On the lack of observability for wave equations: a gaussian beam approach

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Abstract

This paper is devoted to study the property of observability for wave equations guaranteeing that the total energy of solutions may be estimated by means of the energy concentrated on a subset of the domain or of the boundary. We prove that this property fails in three different situations. First, we consider the wave equation with piecewise smooth coefficients when the observation is made in the exterior boundary. We also present a wave equation with highly oscillating Hölder continuous coefficients for which observability fails from any open set that does not contain the origin. Finally, lack of observability is proved for the constant coefficient wave equation when the observation is made from an interior hypersurface. All the counterexamples presented here are constructed using highly localized solutions known as Gaussian Beams.

1 Introduction

This paper is concerned with the analysis of the observability property for solutions of wave equations. This property is established by means of an **observability inequality**, in which the total energy of the solutions is uniformly estimated by a partial measurement. Typically, this measurement is the portion of energy localized in a subset of the domain or of its boundary. This property of observability is relevant, in particular, in the context of control problems (see e.g. J.L. Lions [10]).

It is well known that the failure of the observability property for the wave equation is closely related to the existence of solutions whose energy is localized near certain curves $(t, x(t))$ in space-time. These curves, the so called **rays**, are, in the interior of Ω, the domain of definition of the equation, solutions of a Hamiltonian system of ordinary differential equations which involves the coefficients of the operator (see definition 10). When one of these trajectories hits the boundary $\partial\Omega$ it is reflected according to the law of geometric optics. Given a ray $(t, x(t))$ it is possible to construct a sequence of solutions $(u_k)_{k\in\mathbb{N}}$ of the wave equation such that the amount of their energy outside a ball of radius $k^{-1/4}$ centered at $x(t)$ is of the order of

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k^{-1}. These solutions, called **gaussian beams,** are well-known in Optics but relatively new to mathematicians; in the articles [1], [2] and [11] the reader may find an extensive bibliography and comments on the historical development of this construction.

The existence of these solutions gives sharp necessary conditions for the observability property to hold. As it was remarked by J. Ralston in [11], in order to observe these gaussian beam solutions, the observation set must intersect every ray. If this were not the case, one could construct a gaussian beam along a ray that would not hit the observation set; and clearly this solution could not be observed, since it would be negligible outside an arbitrarily small neighborhood of the ray. Later on, C. Bardos, G. Lebeau and J. Rauch proved in their 1992 paper [3] that this condition is "almost" sufficient. The sharp sufficient condition in [3] requires every ray of geometric optics to intersect the control region in a non-diffractive point.

In this paper we prove the failure of the observability property for equations that, due to the low regularity of their coefficients or to the geometrical setting, cannot be treated in the framework of [3].

We first recall, in Section 1, the construction of gaussian beams in [11], the main result being Theorem 1; this requires some smoothness in the coefficients of the equation (they must be at least twice differentiable). That is also necessary for the rays to be well defined, as they are locally solutions of system (1) which involves the first derivatives of the coefficients in the principal symbol of the wave operator

However, as we shall prove in Section 2, this construction can be generalized to the case of piecewise smooth coefficients. We consider a system of two wave equations with propagation speeds $a, b \in C^2$ defined respectively in an inner domain Ω_i and an outer domain Ω_o. The equations are coupled at the interface $\partial\Omega_i$ by transmission conditions, see system (18), (19), (20). In Theorem 15 we construct gaussian beam solutions for this problem. In fact, we prove that a gaussian beam defined a priori in Ω_i can be extended to a gaussian beam for the transmission problem: when the beam hits the interface $\partial\Omega_i$, a refracted and a reflected component appear. The most noticeable property is that the refracted component (the one lying in Ω_o) can be arbitrarily small when the propagation speeds satisfy the relation $|\sin(\text{incidence angle})| > a/b$ at the incidence point. Thus, **total reflection** occurs. This is a well known fact, but a complete proof has not been given in the literature.

In Section 3, we exploit this fact to prove that the observability property for the transmission problem fails when the observation in made on Ω_o, provided the coefficients satisfy the monotonicity relation $a < b$ near the interface and the inner region Ω_i is strictly convex. We state the corresponding non-controllability result which complements the positive ones already known for the case $a > b$ (see, for instance, [9] and [10]).

In Section 4 we analyze the observability property for wave equations whose coefficients are Hölder continuous yet non smooth. It is known that for the $1-d$ wave equation, observability holds if the coefficients are of bounded variation, and recently, C. Castro and E. Zuazua [5], proved the lack of observability for highly oscillating Hölder continuous coefficients, which are smooth outside an hypersurface. Here we prove a result in the same vein, showing the existence of a function $c \in C^{0,\alpha}\left(\mathbb{R}^d\right)$ for all $\alpha \in (0,1)$ such that the observability property for solutions of $\partial_t^2 u - \text{div}(c\nabla_x u) = 0$ fails when the observation is made in a set that does not contain the origin. The coefficient c is, in fact, smooth outside the origin $x = 0$ and the wave operator associated to it has the property of having periodic rays of arbitrary small radius around the origin. It is then possible to construct gaussian beams concentrated along any of those periodic orbits. This contradicts any observability result made from any open set that does not contain the point $x = 0$.

Finally, in Section 5 we discuss the observability property for the constant coefficient wave equation when the observation is made from a hypersurface. This problem arises in the context of strong stabilization of a singularly damped wave equation studied in [7]. By means of gaussian beams we present several geometric situations in which the observability property fails.

2 Preliminaries on gaussian beams

In this section we shall recall the construction of gaussian beams for the wave equation with C^2 coefficients. The contents of this section are inspired in the approach given in [11]; the reader may consult [2] and the references therein for a slightly different viewpoint.

Let us consider the wave operator

$$\Box := \partial_t^2 - \Delta_g,$$

where $g = (g^{ij})$ is a $d \times d$ matrix with C^2 bounded coefficients which we shall assume to be uniformly elliptic and Δ_g is the laplacian associated to g, that is,

$$\Delta_g u = \sum_{i,j=1}^{d} \partial_{x_i} \left(g^{ij} \partial_{x_j} u \right).$$

The symbol of $\Box$ is $\xi^T \cdot g(x) \cdot \xi - \tau^2$; in what follows, we shall denote $H(x, \xi) := \xi^T \cdot g(x) \cdot \xi$. Recall that a **null bicharacteristic** of $\Box$ is a solution of the system

$$\begin{cases} \dot{t}(s) = -2\tau(s), \\ \dot{x}(s) = \partial_\xi H(x(s), \xi(s)), \\ \dot{\tau}(s) = 0, \\ \dot{\xi}(s) = -\partial_x H(x(s), \xi(s)), \\ H(x(0), \xi(0)) = \tau(0)^2. \end{cases} \tag{1}$$

If $t(0) = t_0$, $x(0) = x_0$, $\tau(0) = \tau_0$, $\xi(0) = \xi_0$ are such that $H(x_0, \xi_0) = \tau_0$ then, since system (1) is Hamiltonian, we have $H(x(s), \xi(s)) = \tau(s)^2$ for all $s \in \mathbb{R}$. In the sequel we shall always take $\tau = -1/2$. This implies that $t(s) = s + t_0$ and $(x(t), \xi(t))$ still satisfy (1) and, since H is homogeneous in ξ, this will not be a restriction. A **ray** for the operator $\Box$ will be a curve $x(t)$ that solves (1) with $H(x(t), \xi(t)) = 1/4$. Observe that $x(t)$ is a geodesic for the Riemannian metric defined by g^{-1}.

Given a ray $x(t)$, we shall describe the construction of approximate solutions of the equation

$$\Box u = 0 \text{ on } (0, T) \times \mathbb{R}^d \tag{2}$$

with energy

$$E_g(u(t, \cdot)) = \frac{1}{2} \int_{\mathbb{R}^d} |\partial_t u(t, x)|^2 + H(x, \nabla_x u(t, x))\, dx,$$

concentrated on $x(t)$ for every $t \in (0, T)$. This construction is by now well-known and can be found, for instance, in [11] and [2].

These solutions will have the structure:

$$u_k(t, x) := k^{d/4-1} a(t, x) e^{ik\psi(t,x)}, \tag{3}$$

with a phase function ψ of the form

$$\begin{cases} \psi(t,x) = \xi(t) \cdot (x - x(t)) + \frac{1}{2}(x - x(t))^T \cdot M(t) \cdot (x - x(t)), \\ \text{where } M(t) \text{ is a } d \times d \textbf{ complex} \text{ symmetric matrix with} \\ \textbf{positive definite imaginary part}. \end{cases} \tag{4}$$

Observe that

$$|u_k(t,x)|^2 = k^{d/2-2} |a(t,x)|^2 e^{-k(x-x(t))^T \cdot \mathrm{Im}\, M(t) \cdot (x-x(t))};$$

so $\mathrm{Im}\, M(t) > 0$ implies that $|u_k|$ is essentially a gaussian profile translated along $x(t)$.

The main result that we recall in this section establishes the existence of functions of the form (3), (4) that are approximate solutions of the wave equation (2):

Theorem 1 ([11]) *Let $x(t)$ be a ray for $\Box$. Then there exist $a, \psi \in C^2(\mathbb{R}_t \times \mathbb{R}^d_x)$ with ψ of the form (4) such that the functions u_k defined by (3) satisfy for any $T > 0$:*

- ***the u_k are approximate solutions of the wave equation**:*

$$\sup_{t \in (0,T)} \|\Box u_k(t,\cdot)\|_{L^2(\mathbb{R}^d_x)} \leq Ck^{-1/2}, \tag{5}$$

- ***the energy of u_k is bounded with respect to k**: more precisely, for $t \in (0,T)$,*

$$\lim_{k \to \infty} E_g(u_k(t,\cdot)) = \frac{\pi^{d/2} |a(t,x(t))|^2}{4\sqrt{\det\left(\mathrm{Im}\, \nabla^2_x \psi(t,x(t))\right)}}, \tag{6}$$

- ***the energy of the u_k is exponentially small off $x(t)$**:*

$$\sup_{t \in (0,T)} \int_{\mathbb{R}^d \setminus B_k(t)} |\partial_t u_k(t,x)|^2 + H(x, \nabla_x u(t,x))\, dx \leq Ce^{-\beta k}. \tag{7}$$

Here $B_k(t)$ denotes the ball centered at $x(t)$ of radius $k^{-1/4}$ and C, β are positive constants that depend on T but not on k. Moreover, the functions a, ψ can be constructed to satisfy $a(t_0,x_0) = a_0$, $M(t_0) = M_0$ for any $t_0, a_0 \in \mathbb{R}$ and any $d \times d$ complex symmetric matrix M_0 with positive definite imaginary part.

We shall not give a complete proof of this theorem, that may be found in [11]; we are just going to highlight the ingredients of the construction that we shall need in the sequel.

First of all, we shall need a technical lemma whose proof is straightforward:

Lemma 2 *Let $b \in L^\infty(\mathbb{R}^d_x)$ be a function satisfying $|x - x_0|^{-\alpha} b(x) \in L^\infty(\mathbb{R}^d_x)$ for some $x_0 \in \mathbb{R}^d$ and some $\alpha \geq 0$, and let A be a symmetric, positive definite, real $d \times d$ matrix. Then*

$$\int_{\mathbb{R}^d} \left| b(x) e^{-kx^T \cdot A \cdot x} \right|^2 dx \leq Ck^{-d/2-\alpha}$$

for some $C > 0$ that does not depend on k.

Proof of Theorem 1. Let u_k be of the form (3). Then one readily sees that

$$\begin{aligned} \Box u_k &= k^{d/4-1} e^{ik\psi} \Box a + \\ &+ k^{d/4} e^{ik\psi} i \left(a \Box \psi + 2 \partial_t a \partial_t \psi - 2 \nabla_x a^T \cdot g \cdot \nabla_x \psi \right) + \\ &+ k^{1+d/4} e^{ik\psi} \left(\nabla_x \psi^T \cdot g \cdot \nabla_x \psi - (\partial_t \psi)^2 \right) a. \end{aligned}$$

Let us write the above expression as

$$\Box u_k =: k^{d/4-1} e^{ik\psi} r_0 + k^{d/4} e^{ik\psi} r_1 + k^{1+d/4} e^{ik\psi} r_2.$$

We are going to construct a and ψ in such a way that the terms of higher order in k, namely r_2 and r_1, vanish on $x(t)$ up to order 2 and 0 respectively on $x(t)$. If so, then, by Lemma 2 (with $\alpha = 3$ for r_2 and $\alpha = 1$ for r_1), we have

$$\|\Box u_k (t, \cdot)\|^2_{L^2(\mathbb{R}^d_x)} \leq C \left(k^{-2} + k^{-1} + k^{-1} \right) \leq C k^{-1},$$

with a constant C uniform in $t \in (0, T)$.

1. Analysis of the r_2 **term:** We want to construct ψ such that $\partial_x^\alpha r_2 (t, x(t)) = 0$ for all $t \in \mathbb{R}$ and all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq 2$; this is equivalent to solving the **eikonal equation**

$$H(x, \nabla_x \psi(t,x)) - (\partial_t \psi(t,x))^2 = 0 \tag{8}$$

up to order 2 on $(t, x(t))$. Next we prove that this can be done if ψ is of the form (4) for a suitable $M(t)$, that can be chosen to satisfy $\operatorname{Im} M(t) > 0$. Denote $R(t,x) := H(x, \nabla_x \psi(t,x)) - (\partial_t \psi(t,x))^2$; since $\nabla_x \psi(t, x(t)) = \xi(t)$, $\partial_t \psi(t, x(t)) = -1/2$ and $(x(t), \xi(t))$ solves (1), we have $R(t, x(t)) = 0$. An easy computation shows

$$\begin{aligned} \nabla_x R(t,x) = \partial_x H(x, \nabla_x \psi(t,x)) + \partial_\xi H(x, \nabla_x \psi(t,x)) \cdot \nabla_x^2 \psi(t,x) \\ -2 \partial_t \psi(t,x) \nabla_x \partial_t \psi(t,x). \end{aligned} \tag{9}$$

Taking into account that

$$\begin{cases} \nabla_x^2 \psi(t, x(t)) = M(t), \\ \nabla_x \partial_t \psi(t, x(t)) = \dot{\xi}(t) - M(t) \cdot \dot{x}(t), \\ \partial_t^2 \psi(t, x(t)) = -\dot{\xi}(t) \cdot \dot{x}(t) - \dot{x}(t)^T \cdot M(t) \cdot \dot{x}(t), \end{cases} \tag{10}$$

we find, $\nabla_x R(t, x(t)) = 0$. Finally, the equation $\nabla_x^2 R(t, x(t)) = 0$ results in a nonlinear ODE for $M(t)$:

$$\frac{d}{dt} M(t) + M(t) C(t) M(t) + B(t) M(t) + M(t) B(t)^T + A(t) = 0, \tag{11}$$

where $C(t)$, $B(t)$ and $A(t)$ are $d \times d$ matrices whose coefficients only depend on the first and second derivatives of H evaluated along $(x(t), \xi(t))$. This is a Riccati equation and it can be shown ([2],[11]) that, given a symmetric $d \times d$ matrix M_0 with $\operatorname{Im} M_0 > 0$, there exist a global solution $M(t)$ of (11) that satisfies $M(t_0) = M_0$, $M(t) = M(t)^T$ and $\operatorname{Im} M(t) > 0$ for all $t \in \mathbb{R}$. This completes the construction of ψ.

2. Analysis of the r_1 **term:** Now we construct a that makes r_1 vanish on $(t, x(t))$. Substituting the values of $\partial_t \psi$, $\nabla_x \psi$ in r_1 and evaluating in $(t, x(t))$, we obtain the following equation for $a(t, x(t))$:

$$\frac{d}{dt} a(t, x(t)) = a(t, x(t)) \Box \psi(t, x(t)).$$

This linear ODE determines $a(t, x(t))$ uniquely from $a(t_0, x(t_0))$.

3. Proof of the energy formula (6): First of all, observe that

$$E_g(u_k(t,\cdot)) = \frac{k^{d/2}}{2}\int_{\mathbb{R}^d} |a|^2 \left(\partial_t\psi^2 + \nabla_x\psi^T \cdot g \cdot \nabla_x\psi\right) e^{-2k\,\mathrm{Im}\,\psi} dx + R_k(t),$$

where $\sup_{t\in(0,T)} |R_k(t)| \to 0$ when $k \to \infty$. By construction we have $\nabla_x\psi^T \cdot g \cdot \nabla_x\psi = \partial_t\psi^2 = 1/4$, and formula (6) follows by a straightforward evaluation of the resulting gaussian integral.

4. Proof of the energy concentration estimate (7): take $\chi \in C^\infty\left(\mathbb{R}^d_x\right)$, $0 \le \chi \le 1$, vanishing for $|x| \le 1/2$ and being identically 1 for $|x| \ge 1$. We have

$$\frac{1}{2}\int_{\mathbb{R}^d\setminus B_k(t)} |\partial_t u_k|^2 + H(x, \nabla_x u(t,x))\,dx \le E\left(\chi\left(k^{1/4}\cdot\right) u_k(t,\cdot)\right) \le Ce^{-\beta k} k^{d/2}\int_{\mathbb{R}^d} e^{-k\,\mathrm{Im}\,\psi(t,x)} dx.$$

with $\beta = \inf\{\mathrm{Im}\,\psi(t,x) : t\in(0,T), |x|\ge 1\} > 0$ and $C > 0$ depending on the L^∞ norm of a, ψ and their derivatives. ■

At this point, it is convenient to introduce some terminology:

Definition 3 *A sequence of functions of the form (3), (4) constructed as in Theorem 1 will be called a* ***gaussian beam along the ray*** *$x(t)$.*

Remark 4 *As a consequence of formulae (10), the quadratic form $\mathrm{Im}\,\nabla^2_{t,x}\psi(t, x(t))$ is positive when restricted to $\{0\}\times\mathbb{R}^d_x$ and null when evaluated along the vector $(1, \dot{x}(t))$. It then follows by elementary linear algebra that $\mathrm{Im}\,\nabla^2_{t,x}\psi(t, x(t))$ is positive in any complement of the space spanned by $(1, \dot{x}(t))$.*

Remark 5 *The above construction applies almost identically to the wave operator $\rho(x)\,\partial_t^2 - \Delta_g$ when $\rho \in C^2\left(\mathbb{R}^d_x\right)$ is bounded from above and below by positive constants.*

Remark 6 *Let $\chi \in C^\infty_c\left(\mathbb{R}_t\times\mathbb{R}^d_x\right)$ be identically equal to one in a neighborhood of the ray $\{(t, x(t) : t\in\mathbb{R})\}$. Then the functions χu_k also satisfy (5), (6), (7).*

Remark 7 *As shown in [11], when the coefficients of g are in C^r, with $r \ge 2$, it is possible to find correcting terms $\tilde{\psi}$, $a_1,..,a_N$ and a cut-off function χ as in the preceding Remark such that the functions*

$$u_k = \chi k^{d/4-1}\left(a + \sum_{j=1}^N a_j k^{-j}\right) e^{ik\left(\psi+\tilde{\psi}\right)}$$

still satisfy the conclusions of Theorem 1 and moreover

$$\sup_{t\in(0,T)} \|\Box u_k(t,\cdot)\|_{H^{r/2-1}\left(\mathbb{R}^d_x\right)} \le Ck^{-1/2}.$$

The construction of Theorem 1 can be adapted to obtain highly localized solutions of the Dirichlet problem

$$\begin{cases} \Box u = 0 \text{ in } (0,T) \times \Omega, \\ u = 0 \text{ in } (0,T) \times \partial\Omega, \\ u|_{t=0} = u^0, \; \partial_t u|_{t=0} = u^1. \end{cases} \tag{12}$$

Obviously, if Ω is bounded there may exist rays that exit Ω in finite time; so for an arbitrary $T > 0$ a gaussian beam will not satisfy in general the Dirichlet boundary condition. In order to overcome this difficulty, one has to superpose two gaussian beams, one reflected of the other at the boundary.

In what follows, Ω will be a domain of $\mathbb{R}^d$ with C^2 boundary and ν will be a field of unit normal vectors of $\partial\Omega$ (with respect to the metric g^{-1}) pointing in the inwards direction. We shall work in a system of **geodesic normal coordinates** (see, for example, C.5 of [8]): for $(y,s) \in \partial\Omega \times [0,\varepsilon)$ let $\gamma(y,s)$ denote the geodesic of g^{-1} defined by $\gamma(y,0) = y$ and $\dot\gamma(y,0) = \nu(y)$. For $\varepsilon > 0$ small enough, the mapping $(y,s) \longmapsto \gamma(y,s)$ defines a system of local coordinates. The change of variables formula for the principal symbol of a differential operator asserts that the laplacian Δ_g in geodesic normal coordinates has principal symbol $H\left(\gamma(y,s), (d\gamma^{-1})^T_{\gamma(y,s)}(\eta,\sigma)\right)$; here we have denoted by (η,σ) the dual variables of (y,s) in the principal symbol; they are related to the "old" variable ξ by $(\eta,\sigma) = \xi^T \cdot d\gamma_{(y,s)}$. Observe that $(d\gamma^{-1})^T_{\gamma(y,s)}(0,\sigma)$ is normal to $\gamma(\partial\Omega \times \{s\})$ at $\gamma(y,s)$ (for the euclidean metric) and $(d\gamma^{-1})^T_{\gamma(y,s)}(\eta,0)$ is tangent (indeed, $(d\gamma^{-1})^T_{\gamma(y,0)}(\eta,0) = \eta$). A simple computation shows that

$$H\left(\gamma(y,s), (d\gamma^{-1})^T_{\gamma(y,s)}(\eta,\sigma)\right) = \sigma^2 + r(y,s,\eta),$$

where $r(y,s,\eta)$ is a polynomial of second order in η and $r(y,0,\eta) = H(y,\eta)$.

Now, let $\left(x^-(t), \xi^-(t)\right)$ be a ray with $x^-(0) \in \Omega$, $y_0 := x^-(t_0) \in \partial\Omega$ for some $t_0 > 0$ and $x^-(t) \in \Omega$ for $t \in (0,t_0)$; suppose that $\xi^-(t_0)$ is (η_0,σ_0) when written in geodesic normal coordinates. Let u_k^- be a gaussian beam along $x^-(t)$. The next result (also to be found in [11]) describes the construction of a reflected gaussian beam u_k^+ which, superposed to u_k^-, achieves the Dirichlet boundary condition on $\mathbb{R}_t \times \partial\Omega$:

Proposition 8 *Let $\left(x^-(t), \xi^-(t)\right)$ and u_k^- be as above, $y_0 := x^-(t_0) \in \partial\Omega$. Moreover, suppose that $\xi^-(t_0)$ is transversal to $\partial\Omega$ at y_0 (i.e. $\sigma_0 \neq 0$). Then there exists a gaussian beam u_k^+, constructed along the ray $\left(x^+(t), \xi^+(t)\right)$ given by*

$$x^+(t_0) = y_0, \; \xi^+(t_0) = (\eta_0, -\sigma_0), \tag{13}$$

which satisfies

$$\left\| u_k^- + u_k^+ \right\|_{H^1((0,T)\times\partial\Omega)} \leq C k^{-1} \tag{14}$$

whenever $T > 0$ is small enough to ensure that $x^+(t)$ remains in Ω if $t \in (t_0,T)$.

Proof. In order to use Theorem 1 to construct the beam $u_k^+ = k^{d/4-1} a^+ e^{ik\psi^+}$ we must specify the values of a^+, $\nabla_x \psi^+$ and $\nabla_x^2 \psi^+$ at (t_0, y_0).

First of all, we impose that the derivatives of ψ^+ involving the tangential and time directions equal those of ψ^- at (t_0,y_0); of course, we have written $u_k^- = k^{d/4-1} a^- e^{ik\psi^-}$. This results

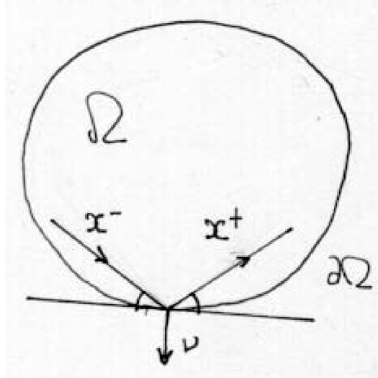

Figure 1: x$^+$ is obtained from x^- by reflection.

in $\nabla_y\psi^+(t_0,y_0,0)=\nabla_y\psi^-(t_0,y_0,0)=\eta_0$, $\nabla_y^2\psi^+(t_0,y_0,0)=\nabla_y^2\psi^-(t_0,y_0,0)$ ($\psi^\pm(t,y,s)$ denotes the expression of $\psi^\pm$ in geodesic normal coordinates). It only remains to define $\partial_s\psi^+$, $\partial_s^2\psi^+$, $\nabla_y\partial_s\psi^+$ at the point (t_0,y_0). This will be done by solving the eikonal equation

$$(\partial_s\psi)^2+r(y,s,\nabla_y\psi)-(\partial_t\psi)^2=0. \tag{15}$$

If $\nabla_y\psi^+(t_0,y_0,0)=(\eta_0,\sigma^+)$ then we obtain $\sigma^+=\pm\sqrt{1/4-\eta_0}=\pm|\sigma_0|$; the only admissible choice, which ensures that $\nabla_y\psi^+(t_0,y_0,0)$ points inside Ω, is $\sigma^+=-\sigma_0$.

The second order derivatives are found by requiring that equation (15) is satisfied at first order in (t_0,y_0); at this point it is essential that $\sigma_0\neq 0$. Observe that, by Remark 4, we still have that $\operatorname{Im}\nabla_x^2\psi^+(t_0,y_0)$ is positive definite.

Finally, we define $a^+(t_0,y_0)=-a^-(t_0,y_0)$. Then, when restricted to $\mathbb{R}_t\times\partial\Omega$ we have

$$\left|u_k^-+u_k^+\right|=k^{d/4-1}\left|a^-+a^+\right|e^{-\operatorname{Im}\psi^\pm},$$

and (14) follows by lemma 2, since $(a^-+a^+)(t_0,y_0)$ vanishes as well as its first derivatives. ■

Remark 9 *Property (13) can be restated as*

$$\dot{x}^+(t_0)=\dot{x}^-(t_0)-2\left(\dot{x}^-(t_0)\cdot g^{-1}(y_0)\cdot\nu(y_0)\right)\nu(y_0). \tag{16}$$

This means that $x^+(t)$ is obtained from $x^-(t)$ by reflection with respect to ν (in the metric g^{-1}), see figure 1. When the metric g^{-1} is conformal to the euclidean metric (i.e. g is a multiple of the identity matrix), equation (16) results in the well-known ***geometric optics law****.*

Now we recall the notion of generalized ray, which is a particular case of definition 24.2.2 in [8]:

Definition 10 *Let I be a bounded interval; a curve $x:I\to\bar{\Omega}$ will be called a* ***generalized ray*** *for $\Box$ in Ω if there exists a finite set $B\subset I$ and a curve $\xi(t):I\to\mathbb{R}^d$ such that:*

i) $(x(t),\xi(t))$ solve (1) for $t\in I\setminus B$.

ii) For $t\in B$, $x(t)\in\partial\Omega$ and $\xi^\pm(t):=\lim_{s\to t^\pm}\xi(s)$ satisfy (13) and $\xi^\pm(t)$ is transversal to $\partial\Omega$ at $x(t)$.

It is now clear that, given a generalized ray $x(t)$ such that $B = \{t_1, ..., t_N\}$ is finite, it is possible to construct functions u_k^i, $i = 0, ..., N$, such that u_k^i is a gaussian beam along the ray $x(t)$ with $t \in (t_i, t_{i+1})$ (we have set $t_0 = \inf I$ and $t_{N+1} = \sup I$) and

$$\left\| \sum_{i=0}^{N} u_k^i \right\|_{H^1(I\times\partial\Omega)} \leq Ck^{-1}.$$

We shall call the function $\sum_{i=0}^{N} u_k^i$ a **gaussian beam along the generalized ray** $x(t)$; we shall denote $a(t, x(t))$ the function defined as the amplitude of the gaussian beam u_k^i (evaluated at $(t, x(t))$) when $t \in (t_i, t_{i+1})$; $M(t)$ will have an analogous meaning.

We now deduce the following properties for the **exact solutions** of the Dirichlet problem for the wave equation (2) whose initial data are those of a gaussian beam:

Corollary 11 *Let $x(t)$ be a generalized ray in Ω defined on $(0, T)$ and $\chi \in C_c^\infty(\Omega)$ with $\chi \equiv 1$ in a neighborhood of $x(0)$. Suppose u_k is a gaussian beam constructed along $x(t)$ and w_k be the solutions of the Cauchy problem:*

$$\begin{cases} \Box w_k = 0 \text{ in } (0,T) \times \Omega, \\ w_k|_{(0,T)\times\partial\Omega} = 0, \\ w_k|_{t=0} = \chi u_k|_{t=0}, \ \partial_t w_k|_{t=0} = \chi \partial_t u_k|_{t=0}. \end{cases}$$

Then we have:

i) $\lim_{k\to\infty} E_g(\mathbf{1}_\Omega w_k(t,\cdot)) = \dfrac{\pi^{d/2}}{4} |a(t, x(t))|^2 |\det(\operatorname{Im} M(t))|^{-1/2}$ *for* $t \in (0,T) \setminus B$,

ii) $\sup_{t\in(0,T)} \int_{\Omega\setminus B_k(t)} |\partial_t w_k(t,x)|^2 + |\nabla_x w_k(t,x)|^2 \, dx \leq Ck^{-1}$.

Here $B_k(t)$ is defined as in Theorem 1.

Proof. Let $\theta(t,x) = \chi(x - x(t) + x(0))$ and denote $f_k := \mathbf{1}_\Omega \Box(\theta u_k)$, $g_k := \theta u_k|_{(0,T)\times\partial\Omega}$. Let v_k be the solution of the problem

$$\begin{cases} \Box v_k = f_k \text{ in } (0,T) \times \Omega, \\ v_k|_{(0,T)\times\partial\Omega} = g_k, \\ v_k|_{t=0} = 0, \ \partial_t v_k|_{t=0} = 0. \end{cases}$$

We recall the well-known estimate

$$\sup_{t\in(0,T)} E_g(v_k(t,\cdot)) \leq C\left(E_g(v_k(0,\cdot)) + \|f_k\|_{L^1(0,T;L^2(\mathbb{R}^n))} + \|g_k\|_{H^1((0,T)\times\partial\Omega)}\right),$$

which in our case results in $\sup_{t\in(0,T)} E_g(v_k(t,\cdot)) \leq Ck^{-1}$. Since $v_k = \chi u_k - w_k$, this proves i).

To prove the part ii) it suffices to observe that

$$\begin{aligned} &\sup_{t\in(0,T)} \int_{\mathbb{R}^d\setminus B_k(t)} |\partial_t w_k(t,x)|^2 + H(x, \nabla_x w_k(t,x)) \, dx \\ \leq\ &\sup_{t\in(0,T)} \int_{\mathbb{R}^d\setminus B_k(t)} |\partial_t u_k(t,x)|^2 + H(x, \nabla_x w_k(t,x)) \, dx + \sup_{t\in(0,T)} E_g(v_k(t,\cdot)) \\ \leq\ &C\left(e^{-\beta k} + k^{-1}\right). \end{aligned}$$

■

Remark 12 *By conservation of energy we have that* $\lim_{k\to\infty} E(w_k(t,\cdot))$ *is constant; thus* $|a(t,x(t))|^2 / \sqrt{\det(\operatorname{Im} M(t))}$ *does not depend on* t.

Remark 13 *If we consider in Corollary 11 a gaussian beam* u_k *corrected as in Remark 7 one can show that* $\|w_k(t,\cdot) - u_k(t,\cdot)\|_{H^{r/2-1}(\Omega)} \to 0$. *This requires* $\partial\Omega$ *and* g *to be* C^r *and a straightforward modification of Proposition 8.*

3 Gaussian beams for a transmission problem

In this section we shall generalize the construction of gaussian beams to a wave equation with coefficients having jump discontinuities. Let Ω be a domain of $\mathbb{R}^d$ with C^2 boundary and consider the problem

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_c u(t,x) = 0 \text{ in } (0,T)\times\Omega, \\ u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \\ u(0,x) = u^0(x), \ \partial_t u(0,x) = u^1(x); \end{cases} \tag{17}$$

where c is a piecewise smooth positive function of the form

$$c(x) = \begin{cases} a(x)^2 \text{ if } x\in\Omega_i, \\ b(x)^2 \text{ if } x\in\Omega\setminus\Omega_i, \end{cases}$$

with $a, b \in C^2(\mathbb{R}^d)$ bounded from below by a positive constant. We have denoted by Δ_c the laplacian associated to the matrix $c(x)Id$.

We shall assume that Ω_i is a subdomain of Ω with C^2 boundary and $\overline{\Omega_i}\subset\Omega$. We shall refer to Ω_i and $\Omega_o := \Omega\setminus\overline{\Omega_i}$ as the **inner and outer regions** respectively, and to $\partial\Omega_i$ as the **interface**.

First of all, observe that rays are no longer well-defined. To have an insight of what curves should be their natural substitutes we look at the following equivalent formulation of our wave equation: every solution u of problem (17) can be written as $u(t,\cdot) = v(t,\cdot)\mathbf{1}_{\Omega_i} + w(t,\cdot)\mathbf{1}_{\Omega_o}$ where (v,w) are solutions of the system:

$$\begin{cases} \partial_t^2 v - \Delta_{a^2} v = 0 \text{ in } (0,T)\times\Omega_i, \\ v(0,\cdot) = u^0|_{\Omega_i}, \ \partial_t v(0,\cdot) = u^1|_{\Omega_i}, \end{cases} \tag{18}$$

$$\begin{cases} \partial_t^2 w - \Delta_{b^2} w = 0 \text{ in } (0,T)\times\Omega_o, \\ w = 0 \text{ on } (0,T)\times\partial\Omega, \\ w(0,\cdot) = u^0|_{\Omega_o}, \ \partial_t w(0,\cdot) = u^1|_{\Omega_o}; \end{cases} \tag{19}$$

coupled at the interface by **transmission conditions**:

$$v = w, \ a^2\partial_\nu v = b^2\partial_\nu w \text{ on } (0,T)\times\partial\Omega_i. \tag{20}$$

From now on, ν will denote a field of normal unit vectors of $\partial\Omega_i$ pointing towards Ω_o.

The techniques developed in section 2 allow us to construct gaussian beam solutions to equations (18) and (19). We now describe how these solutions can be assembled in order to satisfy the transmission conditions (20).

Let $(x(t), \xi(t))$ be a ray for (18). We shall restrict to a certain class of rays:

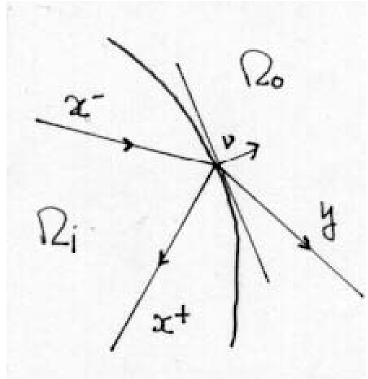

Figure 2: Refraction, $|\sin\theta| > a/b$

Assumption T: $x(0) \in \Omega_i$ and at a time t_0, $\xi_0 := \xi(t_0)$ hits the interface $\partial\Omega_i$ **transversely** at $y_0 := x(t_0)$; moreover, for $t < t_0$, $x(t) \in \Omega_i$.

Let $v_k^- = k^{d/4-1} A^- e^{ik\psi^-}$ be a gaussian beam constructed along $x(t)$. In Theorem 15 below, we prove that there exist gaussian beams $v_k^+ = k^{d/4-1} A^+ e^{ik\psi^+}$, and $w_k = k^{d/4-1} B e^{ik\varphi}$, defined for the operators $\Box_a := \partial_t^2 - \Delta_{a^2}$ and $\Box_b := \partial_t^2 - \Delta_{b^2}$ respectively, such that the pair $\left(v_k^- + v_k^+, w_k\right)$ satisfies approximately (20).

The gaussian beam v_k^+ is constructed along the ray $\left(x^+(t), \xi^+(t)\right)$ obtained from $x(t)$ by reflection at the interface $\partial\Omega_i$, that is

$$x^+(t_0) = y_0,\ \xi^+(t_0) = \xi_0 - \frac{\cos\theta}{a(y_0)} \nu(y_0), \tag{21}$$

where θ is the angle of ξ_0 with respect to the normal ν (hence $\cos\theta = 2a(y_0)(\xi_0 \cdot \nu(y_0))$).

The form of w_k depends on θ; as we shall see below, if $\eta_0 := \nabla_x\varphi(t_0, y_0)$ (recall that φ is the phase of w_k) then

$$\begin{cases} \text{the tangential components of } \xi_0 \text{ and } \eta_0 \text{ are equal,} \\ \eta_0 \cdot \nu(y_0) = \dfrac{1}{2}\sqrt{\dfrac{1}{b(y_0)^2} - \dfrac{\sin^2\theta}{a(y_0)^2}}. \end{cases} \tag{22}$$

Thus, two different kind of phenomena may occur:

1. Refraction: this corresponds to the case $|\sin\theta| < a(y_0)/b(y_0)$ (Figure 2). Then, $\eta_0 \cdot \nu(y_0)$ is real and w_k is a gaussian beam constructed along the ray $(y(t), \eta(t))$ in Ω_o with $y(t_0) = y_0$, $\eta(t_0) = \eta_0$ and the angle ϕ of η_0 with respect to the normal ν at y_0 satisfies **Snell's law:**

$$a(y_0)|\sin\phi| = b(y_0)|\sin\theta|.$$

2. Total reflection: this is the case if $|\sin\theta| > a(y_0)/b(y_0)$ (Figure 3). Now, $\eta_0 \cdot \nu(y_0)$ is purely imaginary and it makes no sense to speak of the ray with $\eta(t_0) = \eta_0$. Indeed, w_k degenerates in a function that is exponentially small off $\partial\Omega_i$; we still make Ansatz (3) to construct w_k, but the phase function φ is no longer of the form (4). The next proposition describes the construction in this case:

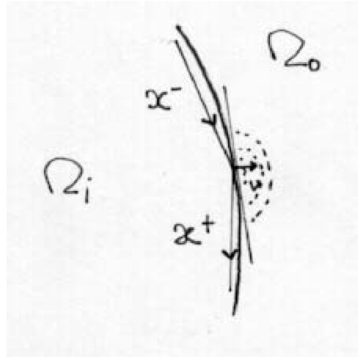

Figure 3: Total reflection, $|\sin\theta| < a/b$

Proposition 14 *Suppose (y, s) is a system of geodesic normal coordinates in $\overline{\Omega}$ near $\partial\Omega$ and $\Box$ a general wave operator as described in section 2. Let $a, \psi \in C^2(\mathbb{R}_t \times \partial\Omega)$ and $(t_0, y_0) \in \mathbb{R}_t \times \partial\Omega$ having the following properties*

$$\begin{cases} \operatorname{Im}\psi(t_0, y_0) = 0, \quad \operatorname{Im} d\psi_{(t_0,y_0)} = 0, \\ r(y_0, 0, \eta_0) - \tau_0^2 > 0, \\ \operatorname{Im}\nabla^2_{(t,y)}\psi(t_0, y_0) > 0. \end{cases}$$

Let $\sigma_0 = i\sqrt{r(y_0, 0, \eta_0) - \tau_0^2}$. Then there exist a phase function φ and an amplitude b with

$$\varphi|_{\mathbb{R}_t\times\partial\Omega} = \psi \text{ at } (t_0, y_0) \text{ up to order 2, } b|_{\mathbb{R}_t\times\partial\Omega} = a \text{ at } (t_0, y_0)$$

that satisfy:

- *ψ is of the form*

$$\varphi(t, y, s) = \psi(t, y) + i\sigma_0 s + O\left(|sy| + |s|^2 + |y|^3\right),$$

and, as a result, $\left|k^{d/4-1}be^{ik\varphi}\right|$ decays exponentially in the (positive) s direction.

- *The functions $k^{d/4-1}be^{ik\varphi}$ are approximate solutions of the wave equation:*

$$\left\|\Box\left(k^{d/4-1}be^{ik\varphi}\right)\right\|_{L^2((0,T)\times\Omega)} \leq Ck^{-1/2}.$$

- *The energy of $k^{d/4-1}be^{ik\varphi}$ in the region $s > 0$ tends to zero as k tends to infinity:*

$$\sup_{t\in(0,T)} E_g\left(\mathbf{1}_{s>0}k^{d/4-1}b(t,\cdot)\,e^{ik\varphi(t,\cdot)}\right) \leq Ck^{-1/2}.$$

The proof of this result, very similar to that of Proposition 8, can be found in [11].

Remark that total reflection is only possible if $a(y_0) < b(y_0)$, while refraction is always the case when $a(y_0) > b(y_0)$. Since critical incidence $|\sin\theta| = a(y_0)/b(y_0)$ cannot be treated with our Ansatz we shall assume:

Assumption NC: The ray $(x(t), \xi(t))$ does not hit the interface with the critical angle, i.e. $|\sin\theta| \neq a(y_0)/b(y_0)$.

We now can state the main result of this section:

Theorem 15 *Let $(x(t), \xi(t))$ be a ray such that assumptions T and NC above hold. Let $v_k^- = k^{d/4-1}A^- e^{ik\psi^-}$ be a gaussian beam along $x(t)$. There exist gaussian beams $v_k^+ = k^{d/4-1}A^+ e^{ik\psi^+}$, $w_k = k^{d/4-1}Be^{ik\varphi}$ such that*

$$\begin{cases} A^+(t_0, y_0) = \dfrac{\left(a(y_0)^2 \xi_0 + b(y_0)^2 \eta_0\right) \cdot \nu(y_0)}{\left(a(y_0)^2 \xi_0 - b(y_0)^2 \eta_0\right) \cdot \nu(y_0)} A^-(t_0, y_0), \\ B(t_0, y_0) = \dfrac{2a(y_0)^2 \xi_0 \cdot \nu(y_0)}{\left(a(y_0)^2 \xi_0 - b(y_0)^2 \eta_0\right) \cdot \nu(y_0)} A^-(t_0, y_0), \end{cases} \tag{23}$$

v_k^+ is a gaussian beam for $\square_a$ constructed along the ray $\left(x^+(t), \xi(t)^+\right)$ defined by (21) and

- *if $|\sin\theta| < a(y_0)/b(y_0)$ then w_k is a gaussian beam for $\square_b$ propagating along the ray $(y(t), \eta(t))$ given by (22),*
- *if $|\sin\theta| > a(y_0)/b(y_0)$ then w_k is constructed as in Proposition 14 with $\square = \square_b$.*

Here, $T > 0$ is small enough in order to guarantee that for $t \in (0, T)$, $x^+(t)$ and $y(t)$ remain respectively in Ω_i and Ω_o. Moreover, setting $v_k := v_k^- + v_k^+$ we have:

$$\begin{cases} \|v_k - w_k\|_{H^1((0,T)\times\partial\Omega_i)} \leq Ck^{-2}, \\ \|a^2\partial_\nu v_k - b^2\partial_\nu w_k\|_{L^2((0,T)\times\partial\Omega_i)} \leq Ck^{-1}. \end{cases}$$

Proof. We shall proceed in two steps:

1. Construction of ψ^+ and φ: In order to apply Theorem 1 we must determine the Taylor series of ψ^+, φ at (t_0, y_0) up to order 2 and the values $A^+(t_0, y_0)$, $B(t_0, y_0)$. We first impose the condition that the time and tangential derivatives up to order 2 of $\psi^\pm$, φ must be equal at (t_0, y_0); it remains to determine the derivatives involving the normal component.

We begin with $\partial_\nu \psi^+$, $\partial_\nu \varphi$: since the phase functions must satisfy the eikonal equations

$$\begin{cases} a^2\left(\nabla_x \psi^\pm\right)^2 - \left(\partial_t \psi^\pm\right)^2 = 0, \\ b^2(\nabla_x \varphi)^2 - (\partial_t \varphi)^2 = 0, \end{cases}$$

at the point (t_0, y_0) and the time derivatives must be equal, we have $\left(\nabla_x \psi^+\right)^2 = |\xi_0|^2$, $(\nabla_x \varphi)^2 = (a/b)^2 |\xi_0|^2$. Taking into account that the tangential components of the gradients are identical, we conclude $\left(\partial_\nu \psi^+\right)^2 = (\xi_0 \cdot \nu(y_0))^2$ and $(\partial_\nu \varphi)^2 = (\xi_0 \cdot \nu(y_0))^2 + (a(y_0)^2/b(y_0)^2 - 1)|\xi_0|^2$. We make the following choices:

$$\begin{cases} \partial_\nu \psi^+(t_0, y_0) = -(\xi_0 \cdot \nu(y_0)), \\ \partial_\nu \varphi(t_0, y_0) = \sqrt{(\xi_0 \cdot \nu(y_0))^2 + 1/4(1/b(y_0)^2 - 1/a(y_0)^2)}. \end{cases} \tag{24}$$

These are made in order to ensure that v_k^+ and w_k propagate inside Ω_i and Ω_o respectively. Remark that (24) is equivalent to $\nabla_x \psi^+(t_0, y_0) = \xi^+(t_0)$ and $\nabla_x \varphi(t_0, y_0) = \eta_0$, where $\xi^+(t_0)$, η_0 were defined in (21) and (22).

2. Construction of the amplitudes A^+, B: we can now compute the values of the amplitudes at (t_0, y_0). One easily obtains $v_k^\pm(t_0, y_0) = k^{d/4-1}A^\pm(t_0, y_0)$, $w_k = k^{d/4-1}B(t_0, y_0)$ and for the normal derivatives

$$\begin{cases} \partial_\nu v_k^\pm(t_0, y_0) = k^{d/4-1}\left(\partial_\nu A^\pm(t_0, y_0) + ikA^\pm(t_0, y_0)\partial_\nu \psi^\pm(t_0, y_0)\right), \\ \partial_\nu w_k(t_0, y_0) = k^{d/4-1}(\partial_\nu B(t_0, y_0) + ikB(t_0, y_0)\partial_\nu \varphi(t_0, y_0)). \end{cases}$$

If we want conditions (20) to be satisfied at first order we must have, at point (t_0, y_0),

$$\begin{cases} A^- + A^+ = B, \\ a^2 \left(A^- \partial_\nu \psi^- + A^+ \partial_\nu \psi^+\right) = b^2 B \partial_\nu \varphi. \end{cases} \tag{25}$$

Substituting $\partial_\nu \psi^\pm$, $\partial_\nu \varphi$ by their previously computed values and solving the resulting system one obtains (23).

Finally, we complete the construction of v_k^+ by applying Proposition 8. If $|\sin\theta| < a(y_0)/b(y_0)$ the beam w_k is also constructed using Proposition 8. In the case $|\sin\theta| > a(y_0)/b(y_0)$ the result follows by Proposition 14. ■

Remark 16 *We could have considered as well the system*

$$\begin{cases} a^{-2}\partial_t^2 v - \Delta v = 0 \text{ in } (0,T) \times \Omega_i, \\ v(0,\cdot) = u^0|_{\Omega_i}, \ \partial_t v(0,\cdot) = u^1|_{\Omega_i}, \end{cases}$$

$$\begin{cases} b^{-2}\partial_t^2 w - \Delta w = 0 \text{ in } (0,T) \times \Omega_o, \\ w = 0 \text{ on } (0,T) \times \partial\Omega, \\ w(0,\cdot) = u^0|_{\Omega_o}, \ \partial_t w(0,\cdot) = u^1|_{\Omega_o}; \end{cases}$$

with transmission conditions

$$v = w, \ \partial_\nu v = \partial_\nu w \text{ on } (0,T) \times \partial\Omega_i,$$

*which correspond to viewing a^{-2} and b^{-2} as **densities**. The results of the preceding theorem are still valid in this case, except for the values of the transmitted and reflected amplitudes A^+, B that can be easily computed by considering the analog of system (25).*

4 On the lack of observability and controllability for the transmission problem

Here we use the results of the preceding section to study the following observability problem:

Let $\omega \subset \Omega_o$ be a neighborhood of $\partial\Omega$ and $T > 0$. Does there exist a constant $C = C(T,\omega) > 0$ such that

$$E_{a^2}(\mathbf{1}_{\Omega_i} v(0,\cdot)) + E_{b^2}(\mathbf{1}_{\Omega_0} w(0,\cdot)) \le C \int_0^T \int_\omega |\partial_t w(t,x)|^2 \, dx dt \tag{26}$$

holds for every finite energy solution (v,w) of (18), (19), (20)?

This problem is relevant in the context of controllability and stabilization of wave equation (17), see, for example [9], [10]. These authors gave sufficient conditions on Ω_o, ω and T for (26) to hold under the monotonicity assumption $a > b$. In particular, (26) holds when ω is a neighborhood of $\partial\Omega$ and T is large enough.

Here we shall concentrate in the case $a < b$. More precisely we shall suppose that a is constant and equal to $a_0 > 0$ in some neighborhood $U \subset \Omega_i$ of $\partial\Omega_i$ and that $a_0 < b_0 := \inf_{\partial\Omega_i} b$.

As a consequence of Theorem 15 we are able to prove that there exist solutions which are essentially trapped in Ω_i, i.e. for which the component w is negligible. More precisely we have the following result:

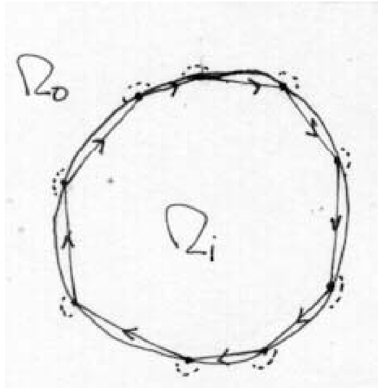

Figure 4: The trapped ray

Theorem 17 *Suppose that Ω_i is strictly convex and that a, b are as above. Then, given $T > 0$ there exist a sequence $(v_k, w_k)_{k\in\mathbb{N}}$ of solutions of (18), (19), (20) such that*

$$E_{a^2}(\mathbf{1}_{\Omega_i} v_k(0,\cdot)) + E_{b^2}(\mathbf{1}_{\Omega_0} w_k(0,\cdot)) = 1 \text{ for all } k \in \mathbb{N}$$

and

$$\lim_{k\to\infty} \int_0^T \int_{\Omega_0} |\partial_t w_k(t,x)|^2 = 0.$$

In particular, inequality (26) does not hold.

Proof. Suppose for the moment that $a \equiv a_0$ in the whole of Ω_i. Rays for the operator $\Box_{a^2}$ are of the form $(x(t), \xi)$ where

$$x(t) = (t - t_0)\, a \frac{\xi}{|\xi|} + x_0,\ \xi = \text{constant},\ \ |\xi| = \frac{1}{2a}.$$

Thus, generalized rays are just segments reflected at the interface $\partial\Omega_i$ following the law of geometric optics. Now, since we have assumed that Ω_i is strictly convex, there exists a generalized ray $(x(t), \xi(t))$ for $\Box_a$ such that at every point of incidence at the interface the angle θ between the corresponding segment of the ray and the outer normal ν satisfies $|\sin\theta| > a_0/b_0$, see Figure 4. Then, iterating the construction in Theorem 15 one obtains functions $v_k^1, ..., v_k^N$, $w_k^1, ..., w_k^{N-1}$ such that $\sum_{i=1}^N v_k^i$ is a gaussian beam along $x(t)$ and, for $i = 1, ..., N-1$, w_k^i is constructed as in Proposition 14. The pair $(v_k, w_k) := \left(\sum_{i=1}^N v_k^i, \sum_{i=1}^{N-1} w_k^i\right)$ clearly satisfies the conclusions of the Theorem. And, due to the well-posedness of system (18), (19), (20), so does the exact solution issued from the initial data corresponding to (v_k, w_k).

In the general case, i.e. $a \equiv a_0$ in a neighborhood $U \subset \Omega_i$ of $\partial\Omega_i$ only, the same argument is valid, since the existence of the generalized ray $x(t)$ that allows us to construct the trapped gaussian beam v_k only depends on the values of a near $\partial\Omega_i$. ∎

Remark 18 *The hypothesis of Ω_i being strictly convex is made only to ensure that there exist a generalized ray in Ω_i such that $|\sin\theta| > a_0/b_0$ holds for every incidence angle θ of the ray on the interface $\partial\Omega_i$. Of course, there are geometrical situations in which this property holds and Ω_i is not convex.*

Remark 19 *The same argument can be used to prove that for any finite time $T > 0$ it is impossible to find a constant $C > 0$ such that*

$$E_{a^2}(\mathbf{1}_{\Omega_i} v(0,\cdot)) + E_{b^2}(\mathbf{1}_{\Omega_o} w(0,\cdot)) \leq C \int_0^T \int_{\partial\Omega} |\partial_\nu w(t,x)|^2 \, dxdt$$

holds for every solution of (18), (19), (20) (here we have denoted by ν the outward unit normal field of $\partial\Omega$).

Remark 20 *A simple modification of the construction of section 3 in the spirit of Remarks 7 and 13 proves that an inequality as (26) is still false even if the $H^1 \times L^2$-energy is replaced by the $H^{s+1} \times H^s$-energy for any $s < 0$.*

We conclude this section by stating the non-controllability result issued from Theorem 17:

Theorem 21 *Let Ω_i, a and b be as in Theorem 17. Given $T > 0$, there exists $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ such that the solution of*

$$\begin{cases} \partial_t^2 u - c^2 \Delta_x u = \mathbf{1}_{\Omega_o} f \text{ in } (0,T) \times \Omega, \\ u = 0 \text{ on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = u^0, \ \partial_t u(0,\cdot) = u^1, \end{cases} \tag{27}$$

satisfies $(u(T), \partial_t u(T)) \neq 0$ whatever $f \in L^2((0,T) \times \Omega_o)$ is.

Proof. The proof of this result from Theorem 17 is classical; we shall only sketch the main ideas involved in it, see [10] for further details.

Suppose that system (27) were exactly controllable in time T from Ω_o, in other words, for every initial data $u = (u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there existed a control function $f_u \in L^2((0,T) \times \Omega_o)$ such that the corresponding solution u of (27) satisfied $u(T) = \partial_t u(T) = 0$. In that case, the closed graph theorem would ensure that the map that to an initial datum associates its least norm control f, would be continuous. By duality, this is equivalent to that fact that an inequality such as (26) holds, and this would contradict Theorem 17. ■

Remark 22 *This result shows that, when Ω_i is strictly convex and a, and b are as above it is impossible to observe and control the solutions of the transmission problem from the outer region. Of course these constructions have local nature and therefore can be easily extended to the case where the coefficients have jump discontinuities along several hypersurfaces.*

5 Localization of gaussian beams for oscillating coefficients

This section is devoted to proving the following result

Theorem 23 *Let $d \geq 2$. Then there exists a bounded, Hölder continuous function $c \in C^{0,\alpha}\left(\mathbb{R}^d\right)$, for all $\alpha \in (0,1)$, such that $c \in C^{\infty}\left(\mathbb{R}^d \setminus \{0\}\right)$, $c(x) \geq 1$ for all $x \in \mathbb{R}^d$ and the following holds:*

Let $\Omega \subset \mathbb{R}^d$ be a smooth domain with $0 \in \Omega$ and let $\omega \subset \Omega$ be a neighborhood of 0. For every $T > 0$, there exists a sequence $(u_k)_{k\in\mathbb{N}}$ of solutions of

$$\begin{cases} \partial_t^2 u_k - \Delta_c u_k = 0 \text{ in } (0,T) \times \Omega, \\ u_k = 0 \text{ in } (0,T) \times \partial\Omega, \end{cases} \tag{28}$$

such that

$$\lim_{k\to\infty} E_c\left(\mathbf{1}_{\Omega} u_k(t,\cdot)\right) > 0 \text{ for all } t \in (0,T)$$

and

$$\lim_{k\to\infty} \int_0^T \int_{\Omega\setminus\omega} \left|\partial_t u_k(t,x)\right|^2 + c(x)\left|\nabla_x u_k(t,x)\right|^2 dxdt = 0.$$

According to this result, it is possible to construct solutions of (28) as concentrated near the origin as wanted. Notice that at the point $x = 0$ the solutions of (1) are not well-defined due to the low regularity of the coefficient; the Theorem suggests that rays starting at $x = 0$ are stationary, that is, propagate with zero velocity. This, as we have shown before, cannot be the case when the coefficients are smooth enough (namely, of class C^2), since solutions can only concentrate near a point propagating along a ray.

In particular, this implies that given $T > 0$ and a neighborhood of the origin $\omega \subset \Omega$ it is impossible to find a constant $C(T,\omega) > 0$ such that the following **observability inequality** holds

$$E_c\left(\mathbf{1}_{\Omega} u(0,\cdot)\right) \leq C \int_0^T \int_{\Omega\setminus\omega} \left|\partial_t u\right|^2 + c\left|\nabla_x u\right|^2 dxdt \tag{29}$$

for all finite energy solution u of (28). As an immediate consequence of this result, one can show that the controllability property of system (28) may not be achieved by means of controls with support in $\Omega \setminus \omega$.

We shall construct explicitly a function c having the following property: for every $j \in \mathbb{N}$ there exists a ray for $\Box_c := \partial_t^2 - \Delta_c$ contained in the corona $2^{-j} < |x| < 2^{-j+1}$, see Figure 5.

If u_k is gaussian beam constructed along this ray then, as we know, the energy of u_k outside the ray tends to zero; taking j large enough, this contradicts any inequality like (29).

A similar result has been recently obtained by C. Castro and E. Zuazua [5]. They construct, for dimension $d = 1$, a $C^{0,\alpha}$ function ρ, $\alpha \in (0,1)$, smooth outside a single point, such that the observability inequality (29) fails. The result for $d > 1$ then follows by considering the tensor product of this function which, necessarily, is singular in a hypersurface. Of course, our construction is not valid for $d = 1$, but provides a function c which is singular only at the origin.

The fact that allows us to carry out the proof is the following:

Lemma 24 *Suppose $\kappa(x) = |x|^2$ for $\varepsilon < |x| < 2\varepsilon$. Then the operator $\Box_\kappa$ has a ray contained in the corona $\varepsilon < |x| < 2\varepsilon$.*

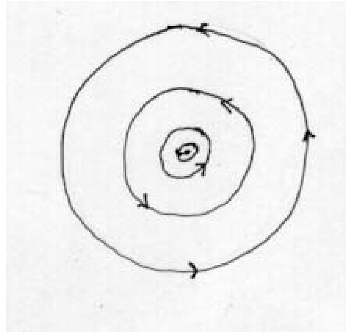

Figure 5: The rays associated to c

Proof. The ray equations are

$$\begin{cases} \dot{x} = 2\kappa(x)\,\xi, \\ \dot{\xi} = -|\xi|^2 \nabla_x \kappa(x), \end{cases}$$

with $|\xi|^2 \kappa(x) \equiv 1/4$. From this one obtains

$$\frac{d}{dt}(x \cdot \xi) = |\xi|^2 (2\kappa(x) - x \cdot \nabla_x \kappa(x))$$

and

$$\frac{d}{dt}|x|^2 = 4\kappa(x)(x \cdot \xi).$$

Since $|x|^2$ solves the equation $2\kappa(x) - x \cdot \nabla_x \kappa(x) = 0$ the result follows choosing $|x(0)| \in (\varepsilon, 2\varepsilon)$ and $x(0) \cdot \xi(0) = 0$. ∎

Now we proceed to construct the function c. Let $I_j := [2^{-j}, 2^{-j} + \delta_j]$ with $\delta_j \in (0, 2^{-j-1})$ such that $\delta_j^\beta 2^{2j} \to 0$ as $j \to \infty$ for every $\beta \in (0,1)$. We define

$$\kappa(r) = \left|2^j r\right|^2 \text{ if } r \in I_j$$

and extend κ to a $C^\infty((0,\infty))$ function that satisfies $1 \leq \kappa(r) \leq 2$ for all $r \in (0,\infty)$ and

$$\sup_{r \in I_j} |\kappa'(r)| = \sup_{r \in [2^{-j}, 2^{-j+1}]} |\kappa'(r)|;$$

this last condition is required in order to ensure that the extension does not produce "extra" oscillations, see figure 6.

Now

$$\begin{aligned} |\kappa|_{C^{0,\alpha}(I_j)} &= \max_{r,s \in I_j} \frac{2^{2j}|r^2 - s^2|}{|r-s|^\alpha} = 2^{2j} \max_{r,s \in I_j} |r-s|^{1-\alpha} |r+s| \\ &= 2^{2j} \delta_j^{1-\alpha} 2\left(2^{-j} + \delta_j\right), \end{aligned}$$

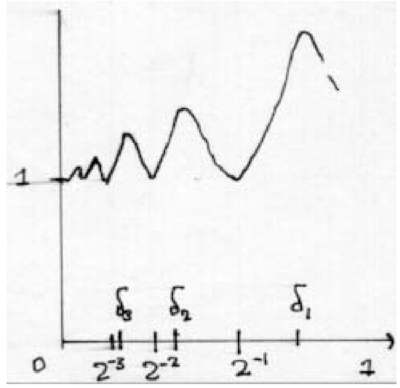

Figure 6: The function κ

and this is enough in order to prove that $\kappa \in C^{0,\alpha}([0,\infty))$. Define $c(x) := \kappa(|x|)$. Then, this highly oscillating coefficient c produces beams which are as localized as one wants near the origin. This clearly contradicts any observability inequality from a subset of Ω that does not contain the origin.

The reader should recall that the observability inequality (29) is true when the coefficients are smooth, under a suitable geometric control condition. For the one-dimensional wave equation, $c \in BV$ suffices, see [6]; in the general case, $c \in C^{1,1}$ gives the inequality, as shown by N. Burq [4]. Notice that this is the weakest regularity assumption for which rays are well defined. The problem of giving sharp conditions for the inequality to hold for coefficients $c \in C^{1,\alpha}$, $\alpha \in (0,1)$ still remains open.

6 Observability of waves from a hypersurface

Consider the wave equation

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 \text{ in } (0,T) \times \Omega, \\ u = 0 \text{ on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = u^0, \ \partial_t u(0,\cdot) = u^1, \end{cases} \tag{30}$$

where Ω is a smooth domain of $\mathbb{R}^d$ and Δ_x is the euclidean laplacian. In this section we shall be concerned with the following observability problem:

Given a smooth hypersurface $M \subset \Omega$ and a time $T > 0$, does there exist a constant $C = C(T,M) > 0$ such that

$$E(\mathbf{1}_\Omega u(0,\cdot)) \leq C \int_0^T \int_M |\partial_t u|^2 \, dSdt \tag{31}$$

holds for every finite energy solution u of (30) with $\int_0^T \int_M |\partial_t u|^2 \, dSdt < \infty$?

This question was addressed in [7] in the context of the study of the asymptotic behavior of the solutions of the following system:

$$\begin{cases} \partial_t^2 u - \Delta_x u + \partial_t u \delta_M = 0 \text{ in } (0,T) \times \Omega, \\ u = 0 \text{ on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = u^0, \ \partial_t u(0,\cdot) = u^1, \end{cases} \tag{32}$$

where δ_M is the Dirac mass supported on M. In [7] it is shown that whenever M is not a nodal set for the Dirichlet laplacian in Ω (i.e. no eingenfunction of the laplacian vanishes on M), the energy of any solution of (32) tends to zero when $t \to \infty$. The observability inequality (31) is then necessary in order to guarantee that the decay rate of solutions of (32) is uniform with respect to the initial data.

Here we shall use the techniques developed so far to give necessary conditions on M for (31) to hold. The following result is stated in [7] in the two-dimensional case, but it holds in any space dimension:

Proposition 25 *Suppose Ω is strictly convex and the distance between M and $\partial\Omega$ is strictly positive. Then (31) fails for every $T > 0$.*

Proof. Let $\varepsilon > 0$ be smaller than the distance between M and $\partial\Omega$. Since Ω is strictly convex there exists a generalized ray γ contained in an ε-neighborhood of $\partial\Omega$. Take $T > 0$ and let $(u_k)_{k\in\mathbb{N}}$ be the gaussian beam along γ with $\sup_{t\in(0,T)} \|\Box u_k(t,\cdot)\|_{H^1(\mathbb{R}^d_x)} \to 0$ (see Remark 7). Clearly

$$\lim_{k\to\infty} \int_0^T \int_M |\partial_t u_k|^2 \, dS dt = 0,$$

since if $\omega \subset \Omega$ is an open set that contains M and that does not intersect γ, one has $\int_0^T \int_M |\partial_t u_k|^2 \, dS dt \leq \int_0^T \|\partial_t u_k(t,\cdot)\|^2_{H^1(\omega)} dt$ and this last quantity tends to zero when $k \to \infty$. The conclusion is still valid for the exact solutions by Remark 13. On the other hand, for $t \in (0,T)$, $\lim_{k\to\infty} E(\mathbf{1}_\Omega u_k(t,\cdot)) > 0$; this contradicts the existence of a constant $C > 0$, independent of u, for which (31) holds. ■

Next we prove that if Ω possesses a diameter, (i.e. there exist points $p, q \in \partial\Omega$ such that the segment $\overline{pq}$ is contained in Ω and is orthogonal to $\partial\Omega$ at p and q) and the hypersurface M intersects this diameter orthogonally, then the observability inequality (31) is false.

Theorem 26 *Suppose that Ω has a diameter l. Let M be a smooth hypersurface such that $M \cap l = \{m_1, .., m_N\}$ and M is orthogonal to l at m_i, $i = 1, .., N$. Moreover, suppose that* $\operatorname{dist}(m_i, \partial\Omega) / |l|$ *is rational for $i = 1, ..., N-1$. Then (31) fails for every $T > 0$.*

Proof. We proceed in several steps:

Step 1: We first show how to produce gaussian beams $u_k^\pm$ such that

$$\lim_{k\to\infty} \int_0^T \int_M \left|\partial_t u_k^+ + \partial_t u_k^-\right|^2 dS dt = 0 \tag{33}$$

for some $T > 0$. This construction does not depend on the geometric properties of M. Take a point a point $x_0 \in M$ and a ray $x^-(t)$ such that $x^-(t_0) = x_0$ and $x^-(t) \notin M$ for $t \in (0, t_0)$. It is always possible to find a ray $x^+(t)$ with $x^+(t_0) = x^-(t_0)$, $x^+(t) \notin M$ for $t \in (0, t_0)$ (see Figure 7) and to construct gaussian beams $u_k^\pm$ along $x^\pm(t)$ in such a way that the superposition

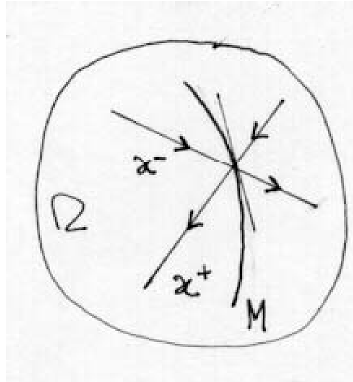

Figure 7: The rays $x^{\pm}$ in the local case

$u_k^+ + u_k^-$ satisfies (33); just argue as in Proposition 8. Then, T is characterized by the condition that both $x(t)^{\pm}$ do not intersect again M for $t \in (t_0, T)$. Notice that, as in Proposition 25, we must require that $\sup_{t\in(0,T)} \left\| \Box u_k^{\pm}(t,\cdot) \right\|_{H^1\left(\mathbb{R}_x^d\right)} \to 0$ in order to ensure that the exact solutions satisfy (33).

Step 2: When Ω and M satisfy the hypotheses of the Theorem, the above argument can be made global in time. First suppose that $M \cap l = \{m\}$ and m is the midpoint of l; the geometric situation is that of Figure 8. Choose as $x^-(t)$ a generalized ray lying in l (this is possible since l intersects M orthogonally); then the ray $x^+(t)$ constructed as in Step 1 also lies in l. Since rays $x^{\pm}(t)$ always intersect M at point m and the amplitudes of $u^{\pm}$ still cancel at m after every bounce at the boundary we can apply the construction above to this case for every $T > 0$, see Figure 9.

Step 3: Now suppose that $M \cap l = \{m_1, .., m_N\}$ and the distance to the border $\partial\Omega$ of every m_i is rational with respect to $|l|$: $\operatorname{dist}(m_i, \partial\Omega) / |l| = p_i/q$. Then a similar construction can be achieved by superposing $2q$ beams, as in shown in Figure 10. Fix a point $r_0 \in \partial\Omega \cap l$ and let r_j, $j = 1, ...q$ be the points located at distance $j|l|/q$ from r_0. Using the construction of Step 2 on can produce beams along rays contained in l, $u_k^{0,-}$, $u_k^{q,+}$ and $u_k^{j,\pm}$, $j = 1, .., q-1$ such that (setting $u_k^{0,+} = u_k^{q,-} = 0$):

- $u_k^{j,\pm}(0,\cdot)$ is concentrated near r_j,
- $u_k^{j,-}$ propagates towards r_q,
- $u_k^{j,+}$ propagates towards r_0,
- $\int_0^T \int_M \left| \sum_{j=0}^q \partial u_k^{j,\pm} \right|^2 dSdt \to 0$, if M is a smooth hypersurface that orthogonally intersects l at the points r_j, $j = 1, ..., q-1$.

Step 4: Finally, for the case $\operatorname{dist}(m_N, \partial\Omega) / |l| =: s_0$ is irrational, we do the following: take a sequence $(\mu_n)_{n\in\mathbb{N}}$ of points in l such that $\mu_n \to m$ as $n \to \infty$ and $\operatorname{dist}(\mu_n, \partial\Omega) / |l| = p_n/q_n$. This points can be chosen to satisfy

$$\left| \frac{p_n}{q_n} - s_0 \right| \le \frac{1}{q_n^2}.$$

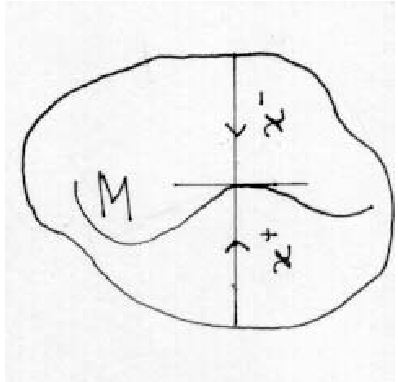

Figure 8: The rays $x^{\pm}$ in the global case

Suppose $\operatorname{dist}(m_i, \partial\Omega) / |l| = p_i/q$ for $i = 1, .., N-1$. The construction of the previous paragraph produces a sequence $(u_k^n)_{n,k\in\mathbb{N}}$ of solutions of the wave equation such that $E\left(\mathbf{1}_\Omega u_k^n (0,\cdot)\right)$ is bounded and

$$\int_0^T \int_{M_n} |\partial_t u_k^n|^2 \, dSdt \to 0 \text{ as } k \to \infty, \tag{34}$$

for any $T > 0$, M_n being a hypersurface which is identical to M outside a small neighborhood of m_N and such that it intersects orthogonally l at the point μ_n. Eventually performing a diagonal extraction argument, one can assume that $\int_0^T \int_{M_n} \left|\partial_t u_{q_n}^n\right|^2 dSdt \to 0$ as $n \to \infty$. Recall that u_k^n is the superposition of $2qq_n$ beams; in order to have a bounded sequence, we consider $v_n := q_n^{-1/2} u_{q_n}^n$.

To prove the result, it suffices to estimate $\partial_t v_n$ near m_N, so we place ourselves in a system of geodesic normal coordinates $\Phi : U \times (s_0 - \varepsilon, s_0 + \varepsilon) \to \mathbb{R}^d$ such that $\Phi(U, s_0)$ is a neighborhood of m_N in M and $\Phi(U, p_n/q_n)$ is a neighborhood of μ_n in M_n (see Section 2); we write this coordinates as (y, s). Then we have

$$\begin{aligned}
&\left|\int_0^T \int_{\Phi(U,p_n/q_n)} |\partial_t v_n|^2 \, dSdt - \int_0^T \int_{\Phi(U,s_0)} |\partial_t v_n|^2 \, dSdt\right| \\
= &\left|\int_0^T \int_U |\partial_t v_n(t,y,p_n/q_n)|^2 \, dSdt - \int_0^T \int_U |\partial_t v_n(t,y,s_0)|^2 \, dSdt\right| \leq \\
\leq &\left|\frac{p_n}{q_n} - s_0\right| \|\partial_s \partial_t v_n(\cdot, \theta_n)\|_{L^2((0,T)\times U)} \leq q_n^{-2} \|\partial_s \partial_t v_n(\cdot, \theta_n)\|_{L^2((0,T)\times U)},
\end{aligned}$$

where θ_n lies between s_0 and p_n/q_n.

But now, for every $\varepsilon > 0$,

$$\|\partial_s \partial_t v_n(\cdot, \theta_n)\|_{L^2((0,T)\times U)} \leq \|v_n\|_{H^{5/2+\varepsilon}((0,T)\times\Omega)}$$

and it is easy to check that

$$q_n^{-2} \|v_n\|_{H^{5/2+\varepsilon}((0,T)\times\Omega)} \leq C q_n^{-1/2+\varepsilon}.$$

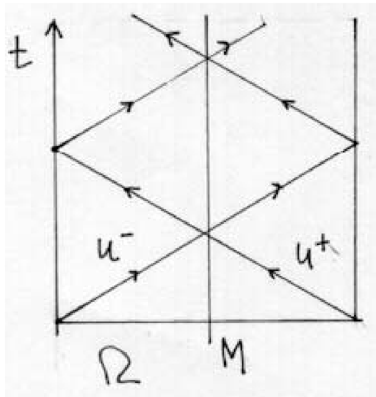

Figure 9: $\partial_t u_k^+ + \partial_t u_k^-$ cancel on M for all t

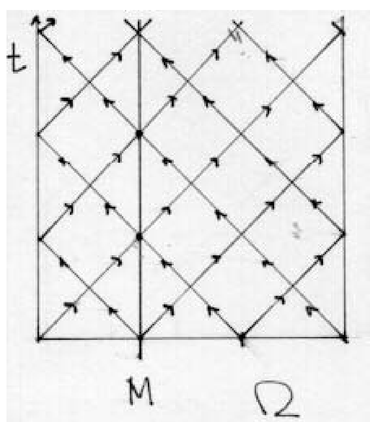

Figure 10: Six beams are needed when $dist\,(m, \partial\Omega)\,/\,|l| = 1/3$

Taking ε sufficiently small, the Theorem is proved. ■

According to this result, there are situations in which all rays intersect the hypersurface M and the observability inequality (31) still fails. The obtention of sharp necessary (or sufficient) conditions for (31) to hold is an open problem. In particular, as far as we know, there are no examples in the literature of domains Ω and hypersurfaces M for which (31) holds.

It is interesting to compare this result with its one-dimensional version (see [7]): if Ω is an interval and M is a single point, the observability inequality (31) never holds. However, when the ratio between M and the length of Ω is irrational, there are instances in which an observability inequality holds, provided the energy E is replaced by a weaker one.

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