small. Integrating now this inequality in $x \in (0, \varepsilon)$ we get

$$u(x, t) \leq c|x|^{-2/(p-m)}, \quad \text{in } (0, \varepsilon) \times (t_0, T),$$

and the only blow-up point is the origin.

We end with the case $p = m$. This is done by using the convergence to a compactly supported self-similar profile. Indeed, the support of this limit profile is contained in the blow-up set of u , $B(u) \supseteq [-L_1, L_1]$; outside this support f vanishes exponentially fast, $f(x, \tau) \leq Ce^{-\alpha\tau}$, see [CDE]. This implies that u is bounded for every $|x| > L_1$ and finally $B(u) = [-L_1, L_1]$. \square

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