On the topology of germs of meromorphic functions

S.M. Gusein-Zade   I. Luengo   A. Melle-Hernández [*]

Topology of holomorphic germs has been intensively studied after the celebrated book of J. Milnor [6]. The main topological object related with a germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ of a holomorphic function is the *Milnor fibration* $f^{-1}(S_\delta^1) \cap B_\varepsilon^{2n} \to S_\delta^1$, $(0 < \delta << \varepsilon)$. Its fibre $f^{-1}(z) \cap B_\varepsilon^{2n}$ is called *the Milnor fibre* of the germ f.

Some ideas have been applied to study the topology of the fibration defined by polynomial functions on complex affine spaces. A polynomial P of n variables defines a map $P : \mathbb{C}^n \to \mathbb{C}$ which is a locally trivial fibration over the complement to a finite set in the target $\mathbb{C}$. At a point $a \in \mathbb{C}^n$ the polynomial P defines a holomorphic germ which can be studied with the use of the Milnor fibration. In particular, P is not a locally trivial fibration over its critical values.

However the global topology of the fibration defined by a polynomial function is not determined only by its local behaviour at points in $\mathbb{C}^n$. For example the polynomial function $P(x, y) = x(xy - 1)$ on $\mathbb{C}^2$ has no critical points, however it does not determine a locally trivial fibration over $\mathbb{C}$: the level set $P^{-1}(0)$ is topologically different from other level sets $P^{-1}(c)$, $(c \neq 0)$. One can say that the global topology of a polynomial function depends on its behaviour at infinite points.

To give sense to this statement one can consider the projective closure $\mathbb{CP}^n$ of the complex affine space $\mathbb{C}^n$. A polynomial P defines a meromorphic function of $\mathbb{CP}^n$. At points of the infinite hyperplane $\mathbb{CP}_\infty^{n-1} \subset \mathbb{CP}^n$ this function defines not a holomorphic but a meromorphic germ. Thus to study the behaviour of the polynomial P near an infinite point one has to understand topological properties of meromorphic germs.

The goal of the paper is to give a sketch of some notions and constructions related to the topology of meromorphic germs and to formulate criteriums for a value to be typical for a germ of meromorphic function of two variables.

Definition: *A germ of a meromorphic function on* $(\mathbb{C}^n, 0)$ *is a fraction* $f = \frac{P}{Q}$, *where* P and Q are germs of holomorphic functions on $(\mathbb{C}^n, 0)$.

It appears that in the framework of the described study the following equivalence relation is adequate: germs of meromorphic functions $f = \frac{P}{Q}$ and $f' = \frac{P'}{Q'}$ are *equal* if and

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only if $P' = U \cdot P$ and $Q' = U \cdot Q$ for a holomorphic germ $U : (\mathbb{C}^n, 0) \to \mathbb{C}$ such that $U(0) \neq 0$.

A meromorphic germ $f = \frac{P}{Q}$ defines a map from the complement $B_\varepsilon^{2n} \setminus \{P = Q = 0\}$ to the indeterminacy set $\{P = Q = 0\}$ in a small ball B_ε^{2n} to the complex projective line $\mathbb{CP}^1$. To a regret this map is not a locally trivial fibration over the complement of a finite set in $\mathbb{CP}^1$. Roughly speaking f fails to be a locally trivial fibration over values c for which the level set $f^{-1}(c)$ is not transversal to the sphere $S_\varepsilon^{2n-1} = \partial B_\varepsilon^{2n}$. This is a "real condition" and thus it can appear at points of $\mathbb{CP}^1$ from a set of real codimension 1. It means that in this way one cannot define a generic fibre of a meromorphic germ.

Example 1. Let $f = \frac{x^2 - y^3}{y^2}$. One can see that $f : B_\varepsilon^4 \setminus \{0\} \to \mathbb{CP}^1$ is not a locally trivial fibration over neighbourhoods of 0, ∞, and points $c(= (c:1))$ such that $\|c\| = \frac{3}{2}\varepsilon$.

However if one fixes a value c in $\mathbb{CP}^1$, one cannot meet this effect in a neighbourhood of the value c if the radius ε is small enough.

THEOREM 1 [3] *For any value $c \in \mathbb{CP}^1$, there exists $\varepsilon_0 > 0$ ($\varepsilon_0 = \varepsilon_0(c)$) such that for any positive $\varepsilon \leq \varepsilon_0$ the map $f : B_\varepsilon^{2n} \setminus \{P = Q = 0\} \to \mathbb{CP}^1$ is a locally trivial fibration over a punctured neighbourhood of c.*

Definition: The described fibration is called *c-Milnor fibration* of the meromorphic germ f.

Definition: The fibre of the c-Milnor fibration, i.e.

$$\mathcal{M}_f^c = \{z \in B_\varepsilon^{2n} \ : \ f(z) = \frac{P(z)}{Q(z)} = c'\}$$

for ε small enough and for c' close to c enough (in $\mathbb{CP}^1$) is called the *c-Milnor fibre* of the meromorphic germ f.

Example 2. For f from the Example 1, one has: $\mathcal{M}_f^0$ is the (2-dimensional) disk minus two points; for $c \neq 0$, $\mathcal{M}_f^c$ is the disjoint union of two punctured disks.

For a fibration over a punctured neighbourhood of a point in $\mathbb{CP}^1$, there is defined a monodromy transformation which is a diffeomorphism of the fibre (well defined up to isotopy).

Definition: The monodromy transformation $h_f^c : \mathcal{M}_f^c \to \mathcal{M}_f^c$ of the c-Milnor fibration is called the *c-monodromy transformation* of the meromorphic germ f.

Example 3. For f from the Example 1, h_f^c is trivial (i.e. isotopic to the identity) for all $c \neq 0, \infty$. the 0-monodromy transformation is a natural transformation of the disk without two points which interchanges these points. The ∞-monodromy transformation interchanges two punctured disks.

One can show that for almost all values c (i.e. for all but a finite number) the c-monodromy transformation h_f^c is trivial, i.e. isotopic to the identity. Moreover the following takes place.

Definition: A value $c \in \mathbb{CP}^1$ is called a *typical* value of the meromorphic germ f if the map $f : B_\varepsilon^{2n} \setminus \{P = Q = 0\} \to \mathbb{CP}^1$ is a locally trivial (and thus a trivial) fibration over a neighbourhood (not punctured) of the point c. Otherwise the value c is called *atypical*.

If a value c is typical, then the corresponding monodromy transformation h_f^c is isotopic to identity.

THEOREM 2 [**4**] *A meromorphic germ has a finite number of atypical values.*

Example 4. The meromorphic germ f from the Example 1 has two atypical values: 0 and ∞.

Definition: For a transformation $h : X \to X$ of a topological space X its *zeta function* $\zeta_h(t)$ is the rational function defined by

$$\zeta_h(t) = \prod_{q \geq 0} \{\det [\, id - t\, h_*|_{H_q(X;\mathbb{C})}]\}^{(-1)^q}.$$

This definition coincides with that in [2] and differs by minus sign in the exponent from that in [1].

Let $\zeta_f^c(t)$ be the zeta-function of the c-monodromy transformation h_f^c of the meromorphic germ f. The degree of the rational function $\zeta_f^c(t)$ (i.e., the degree of the numerator minus the degree of the denominator) is equal to the Euler characteristic of the c-Milnor fibre $\mathcal{M}_f^c$. One has the following statement.

THEOREM 3 [**4**] *If a value c is typical then the Euler characteristic of the c-Milnor fibre is equal to 0 and its zeta-function $\zeta_f^c(t)$ is equal to 1.*
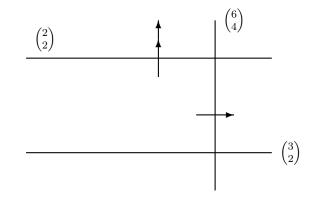
Example 5. For f from the first example $\zeta_f^0(t) = \frac{1}{1+t}$ and $\zeta_f^c(t) = 1$ for $c \neq 0$.

Remark. This example shows that the inverse of Theorem 3 is not true: the ∞-zeta function $\zeta_f^\infty(t) = 1$, but ∞ is not typical and the ∞-monodromy transformation is not isotopic to identity. One can show that the inverse of Theorem 3 is true if the holomorphic germ $P + cQ$ has an isolated critical point at the origin ([4]).

One method to study the topology of a holomorphic germ is through its resolution. A version of it is suitable for meromophic germs as well.

Definition: *A resolution of a meromorphic germ $f = \frac{P}{Q}$ is a modification of $(\mathbb{C}^n, 0)$, i.e. a proper analytic map $\pi : \mathcal{X} \to \mathcal{U}$ of a smooth analytic manifold $\mathcal{X}$ onto a neighbourhood $\mathcal{U}$ of the origin in $\mathbb{C}^n$, which is an isomorphism outside of the hypersurface $H = \{P = 0\} \cup \{Q = 0\}$, such that the total transform $\pi^{-1}(H)$ of the hypersurface H is a normal crossing divisor at each point of $\mathcal{X}$.*

In terms of a resolution it is possible to express the zeta-functions of the 0 and ∞-monodromy transformations of the meromorphic germ f. The fact that the preimage $\pi^{-1}(H)$ is a divisor with normal crossings means that at any point of it, there exists a local system of coordinates $y_1, \ldots, y_n$ such that the liftings $\widetilde{P} = P \circ \pi$ and $\widetilde{Q} = Q \circ \pi$ of

$\binom{2}{2}$ $\binom{6}{4}$ $\binom{3}{2}$

Figure 1:

the functions P and Q to the space $\mathcal{X}$ of the modification are equal to $u \cdot y_1^{k_1} \cdot y_2^{k_2} \cdot \ldots \cdot y_n^{k_n}$ and $v \cdot y_1^{\ell_1} \cdot y_2^{\ell_2} \cdot \ldots \cdot y_n^{\ell_n}$ respectively, where $u(0) \neq 0$ and $v(0) \neq 0$.

Let $\mathcal{D} = \pi^{-1}(0)$ be the preimage of the origin, and let $S_{k,l}$ be the set of points of $\mathcal{D}$ in a neighbourhood of which the functions $P \circ \pi$ and $Q \circ \pi$ in some local coordinates have the form $u \cdot y_1^k$ and $v \cdot y_1^\ell$ respectively ($u(0) \neq 0$, $v(0) \neq 0$).

THEOREM 4 [**3**]

$$\zeta_f^0(t) = \prod_{k > \ell} (1 - t^{k-\ell})^{\chi(S_{k,\ell})},$$

$$\zeta_f^\infty(t) = \prod_{k < \ell} (1 - t^{\ell-k})^{\chi(S_{k,\ell})}.$$

For a meromorphic germ of two variables a resolution can be obtained by a sequence of blow-ups at points.

Example 6. The minimal resolution of the germ f from example 1 can be described by the picture on Fig. 1. Here lines correspond to components of the exceptional divisor $\mathcal{D}$. Each of them is isomorphic to the complex projective line $\mathbb{CP}^1$. Pairs of numbers near them are the multiplicities of the liftings of the numerator P and of the denominator Q along these components. The arrow (respectively the double arrow) corresponds to the strict transform of the curve $\{P = 0\}$ (respectively of the curve $\{Q = 0\}$). Then $S_{2,2}$ (respectively $S_{3,2}$ and $S_{6,4}$ is the complex projective line minus two points (minus one and three points respectively). Thus

$$\zeta_f^0(t) = (1 - t)(1 - t^2)^{-1} = \frac{1}{1 + t},$$

$$\zeta_f^\infty(t) = 1.$$

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Now let $f = \frac{P}{Q}$ be a meromorphic germ of two variables (i.e. P and Q are germs of holomorphic functions on $(\mathbb{C}^2, 0)$). Suppose that the curve $\{P = 0\}$ is reduced (i.e. has no multiple components) and has no common component with the curve $\{Q = 0\}$. In [4] we showed that 0 is a typical value of f if and only if the Euler characteristic $\chi(\mathcal{M}_f^0)$ of the 0-Milnor fibre is zero. This condition is equivalent to the fact that the family $P + cQ$ (c small enough) is a μ-constant family. In particular this shows that the notion of typical (respectively atypical) value is the same as the notion of generic (respectively special) value described by Lê and Weber. In [5], Lê and Weber gave a criterium for a value to be typical in terms of the minimal resolution of the pencil described by the meromorphic germ $f = \frac{P}{Q}$. Here we formulate a criterium for 0 be typical in terms of the minimal resolution of the curve $\{P = 0\}$ defined by the numerator.

Let $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ be the minimal (embedded) resolution of the curve $\{P = 0\}$. Each component of the exceptional divisor $\mathcal{D}$ is isomorphic to $\mathbb{CP}^1$. For a component E, let $k(E)$ and $l(E)$ be the multiplicities of the liftings $\widetilde{P} = P \circ \pi$ and $\widetilde{Q} = Q \circ \pi$ of the functions P and Q along E.

THEOREM 5 *In this case, 0 is a typical value for the meromorphic germ $f = \frac{P}{Q}$ if and only if the strict transform of the curve $\{P = 0\}$ intersects only components of the exceptional divisor $\mathcal{D}$ with $k(E) \leq l(E)$.*

Proof. According to Theorems 2 and 3 from [4] the value 0 is typical if and only if the family $P + cQ$ is μ-constant (for c from a neighbourhood of 0).

If a family P_c of functions of two variables ($c \in (\mathbb{C}, 0)$) is μ-constant then the embedded resolution of the curves $\{P_c = 0\}$ are combinatorially equivalent. However these resolutions are obtained by blow-ups of different points and thus the (minimal) resolution of the curve $\{P_0 = 0\}$ can be not a resolution of the curve $\{P_c = 0\}$ (see, e.g., the μ-constant family $(x + cy)^2 + y^3$).

Lemma. If a linear family $P + cQ$ of functions of two variables is μ-constant, then the minimal resolution of $\{P = 0\}$ is at the same time the minimal resolution of the curve $\{P + cQ = 0\}$ for c small enough.

Proof. Let $\pi_1 : (\mathcal{X}, E_1) \to (\mathbb{C}^2, 0)$ be the blowing-up of the origin in the plane $\mathbb{C}^2$, and let $\widetilde{C}_c$ be the strict transform of the curve $C_c = \{P + cQ = 0\}$. We shall prove that the intersection points of the curve $\widetilde{C}_c$ with the exceptional divisor E_1, at which the curve is not resolved by π_1, (i.e., those at which the intersection index $(\widetilde{C}_c \circ E_1)$ is greater than 1) do not depend on c. The reduced singularities of the total transform $\pi_1^{-1}(C_c)$ (i.e., of $\widetilde{C}_c \cup E_1$) at these points are μ–constant (since the family $\{C_c\}$ is μ–constant) and are determined by linear families of equations (since the family $\{C_c\}$ is determined by a linear family). Therefore one proves Lemma using the induction on the length of the resolution.

The intersection points of the strict transform $\widetilde{C}_c$ of the curve C_c with the exceptional divisor E_1 are those which belong to the projectivization of the tangent cone of the curve C_c. The tangent cone of the curve C_c is determined by the equation $P^{(d)} + cQ^{(d)}$, where d is the multiplicity of the curve $C = C_0$ at the origin (equal to the multiplicity of the

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curve C_c for c small enough), $P^{(d)}$ and $Q^{(d)}$ are homogeneous parts of degree d of the germs P and Q. Let $p(t)$ and $q(t)$ be the corresponding polynomials of one variable: $p(t) = P^{(d)}(t,1)$, $q(t) = Q^{(d)}(t,1)$ (for a corresponding choice of coordinates in the plane $\mathbb{C}^2$: so that $P(1,0) \neq 0$). The intersection points of the curve $\tilde{C}_c$ with the exceptional divisor E_1 are roots of the polynomial $p_c(t) = p(t) + c \cdot q(t)$. The multiplicity of a root is just the intersection index of $\tilde{C}_c$ and E_1 at the corresponding point.

Let t_0 be a root of the polynomial $p(t)$ of multiplicity $s > 1$. Then $p(t) = a \cdot (t - t_0)^s +$ terms of higher degree in $(t - t_0)$ ($a \neq 0$). Suppose that t_0 is a root of the polynomial $q(t)$ of multiplicity $r < s$, i.e., $q(t) = b \cdot (t - t_0)^r +$ terms of higher degree in $(t - t_0)$, where $b \neq 0$, $0 \leq r < s$. If $r > 0$, then, for c small enough, the polynomial $p_c(t) = p(t) + c \cdot q(t)$ has $s - r + 1$ different roots in a neighbourhood of the point t_0. If $r = 0$, then, for c small enough, the polynomial p_c has s different roots in a neighbourhood of the point t_0. Therefore in any case the root t_0 of the polynomial p splits into several (more than 1) different roots of the polynomial p_c. This contradicts the supposition that the family $P + cQ$ is μ–constant. $\square$

To finish the proof of Theorem 5, let us suppose that the strict transform of a branch of the curve $\{P = 0\}$ intersects a component E of the exceptional divisor with $k(E) > l(E)$. In this case in local coordinates at the point of intersection $\tilde{P} = u \cdot x^k \cdot y$ and $\tilde{Q} = v \cdot x^l$ ($u(0) \neq 0$, $v(0) \neq 0$). The lifting $\tilde{P} + c\tilde{Q}$ of the function $P + cQ$ is equal to $x^l(u \cdot x^{k-l} \cdot y + c \cdot v)$. Therefore its multiplicity along the component E is equal to l and it is less than that one of the function P. Thus π is not the minimal resolution of $\{P + cQ = 0\}$ for $c \neq 0$. Contradiction.

If the strict transform of the curve $\{P = 0\}$ intersect only components of the exceptional divisor with $k(E) \leq l(E)$, then the family $\tilde{P} + c\tilde{Q}$ in a neighbourhood of the intersection has the form $x^k(u \cdot y + c\dot{v} \cdot x^{l-k})$ with $l - k \geq 0$. Thus the strict transform of the corresponding branch of the curve $\{P + cQ = 0\}$ ($\{u \cdot y + c \cdot v \cdot x^{l-k} = 0\}$) is smooth and transversal to the exceptional divisor. Therefore $\{P + cQ = 0\}$ has the same resolution as $\{P = 0\}$ and the family is μ-constant. $\square$

The example $f = \frac{P}{Q} = \frac{xy}{x}$ shows that in general the condition $\chi(\mathcal{M}_f^0) = 0$ is not equivalent to the fact that the family $\{P + cQ = 0\}$ is μ constant at the origin. Nevertheless one has.

THEOREM 6 *Let $f = \frac{P}{Q}$ be a germ of meromorphic function of two variables. Then*

(i) *If the germ of the curve $\{P = 0\}$ at 0 has a non-isolated singularity but $\{P + cQ = 0\}$ has an isolated singularity (for c small enough) then the value 0 is atypical.*

(ii) *If $P = R \cdot P_1$ and $Q = R \cdot Q_1$ where $R = g.c.d.(P, Q)$ and the curve $\{P_1 = 0\}$ has an isolated singularity at the origin then 0 is a typical value for the meromorphic germ f if and only if $\chi(\mathcal{M}_f^0) = 0$.*

Proof. The first part follows from the definition of typical value.

The *if* part of (ii) follows from Theorem 3. For the "only if" part, let us assume that $\{P = 0\}$ has an isolated singularity at the origin.

If $Q_1(0) \neq 0$ then

$$\chi(\mathcal{M}_f^0) = \chi\left(\{(x,y) \in B_\varepsilon : P_1 = c\} \setminus \{R = 0\}\right) = 1 - \mu(P_1, 0) - (P_1, R)_0,$$

where $(P_1, R)_0$ is the intersection multiplicity of both curves at the origin. Therefore the Euler characteristic $\chi(\mathcal{M}_f^0)$ is equal to zero if and only if P_1 has no critical point at the origin and $(P_1, R)_0 = 1$. It means that we are in the case $f = \frac{P}{Q} = \frac{xy}{x}$.

If $Q_1(0) = 0$ then it follows from the proof of Theorem 2 in [4] that the Euler characteristic $\chi(\mathcal{M}_f^0)$ is equal to

$$-\mu(P, 0) + \sum_{A \in \{P+cQ=0\} \cap B_\varepsilon} \mu(P + cQ, A).$$

Let k (respectively s) be the intersection multiplicity at the origin of the curve $\{R = 0\}$ with the curve $\{P_1 = 0\}$ (respectively with the curve $\{P_1 + cQ_1 = 0\}$). At any other intersection point $A \in \{P_1 + cQ_1 = 0\} \cap \{R = 0\} \cap B_\varepsilon\} \setminus \{0\}$ the curve $\{P + cQ = 0\}$ has a nondegenerate critical point with Milnor number equals to 1. Let l be the number of such points. The conservation law of the intersection multiplicity gives $k = (R, P_1)_0 = (R, P_1 + cQ_1)_0 + l = s + l$.

Using the following formula for the Milnor number

$$\mu(RP_1, 0) = \mu(R, 0) + \mu(P_1, 0) + 2(R, P_1)_0 - 1$$

and the vanishing of the Euler characteristic $\chi(\mathcal{M}_f^0)$ one has

$$0 = \chi(\mathcal{M}_f^0) = (\mu(P_1 + cQ_1, 0) - \mu(P_1, 0)) + (R, P_1 + cQ_1)_0 - (R, P_1)_0.$$

Since the Milnor number and the intersection multiplicity are semicontinuous, the family $P_1 + cQ_1$ has to be μ-constant and $(R, P_1 + cQ_1)_0 = (R, P_1)_0$. Notice that these two last conditions are equivalents to the fact that the family $P + cQ$ is μ-constant. Now the proof that 0 is typical follows from the theorem of Parusinski ([7]) as in [4].

The proof of the "only if" part in the general case follows from the fact that if $R = g.c.d(P, Q) = R_1^{n-1} \cdots R_s^{n_s}$ and $P = R \cdot P_1$ and $Q = R \cdot Q_1$ then the meromorphic germ f defines the same fibration as the meromorphic germ $f = \frac{R_1 \cdots R_s \cdot P_1}{R_1 \cdots R_s \cdot Q_1}$ for which the Theorem has been proved. $\square$

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Addresses:

Sabir M. Gusein-Zade,
Moscow State University, Faculty of Mathematics and Mechanics.
e-mail: sabir@mccme.ru

Ignacio Luengo and Alejandro Melle-Hernández,
Departamento de Álgebra, Facultad de Matemáticas, Universidad Complutense de Madrid.
Ciudad Universitaria, 28040 Madrid, Spain.
e-mail: iluengo@eucmos.sim.ucm.es, amelle@eucmos.sim.ucm.es