Open subgroups, compact subgroups and Binz–Butzmann reflexivity

M. Montserrat Bruguera [a,1], Elena Martín-Peinador [b,*]

[a] *Departamento de Matemática Aplicada I, Universidad Politécnica de Cataluña, Avda. Dr. Gregorio Marañón, 44-50, 08028 Barcelona, Spain*

[b] *Departamento de Geometría y Topología, Universidad Complutense de Madrid, Avda. Complutense s/n, 28040 Madrid, Spain*

Abstract

A number of attempts to extend Pontryagin duality theory to categories of groups larger than that of locally compact abelian groups have been made using different approaches. The extension to the category of topological abelian groups created the concept of reflexive group. In this paper we deal with the extension of Pontryagin duality to the category of convergence abelian groups. Reflexivity in this category was defined and studied by E. Binz and H. Butzmann. A convergence group is reflexive (subsequently called BB-reflexive by us in our work) if the canonical embedding into the bidual is a convergence isomorphism.

Topological abelian groups are, in an obvious way, convergence groups; therefore it is natural to compare reflexivity and BB-reflexivity for them. Chasco and Martín-Peinador (1994) show that these two notions are independent. However some properties of reflexive groups also hold for BB-reflexive groups, and the purpose of this paper is to show two of them. Namely, we prove that if an abelian topological group G contains an open subgroup A, then G is BB-reflexive if and only if A is BB-reflexive. Next, if G has sufficiently many continuous characters and K is a compact subgroup of G, then G is BB-reflexive if and only if G/K is BB-reflexive.

Keywords: Reflexive group; Continuous convergence structure; Character; Dual group

AMS classification: 43A40; 20K27; 22A05

Notations, definitions and remarks

Let $\mathbb{T}$ be the unidimensional torus group, i.e., $\mathbb{R}/\mathbb{Z}$ with the natural group structure and the quotient topology. We shall denote by ΓG the set of all continuous homomorphisms

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* Corresponding author. E-mail: peinador@sungt1.mat.ucm.es. Supported by D.G.I.C.Y.T., project # BE91-031.

[1] E-mail: montse@ma1.upc.es.

(i.e., continuous characters) from an abelian topological group G into $\mathbb{T}$. If addition in ΓG is defined pointwise, then ΓG endowed with the compact open topology is a topological abelian group, which will be called $G^\wedge$.

The group G is called *reflexive* if the natural embedding α_G from G into the bidual $G^{\wedge\wedge} := (G^\wedge)^\wedge$ is a topological isomorphism. The classical Pontryagin duality theorem states that every locally compact topological abelian group (LCA) is reflexive.

Examples of reflexive groups which are not locally compact are known from the late 1940s. In [9] it is proved that arbitrary products of LCA groups are reflexive, whilst they may not be locally compact, like $\mathbb{R}^\omega$ or $\mathbb{R}^c$. In [15] it is proved that any infinite dimensional Banach space considered in its additive structure is a reflexive group.

Reflexive groups do not share some of the nice properties of locally compact groups. For instance, closed subgroups and Hausdorff quotients of a reflexive group are not necessarily reflexive [1]. Some additional conditions must be required for these facts to hold. The following results are proved in [3]: (1) If A is an open subgroup of a topological group G, then G is reflexive if and only if A is reflexive. (2) If K is a compact subgroup of a group G with sufficiently many continuous characters, then G is reflexive iff G/K is reflexive. Our aim here is to prove that analogous statements hold for another sort of reflexivity, which we define below.

For the sake of completeness we include here the definition of a convergence structure. The reader should consult [5] for further details.

Let X be a set and suppose that to each x in X is associated a collection $\Xi(x)$ of filters on X satisfying:

(i) the ultrafilter $\{A \subset X\colon\ x \in A\}$ is in $\Xi(x)$,

(ii) if $\mathcal{F} \in \Xi(x)$ and $\mathcal{G} \in \Xi(x)$, then the filter $\mathcal{F} \cap \mathcal{G} = \{F \cup G\colon\ F \in \mathcal{F}, G \in \mathcal{G}\}$ also belongs to $\Xi(x)$,

(iii) if $\mathcal{F} \in \Xi(x)$ and $\mathcal{G} \supset \mathcal{F}$ then $\mathcal{G} \in \Xi(x)$.

The totality Ξ of filters $\Xi(x)$ for x in X is called a *convergence structure* for X, the pair (X, Ξ) a *convergence space* and the filters $\mathcal{F}$ in $\Xi(x)$ will be called *convergent* to x. We write $\mathcal{F} \to x$ instead of $\mathcal{F} \in \Xi(x)$. A mapping $f\colon X \to Y$ between two convergence spaces X and Y is *continuous* if $f(\mathcal{F}) \to f(x)$ in Y whenever $\mathcal{F} \to x$ in X.

A *convergence group* (G, Ξ), or briefly G, is a group for which the convergence structure Ξ is compatible with addition. If G is a convergence group, we also call ΓG the set of all continuous homomorphisms from G into $\mathbb{T}$ and the *continuous convergence structure* Λ, in ΓG, is defined in the following way: a filter $\mathcal{F}$ in ΓG converges in Λ to an element $\xi \in \Gamma G$ if for every $x \in G$ and every filter $\mathcal{H}$ in G, convergent to x, $w(\mathcal{F} \times \mathcal{H})$ converges to $\xi(x)$ in $\mathbb{T}$. ($\mathcal{F} \times \mathcal{H}$ denotes the filter generated by the products $F \times H$, $F \in \mathcal{F}$ and $H \in \mathcal{H}$, and $w(F \times H) := \{f(x)\colon\ f \in F \text{ and } x \in H\}$.)

It can be said that Λ is the coarsest convergence structure in ΓG for which w is continuous. Obviously, if Ξ is a convergence structure in ΓG, such that the evaluation mapping $\omega\colon \Gamma G \times G \to \mathbb{T}$ is continuous (the first space has the natural product structure), then every Ξ-convergent filter is Λ-convergent, i.e., $\Lambda \leqslant \Xi$.

As is well known every filter defines canonically a net, and conversely, the sections of a net constitute a filter. In the realm of topological spaces, convergence of a filter (net) is equivalent to convergence of its associated net (filter), [10]. Thus, depending on personal preferences, one can choose nets or filters indistinctly. We state the definition of the continuous convergence structure in terms of nets, as derived from that of filters, because it will be more convenient for some of our proofs.

A net $(f_\alpha)_{\alpha\in D}$ in ΓG is Λ-convergent to $f \in \Gamma G$ if for every net $(x_\beta)_{\beta\in E}$ in G converging to $x \in G$, the net $(f_\alpha(x_\beta))_{(\alpha,\beta)\in D\times E}$ converges to $f(x)$ in $\mathbb{T}$.

In the literature the above definition often appears with the directed sets D and E identical, and requiring only that $(f_\alpha(x_\alpha))_{\alpha\in D}$ be convergent. Certainly this should be enough to assure the continuity of the evaluation mapping if ΓG would have been assigned previously a topology. However, as an axiomatic definition of convergence it is not satisfactory, because a subnet of a convergent net may not converge. Furthermore it is not equivalent to the above definition, as can be seen in [16].

E. Binz and H. Butzmann have succeeded to extend Pontryagin duality theory to the category of convergence abelian groups and continuous (in the sense of convergence) homomorphisms, CONABGRP. They first define the "*convergence dual*" $\Gamma_c G$ of a group $G \in$ CONABGRP, as the set of all continuous characters endowed with the "continuous convergence structure". If G is an LCA group, the continuous convergence structure in ΓG is precisely the convergence given by the compact open topology [5]; thus, the "convergence dual" and the ordinary dual are identical. They call G *reflexive* if the natural embedding $\kappa_G : G \to \Gamma_c\Gamma_c G$ is an isomorphism in the category CONABGRP. They have studied many features of this concept of reflexivity, which we have called *BB-reflexivity*. To mention one, a topological vector space, regarded as an abelian group, is BB-reflexive if and only if it is locally convex and complete [6].

The continuous convergence structure is a basic tool to develop an adequate theory of differential calculus, in nonnormable spaces [11]. It has also been used in the theory of distributions [4]. The fact that it makes evaluation continuous, and that in general there are no topologies meeting this condition makes its uses advantageous. By means of it we proved that every reflexive, admissible group must be locally compact [12]. The fact that convergence structures appear in physical proccesses more likely than topologies, calls for a development of convergence spaces.

Although every topological abelian group can be viewed as a convergence group, not every convergence (group) structure can be derived from a (group) topology. For instance, in "Measure Theory" it is well known that "convergence almost everywhere" in the set of all real measurable functions defined on $[0, 1]$ does not derive from any topology on the mentioned set.

In other words, the natural forgetful functor from TOPABGRP into CONABGRP is not surjective. We have proved the results mentioned in the abstract for the image of this functor. Our proofs are not suitable for the whole category CONABGRP, because Theorem 1.5 relies heavily on the topological structure. We think however that this theorem is interesting in itself.

We divide the paper into two sections. In the first one we prove some properties of convergence groups needed later. The second section contains the main theorems.

We will only deal with abelian Hausdorff groups; therefore when we speak of convergence groups or topological groups, we mean Hausdorff abelian groups.

1. Some properties of open subgroups and compact subgroups of a convergence group

We recall that a subgroup A of a topological group G is *open* iff $A \in \mathcal{F}$, for every filter $\mathcal{F}$ in G, convergent to some point of A. Equivalently, A is open iff every net $(x_d)_{d \in D}$ of G, convergent to an element x of A, is eventually in A. These facts can be used to define the open subgroups of a convergence group; namely, if (G, Ξ) is a convergence group, a subgroup A of G is said to be open if it verifies the mentioned properties of convergence. Propositions 1.1, 1.2 and 1.3 show that to some extent they behave like open subgroups of a topological group.

Proposition 1.1. *If G is a convergence group, then every open subgroup of G must be closed.*

Proof. Let A be an open subgroup of G. We shall prove that if $x \notin A$ then $x \notin \bar{A}$. Take a filter $\mathcal{F}$ convergent to x; then $\mathcal{F} - x$ converges to 0 and $A \in \mathcal{F} - x$ since A is an open subgroup. So $A + x \in \mathcal{F}$ and $(x + A) \cap A = \emptyset$ for otherwise $x + a = b$ with $a, b \in A$ and then $x = b - a \in A$. Therefore $\mathcal{F}$ does not have a trace on A, which implies $x \notin \bar{A}$. □

Let M be a closed subgroup of a convergence group (G, Ξ) and let $p: G \to G/M$ be the canonical projection. The *quotient structure* Ξ_q in G/M is defined as the finest convergence structure for which the canonical projection p is continuous. Or to say it in a different way, a filter $\mathcal{F}$ in G/M converges in Ξ_q to $[x] \in G/M$ if and only if it contains some filter of the form $p(\phi_1) \cap \cdots \cap p(\phi_n)$, where ϕ_i is a filter in G, Ξ-convergent to some preimage $z_i \in p^{-1}[x]$.

Proposition 1.2. *The quotient of a convergence group by an open subgroup is a discrete group.*

Proof. Let A be an open subgroup of a convergence group G and $\mathcal{F}$ a filter in G/A convergent to $[A]$. Then $\mathcal{F}$ contains some filter of the form $\bigcap_{i=1}^{n} p(\phi_i)$, where ϕ_i is a filter in G, convergent to $a_i \in A$. Since A is open, $A \in \phi_i$ for every $i \in \{1, \ldots, n\}$. Thus $[A] \in \bigcap_{i=1}^{n} p(\phi_i) \in \mathcal{F}$, therefore $\mathcal{F}$ is the filter generated by $\{[A]\}$. Observe that G/A, being discrete, is a topological group. □

Proposition 1.3. *Let A be an open subgroup of a convergence group G. A homomorphism φ from G into a topological group G' is continuous if and only if the restriction to A, $\varphi|_A$, is continuous.*

Proof. Suppose $\varphi|_A$ is continuous, and consider a filter $\mathcal{F}$ in G convergent to 0. Then $A \in \mathcal{F}$ because A is an open subgroup, and so $\mathcal{F}$ has a trace $\mathcal{F}_A$ on A ($\mathcal{F}_A$ is a filter).

Now $\varphi|_A(\mathcal{F}_A)$ is convergent to 0 in G'. Thus for every $\mathcal{V}$ neighborhood of zero in G', there exists F in $\mathcal{F}$ such that $\mathcal{V} \supset \varphi(F \cap A)$. Since $F \cap A \in \mathcal{F}$, the filter $\varphi(\mathcal{F})$ converges to 0. □

A convergence group K is said to be *compact* if it is Hausdorff and every ultrafilter in K is convergent [5].

A subgroup M of a convergence group (G, Ξ) is said to be *dually closed* if, for every element x of $G - M$, there is a continuous character φ in $\Gamma_c G$ such that $\varphi(M) = 0$ and $\varphi(x) \neq 0$. It is said to be *dually embedded* if every continuous character defined on M can be extended to a continuous character on G. The *annihilator of M* is defined as the subgroup $M^o := \{\varphi \in \Gamma_c G,$ such that $\varphi(M) = 0\}$ and we will denote by $M^{oo} := \{\phi \in \Gamma_c \Gamma_c G,$ such that $\phi(M^o) = 0\}$. It is well known that an open subgroup of a topological group G is dually closed and dually embedded [14, Lemma 3.3]. We prove now that with a further assumption on the group G, the same holds for a compact subgroup K.

Proposition 1.4. *Let K be a compact subgroup of a topological Hausdorff abelian group G with sufficiently many continuous characters. Then K is dually closed and dually embedded.*

Proof. First we prove that K is dually embedded.

(1) Since G has sufficiently many continuous characters, the subgroup of $K^\wedge$ defined by $L := \{\varphi \in K^\wedge \mid \varphi$ extends to $G\}$ separates points of K. Furthermore, every nontrivial Hausdorff quotient of $K^\wedge$ has a nontrivial character. By [13, Proposition 31] L is dense in $K^\wedge$ and as $K^\wedge$ is discrete, $L = K^\wedge$.

(2) The fact that K is dually closed will follow from the equality $\alpha_G(K) = K^{oo}$, because α_G is injective. So, let $i^\wedge : G^\wedge \to K^\wedge$ be the restriction mapping. It is surjective by (1), and its kernel is K^o. Therefore it induces a continuous isomorphism $\psi : G^\wedge / K^o \to K^\wedge$, which is open since $K^\wedge$ is discrete. Its dual mapping $\psi^\wedge : K^{\wedge\wedge} \to (G^\wedge/K^o)^\wedge$ is a topological isomorphism. If $p : G^\wedge \to G^\wedge/K^o$ is the canonical projection, $p^\wedge : (G^\wedge/K^o)^\wedge \to G^{\wedge\wedge}$ is injective, and its image is K^{oo}. Since $(G^\wedge/K^o)^\wedge$ is compact we have that $p^\wedge : (G^\wedge/K^o)^\wedge \to K^{oo}$ is a topological isomorphism. Thus, the composite $\varphi := p^\wedge \psi^\wedge \alpha_K$ is a topological isomorphism from K onto K^{oo}.

Let $h \in K^{oo}$, and let $h' \in K$ be such that $h = p^\wedge \psi^\wedge \alpha_K(h')$. Then $h = \alpha_G(h') \in \alpha_G(K)$. In fact, if $\chi \in G^\wedge$ we have $\alpha_G(h')(\chi) = \chi(h')$ and $(p^\wedge \psi^\wedge \alpha_K(h'))(\chi) = \psi^\wedge \alpha_K(h') p(\chi) = \alpha_K(h')(\psi p \chi) = \alpha_K(h')(\chi|_K) = \chi|_K(h')$. This proves that $K^{oo} \subset \alpha_G(K)$. The converse is straightforward. □

Remark 1. The last proposition holds also when G is a convergence group. In order to prove it, the following fact should be taken into account:

If K is a compact subgroup of G, then $\Gamma_c K$ is topological; i.e., the continuous convergence structure on ΓK is derived from a topology, which necessarily is the compact open topology [5].

Remark 2. The assumption that G may have sufficiently many continuous characters is essential in Proposition 1.4, as the following example shows:

Let E be an infinite dimensional separable Banach space. There exists a free, discrete subgroup K of E such that $(E/K)^\wedge = \{0\}$ [1]. Let a be a generator of K; the linear span of a is a closed subgroup of E, say $N := \mathbb{R}a$. If $p: E \to E/K$ is the canonical projection then $p(N)$ can be identified with $p\left([0,a]\right)$ which is compact and isomorphic to S^1, for $p(0) = p(a)$. It is not dually embedded since $(E/K)^\wedge = \{0\}$.

Remark 3. It is easy to see that if A is an open subgroup of a convergence group then A is also dually closed and dually embedded. The proofs follow respectively from the facts:

(1) G/A is discrete, therefore reflexive and G/A has sufficiently many continuous characters, and

(2) every character φ in $\Gamma_c A$ can be extended to an algebraic homomorphism $\widetilde{\varphi}$ of G into $\mathbb{T}$, which by Proposition 1.3 is continuous.

We establish now the fact that every convergent net of characters defined on an open subgroup can be lifted to a convergent net of characters of the group.

Theorem 1.5. *Let A be an open subgroup of a Hausdorff abelian topological group G. Let $(\xi_\alpha)_{\alpha\in D}$ be a convergent net in $A^\wedge$, say $\xi_\alpha \to \xi$. Then there exist $\widetilde{\xi}_\alpha$, $\widetilde{\xi}$ in $G^\wedge$, extensions of ξ_α and ξ respectively, such that $\widetilde{\xi}_\alpha \to \widetilde{\xi}$.*

Proof. We will prove that the standard procedure to extend continuous characters yields a convergent net, when applied to a convergent net of continuous characters. For the sake of completeness we reproduce here how a character φ defined on A can be extended to the group $Gp\{A, x\}$, generated by A and $x \in G - A$.

Two cases arise:

(a) $nx \notin A$ for any $n \in \mathbb{N}$. Then define $\widetilde{\varphi}(nx+a) = \varphi(a)$ for every $a \in A$ and $n \in \mathbb{N}$.

(b) $mx \in A$ and m is the least positive integer with that condition. Then define $\widetilde{\varphi}(x) = \varphi(mx)/m$ and extend it to an algebraic homomorphism.

Since $\widetilde{\varphi}|_A = \varphi$ is continuous, and since A is an open subgroup, we have that $\widetilde{\varphi}$ is continuous.

Take now $(\xi_\alpha)_{\alpha\in D} \subset A^\wedge$ such that $\xi_\alpha \to \xi$. We call $\xi_\alpha^{(1)}$, $\xi^{(1)}$ the extensions of ξ_α, ξ to $A_1 := Gp\{A, x\}$ defined as above. We shall prove that $\xi_\alpha^{(1)} \to \xi^{(1)}$.

To this end denote by (S, V) a neighborhood of zero in $A_1^\wedge$, where $S \subset A_1$ is compact, and V a zero-neigborhood in $\mathbb{T}$. We must determine α_0 such that $\xi_\alpha^{(1)} - \xi^{(1)} \in (S, V)$ for every $\alpha \geqslant \alpha_0$.

Let α_1 be such that $\xi_\alpha - \xi \in (S - \{x\}, V)$, $\forall \alpha \geqslant \alpha_1$. This is possible since $S - \{x\} = S \cap A$ is compact.

If x is as in (a), let $\alpha_0 = \alpha_1$.

If x is as in (b), we consider the following possibilities:

(i) $S \subset A$; then trivially $\xi_\alpha^{(1)}(s) = \xi_\alpha(s)$ and $\xi^{(1)}(s) = \xi(s)$, $\forall s \in S$, and the previous α_1 should do.

(ii) $S - A$ has only one element, say $y = lx + a$, for some integer l and some $a \in A$. Let $W \in E_{\mathbb{T}}(0)$ be such that $W + W \subset V$. Since

$$\xi_\alpha^{(1)}(lx) = \frac{l}{m}\xi_\alpha(mx),$$

it converges to

$$\xi^{(1)}(lx) = \frac{l}{m}\xi(mx),$$

so we can determine α_0 in D such that $(\xi_\alpha^{(1)} - \xi^{(1)})(s) \in W + W \subset V$, for every s in S.

A similar argument holds if $S - A$ is finite.

(iii) $H := S - A$ is infinite. Since $H \subset S$ is compact, the open covering $\{y + A, y \in H\}$ must have a finite subcovering, say

$$\bigcup_{i=1}^{n}(y_i + A) \supset H,$$

with $y_i \in H$. Call $S_i' := (y_i + A) \cap H$, $S_i := S_i' - y_i$ and $F := \bigcup_{i=1}^n S_i \subset A$. F is compact and also $H \subset \bigcup_{i=1}^n (y_i + F)$. For $i \in \{1, 2, \dots, n\}$ determine α_i, proceeding as in (ii), such that $(\xi_\alpha^{(1)} - \xi^{(1)})(y_i) \in W$, for every $\alpha \geqslant \alpha_i$. Now if $\alpha_0 \geqslant \alpha_1, \dots, \alpha_n$ we have: any z in H is in some $y_i + F$, say $z = y_i + f$ thus $(\xi_\alpha^{(1)} - \xi)(y_i + f) \in W + W \subset V$, for every $\alpha \geqslant \alpha_0$. This proves that $\xi_\alpha^{(1)} \to \xi^{(1)}$ in $A_1 = Gp\{A, x\}$. Take now A_1 in the place of A, and repeat this process with a new $z \in G - A_1$ and do it on and on. We obtain a transfinite sequence of nested groups A_j and characters, say $\xi_\alpha^{(j)}$, and if we call $\widetilde{\xi}_\alpha := \bigcup \xi_\alpha^{(j)}$, and $\widetilde{\xi} := \bigcup \xi^{(j)}$, we have the desired extensions.

In fact $\widetilde{\xi}_\alpha$, $\widetilde{\xi}$ are defined on G and $\widetilde{\xi}_\alpha \to \widetilde{\xi}$. To prove the last statement take (S, V), $S \subset G$ compact. The family $\{x + A, x \in S\}$ is an open covering of S. Let $x_1, \dots, x_n \in S$ be such that $\bigcup_{i=1}^n (x_i + A) \supset S$. Every x_i must belong to a subgroup A_{ι_i}; denote by A_j the biggest of $A_{\iota_1}, \dots, A_{\iota_n}$ and the proof is reduced to that stage, i.e., determine α_j such that $\xi_\alpha^{(j)} - \xi^{(j)} \in (S, V)$, for all $\alpha > \alpha_j$. □

Proposition 1.6. *If A is an open subgroup of a Hausdorff abelian topological group G, every convergent net in $\Gamma_c A$ can be lifted to a convergent net in $\Gamma_c G$.*

Proof. If a net $(\xi_\alpha)_{\alpha \in D} \subset \Gamma_c A$ is Λ-convergent to $\xi \subset \Gamma_c A$, then it is straightforward to prove that it also converges to ξ in the compact open topology on ΓA, τ_{co}. By the previous theorem there exist extensions $\widetilde{\xi_\alpha}$, $\widetilde{\xi} \in G^\wedge$ with $\widetilde{\xi_\alpha} \xrightarrow{\tau_{co}} \widetilde{\xi}$.

In order to prove that $\widetilde{\xi_\alpha} \xrightarrow{A} \widetilde{\xi}$, take a convergent net $(x_\beta)_{\beta \in E}$ in G; say $x_\beta \to x$. Since A is an open subgroup, $(x_\beta - x)_{\beta \in E}$ is eventually in A, i.e., there exists β_0 such that for every $\beta \geqslant \beta_0$, $x_\beta \in x + A$, so take $(a_\beta)_{\beta \in E} \subset A$ with $x_\beta = x + a_\beta$, $a_\beta \to 0$. We have $\widetilde{\xi_\alpha}(x_\beta) = \widetilde{\xi_\alpha}(x) + \widetilde{\xi_\alpha}(a_\beta) = \widetilde{\xi_\alpha}(x) + \xi_\alpha(a_\beta)$. But $\xi_\alpha(a_\beta) \to 0$ and $\widetilde{\xi_\alpha}(x) \to \widetilde{\xi}(x)$, since $\widetilde{\xi_\alpha} \xrightarrow{\tau_{co}} \widetilde{\xi}$. Consequently $\widetilde{\xi_\alpha}(x_\beta) \to \widetilde{\xi}(x)$. □

Now it is easy to prove the following:

Corollary 1.7. *Let A be an open subgroup of a Hausdorff abelian topological group G. The canonical homomorphism $\Psi_A : \Gamma_c G/A^o \to \Gamma_c A$ is a bicontinuous isomorphism.*

2. The main theorems

Theorem 2.1. *Let A be an open subgroup of a Hausdorff abelian topological group G. Then G is BB-reflexive if and only if A is BB-reflexive.*

Proof. Let $\kappa_G : G \to \Gamma_c\Gamma_c G$ and $\kappa_A : A \to \Gamma_c\Gamma_c A$ be the natural embeddings into the bidual. By the definition of the continuous convergence structure it is straightforward to see that κ_G and κ_A are continuous.

($\Longrightarrow$) For the necessity of the condition it is not required that G be topological. So suppose G is a BB-reflexive convergence group; we shall prove that κ_A is an isomorphism of convergence groups.

(1) κ_A *is injective.* Let $a \neq b$ in A. Since G is BB-reflexive, G has sufficiently many continuous characters, and we can find χ in $\Gamma_c G$ such that $\chi(a) \neq \chi(b)$. Thus $\kappa_A(a)(\chi|_A) = \chi|_A(a) \neq \chi|_A(b) = \kappa_A(b)(\chi|_A)$.

(2) κ_A *is surjective.* The following diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & G \\ \kappa_A \downarrow & & \downarrow \kappa_G \\ \Gamma_c\Gamma_c A & \xrightarrow{i^{**}} & \Gamma_c\Gamma_c G \end{array}$$

is commutative. By the symbol * we denote the dual of a homomorphism in the category CONABGRP. Taking into account that κ_G is surjective, for every $\chi \in \Gamma_c\Gamma_c A$, there is $g \in G$ such that $i^{**}\chi = \kappa_G(g)$. If $g \in A$, then $\kappa_A(g) = \chi$, because A is dually embedded and therefore i^{**} is injective. Suppose that $g \notin A$; as A is dually closed there exists $\beta \in \Gamma_c G$ such that $\beta(A) = 0$ and $\beta(g) \neq 0$. Then $(i^{**}\chi)(\beta) = \chi(i^*\beta) = \chi(\beta|_A) = 0$, but also $(i^{**}\chi)(\beta) = (\kappa_G(g))(\beta) = \beta(g) \neq 0$, which is a contradiction.

(3) κ_A^{-1} *is continuous.* Take a convergent net in $\Gamma_c\Gamma_c A$. It can be written as $\kappa_A(x_\alpha)$ for some net $(x_\alpha)_{\alpha\in D}$ in A, and denote by $\kappa_A(x)$ the limit point. We shall prove that (x_α) converges to x in A. Since i^{**} is continuous, $\kappa_G(x_\alpha) = \kappa_G i(x_\alpha) = i^{**}\kappa_A(x_\alpha) \to i^{**}\kappa_A(x) = \kappa_G i(x) = \kappa_G(x)$ and, as κ_G^{-1} is continuous, $x_\alpha \to x$.

($\Longleftarrow$) Suppose now that κ_A is a bicontinuous isomorphism.

(1) κ_G *is injective.* Let $g, g' \in G$ with $g \neq g'$. We distinguish two cases:

(a) If $g - g' \notin A$, we consider $[g - g'] \neq [0]$ in G/A. As G/A is discrete, we can find $\chi \in \Gamma_c(G/A)$ such that $\chi[g] \neq \chi[g']$. Take $\chi p \in \Gamma_c G$ and

$$(\kappa_G(g))(\chi p) = \chi p(g) = \chi[g] \neq \chi[g'] = (\kappa_G(g'))(\chi p) \Rightarrow \kappa_G(g) \neq \kappa_G(g').$$

(b) If $g - g' \in A$, let $\beta \in \Gamma A$ be such that $\beta(g - g') \neq 0$. Since A is dually embedded, β can be extended to a character $\widetilde{\beta} \in \Gamma_c G$. So $\widetilde{\beta}(g) \neq \widetilde{\beta}(g') \Rightarrow \kappa_G(g) \neq \kappa_G(g')$.

(2) κ_G *is surjective*. Consider the canonical commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{i} & G & \xrightarrow{p} & G/A \\ \kappa_A \downarrow & & \kappa_G \downarrow & & \downarrow \kappa_{G/A} \\ \Gamma_c\Gamma_c A & \xrightarrow{i^{**}} & \Gamma_c\Gamma_c G & \xrightarrow{p^{**}} & \Gamma_c\Gamma_c(G/A) \end{array}$$

where $\kappa_{G/A}$ is surjective because G/A is discrete and p is also surjective.

If $\chi \in \Gamma_c\Gamma_c G$, there exists $g \in G$ such that

$$p^{**}\chi = \kappa_{G/A}[g] = \kappa_{G/A}p(g) = p^{**}\kappa_G(g).$$

Then $p^{**}(\chi - \kappa_G(g)) = 0$, so $\chi - \kappa_G(g) \in \mathrm{Ker}\, p^{**}$. We will show that $\mathrm{Ker}\, p^{**} \subseteq \mathrm{Im}\, i^{**}$, and therefore $\chi - \kappa_G(g) = i^{**}(\kappa_A(a)) = \kappa_G i(a)$, for some $a \in A$. Thus $\chi = \kappa_G(g+a)$ with $g + a \in G$.

In order to prove that $\mathrm{Ker}\, p^{**} \subseteq \mathrm{Im}\, i^{**}$, for every $\varphi \in \mathrm{Ker}\, p^{**}$, define ψ in $\Gamma_c\Gamma_c A$ just requiring the diagram

$$\begin{array}{ccc} \Gamma_c G & \xrightarrow{i^*} & \Gamma_c A \\ \varphi \downarrow & \swarrow \psi & \\ T & & \end{array}$$

to be commutative. It is well defined, because $i^*(\xi) = i^*(\xi') \Rightarrow \xi - \xi' \in \mathrm{Im}\, p^*$. If $\chi \in \Gamma_c(G/A)$ is such that $p^*(\chi) = \xi - \xi'$, we have $\varphi(\xi - \xi') = \varphi(p^*\chi) = p^{**}\varphi(\chi) = 0$, i.e., $\varphi(\xi) = \varphi(\xi')$. Clearly $i^{**}(\psi) = \varphi$.

The continuity of ψ follows from Proposition 1.6. Take $\xi_\alpha \to \xi$ in $\Gamma_c A$ and extend them to $\widetilde{\xi_\alpha}$, $\widetilde{\xi}$ in $\Gamma_c G$, such that $\widetilde{\xi_\alpha} \xrightarrow{A} \widetilde{\xi}$. Now,

$$\psi(\xi_\alpha) = \psi i^*(\widetilde{\xi_\alpha}) = \varphi(\widetilde{\xi_\alpha}) \to \varphi(\widetilde{\xi}) = \psi i^*(\widetilde{\xi}) = \psi(\xi).$$

(3) κ_G^{-1} *is continuous*. An easy argument shows that this follows from the fact that $(\kappa_G|_A)^{-1}$ is continuous. In order to prove the latter, take a convergent net in $\Gamma_c\Gamma_c G$, say $\kappa_G(h_\alpha) \to \kappa_G(h)$, with $(h_\alpha)_{\alpha\in D} \subset A$. Since A is dually closed, h must be also in A. We must see that $h_\alpha \to h$, but this is equivalent to $\kappa_A(h_\alpha) \to \kappa_A(h)$, being κ_A a bicontinuous isomorphism.

In fact, if $(\chi_\beta)_{\beta\in E}$ is a net in $\Gamma_c A$ with $\chi_\beta \to \chi$, by Proposition 1.6 it can be lifted to a convergent net in $\Gamma_c G$, say $\widetilde{\chi_\beta} \to \widetilde{\chi}$. We have

$$\begin{aligned} \kappa_A(h_\alpha)(\chi_\beta) &= i^{**}\kappa_A(h_\alpha)(\widetilde{\chi_\beta}) = \kappa_G(h_\alpha)(\widetilde{\chi_\beta}) \to \kappa_G(h)(\widetilde{\chi}) \\ &= i^{**}\kappa_A(h)(\widetilde{\chi}) = \kappa_A(h)(\chi). \end{aligned}$$

Thus $\kappa_A(h_\alpha) \to \kappa_A(h)$ as stated. □

Theorem 2.2. *Let K be a compact subgroup of a Hausdorff abelian topological group G, which has sufficiently many continuous characters. Then G is BB-reflexive iff G/K is BB-reflexive.*

Proof. The proof is similar to that of Theorem 2.6 of [3], which in turn can be re-stated in a way analogous to 2.2.

($\Longrightarrow$) Suppose that G is BB-reflexive. According to Proposition 1.4, K is dually closed and $\kappa_G(K) = K^{oo}$. We may identify G/K with $\Gamma_c\Gamma_c G/K^{oo}$. Since K^o is an open subgroup of the reflexive group $\Gamma_c G$, by Theorem 2.1, K^o is BB-reflexive, and so is also its dual $\Gamma_c K^o$. Taking into account that K^o is dually embedded in $\Gamma_c G$, straightforward computations show that the natural map $\Psi_{K^o} : \Gamma_c\Gamma_c G/K^{oo} \to \Gamma_c K^o$ is a bicontinuous isomorphism. So, G/K is isomorphic to the reflexive group $\Gamma_c K^o$.

($\Longleftarrow$) (1) κ_G *is injective* due to the fact that G has sufficiently many continuous characters.

(2) κ_G *is surjective*. We argue as in Theorem 2.1. Consider the canonical commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & K & \xrightarrow{i} & G & \xrightarrow{p} & G/K & \longrightarrow 0 \\ & \kappa_K \downarrow & & \kappa_G \downarrow & & \downarrow \kappa_{G/K} & \\ & \Gamma_c\Gamma_c K & \xrightarrow{i^{**}} & \Gamma_c\Gamma_c G & \xrightarrow{p^{**}} & \Gamma_c\Gamma_c(G/K) & \end{array}$$

where its upper row is exact. If $\chi \in \Gamma_c\Gamma_c G$, there exists $g \in G$ such that $p^{**}(\chi - \kappa_G(g)) = 0$. To see that $\mathrm{Ker}\, p^{**} \subseteq \mathrm{Im}\, i^{**}$, take $\varphi \in \mathrm{Ker}\, p^{**}$, and define ψ in $i^*(\Gamma_c G) = \Gamma_c K$ requiring the commutativity of the diagram

$$\begin{array}{ccc} \Gamma_c G & \xrightarrow{i^*} & \Gamma_c K \\ \varphi \downarrow & \swarrow \psi & \\ T & & \end{array}$$

As $\Gamma_c K$ is discrete, ψ is continuous.

(3) κ_G^{-1} *is continuous*. Otherwise there would exist a neighbourhood U of zero in G and a net $(g_\alpha)_{\alpha\in D}$ in G such that $g_\alpha \notin U$ and $\kappa_G(g_\alpha) \to 0$. Hence

$$\kappa_{G/K}[g_\alpha] = \kappa_{G/K} p(g_\alpha) = p^{**}\kappa_G(g_\alpha) \to 0$$

and, consequently, $[g_\alpha] \to 0$ because $\kappa_{G/K}$ is a bicontinuous isomorphism. An easy argument shows that we can find a finer net $(g_\beta)_{\beta\in D'}$ converging to some $g \in G$. As $g_\beta \notin U$, we have $g \neq 0$. Since G has sufficiently many continuous characters, $\chi(g) \neq 0$ for some $\chi \in \Gamma_c G$. On the other hand, we have $\chi(g_\beta) = \kappa_G(g_\beta)(\chi) \to 0$; but due to continuity, $\chi(g_\beta) \to \chi(g) \neq 0$, which is a contradiction. □

Acknowledgement

The authors wish to thank W. Banaszczyk for very helpful suggestions and to the referee, whose observations have contributed to improve the global view of the paper.

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