Tropical linear mappings on the plane

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Abstract

In this paper we fully describe all *tropical linear mappings in the tropical projective plane* $\mathbb{TP}^2$, that is, maps from the tropical plane to itself given by tropical multiplication by a 3×3 matrix A with entries in $\mathbb{T}$. First we will allow only real entries in the matrix A and, only at the end of the paper, we will allow some of the entries of A equal $-\infty$. The mapping f_A is continuous and piecewise–linear in the classical sense. In some particular cases, the mapping f_A is a parallel projection onto the set spanned by the columns of A. In the general case, after a change of coordinates, the mapping collapses at most three regions of the plane onto certain segments, called antennas, and is a parallel projection elsewhere (theorem 3).

In order to study f_A, we may assume that A is normal, i.e., $I \leq A \leq 0$, up to changes of coordinates. A given matrix A admits infinitely many normalizations. Our approach is to define and compute a unique normalization for A (which we call *canonical normalization*) (theorem 1) and then always work with it, due both to its algebraic simplicity and its geometrical meaning.

On $\mathbb{R}^n$, any $n \in \mathbb{N}$, some aspects of tropical lineal maps have been studied in [5]. We work in $\mathbb{TP}^2$, adding a geometric view and doing everything explicitly. We give precise pictures.

Inspiration for this paper comes from [3, 5, 7, 11, 24]. We have tried to make it self–contained. Our preparatory results present noticeable relationships between the algebraic properties of a given matrix A (normal idempotent matrix, permutation matrix, etc.) and classical geometric properties of the points spanned by the columns of A (classical convexity and others); see theorem 2 and corollary 1. As a by–product, we compute all the tropical square roots of normal matrices of a certain type; see corollary 3. This is, perhaps, a curious result in tropical algebra. Our final aim is, however, to give a precise description of the mapping $f_A : \mathbb{TP}^2 \to \mathbb{TP}^2$. This is particularly easy when two tropical triangles arising from A (denoted $\mathcal{T}_A$ and $\mathcal{T}^A$) fit as much as possible. Then the action of f_A is easily described on each cell of the cell decomposition $\mathcal{C}^A$; see theorem 3.

Normal matrices play a crucial role in this paper. The tropical powers of normal matrices of size $n \in \mathbb{N}$ satisfy $A^{\odot n-1} = A^{\odot n} = A^{\odot n+1} = \cdots$. This statement can be traced back, at least, to [24], and appears later many times, such

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as [1, 2, 5, 8, 9]. In lemma 1, we give a direct proof of this fact, for $n = 3$. But now the equality $A^{\odot 2} = A^{\odot 3}$ means that the columns of $A^{\odot 2}$ are three fixed points of f_A and, in fact, *any point spanned by the columns of* $A^{\odot 2}$ *is fixed by* f_A. Among 3×3 normal matrices, the *idempotent* ones (i.e., those satisfying $A = A^{\odot 2}$) are particularly nice: we prove that the columns of such a matrix tropically span a set which is classically compact, connected and *convex* (lemma 2 and corollary 1). In our terminology, it is a *good tropical triangle*.

1 Introduction, Notations and Background on Tropical Mathematics

Many results on finite dimensional tropical linear algebra (spectral theory, etc.) have been published over the last 40 years and more; they are summarized in [1, 9, 13], where a wide bibliography can also be found. In this paper we will use the adjective *classical* as opposed to *tropical*. Most definitions in tropical mathematics just mimic the classical ones. However, tropical geometry is a peculiar one. Say an inhabitant of the tropical plane is disoriented. He/she takes a look at a compass and tries to spot the tropical cardinal points. There are only three: *east, north and south–west*! Accordingly, he/she will set the positive part of the three coordinate axes in the given directions, when doing geometry on the plane. He/she will find out that a generic tropical line in the tropical plane looks like a tripod (they have a vertex!), although some particular tropical lines look just like classical lines, see figure 1.

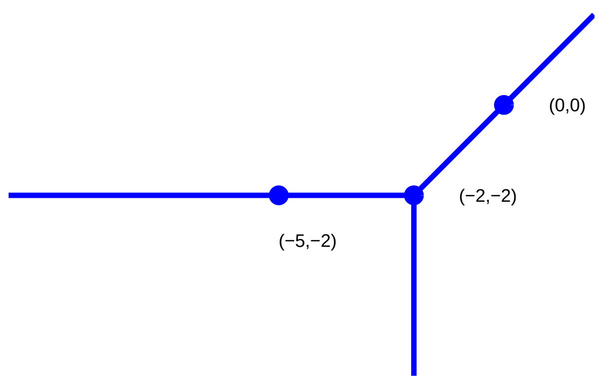

Figure 1: Tropical line with vertex at the point $(-2, -2)$.

If we happen to go down–town in a city designed by a tropical geometer, we will find out that the shape of most blocs is that of a classical hexagon, with parallel opposite sides of slopes $0, 1, \infty$, see figure 2.

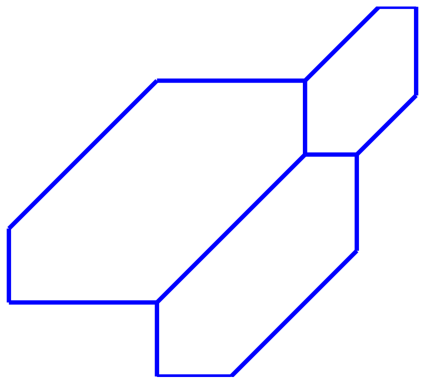

Figure 2: Downtown blocs in a tropical city.

The shortest path between two given points is made up of, at most, two classical segments with slopes $0, 1, \infty$. Moreover, the distance between the given points is the sum of the *integer lengths* (also called lattice length) of these segments. For instance, the integer length between the points $(-2, -2)$ and $(0, 0)$ is 2 (not $2\sqrt{2}$!) and the integer length between the points $(-5, -2)$ and $(0, 0)$ is $3 + 2 = 5$. This is indeed, a sort of Manhattan distance.

So, plane tropical geometry is a funny looking piecewise–linear geometry. And, by the way, why is it called tropical? Well, the explanation appears in [12, 14], etc. and we must add that some other names have also been used (for this or akin mathematics): max–plus, dioids, path algebra, extremal algebra, idempotent mathematics, etc.

Consider the set $\mathbb{R} \cup \{-\infty\}$ endowed with tropical addition $\oplus$ and tropical multiplication $\odot$, where these operations are defined as follows:

$$a \oplus b = \max\{a, b\}, \qquad a \odot b = a + b,$$

for $a, b \in \mathbb{R} \cup \{-\infty\}$. Here, $-\infty$ is the neutral element for tropical addition and 0 is the neutral element for tropical multiplication. Notice that $a \oplus a = a$, for all a, i.e., tropical addition is *idempotent*. Notice also that a has no inverse with respect to $\oplus$.

We will work with $\mathbb{R} \cup \{-\infty\}$, which will be denoted $\mathbb{T}$ and will be called the *tropical semi–field*. We will write $\oplus$ or $\max$, (resp. $\odot$ or $+$) at our convenience.

In classical mathematics, we have a choice in geometry: affine or projective. The *tropical affine plane* is $\mathbb{T}^2$, where addition and multiplication are defined coordinatewise. In the space $\mathbb{T}^3 \setminus \{(-\infty, -\infty, -\infty)\}$ we define an equivalence relation $\sim$ by letting $(p_1, p_2, p_3) \sim (q_1, q_2, q_3)$ if there exists $\lambda \in \mathbb{R}$ such that

$$\lambda \odot (p_1, p_2, p_3) = (\lambda + p_1, \lambda + p_2, \lambda + p_3) = (q_1, q_2, q_3).$$

The equivalence class of (p_1, p_2, p_3) is denoted $[p_1, p_2, p_3]$. The *tropical projective plane* is the set, $\mathbb{TP}^2$, of such equivalence classes. Notice that, at least, one of the coordinates of any point in $\mathbb{TP}^2$ must be finite.

We endow the tropical plane (either affine or projective) with the topology induced by the *Euclidean topology*. Thus, topological notions of a set S such as closure $\overline{S}$ will refer to this topology. It can be easily proved that $\mathbb{TP}^2$ is compact. In p. 9 below, we also define a *tropical norm* in the projective tropical plane. This norm gives rise to the Euclidean topology.

$\mathbb{TP}^2$ is a *compactification* of $\mathbb{T}^2$. Indeed, we have an injective mapping $\varphi : \mathbb{T}^2 \to \mathbb{TP}^2$ given by $(x, y) \mapsto [x, y, 0]$. The image of φ is open and dense. Now, for any $p = [x, y, z]$, we have $\varphi^{-1}(p) = (x-z, y-z)$, whenever $z \neq -\infty$. Taking $(x-z, y-z, 0)$ as a representative of p will be expressed by saying that *we work in* $Z = 0$. In other words, it is just a way of passing from the projective to the affine tropical plane. $\mathbb{TP}^2$ has three *boundary components*: these are the sets of points $[x, y, z]$ in $\mathbb{TP}^2$ where either $x = -\infty$ or $y = -\infty$ or $z = -\infty$.

The simplest objects in the tropical plane are lines. Given a tropical linear form

$$p_1 \odot X \oplus p_2 \odot Y \oplus p_3 \odot Z = \max\{p_1 + X, p_2 + Y, p_3 + Z\}$$

a *tropical line* consists of the points $[x, y, z] \in \mathbb{TP}^2$ where *the maximum is attained, at least, twice*, (this is the tropical analog of the classical vanishing point set). Denote this line by L_p, where $p = [p_1, p_2, p_3] \in \mathbb{TP}^2$.

Most lines in the tropical plane look like *tripods*. Indeed, if two coefficients are equal to $-\infty$, then L_p is a boundary component of $\mathbb{TP}^2$. If $p_j = -\infty$ for just one j then, in $Z = 0$, L_p is nothing but a classical slope–one line. If all p_j are real, the L_p is the union of three rays. The directions of these rays are West, South and North–East (just opposite to the cardinal directions of the tropical plane!) and these rays are emanating from the point $-p$, called the *vertex* of L_p. The latter is the generic case.

Let two points p, q in the tropical plane be given. The tropical *stable join* of p, q is defined as the limit, as ϵ tends to zero, of the tropical lines going through perturbed points $p^{v_\epsilon}, q^{v_\epsilon}$. Here, p^{v_ϵ} denotes a translation of p by a length–ϵ vector v_ϵ, see [12, 21]. We denote this line by pq.

Now, given two tropical lines L_p, L_q in the plane, the *stable intersection* of L_p, L_q, denoted $L_p \cap_{\rm st} L_q$, is defined as the limit point, as ϵ tends to zero, of the intersection of perturbed lines $L_p^{v_\epsilon}, L_q^{v_\epsilon}$. Here, $L_p^{v_\epsilon}$ denotes a translation of L_p by a length–ϵ vector v_ϵ.

There exists a *duality* between lines and points since

$$q \in L_p \iff p \in L_q,$$

meaning that *the maximum* $\max\{p_1 + q_1, p_2 + q_2, p_3 + q_3\}$ *is attained, at least, twice.* This duality transforms stable join into stable intersection and conversely, i.e.,

$$L_p \cap_{\rm st} L_q = r \iff pq = L_r,$$

for p, q, r in $\mathbb{TP}^2$.

The *tropical version of Cramer's rule* (see [21]) goes as follows: the stable intersection of the lines L_p and L_q is the point

$$[\max\{p_2 + q_3, q_2 + p_3\}, \max\{p_1 + q_3, q_1 + p_3\}, \max\{p_1 + q_2, q_1 + p_2\}].$$

Since the computation of this point is nothing but a *tropical version of the cross–product* of the triples p and q, we will denote it by $p \otimes q$ (this is not to be mixed up with $p \odot q = p + q$). Notice that $p \otimes q = q \otimes p$. In other words, the *tropical version of Cramer's rule* in the plane can be written as

$$L_p \cap_{\text{st}} L_q = p \otimes q \qquad \text{and} \qquad pq = L_{p \otimes q},$$

by duality. In particular, $-(p \otimes q)$ *is the vertex of the line* pq, a crucial fact that we use again and again.

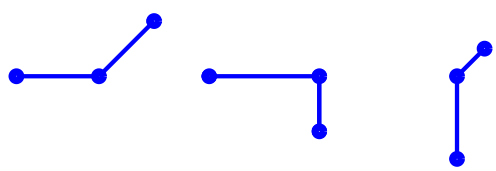

Figure 3: Tropical line segments.

Given a subset U of points in $\mathbb{TP}^2$ (resp. $\mathbb{T}^2$), we can consider the *tropical span of* U, denoted $\text{span}(U)$, meaning the set of points $u \in \mathbb{TP}^2$ (resp. $\mathbb{T}^2$) which can be written as

$$u = \lambda_1 \odot u_1 \oplus \cdots \oplus \lambda_s \odot u_s = \max\{\lambda_1 + u_1, \dots, \lambda_s + u_s\},$$

for some $s \in \mathbb{N}$, $u_1, \dots, u_s \in U$, $\lambda_1, \dots, \lambda_s \in \mathbb{T}$, and not all λ_j equal to $-\infty$. The tropical span of two points p, q is called a *tropical segment*. We know that $\text{span}(p, q)$ is the union of the classical segments $\overline{p, -(p \otimes q)}$ and $\overline{-(p \otimes q), q}$. Similarly, the co–span of p, q is the union of the classical segments $\overline{p, (-p) \otimes (-q)}$ and $\overline{(-p) \otimes (-q), q}$,

The *tropical co–span of* U, denoted $\text{co–span}(U)$, is the set of points u which can be written as

$$u = \min\{\lambda_1 + u_1, \dots, \lambda_s + u_s\},$$

for some $s \in \mathbb{N}$, $u_1, \ldots, u_s \in U$, $\lambda_1, \ldots, \lambda_s \in \mathbb{R} \cup \{+\infty\}$, and not all λ_j equal to $+\infty$.

Given two points $p, q \in \mathbb{TP}^2$, we know that $-(p \otimes q)$ represents the vertex of the stable tropical line joining p and q. On the other hand, the stable intersection of the tropical lines with vertices at p, q is represented by the point $(-p) \otimes (-q)$. It turns out that the points $p, -(p \otimes q), q, (-p) \otimes (-q)$ are the vertices of a classical parallelogram, see figure 4.

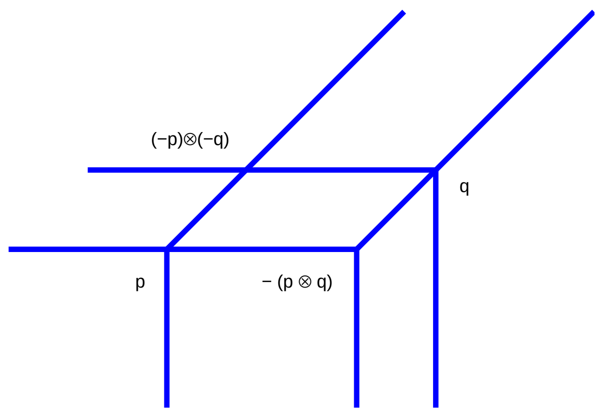

Figure 4: Span and co–span of points p, q.

Another sort of duality is taking place here. Indeed, we may consider $\mathbb{R} \cup \{+\infty\}$ endowed with tropical addition $\ominus = \min$ and the same tropical multiplication $\odot$. The relationship between these two operations is $\max\{p, q\} = -\min\{-p, -q\}$, whence

$$p \oplus q = -(-p) \ominus (-q),$$

for $p, q \in \mathbb{R}$. This *max–min duality* appears in the literature, see [4, 8, 7], etc.

Why do we care about the co–span? A tropical triangle can be determined by three points, or by three lines. First, a tropical triangle $\mathcal{T}$ is determined by three points a, b, c. If the points are tropically collinear then $\mathcal{T}$ is not generic. We have

$$\mathcal{T} = \operatorname{span}(a, b, c).$$

The *sides* of $\mathcal{T}$ are, by definition, the tropical lines ab, bc and ca. The vertices of the sides of $\mathcal{T}$ (as tropical lines) are $-(a \otimes b), -(b \otimes c)$ and $-(c \otimes a)$, again by the *tropical* *version of Cramer's rule*. The properties of the triangle $\mathcal{T}$ depend on the configuration of the six points

$$a, b, c, -(a \otimes b), -(b \otimes c), -(c \otimes a), \tag{1}$$

which are all different, generically.

Three tropical lines L_p, L_q, L_r also determine a *tropical triangle*, $\mathcal{T}'$, which is generic if the lines are not tropically concurrent. We can write

$$\mathcal{T}' = \text{co–span}(-p, -q, -r).$$

The stable intersections (by pairs) of the lines L_p, L_q, L_r are called *vertices* of $\mathcal{T}'$. These points should not be mixed up with the vertices $-p, -q, -r$ of the lines. By the *tropical version of Cramer's rule*, the coordinates of the vertices of $\mathcal{T}'$ are $p \otimes q, q \otimes r$ and $r \otimes p$. The properties of $\mathcal{T}'$ depend on the configuration of the six points

$$p \otimes q, q \otimes r, r \otimes p, -p, -q, -r,$$

which are all different, in the generic case.

Tropical triangles are, in general, infinite unions of tropical segments. Indeed,

$$\mathcal{T} = \text{span}(a, b, c) = \bigcup_{s \in \text{span}(b,c)} \text{span}(a, s). \tag{2}$$

Therefore, tropical triangles are, in general, connected non–pure 2–dimensional sets. The non–generic case arises when the points a, b, c are tropically collinear. In addition, it is easy to check that tropical triangles are classically compact, both in $\mathbb{TP}^2$ and in $\mathbb{T}^2$.

It is not true, in general, that the stable intersection of the tropical lines ab and bc gives back the point b, and this makes *tropical triangles trickier than classical triangles*. For example, take $a = [3, 4, 6], b = [-2, 0, 8]$ and $c = [1, 1, 0]$ Then $a \otimes b = [12, 11, 3], b \otimes c = [9, 9, 1]$ and $ab \cap_{\text{st}} bc = [12, 13, 21] = [-1, 0, 8] \neq b$. The reader is encouraged to draw this example, in $Z = 0$.

This anomalous situation for tropical triangles has been studied in [3], where the definition of *good tropical triangle* has been given. Three points a, b, c define a *good tropical triangle* if, by stable join, they give rise to three tropical lines ab, bc, ca which, stably intersected by pairs, yield the original points a, b, c, i.e.,

$$ca \cap_{\text{st}} ab = a, \quad ab \cap_{\text{st}} bc = b, \quad bc \cap_{\text{st}} ca = c.$$

Good tropical triangles are characterized by six slack inequalities. Indeed, write the coordinates of (representatives of) a, b, c as the columns of a matrix $A = (a_{ij})$ so that c occupies the first column and a occupies the third. Write

$$\mathcal{T}_A = \text{span}(A). \tag{3}$$

Then theorem 2 in [3] tells us that $\mathcal{T}_A \subseteq \mathbb{TP}^2$ is a good tropical triangle if and only if

$$a_{12} - a_{22} \leq a_{13} - a_{23} \leq a_{11} - a_{21},$$

$$
\begin{aligned}
a_{23} - a_{33} \leq a_{21} - a_{31} \leq a_{22} - a_{32}, \\
a_{31} - a_{11} \leq a_{32} - a_{12} \leq a_{33} - a_{13}.
\end{aligned} \tag{4}
$$

Write

$$
A_0 = \begin{bmatrix} a_{11} - a_{31} & a_{12} - a_{32} & a_{13} - a_{33} \\ a_{21} - a_{31} & a_{22} - a_{32} & a_{23} - a_{33} \\ 0 & 0 & 0 \end{bmatrix}
$$

in order to make drawings in $Z = 0$. It is easy to check that the six inequalities (4) imply the following *cardinal points condition in* $Z = 0$: $\operatorname{col}(A_0, 1)$ represents the most eastwards point, $\operatorname{col}(A_0, 2)$ represents the most northwards one, and $\operatorname{col}(A_0, 3)$ represents the most south–westwards one, among the columns of A_0. The converse is not true.

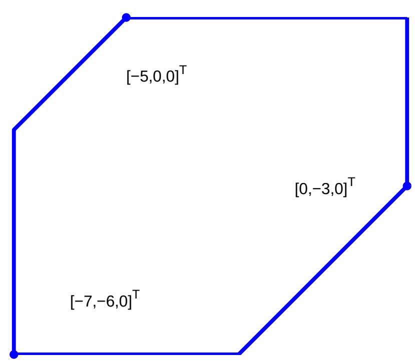

Figure 5: Good triangle determined by the matrix A.

In figure 5 we see the good tropical triangle determined by the matrix

$$
A = A_0 = \begin{bmatrix} 0 & -5 & -7 \\ -3 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}.
$$

Actually, in $Z = 0$, good tropical triangles are nothing but classical hexagons, pentagons, quadrangles or triangles having slopes $0, 1$ and ∞, where the inequalities (4) provide the integer length of the sides. They are obtained by chopping off two corners, in a classical rectangle, see figure 6. In figure 7 we see a few good triangles for which some inequalities are equalities.

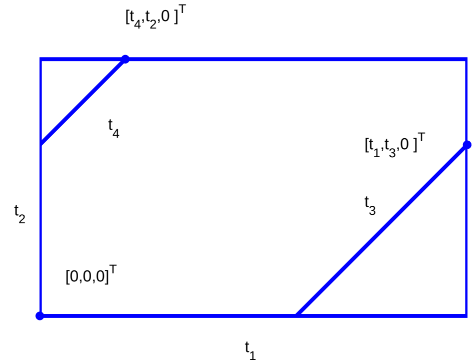

Figure 6: A good tropical triangle is a classical rectangle with two corners chopped off.

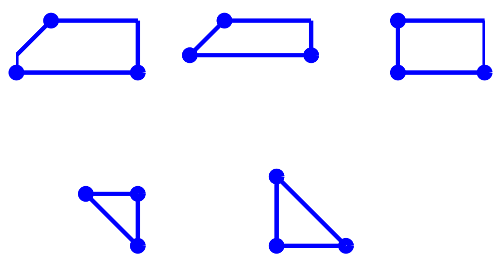

Figure 7: Some good tropical triangles.

In the classical plane $\mathbb{R}^2$ we have the following norm

$$\|p\| = \max\{\|p_1\|, \|p_2\|, \|p_1 - p_2\|\}, \qquad p \in \mathbb{R}^2.$$

It is easy to check that $\|p\|$ is the integer length of the tropical segment $\operatorname{span}(p, 0)$, if we identify $p = (p_1, p_2)$ with $[p_1, p_2, 0]$. For instance, $\|(-5, -2)\| = 5$, $\|(-3, 5)\| = 8$. The unit ball and some radii in it are shown in figure 8. Given real points $p, q \in \mathbb{R}^2$ the *tropical distance* between p and q is $\|p - q\|$, by definition. It is the integer length of the tropical segment $\operatorname{span}(p, q)$. This is connected with the generalized Hilbert projective metric appearing in [7, 11, 15] and to the range seminorm of [10].

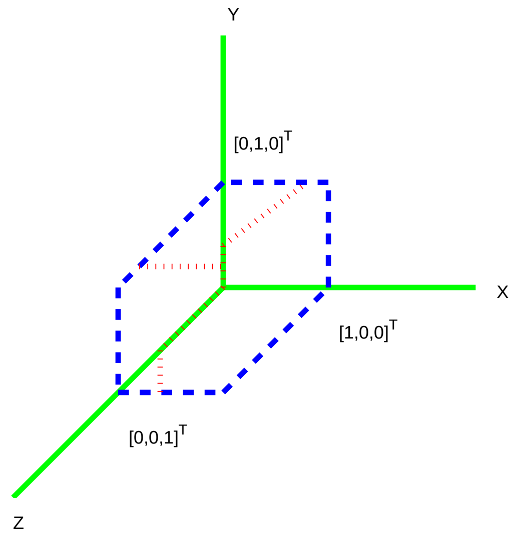

Figure 8: Axes and unit ball in the tropical the plane. Some rays are shown in dotted lines.

In $\mathbb{TP}^2$, let V be the tropical span of a finite family of points. The *projector map* (or *nearest point map*) $\rho : \mathbb{TP}^2 \longrightarrow V$ satisfies $\rho \circ \rho = \rho$ and $\rho|_V = \mathrm{id}_V$. For a point $p \in \mathbb{TP}^2 \setminus V$, the image $\rho(p)$ is computed as follows: fix a representative $p' \in \mathbb{T}^3$ of p and, for each generator v of V, choose a representative $v' \in \mathbb{T}^3$, optimal for the condition $v' \le p'$ (meaning $v'_j \le p'_j$, for $j = 1, 2, 3$ and equality is attained for, at least, one j). Tropically add all such $v's$ and then, take $\rho(p)$ to be the point in $\mathbb{TP}^2$ represented by the sum. In [7, 11, 15] it is shown that $\rho(p) \in V$ minimizes the tropical distance $\|p - q\|$, when q runs through V. In general, there are infinitely many points q in V minimizing such a distance, in addition to $\rho(p)$. Indeed, we consider tropical balls $B(d)$ centered at p of increasing radius d and take the minimum $d > 0$ such that the intersection $B(d) \cap V$ is non–empty. Then $B(d) \cap V$ is the set of minimizing points.

2 Matrices, mappings and pictures in $Z = 0$

All arrays will have entries in $\mathbb{T}$. Arrays will be denoted by capital letters A, B, C, N, P, Q, etc. Tropical matrix addition and multiplication are defined in the usual way, but using the tropical operations $\oplus$ and $\odot$, instead of the classical ones. Any array all whose entries are zero will be denoted by 0. Given two arrays of the same size $A = (a_{ij})$, $B = (b_{ij})$, we will write $A \le B$ if $a_{ij} \le b_{ij}$, for all i, j.

Our square matrices will have entries in $\mathbb{T}$, but *at least one entry in each row and in each column will be real.* If all the entries of a matrix A are real, we will say that A is a *real matrix*. We will deal with 3×3 matrices. The *tropical determinant* of a 3×3

matrix $A = (a_{ij})$ (also called *tropical permanent*) is defined as

$$|A|_{trop} = \max_{\sigma \in \Sigma_3} \{a_{1\sigma(1)} + a_{2\sigma(2)} + a_{3\sigma(3)}\},$$

where Σ_3 denotes the symmetric group in 3 symbols. A matrix is *tropically singular* if *the maximum in the tropical determinant is attained, at least, twice*. Otherwise the matrix is *tropically regular*, or it is said to have a *strong permanent*. These are all standard definitions.

Given a matrix A, the j–th column (resp. row) of A will be denoted $\operatorname{col}(A, j)$ (resp. $\operatorname{row}(A, j)$). The triple of diagonal entries of A will be denoted $\operatorname{diag}(A)$. Moreover, if $t \in \mathbb{R}^3$, then $\operatorname{diag}(t)$ will denote the matrix whose diagonal is t, the rest of entries being equal to $-\infty$; such matrices will be called *diagonal matrices*. A *permutation matrix* is a matrix obtained from a diagonal matrix, by permuting some of its rows or permuting some of its columns. A particular case is the *tropical identity matrix*, $I = \operatorname{diag}(0)$. Another example is

$$P_{12} = \begin{bmatrix} -\infty & 0 & -\infty \\ 0 & -\infty & -\infty \\ -\infty & -\infty & 0 \end{bmatrix}.$$

Any permutation matrix P has a tropical inverse $P^{\odot -1}$, meaning $P \odot P^{\odot -1} = P^{\odot -1} \odot P = I$.

From now on, *points in* $\mathbb{TP}^2$ *will be denoted by columns*, for convenience. We often *identify a* 3×3 *matrix* A *with the three points in* $\mathbb{TP}^2$ *represented by its columns.*

The reader can easily check that left–multiplication by the matrix P_{12} exchanges coordinates X and Y:

$$P_{12} \odot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ x \\ z \end{bmatrix}.$$

A triple $t = (t_1, t_2, t_3) \in \mathbb{R}^3$ gives rise to a *translation* in $\mathbb{TP}^2$:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \mapsto \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} t_1 + X \\ t_2 + Y \\ t_3 + Z \end{bmatrix} = \operatorname{diag}(t) \odot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix},$$

By a *change of projective coordinates* in the tropical projective plane $\mathbb{TP}^2$ we mean left–multiplying coordinates by a permutation matrix. Therefore, a change of projective coordinates amounts to the composite of a translation and a permutation of coordinates. Notice that right–multiplying A by a diagonal matrix does not change the columns of A in $\mathbb{TP}^2$; it only changes the representatives of them.

All pictures will be done in the affine tropical plane $Z = 0$. In order to do so, from a given matrix A we compute the matrix

$$A_0 = A \odot (\operatorname{diag}(-\operatorname{row}(A, 3))). \tag{5}$$

Conversely, if A_0 is a matrix having $\operatorname{row}(A_0, 3) = 0$, then we recover A as follows:

$$A = A_0 \odot \operatorname{diag}(-A_0). \tag{6}$$

Our aim is to describe the mapping $f_A : \mathbb{TP}^2 \to \mathbb{TP}^2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto A \odot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \max\{a_{11}+x, a_{12}+y, a_{13}+z\} \\ \max\{a_{21}+x, a_{22}+y, a_{23}+z\} \\ \max\{a_{31}+x, a_{32}+y, a_{33}+z\} \end{bmatrix}.$$

First, notice that tropically proportional matrices A and $\lambda \odot A$ determine the same mapping $f_A = f_{\lambda \odot A}$, any $\lambda \in \mathbb{R}$. The simplest example of mapping f_A arises for $A = 0$, the mapping being constant. This is also true for $f_{A \otimes 0}$ and $f_{0 \otimes A}$.

The mapping f_A is obviously continuous and piecewise–linear. The image $\operatorname{im} f_A$ is the tropical triangle spanned by A, meaning that it is spanned by the columns of A:

$$\mathcal{T}_A = \operatorname{im} f_A = \operatorname{span}(A). \tag{7}$$

If A is real, then *the mapping f_A is not surjective*, since no finite family of points with finite coordinates span the whole $\mathbb{TP}^2$; this is well–known (see, e.g., [23]). Moreover, if $r, s \in \mathbb{R}$ are negative and big enough, we have

$$A \odot \begin{bmatrix} 0 \\ r \\ s \end{bmatrix} = \operatorname{col}(A,1),\ A \odot \begin{bmatrix} s \\ 0 \\ r \end{bmatrix} = \operatorname{col}(A,2),\ A \odot \begin{bmatrix} r \\ s \\ 0 \end{bmatrix} = \operatorname{col}(A,3).$$

Therefore, f_A is *locally constant* on three big chunks of $\mathbb{TP}^2$, called *corners*. In particular, *f_A is not injective.*

Let us see how do these corners arise. First, the matrix A defines three tropical lines A_1, A_2, A_3, because the j–th row of A provides a tropical linear form

$$a_{j1} \odot X \oplus a_{j2} \odot Y \oplus a_{j3} \odot Z = \max\{a_{j1}+X, a_{j2}+Y, a_{j3}+Z\}.$$

The vertices of A_1, A_2, A_3 are (represented by) the rows of $-A$, i.e., the columns of $-A^T$. Thus we have another tropical triangle here, namely

$$\mathcal{T}^A = \text{co–span}(-A^T). \tag{8}$$

Moreover, the lines A_1, A_2, A_3 (or, rather, the matrix A) induce a *cell decomposition of* $\mathbb{TP}^2$, denoted $\mathcal{C}^A$ (see [11] for an isomorphic cell decomposition). The decomposition $\mathcal{C}^A$ consists of, at most, 31 cells, and this is the generic case.

It has:

- ten 2–dimensional cells: one bounded cell, denoted B^A, the three already mentioned *corners* (denoted C_1^A, C_2^A, C_3^A), six unbounded cells (parallel to some tropical coordinate axis X, Y or Z),
- fifteen 1–dimensional cells: nine unbounded cells (parallel to some coordinate axis) and six bounded cells,
- six 0–dimensional cells or points.

Notice that the union of the bounded cells above is nothing but

$$\overline{B^A} = \mathcal{T}^A. \tag{9}$$

In figure 9 we find the 31 cells described above, and figure 10 represents the cell decomposition induced by the matrix

$$A = \begin{bmatrix} 0 & -1 & -5 \\ -4 & 0 & -2 \\ -1 & -4 & 0 \end{bmatrix}, \ (-A^T)_0 = \begin{bmatrix} -5 & 2 & 1 \\ -4 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

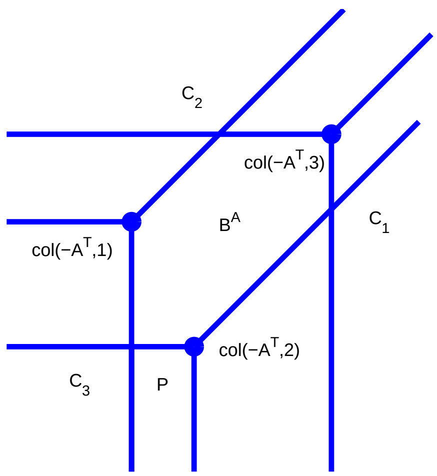

Figure 9: The 31 cells in the cell decomposition $\mathcal{C}^A$ induced by a matrix A.

The description of the mapping $f_A : \mathbb{TP}^2 \to \mathbb{TP}^2$ is particularly easy when the tropical triangles $\mathcal{T}_A$ and $\mathcal{T}^A$ fit as much as possible: then the action of f_A is easily described on each cell of the cell decomposition $\mathcal{C}^A$; see theorem 3.

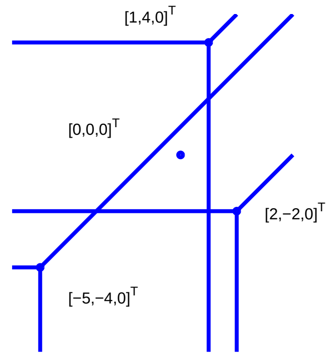

Figure 10: Cell decomposition $\mathcal{C}^A$ induced by matrix A.

3 Normal matrices

By definition, a matrix A is *normal* if $\mathrm{diag}(A) = 0$ and $A \leq 0$; in symbols,

$$I \leq A \leq 0 \tag{10}$$

see [5]. For any matrix A there exist permutation matrices P, Q such that the product

$$N = P \odot A \odot Q \tag{11}$$

is normal. The matrix N is called ***a normalization of*** A. The *Hungarian method* (see [5, 16, 20]) is an algorithm to obtain such N, P, Q. A matrix A admits several normalizations. Notice that the columns of A and the columns of $A \odot Q$ represent the same points in $\mathbb{TP}^2$, given perhaps in a different order. And the columns of N are a just a translation of those points.

As in classical mathematics, the product of matrices corresponds to the composite of mappings:

$$f_N = f_P \circ f_A \circ f_Q.$$

Now, f_P and f_Q are changes of projective coordinates, so that *in order to study the mapping f_A, we may assume that A is normal, up to changes of coordinates.*

A normal matrix A satisfies $I \leq A \leq 0$, and therefore

$$I \leq A \leq A^{\odot 2} \leq A^{\odot 3} \leq \cdots \tag{12}$$

since tropical multiplication by any matrix is monotonic (because max and $+$ are monotonic). Moreover, for any natural m, $A^{\odot m+1} \leq A^{\odot m} \odot 0$ and the mapping $f_{A^{\odot m} \odot 0}$ is constant. In corollary 3 we see that the tropical powers of A are simpler than A (in the sense that they depend on fewer parameters), when A belongs to a particular class of normal matrices. This simplification carries over to the corresponding mappings

$$\mathrm{id} = f_I, f_A, f_{A^{\odot 2}}, f_{A^{\odot 3}}, \ldots, \mathrm{const}\,.$$

In $\mathbb{TP}^2$, we consider the cell decomposition $\mathcal{C}^0$ induced by the zero matrix, i.e., the cell decomposition given by the tropical line L_0. It has only three 2–dimensional cells (corners), which have the following description in $Z = 0$:

$$C_1^0 = \{0 < x, y < x\}, \qquad C_2^0 = \{0 < y, x < y\}, \qquad C_3^0 = \{x < 0, y < 0\}.$$

The *geometric meaning of normality* is the following: if A is a 3×3 normal matrix then,

$$\operatorname{col}(A_0, j) \in \overline{C_j^0}, \quad \text{for all} \quad j = 1, 2, 3. \tag{13}$$

Conversely, if A_0 is a 3×3 matrix with $\operatorname{row}(A_0, 3) = 0$ satisfying (13), then A in (6) is normal.

Next we define several operators on matrices and then we study the relationship among them. Of course, we are particularly interested in these operators acting on normal matrices.

For any $k \in \mathbb{N}$, the tropical k–th power of A, denoted $A^{\odot k}$, takes normal matrices to normal matrices. The transpose A^T of a normal matrix A is a normal matrix. These operators commute with each other. Warning: $(-A)^{\odot 2} \neq A^{\odot 2}$, in general. Also, $(A_0)^{\odot 2} \neq (A^{\odot 2})_0$, in general.

We introduce the *tropical adjoint* of A, denoted $\hat{A}$. By definition, $\hat{A} = (\alpha_{ij})$, where α_{ij} is the tropical cofactor of a_{ji}. In other words,

$$\operatorname{row}(\hat{A}, j) = \operatorname{col}(A, j-1) \otimes \operatorname{col}(A, j+1), \tag{14}$$

for $j = 1, 2, 3, \mod 3$. Last, we define an auxiliary matrix operator, $\breve{A} = (\beta_{ij})$, by the formulas

$$\beta_{ii} = 0, \qquad \beta_{ij} = a_{ik} + a_{kj},$$

if $i \neq j$ and $\{i, j, k\} = \{1, 2, 3\}$.

Lemma 1. *If A is 3×3 and normal, then*

1. *$\breve{A}$ is normal and $\hat{A} = A \oplus \breve{A} = A^{\odot 2}$,*
2. *$\hat{A}$ is normal,*
3. *$A^{\odot 2} = A^{\odot 3}$,*
4. *the columns of $A^{\odot 2}$ represent fixed points of f_A,*
5. *zero (the neutral element for tropical multiplication) is an eigenvalue of A.*

Proof. A straightforward computation yields (1) and then (2) follows. Now, multiplication by A is a monotonic operator; so that the equality in (1) implies $A^{\odot 3} = \max\{A^{\odot 2}, A \odot \breve{A}\}$. Now, a simple computation shows that $A \odot \breve{A} = A \oplus \breve{A}$, whence $A^{\odot 3} = A^{\odot 2}$ follows. Finally, (4) follows from (3) and (5) follows from (4). □

Lemma 1 follows from [24], where real matrices of any size n are considered. The so called *Kleene star of* A (or *strong closure of* A) is defined as

$$A^* = I \oplus A \oplus A^{\odot 2} \oplus A^{\odot 3} \oplus \cdots ,$$

if the limit exists, see[1, 6]. If A is a 3×3 normal matrix, then $A^* = A^{\odot 2}$, but we will not use this.

Lemma 2. *For a* 3×3 *normal matrix* A*, the following are equivalent:*

1. $\breve{A} \leq A$,
2. $A = A^{\odot 2}$,
3. $\mathcal{T}_A$ *is good.*

Proof. The equivalence follows from lemma 1 and the six inequalities (4), letting $a_{jj} = 0$, for $j = 1, 2, 3$. Indeed, we obtain

$$\begin{aligned} a_{23} + a_{31} &\leq a_{21}, & a_{32} + a_{21} &\leq a_{31}, \\ a_{13} + a_{32} &\leq a_{12}, & a_{31} + a_{12} &\leq a_{32}, \\ a_{12} + a_{23} &\leq a_{13}, & a_{21} + a_{13} &\leq a_{23}. \end{aligned} \tag{15}$$

□

If $A = (a_{ij})$ is normal, we have

$$A_0 = \begin{bmatrix} -a_{31} & a_{12} - a_{32} & a_{13} \\ a_{21} - a_{31} & -a_{32} & a_{23} \\ 0 & 0 & 0 \end{bmatrix}.$$

By a translation, we can assume that $a_{13} = a_{23} = 0$, so that $\mathrm{col}(A_0, 3) = 0$. Write

$$t_1 = -a_{31},\ t_2 = -a_{32},\ t_3 = a_{21} - a_{31},\ t_4 = a_{12} - a_{32}. \tag{16}$$

Then

$$t_1, t_2 \geq 0, \quad 0 \leq t_3, t_4 \leq \min\{t_1, t_2\} \tag{17}$$

and the t_j provide a *parameter space* for good tropical triangles, up to translation; see figure 6.

4 Canonical normalization

The geometric idea of canonical normalization is to center the figures at the origin of $Z = 0$. For each $d, d_1, d_2, d_3 \in \mathbb{R}$ with $d \geq 0$ and $-d \leq d_j$, for $j = 1, 2, 3$, the matrix

$$D(d, d_1, d_2, d_3) = \begin{bmatrix} 0 & -d - d_2 & -2d - d_3 \\ -2d - d_1 & 0 & -d - d_3 \\ -d - d_1 & -2d - d_2 & 0 \end{bmatrix} \tag{18}$$

is normal and it is easy to check that $D = D^{\odot 2}$. Notice the symmetric role played by d_1 with respect to X, d_2 with respect to Y and d_3 with respect to Z. We will use the matrices

$$D(d, d_1, d_2, d_3)_0 = \begin{bmatrix} d + d_1 & d & -2d - d_3 \\ -d & 2d + d_2 & -d - d_3 \\ 0 & 0 & 0 \end{bmatrix}, \tag{19}$$

$$(-D(d, d_1, d_2, d_3)^T)_0 = \begin{bmatrix} -2d - d_3 & d + d_1 - d_3 & d + d_1 \\ -d + d_2 - d_3 & -d - d_3 & 2d + d_2 \\ 0 & 0 & 0 \end{bmatrix}. \tag{20}$$

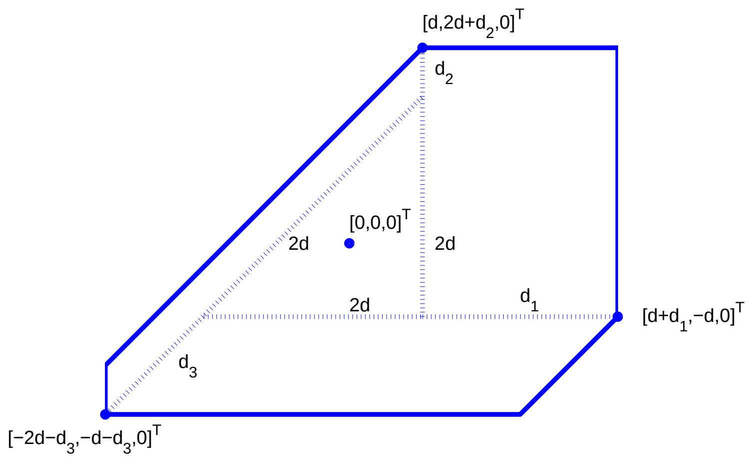

Figure 11: Tropical triangle given by the matrix $D(d, d_1, d_2, d_3)$, in $Z = 0$. Dotted lines are auxiliary.

Lemma 3 (Canonical normalization for an idempotent normal matrix)**.** *If A is a 3×3 normal matrix such that $A = A^{\odot 2}$, then there exist unique $d, d_1, d_2, d_3 \geq 0$ and there exist unique permutation matrices P, Q such that $D(d, d_1, d_2, d_3) = P \odot A \odot Q$.*

Proof. By the geometric meaning of normality (13), a translation allows us to assume that $\operatorname{col}(A_0, 3) = 0$. Then the triangle $\mathcal{T}_A$ is determined by the parameters $t_1, t_2, t_3, t_4 \geq 0$ as in (17).

$$A_0 = \begin{bmatrix} t_1 & t_4 & 0 \\ t_3 & t_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & t_4 - t_2 & 0 \\ t_3 - t_1 & 0 & 0 \\ -t_1 & -t_2 & 0 \end{bmatrix}.$$

We take $d = \frac{1}{3}|t_3 - t_4|$, $d_3 = \min\{t_3, t_4\}$, $m = \max\{t_3, t_4\}$, $d_j = t_j - m$, for $j = 1, 2$ and $Q = \operatorname{diag}(d_3 + 2d, d_3 + d, 0)$. Then $D(d, d_1, d_2, d_3) = Q^{\odot -1} \odot A \odot Q$. The uniqueness follows from the geometric meaning of the parameters d, d_1, d_2, d_3. $\square$

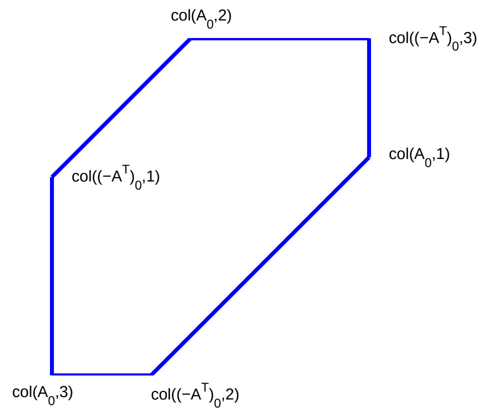

Figure 12: Tropical triangle associated to a normal idempotent matrix.

Corollary 1. *A good tropical triangle is classically convex in $Z = 0$.*

Proof. Let $\mathcal{T}_A$ be a good tropical triangle, for some matrix A. By the paragraph after (11), a translation allows us to assume that A is normal. By lemma 3, we can assume that $A = D(d, d_1, d_2, d_3)$, for some $d, d_1, d_2, d_3 \geq 0$. To work in $Z = 0$, we consider the matrix A_0 shown above in (19). The columns of A_0 span $\mathcal{T}_A$, which is a classical hexagon, pentagon, rectangle or triangle of slopes $0, 1$ and ∞ in $Z = 0$. By the tropical version of Cramer's rule, equality (14) and lemma 2, $\mathcal{T}_A$ it is actually the convex hull of the columns of A_0 and columns of $(-\hat{A}^T)_0 = (-(A^{\odot 2})^T)_0 = (-A^T)_0$ shown in (20). $\square$

Lemma 4. *Given numbers $d, d_1, d_2, d_3 \in \mathbb{R}$, with $d \geq 0$ and $-d \leq d_j$, for $j = 1, 2, 3$, consider the normal matrix $A = D(d, d_1, d_2, d_3)$. The following are equivalent:*

1. *$A \neq A^{\odot 2}$,*
2. *$d_j < 0$, for some $j = 1, 2, 3$,*
3. *in $Z = 0$, $\mathcal{T}_A$ is not classically convex.*

Proof. To check for convexity in $Z = 0$, consider the matrix A_0 in expression (19). Say, $j = 1$. If $d_1 < 0$, then any point in the classical segment $a = \overline{p, q}$, with $p = [d + d_1, 2d + d_1 + d_2, 0]^T$ and $q = \mathrm{col}(A_0, 2) = [d, 2d + d_2, 0]^T$, prevents $\mathcal{T}_A$ from being convex; see figure 13. $\square$

We will say that a is an *antenna* of $\mathcal{T}_A$.

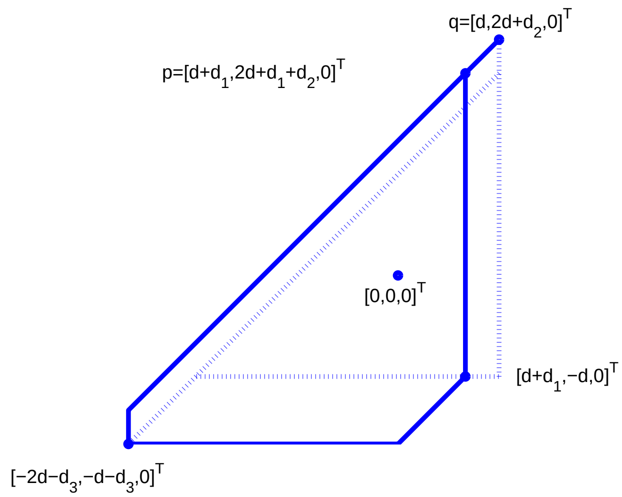

Figure 13: Tropical triangle with antenna due to $d_1 < 0$.

In the hypothesis of the former lemma, $\mathcal{T}_A$ admits a *cell decomposition* having, at most, 13 cells, and this is the generic case:

- one 2–dimensional cell,
- six 1–dimensional cells,
- six 0–dimensional cells.

The closure of any 1–dimensional cell disrupting the convexity of $\mathcal{T}_A$ in $Z = 0$ is called an *antenna*, as in [7]. The union of points in the antennas of $\mathcal{T}_A$ will be denoted $\operatorname{ant}(\mathcal{T}_A)$. Each j with $d_j < 0$ yields an antenna in $\mathcal{T}_A$. The integer length of this antenna is, at most, d.

Corollary 2. *Given $d, d_1, d_2, d_3 \in \mathbb{R}$, $d \geq 0$, $-d \leq d_j$, for $j = 1, 2, 3$, let $A = D(d, d_1, d_2, d_3)$. Then $\mathcal{T}_{A^{\odot 2}}$ is the maximal convex set contained in $\mathcal{T}_A$, in $Z = 0$.*

Proof. We know that A is normal. If $A = A^{\odot 2}$, we just have corollary 1. Otherwise $A^{\odot 2} = (A^{\odot 2})^{\odot 2}$, by lemma 1, so that $\mathcal{T}_{A^{\odot 2}}$ is convex in $Z = 0$, by lemma 2 and corollary 1.

Now we compute $A^{\odot 2} = (b_{ij})$ using (1) of lemma 1, obtaining $a_{ij} = b_{ij}$ unless $(i, j) = (1, 3), (2, 1)$ or $(3, 2)$. If, say, $d_1 < 0$, then

$$b_{32} = -2d - d_1 - d_2 > a_{32} = -2d - d_2.$$

Then, $\operatorname{col}((A^{\odot 2})_0, 2) = [d + d_1, 2d + d_1 + d_2, 0]^T$ and $\operatorname{col}(A_0, 1) = [d + d_1, -d, 0]^T$ so that both points belong to the same classical line $X = d + d_1$, meaning that the antenna in $\mathcal{T}_A$ caused by the inequality $d_1 < 0$ no longer appears in $\mathcal{T}_{A^{\odot 2}}$. $\square$

Suppose $d \geq 0$ and $-d \leq d_j$, all j. The former corollary tells us that *squaring the normal matrix $A = D(d, d_1, d_2, d_3)$ corresponds to chopping off the antennas of $\mathcal{T}_A$, if any*. The tropical triangle $\mathcal{T}_{A^{\odot 2}}$ will be called the *soma* of $\mathcal{T}_A$, denoted $\mathrm{soma}(\mathcal{T}_A)$. Then

$$\mathcal{T}_A = \mathrm{soma}(\mathcal{T}_A) \cup \mathrm{ant}(\mathcal{T}_A). \tag{21}$$

Notice the following:

- $\mathrm{soma}(\mathcal{T}_A)$ reduces to one point when $A^{\odot 2} = 0$ (warning: $A^{\odot 2} = 0$ does not imply $A = 0$),
- if $A^{\odot 2} \neq 0$ then $\mathrm{soma}(\mathcal{T}_A)$ is a 2–dimensional convex, compact connected set, in $Z = 0$,
- the antennas of $\mathcal{T}_A$ have integer length d, at most, if $A = D(d, d_1, d_2, d_3)$.

There exist tropical triangles with antennas of arbitrary length. In order to find a canonical normalization for the matrices describing these triangles, we consider

$$E(d, d_1, d_2, d_3, h_1, h_2, h_3) = \begin{bmatrix} 0 & -d - d_2 & -2d - d_3 - h_3 \\ -2d - d_1 - h_1 & 0 & -d - d_3 \\ -d - d_1 & -2d - d_2 - h_2 & 0 \end{bmatrix}. \tag{22}$$

If $d, d_1, d_2, d_3, h_1, h_2, h_3 \geq 0$, then the matrix $E(d, d_1, d_2, d_3, h_1, h_2, h_3)$ is normal. Notice that $E(d, d_1, d_2, d_3, 0, 0, 0) = D(d, d_1, d_2, d_3)$. For pictures in $Z = 0$, we will use

$$E(d, d_1, d_2, d_3, h_1, h_2, h_3)_0 = \begin{bmatrix} d + d_1 & d + h_2 & -2d - d_3 - h_3 \\ -d - h_1 & 2d + d_2 + h_2 & -d - d_3 \\ 0 & 0 & 0 \end{bmatrix}, \tag{23}$$

$$-E(d, d_1, d_2, d_3, h_1, h_2, h_3)^T = \begin{bmatrix} 0 & 2d + d_1 + h_1 & d + d_1 \\ d + d_2 & 0 & 2d + d_2 + h_2 \\ 2d + d_3 + h_3 & d + d_3 & 0 \end{bmatrix}, \tag{24}$$

$$(-E(d, d_1, d_2, d_3, h_1, h_2, h_3)^T)_0 =$$

$$\begin{bmatrix} -2d - d_3 - h_3 & d + d_1 - d_3 + h_1 & d + d_1 \\ -d + d_2 - d_3 - h_3 & -d - d_3 & 2d + d_2 + h_2 \\ 0 & 0 & 0 \end{bmatrix}, \tag{25}$$

Write $A = E(d, d_1, d_2, d_3, h_1, h_2, h_3)$. Notice that if $h_{j+1} - d_j > 0$, for some j (subscripts modulo 3), the $\mathcal{T}_A$ is not convex in $Z = 0$

Theorem 1 (Canonical normalization). *If A is any 3×3 matrix, then there exist unique $d, d_1, d_2, d_3, h_1, h_2, h_3 \geq 0$, and there exist unique permutation matrices P, Q such that $E(d, d_1, d_2, d_3, h_1, h_2, h_3) = P \odot A \odot Q$. Moreover, $h_{j+1} > 0$ implies $d_j = 0$, (subscripts modulo 3).*

Proof. To prove existence, we may assume that A is normal. If $A = A^{\odot 2}$, we take $h_j = 0$, all j, and apply lemma 3. Now assume that $A \neq A^{\odot 2}$. Again, by lemma 3, we can assume that $A^{\odot 2} = D(d, d_1, d_2, d_3)$, for some $d, d_1, d_2, d_3 \geq 0$. Moreover, A is a *tropical square root of* $D(d, d_1, d_2, d_3)$. The geometric meaning of this assertion is that $\mathcal{T}_A$ *is obtained from* $\mathcal{T}_{A^{\odot 2}}$ *by addition of antennas*. But for $\mathcal{T}_{A^{\odot 2}}$ to admit adjunction of antennas, d_j must vanish for some j. For each $d_j = 0$ consider any $h_{j+1} > 0$ (subscripts modulo 3). Now, it is easy to verify that the matrix $A = E(d, d_1, d_2, d_3, h_1, h_2, h_3)$ satisfies $A^{\odot 2} = D(d, d_1, d_2, d_3)$.

The uniqueness follows from the geometric meaning of the parameters $d, d_1, d_2, d_3, h_1, h_2, h_3$. □

Corollary 3. *If $d, d_1, d_2, d_3, h_1, h_2, h_3 \geq 0$, and $h_{j+1} > 0$ implies $d_j = 0$, (subscripts modulo 3), then $E(d, d_1, d_2, d_3, h_1, h_2, h_3)$ is a tropical square root of $D(d, d_1, d_2, d_3)$.* □

Example 1. *Let us compute the canonical normalization of*

$$A = A_0 = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix},$$

the example in [7], p. 409. Consider the matrices $U = \operatorname{diag}(-1, 0, 0)$,

$$Q = \begin{bmatrix} -\infty & -\infty & 0 \\ 0 & -\infty & -\infty \\ -\infty & 0 & -\infty \end{bmatrix}$$

and obtain

$$N_0 = U \odot P_{12} \odot A \odot Q = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$N = N_0 \odot \operatorname{diag}(-2, -3, 0) = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ -2 & -3 & 0 \end{bmatrix}$$

is normal. Then

$$N^{\odot 2} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix}, \qquad (N^{\odot 2})_0 = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In figure 14, we find, from left to right, the tropical triangles corresponding to the matrices $A, P_{12} \odot A, T \odot P_{12} \odot A, T \odot P_{12} \odot A \odot Q$ and N (the last three matrices yield the same triangle), all in $Z = 0$. Here $3d = 1$, $d_1 = d_2 = 0$ and $d_3 = 1$ so that the canonical normalization of $N^{\odot 2}$ is

$$D\left(1/3, 0, 0, 1\right) = \begin{bmatrix} 0 & -1/3 & -5/3 \\ -2/3 & 0 & -4/3 \\ -1/3 & -2/3 & 0 \end{bmatrix} = S^{\odot -1} \odot N^{\odot 2} \odot S,$$

with $S = \operatorname{diag}(5/3, 4/3, 0)$*. Thus, the canonical normalization of* A *is*

$$S^{\odot -1} \odot N \odot S = E\left(1/3, 0, 0, 1, 0, 1, 1\right) = \begin{bmatrix} 0 & -1/3 & -8/3 \\ -2/3 & 0 & -4/3 \\ -1/3 & -5/3 & 0 \end{bmatrix}.$$

We have $h_1 = 0$*,* $h_2 = h_3 = 1$*. For pictures in* $Z = 0$*, we consider*

$$D\left(1/3, 0, 0, 1\right)_0 = \begin{bmatrix} 1/3 & 1/3 & -5/3 \\ -1/3 & 2/3 & -4/3 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$E\left(1/3, 0, 0, 1, 0, 1, 1\right)_0 = \begin{bmatrix} 1/3 & 4/3 & -8/3 \\ -1/3 & 5/3 & -4/3 \\ 0 & 0 & 0 \end{bmatrix}.$$

In figure 15 we see the triangles corresponding to the matrix $N^{\odot 2}$ *and its canonical normalization, while in figure 16 we see the triangles of the matrix* N *and its canonical normalization.*

Notice that the matrices $S^{\odot -1}$ and S provide the canonical normalization of $N^{\odot 2}$ and also of N. This holds in general, due to the equality (28) below.

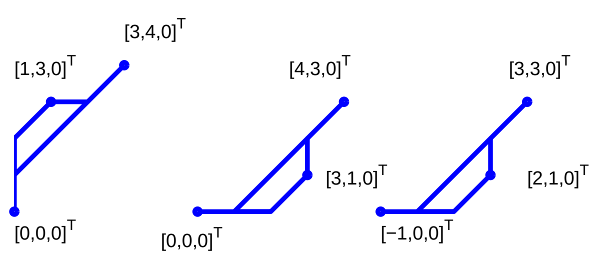

Figure 14: Tropical triangles $\mathcal{T}_A$, $\mathcal{T}_{P_{12} \odot A}$ and $\mathcal{T}_N$.

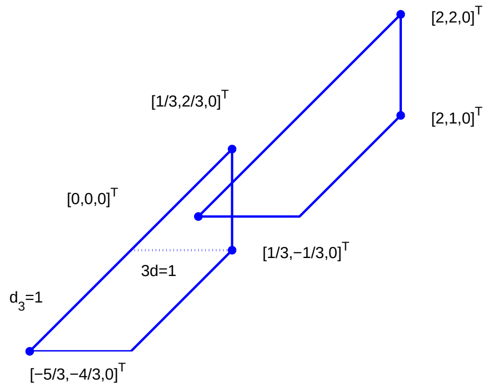

Figure 15: Tropical triangle corresponding to $N^{\odot 2}$ and to its canonical normalization.

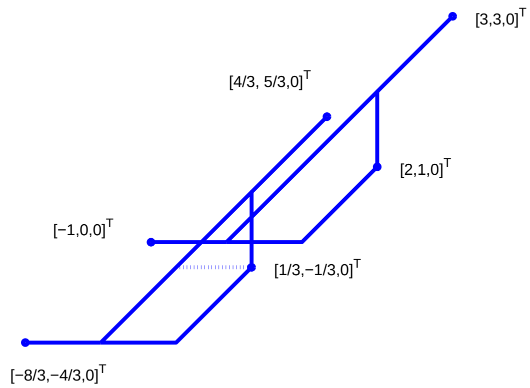

Figure 16: Tropical triangle corresponding to N and to its canonical normalization.

Consider $A = E(d, d_1, d_2, d_3, h_1, h_2, h_3)$, such that $d, d_1, d_2, d_3, h_1, h_2, h_3 \geq 0$ and $h_{j+1} > 0$ implies $d_j = 0$, (subscripts modulo 3). Now, a definition of *soma and antennas of* $\mathcal{T}_A$ can be given, as in p. 19. The soma of $\mathcal{T}_A$ is $\mathcal{T}_{D(d,d_1,d_2,d_3)}$. The antennas of $\mathcal{T}_A$ have tropical length h_j, if $h_j > 0$.

For any 3×3 real matrix A let $N = E(d, d_1, d_2, d_3, h_1, h_2, h_3)$ be the canonical normalization of A, for some $d, d_1, d_2, d_3, h_1, h_2, h_3 \geq 0$, and $h_{j+1} > 0$ implies $d_j = 0$, (subscripts modulo 3). There exist permutation matrices P, Q such that $N = P \odot A \odot Q$. The map f_P is a translation, so that the triangle $\mathcal{T}_A = f_{P^{\odot -1}}(\mathcal{T}_N)$ is just a translated of $\mathcal{T}_N$. Then we define the *soma and antennas of* $\mathcal{T}_A$ as follows:

$$\mathrm{soma}(\mathcal{T}_A) = f_{P^{\odot -1}}(\mathrm{soma}\, \mathcal{T}_N), \qquad \mathrm{ant}(\mathcal{T}_A) = f_{P^{\odot -1}}(\mathrm{ant}\, \mathcal{T}_N), \tag{26}$$

so that equality (21) holds true, for any A.

Remark 1. *Sometimes a non–idempotent normal matrix* $E(e, e_1, e_2, e_3, h_1, h_2, h_3)$ *can be further normalized, to obtain a (simpler) normal matrix* $D(d, d_1, d_2, d_3)$*, with* $d \geq 0$*,* $-d \leq d_j$*, all* j*.*

Say $h_2 > 0$*,* $e_1 = 0$*,* $e_2, e_3 \geq 0$*,* $h_1 = h_3 = 0$*. Then*

$$E(e, e_1, e_2, e_3, h_1, h_2, h_3) = \begin{bmatrix} 0 & -e-e_2 & -2e-e_3 \\ -2e & 0 & -e-e_3 \\ -e & -2e-e_2-h_2 & 0 \end{bmatrix}.$$

Then we take $d = e + \frac{h_2}{3}$*,* $d_1 = -h_2$*,* $d_2 = e_2$*,* $d_3 = e_3$ *and* $P = \operatorname{diag}(d-e-h_2, -d+e, 0)$ *to obtain*

$$P \odot E(e, e_1, e_2, e_3, h_1, h_2, h_3) = D(d, d_1, d_2, d_3),$$

but the matrix $D(d, d_1, d_2, d_3)$ *will be normal if and only if* $d + d_1 = e - \frac{2h_2}{3} \geq 0$*.*

Example 2. *Back to example 1, notice that the matrices*

$$D(1, -1, -1, 1) = \begin{bmatrix} 0 & 0 & -3 \\ -1 & 0 & -2 \\ 0 & -1 & 0 \end{bmatrix}$$

and $H = \operatorname{diag}(\frac{1}{3}, \frac{2}{3}, 0)$ *satisfy*

$$H^{\odot -1} \odot E\left(\frac{1}{3}, 0, 0, 1, 0, 1, 1\right) \odot H = D(1, -1, -1, 1).$$

The following theorem is a simple geometric characterization of normality.

Theorem 2. *The* 3×3 *matrix* A *is normal if and only if the origin belongs to* $\operatorname{soma}(\mathcal{T}_A)$*, in* $Z = 0$*.*

Proof. If A is normal then, expression (13) tells us that $\operatorname{col}(A_0, j) \in \overline{C_j^0}$, for $j = 1, 2, 3$. Working in $Z = 0$, let S_j be the tropical segment joining $\operatorname{col}(A_0, j-1)$ and $\operatorname{col}(A_0, j+1)$ and let t_j be any classical intersection point of S_j with the tropical line L_0. Then $\operatorname{span}(t_1, t_2, t_3)$ is contained in L_0 and passes through the origin. Let $\tilde{t_3} \in S_3$ be a perturbation of t_3 such that $t_1, t_2, \tilde{t_3}$ are not tropical collinear and write $\mathcal{T} = \operatorname{span}(t_1, t_2, \tilde{t_3})$. Then $\mathcal{T}$ is 2–dimensional, by equality (2) and $\mathcal{T} \subset \mathcal{T}_A$. Moreover, the origin belongs to $\operatorname{span}(t_1, t_2) \subset \mathcal{T}$. Perturbing now t_2 and t_1 in S_2 and S_1 provides the result.

If, in $Z = 0$, the origin belongs to $\operatorname{soma}(\mathcal{T}_A)$, a good tropical triangle, then $\operatorname{col}((A^{\odot 2})_0, j) \in \overline{C_j^0}$, for $j = 1, 2, 3$. Therefore, $\operatorname{col}(A_0, j) \in \overline{C_j^0}$, for $j = 1, 2, 3$, showing normality of A. □

Example 3. *Let us look at example 1 again. A normalization of* A *is* $N = P \odot A \odot Q$*, where* $P = \operatorname{diag}(-2, -3, 0)$*,*

$$N = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -3 \\ -1 & 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} -\infty & -\infty & 0 \\ -\infty & 0 & -\infty \\ -1 & -\infty & -\infty \end{bmatrix}.$$

We have

$$N_0 = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}, \qquad N^{\odot 2} = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} = (N^{\odot 2})_0.$$

Another normalization of A is $N = P \odot A \odot Q$, where $P = \mathrm{diag}(-1, -3, 0)$,

$$N = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & -3 \\ -2 & 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} -\infty & -\infty & 0 \\ -\infty & 0 & -\infty \\ -2 & -\infty & -\infty \end{bmatrix};$$

in this case,

$$N_0 = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad N^{\odot 2} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & -3 \\ -1 & 0 & 0 \end{bmatrix},$$

$$(N^{\odot 2})_0 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

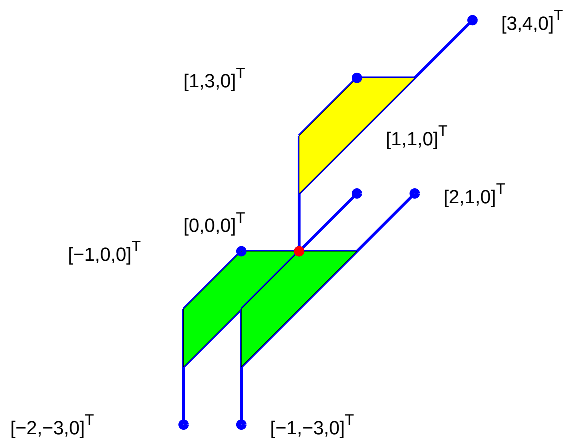

Figure 17: Tropical triangles $\mathcal{T}_A$ and $\mathcal{T}_N$, for two different normalizations N.

Next, we want to define soma and antennas of a co–spanned tropical triangle. Regarding co–span, we choose to work with *co–normal* matrices, i.e., matrices A having non–negative entries and zero diagonal. We can achieve a canonical co–normalization and then define *soma and antennas of a co–spanned tropical triangle*, in a similar fashion to theorem 1 and definition in p. 23. Then, what is the relationship between $\mathcal{T}_N$ and $\mathcal{T}^N$, $\mathrm{soma}(\mathcal{T}_N)$ and $\mathrm{soma}(\mathcal{T}^N)$, $\mathrm{ant}(\mathcal{T}_N)$ and $\mathrm{ant}(\mathcal{T}^N)$, for a given canonical normal matrix $N = E(d, d_1, d_2, d_3, h_1, h_2, h_3)$?

- If $h_1 = h_2 = h_3 = 0$, then $N = N^{\odot 2}$ is normal and idempotent. By lemma 2, $\mathcal{T}_N$ is good. Repeating the argument in the proof of corollary 1, we have

$$N^T = (N^{\odot 2})^T = \widehat{N}^T,$$

meaning that *the columns of $-N^T$ are precisely the vertices of the sides of $\mathcal{T}_N$*, see figure 12. By the max–min duality, the columns of N are the stable intersection points of the tropical lines N_1, N_2, N_3. Therefore,

$$\mathcal{T}_N = \operatorname{span}(N) = \text{co–span}(-N^T) = \mathcal{T}^N. \tag{27}$$

- If $h_{j+1} > 0$ and $d_j = 0$ for some j, write $N' = N^{\odot 2}$. By the previous item,

$$\operatorname{soma}(\mathcal{T}_N) = \mathcal{T}_{N'} = \mathcal{T}^{N'} = \operatorname{soma}(\mathcal{T}^N), \tag{28}$$

even if $\mathcal{T}_N \neq \mathcal{T}^N$. Moreover, there is a *bijection between the sets of antennas of $\mathcal{T}_N$ and of $\mathcal{T}^N$* (at most, three antennas each) so that we can talk of *corresponding antennas.* For every antenna a of $\mathcal{T}_N$ and corresponding antenna a' of $\mathcal{T}^N$, there exists a unique cell P_a in the cell decomposition $\mathcal{C}^N$ such that $a' \cup a \subset \overline{P_a}$. Indeed, suppose that $a = \overline{s, q}$ is the antenna of $\mathcal{T}_N$ coming from $h_2 > 0$, $d_1 = 0$, with $s = \operatorname{col}((N^{\odot 2})_0, 2) = [d, 2d + d_2, 0]^T$ and $q = \operatorname{col}(N_0, 2) = [d + h_2, 2d + d_2 + h_2, 0]^T$. Then $a' = \overline{s, r}$ is corresponding antenna in $\mathcal{T}^N$, with $r = \operatorname{col}((-N^T)_0, 3) = [d, 2d + d_2 + h_2, 0]^T$. In $Z = 0$, the cell P_a is determined by

$$d < X, \qquad X + d + d_2 < Y < X + d + d_2 + h_2. \tag{29}$$

5 The mapping f_A for a $A = E(d, d_1, d_2, d_3, h_1, h_2, h_3)$

Recall that each antenna a of $\mathcal{T}_A$ gives rise to a cell P_a in the cell decomposition $\mathcal{C}^A$ of $\mathbb{TP}^2$, induced by a matrix A, see p. 12.

Theorem 3. *Let $A = E(d, d_1, d_2, d_3, h_1, h_2, h_3)$, with $d, d_1, d_2, d_3, h_1, h_2, h_3 \geq 0$, and $h_{j+1} > 0$ implies $d_j = 0$, (subscripts modulo 3). Then*

1. $\operatorname{soma}(\mathcal{T}_A)$ *is the set of fixed points of f_A,*
2. $f_A|_{P_a} = a$, *for every antenna a of $\mathcal{T}_A$,*
3. $f_A|_P$ *is the classical projection onto $\mathcal{T}_A$, in the direction of P, for every unbounded 2–dimensional cell P of the cell decomposition $\mathcal{C}^A$, if $P \neq P_a$, $a \in \operatorname{ant}(\mathcal{T}_A)$.*

Proof. First suppose that $h_1 = h_2 = h_3 = 0$. Then $A = D(d, d_1, d_2, d_3)$ is normal idempotent and so, by equalities (9) and (27),

$$\operatorname{soma}(\mathcal{T}_A) = \mathcal{T}_A = \mathcal{T}^A = \overline{B^A}. \tag{30}$$

By lemma 1, the set of fixed points of f_A is $\operatorname{span}(A^{\odot 2}) = \operatorname{span}(A) = \mathcal{T}_A$ and this proves part (1). Since $\mathcal{T}_A$ is a tropical triangle without antennas, then part (2) does not apply. In order to prove part (3), let us work in $Z = 0$. We have A_0 and $(-A^T)_0$ as in equalities (19) and (20). Let P be an unbounded cell of the cell decomposition $\mathcal{C}^A$ parallel, say, to the Y direction and let $p = [x, y, 0]^T$ be a point in P. Then y is a big negative real number or $y = -\infty$, so that $A \odot p$ is a tropical linear combination of the first and third columns of A. In particular, $f_A(p)$ belongs to the (lower part of the) boundary of $\overline{B^A}$ and the point $f_A(p)$ is independent of y. More precisely, if P is determined by the conditions

$$-2d - d_3 < X < d + d_1 - d_3, \qquad Y < -d - d_3$$

then $f_A(p) = [x, -d - d_3, 0]^T$. And if P is determined by the conditions

$$d + d_1 - d_3 < X < d + d_1, \qquad Y < X - 2d - d_1$$

then $f_A(p) = [x, x - 2d - d_1, 0]^T$. This proves part (3).

Now, suppose that $h_{j+1} > 0$ and $d_j = 0$, for some j, say $j = 2$. The associated antenna a and cell P_a have been described in (29). If $p = [x, y, 0]^T \in \overline{P_a}$ then $f_A(p) = [y - d - d_2, y, x - d]^T = [y - x - d_2, y - x + d, 0]^T \in a$, proving assertion (2). In particular, $f_A(a') = a$, where a' is the antenna of $\mathcal{T}^A$ corresponding to a. □

If $A = D(d, d_1, d_2, d_3)$ for $d, d_1, d_2, d_3 \geq 0$, then $f_A = f_A \circ f_A$, by theorem 3, meaning that f_A is some kind of projection. But, in general, f_A is different from the projector map onto $\mathcal{T}_A$ in p. 10. For instance, consider the matrix $A = D(3, 9, 2, 4)$, i.e.,

$$A = \begin{bmatrix} 0 & -5 & -10 \\ -15 & 0 & -7 \\ -12 & -8 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 12 & 3 & -10 \\ -3 & 8 & -7 \\ 0 & 0 & 0 \end{bmatrix}, \quad p = \begin{bmatrix} -12 \\ 0 \\ 0 \end{bmatrix}.$$

Then

$$\rho(p) = \begin{bmatrix} -12 \\ -27 \\ -24 \end{bmatrix} \oplus \begin{bmatrix} -12 \\ -7 \\ -15 \end{bmatrix} \oplus \begin{bmatrix} -12 \\ -9 \\ -2 \end{bmatrix} = \begin{bmatrix} -12 \\ -7 \\ -2 \end{bmatrix} = \begin{bmatrix} -10 \\ -5 \\ 0 \end{bmatrix}$$

$$f_A(p) = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix} \neq \rho(p).$$

Finally, we can describe the mapping f_A, for any 3×3 real matrix A. First, we find the canonical normalization $N = E(d, d_1, d_2, d_3, h_1, h_2, h_3) = P \odot A \odot Q$ to obtain $f_N = f_P \circ f_A \circ f_Q$; then we apply theorem 3, knowing that f_P and f_Q are just changes of coordinates.

What happens to f_A, if some entries of the matrix A become $-\infty$, under the assumption that *at least one entry in each row and in each column is real*? All the results in the paper still hold true, *except for the non–injectivity and non–surjectivity assertions* in p. 12. Indeed, write $N = E(d, d_1, d_2, d_3, h_1, h_2, h_3) = (n_{ij})$ with $d, d_1, d_2, d_3, h_1, h_2, h_3 \geq 0$ and $h_{j+1} > 0$ implies $d_j = 0$, (subscripts modulo 3). Suppose that $n_{ij} = -\infty$, for some $i \neq j$. Then for some k, the point represented by $\operatorname{col}(N, k)$ lies on a boundary component of $\mathbb{TP}^2$. And if $n_{ij} = -\infty$, for all $i \neq j$, then $N = I$ and $\mathcal{T}_N = \mathbb{TP}^2$. Suppose now that $N = P \odot A \odot Q$ is the canonical normalization of A. Then the following are equivalent:

- $N = I$,
- A is a permutation matrix,
- the columns of A represent $[0, -\infty, -\infty]^T$, $[-\infty, 0, -\infty]^T$ and $[-\infty, -\infty, 0]^T$,
- $\mathcal{T}_A = \mathbb{TP}^2$, i.e., f_A is surjective,
- f_A is injective.

Corollary 4. *A tropical linear mapping on $\mathbb{TP}^2$ transforms tropical collinear points into tropical collinear points.* □

For any real 3×3 matrix A, consider the set of points where f_A is injective,

$$S_A = \{q \in \mathbb{TP}^2 : \exists! p \in \mathbb{TP}^2, f_A(p) = q\}.$$

Working on $\mathbb{R}^n$, the set S_A plays an important role in [5]. If $N = E(d, d_1, d_2, d_3, h_1, h_2, h_3)$, then $S_N = B^N$, by theorem 3 and equality (9). And if $N = P \odot A \odot Q$ is the canonical normalization of A, then $S_A = S_{A \odot Q} = f_{P^{\odot -1}}(S_N)$, meaning that S_A is a translated of S_N.

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