Perturbation of the exponential type of linear nonautonomous parabolic equations and applications to nonlinear equations

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1 Introduction

Let Ω be a bounded domain in $I\!R^N, N \geq 1$, with smooth boundary $\partial\Omega$. Consider the linear non autonomous model equation

$$\left\{ \begin{array}{rcll} u_t - \Delta u &=& C(t,x)u & \text{in } \Omega, \quad t > s \\ \mathcal{B}u &=& 0 & \text{on } \partial\Omega \\ u(s) &=& u_0 & \end{array} \right. \tag{1.1}$$

with the boundary operator

$$\mathcal{B}u = u, \quad \text{Dirichlet case,} \quad \text{or} \quad \mathcal{B}u = \frac{\partial u}{\partial \vec{n}}, \quad \text{Neumann case,} \quad \mathcal{B}u = \frac{\partial u}{\partial \vec{n}} + b(x)u, \quad \text{Robin case,}$$

being $\vec{n}$ the outward normal vector-field to $\partial\Omega$ and $b(x)$ a C^1 function.

Note that if $C \in C^\theta(I\!R, L^p(\Omega))$, with $0 < \theta \leq 1$ and some $p > N/2$, then (1.1) is well posed for every initial data $u_0 \in L^q(\Omega)$ for $1 \leq q \leq \infty$; see [9] and [10]. Hence, (1.1) defines an evolution operator in $L^q(\Omega)$, $1 \leq q \leq \infty$, $U_C(t,s)$, as $U_C(t,s)u_0 := u(t,s;u_0)$.

Moreover, there exist $M > 0$ and $\beta \in I\!R$ such that

$$\|U(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M e^{\beta(t-s)} \quad \text{for all} \quad t > s. \tag{1.2}$$

Therefore, we can define the **exponential type**

$$\beta_0(C) = \inf\{\beta \in I\!R, \text{ such that (1.2) holds for some } M > 0\}, \tag{1.3}$$

which measures the growth/decay of solutions of (1.1).

In this paper we want to discuss the question of the minimal amount of perturbation needed to change the exponential type of the evolution equation.

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Observe that in the autonomous case, that is, when $C = C(x)$, with $C \in L^p(\Omega)$, and $p > N/2$, the exponential type of the associated semigroup is determined by the first eigenvalue of the associated eigenvalue problem

$$\begin{cases} -\Delta u = C(x)u + \lambda u & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega \end{cases}$$

which is given by

$$\lambda_1(C) = \min_{\phi} \frac{\int_\Omega |\nabla \phi|^2 - \int_\Omega C(x)|\phi|^2}{\int_\Omega |\phi|^2}$$

and

$$\beta_0(C) = -\lambda_1(C). \tag{1.4}$$

Note that the minimum above is attained over a suitable set of test functions, depending on the boundary conditions. Therefore, if $0 \leq P \in L^p(\Omega)$, with $p > N/2$, it is clear that $\lambda_1(C+P) \leq \lambda_1(C)$ and using that the minima are attained and that $P \neq 0$ then we get

$$\lambda_1(C+P) < \lambda_1(C).$$

Hence any signed, no zero, perturbation actually modifies the exponential type.

In the T–periodic case, that is when $C(t,x)$ in (1.1) is a T periodic function, using the Poincarè map associated to (1.1), using the positivity properties of the parabolic equation and the Krein–Rutman theorem, the exponential type (1.3) can be determined in terms of the periodic–parabolic eigenvalue problem

$$\left\{ \begin{array}{rcll} u_t - \Delta u &=& C(t,x)u + \mu u & \text{in } \Omega, \quad 0 < t < T \\ \mathcal{B}u &=& 0 & \text{on } \partial\Omega \\ u(T) &=& u(0). & \end{array} \right. \tag{1.5}$$

In fact, if μ is such that the solution u of (1.5) is positive in $\Omega \times (0,T)$ then from Proposition 14.4 in [4] we have

$$\beta_0(C) = \mathrm{e}^{-\mu T}.$$

See [4] for precise assumptions on the regularity of the coefficients and boundary conditions and further details. In particular note that $C(t,x)$ is assumed to be Hölder continuous in space and time and, for the case of Robin boundary conditions, it is assumed that $b \geq 0$ on the boundary of Ω.

Using these tools, Lemma 15.5 in [4] implies that for $P(t,x) \geq 0$, T–periodic, not identically zero and satisfying certain regularity properties, we have again

$$\beta_0(C) < \beta_0(C+P)$$

and the exponential type is actually modified.

In the general case, that is, when no assumption is made on the time behavior of the coefficients, we have no associated eigenvalue problems anymore. In fact there is no complete spectral theory as for the finite dimensional case, [12]. Thus, a different approach must be explored. Therefore, our goal here is to give sharp conditions on time dependent perturbations $P(t,x)$ of $C(t,x)$ in (1.1) to ensure that the exponential type of the perturbed equation

$$\left\{ \begin{array}{rcll} u_t - \Delta u &=& C(t,x)u + P(t,x)u & \text{in } \Omega, \quad t > s \\ \mathcal{B}u &=& 0 & \text{on } \partial\Omega \\ u(s) &=& u_0 & \end{array} \right.$$

is either increased or decreased.

As will be seen below, our results state that the exponential type is decreased provided the favorable part in the perturbation is “effectively positive” and the defavorable part is not too big. In doing this, we will only require conditions on the asymptotic values of the perturbation as $t \to \pm\infty$. Also, we will show that the good part of the perturbation must be “sustained” at infinity, that is it must be active for large times and on the whole domain; see Theorems 4.4 and 4.5. Otherwise we can not change the exponential type; see Remark 4.6. In particular, it is not true that any non–negative nontrivial perturbation changes the exponential type. Note in particular, that we can change the exponential type with periodic perturbations even if the original problem is not periodic. In particular the exponential type is decreased if the T–periodic perturbation $P(t,x)$ satisfies

$$\frac{1}{T}\int_0^T \inf_{x\in\Omega} P(r,x)\,dr > 0.$$

A particular important case is when the original system (1.1) is at the limit of stability (or neutrally stable) in the sense that the norm of the evolution operators are bounded above and below (in particular, the exponential type is $\beta_0 = 0$). Then our results give qualitative and quantitative threshold values on the perturbations that can stabilize the system, that is, to have solutions that decay exponentially.

In summary, our results, which are of perturbative nature, do not assume any kind of periodicity or almost–periodicity in the equation. Also, no sign conditions are imposed in the boundary coefficient in the case of Robin boundary conditions. Finally, perturbations are only assumed to be in the class $C^\theta(I\!R, L^p(\Omega))$, for some $p > N/2$. Indeed all the results here apply for much more general linear non–autonomous parabolic problems than (1.1), of the form

$$\left\{\begin{array}{rcll} u_t + A(t)u &=& C(t,x)u & \text{in } \Omega, \quad t > s\\ \mathcal{B}(t)u &=& 0 & \text{on } \partial\Omega\\ u(s) &=& u_0 & \end{array}\right.$$

with time dependent elliptic part of the form

$$A(t,D)u = -\sum_{i,j=1}^N a_{ij}(t,x)\partial_i\partial_j u + \sum_{i=1}^N a_i(t,x)\partial_i u + a(t,x)u$$

with suitable smooth coefficients and either Dirichlet boundary conditions or time–dependent boundary conditions of Robin type

$$\mathcal{B}(t)u = \frac{\partial u}{\partial \vec{\eta}} + b(t,x)u,$$

for suitable exterior (oblique) vector $\vec{\eta}$; see (6.1).

Using these results, we also analyze the asymptotic behavior of the positive solutions of the nonlinear equation

$$\left\{\begin{array}{rcll} u_t - \Delta u &=& f(t,x,u) & \text{in } \Omega, \quad t > s\\ \mathcal{B}u &=& 0 & \text{on } \partial\Omega\\ u(s) &=& u_0 \geq 0 & \end{array}\right. \tag{1.6}$$

with the conditions $f(t,x,0) \geq 0$ and

$$\frac{f(t,x,u)}{u} \quad \text{decreasing for} \quad u \geq 0 \tag{1.7}$$

and improve some results in [11]. In fact, in [9] there were given conditions on the nonlinear term $f(t,x,u)$ ensuring the existence of some special complete positive solutions of (1.6), that is, which are defined for all times; see Definition 3.6. Condition (1.7) guarantees the uniqueness of such solution $\varphi(t,x)$, see [11].

This special solution describes the asymptotic behavior of all positive solutions of (1.6) in a pullback sense, that is, for any positive initial data u_0, for $s \le t_0$ and for any $t \in I\!R$, we have that

$$u(t,s;u_0) - \varphi(t) \to 0, \quad \text{as } s \to -\infty \quad \text{in} \quad C(\overline{\Omega}).$$

Furthermore, $\varphi(t,x)$ also describes the forwards behavior of positive solutions of (1.6), since in fact it was also shown in [11] that for any $s \in I\!R$ and for any two positive solutions of (1.6), for $t > s$, we have,

$$u_1(t,x) - u_0(t,x) \to 0 \quad \text{as } t \to \infty \quad \text{in} \quad C(\overline{\Omega}).$$

Our goal here is to show that such convergences are actually exponential, see Theorems 5.3 and 5.4.

An important particular example considered in [11] are logistic equations, for which

$$f(t,x,u) = m(t,x)u - n(t,x)u^{\rho}, \qquad \rho \ge 2$$

where $m \in C^{\theta}(I\!R, L^p(\Omega))$ for certain $p > N/2$ and $0 < \theta \le 1$ and $n \ge 0$ is continuous and locally Hölder in t, not identically zero. Our results here also apply to these models; see Remark 5.2.

The paper is organized as follows. In Section 2 we present some of the basic estimates on the solutions of (1.1) that will be used as building blocks for the rest of the results. We also define the exponential type of the evolution operator at $\pm\infty$ which reflects the possible different behavior of solutions for large positive and very negative times. We show then that the exponential type is independent of the space in which we look at the solutions. Some relationship with the principal spectrum, as defined in [7], [6] is also given. In particular the exponential type coincides with the so called principal Lyapunov exponent. Also, some estimate on the exponential type is derived form the elliptic part of the equation.

In Section 3 we take advantage of the order preserving properties of the solutions, that is, of the maximum principle, and relate the exponential type with the behavior of positive solutions. In particular we show how the exponential type at infinity can be estimated by observing the forwards behavior of a given particular positive solution. On the other hand, we also show how the exponential type at minus infinity is related to the behavior of complete positive solutions. The existence of such objects has been studied in [6, 7] and [8].

In Section 4 we give our main results on the linear problem above, giving conditions on the perturbations that guarantee the change in the exponential type of the evolution operators; see Theorems 4.4 and 4.5. As mentioned before this is done by only imposing conditions only in the asymptotic values of the perturbations as $t \to \pm\infty$. As a by product we also show that if the perturbation is not sustained enough at $\pm\infty$ then, actually, no change in the exponential type is achieved. Some particular easy–to–apply cases are also given in Propositions 4.7 and 4.9. Note in particular that in the latter result we allow sing changing perturbations with very large bad values in small time–wandering sets in Ω.

In Section 5, we apply our previous results to the nonlinear non-autonomous problem (1.6). In particular we first show that the special solution $\varphi(t,x)$ mentioned above, is linearly exponentially stable, both forwards and in pullback senses; see Proposition 5.1. Then we show that

$\varphi(t,x)$ attracts the dynamics of positive solutions of (1.6) exponentially fast; see Theorems 5.3 and 5.4.

Finally, in Section 6 we discuss how all the previous results on linear equations apply for much more general classes of parabolic equations including time–dependent coefficients and boundary conditions.

2 Exponential type of evolution equations

We consider the problem

$$\left\{\begin{array}{rcll} u_t - \Delta u &=& C(t,x)u, & \text{in} \quad \Omega, \quad t>s \\ \mathcal{B}u &=& 0, & \text{on} \quad \partial\Omega, \quad t>s \\ u(s) &=& u_0 & \end{array}\right. \tag{2.1}$$

posed in $X = L^q(\Omega)$ with $1 < q < \infty$ or in $X = C(\overline{\Omega})$. Then, quoting results from [10], we have that if $C \in C^\theta(I\!R, L^p(\Omega))$, with $0 < \theta \leq 1$ and some $p > N/2$, then (2.1) defines an order preserving evolution operator in X. We denote this evolution operator by $U_C(t,s)$, i.e. $u(t,s;u_0) = U_C(t,s)u_0$ is the solution of (2.1).

Moreover for each q and r with $1 \leq q \leq r \leq \infty$ and $R_0 > 0$ there exist $L = L(R_0, r, q) > 0$ and $\delta = \delta(R_0, r, q) > 0$ such that the evolution operator $U_C(t,s)$ satisfies

$$\|U_C(t,s)u_0\|_{L^r(\Omega)} \leq L \frac{e^{\delta(t-s)}}{(t-s)^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}} \|u_0\|_{L^q(\Omega)}, \qquad t>s \tag{2.2}$$

for every $C \in C^\theta(I\!R, L^p(\Omega))$, with $0 < \theta \leq 1$ and some $p > N/2$, such that $\|C\|_{L^\infty(I\!R, L^p(\Omega))} \leq R_0$.

Also, the evolution operator smoothes the solutions. More precisely, for every $u_0 \in L^q(\Omega)$ and $t > s$ we have

$$(s,\infty) \ni t \longmapsto u(t,s;u_0) := U_C(t,s)u_0 \in \left\{\begin{array}{ll} C^\nu_{\mathcal{B}}(\overline{\Omega}) & \text{if } p > N/2 \\ C^{1,\nu}_{\mathcal{B}}(\overline{\Omega}) & \text{if } p > N \end{array}\right.$$

is continuous for some $\nu > 0$. Here $C^{j,\nu}_{\mathcal{B}}(\overline{\Omega}) = \left\{\begin{array}{ll} C^{j,\nu}_0(\overline{\Omega}) & \text{for Dirichlet} \\ C^{j,\nu}(\overline{\Omega}) & \text{for Neumann or Robin} \end{array}\right.$, see e.g. [10].

Note that (2.2) implies that, with $r = q$, the evolution operator satisfies

$$\|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq L e^{\delta(t-s)}$$

with $L = L(R_0, q)$ and $\delta = \delta(R_0, q)$ if $\|C\|_{L^\infty(I\!R, L^p(\Omega))} \leq R_0$.

We will see now that, in fact, for a given $C(t,x)$ an exponent in such an estimate can be taken independent of q; see Lemma 3.1 in [9].

Lemma 2.1 *Assume that $U = U_C$, as above, as an evolution operator in $L^q(\Omega)$, $1 \leq q \leq \infty$, satisfies*

$$\|U(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M e^{\beta(t-s)} \tag{2.3}$$

for some constants $M > 0$ and $\beta \in I\!R$ and for all $t > s \geq s_0$ or $t_0 \geq t > s$ respectively.

Then, as an operator in $L^r(\Omega)$, with $1 < r \leq \infty$, $U(t,s)$ satisfies, for all $t > s \geq s_0$ or $t_0 \geq t > s$

$$\|U(t,s)\|_{\mathcal{L}(L^r(\Omega))} \leq K e^{\beta(t-s)} \quad \textit{for all} \quad t > s,$$

respectively, for $K = L\mathrm{e}^{\delta}\max\{M\mathrm{e}^{-\beta}c(|\Omega|), \mathrm{e}^{\beta_-}\}$*, where* $\beta_- = \max\{-\beta, 0\} \geq 0$ *denotes the negative part of* β *and* L, δ *as in (2.2).*

Proof. Suppose that $r \geq q$, so that $L^r(\Omega) \subseteq L^q(\Omega)$. Then, if $t - s > 1$, since $U(t,s) = U(t,t-1)U(t-1,s)$,

$$\|U(t,s)u_0\|_{L^r(\Omega)} \leq \|U(t,t-1)\|_{\mathcal{L}(L^q(\Omega),L^r(\Omega))}\|U(t-1,s)u_0\|_{L^q(\Omega)}.$$

Using now (2.3) and (2.2) we have

$$\|U(t,s)u_0\|_{L^r(\Omega)} \leq LM\mathrm{e}^{\delta-\beta}\mathrm{e}^{\beta(t-s)}\|u_0\|_{L^q(\Omega)} \leq LM\mathrm{e}^{\delta-\beta}c(|\Omega|)\mathrm{e}^{\beta(t-s)}\|u_0\|_{L^r(\Omega)}.$$

Thus

$$\|U(t,s)\|_{\mathcal{L}(L^r(\Omega))} \leq K_0\mathrm{e}^{\beta(t-s)}$$

for all $t - s > 1$ with $K_0 = LM\mathrm{e}^{\delta-\beta}c(|\Omega|)$.

Suppose now that $1 \leq r < q$, and therefore $L^q(\Omega) \subset L^r(\Omega)$. Now, if $t - s > 1$ we remark that $U(t,s) = U(t,s+1)U(s+1,s)$. So, using (2.3) and (2.2)

$$\begin{aligned}
\|U(t,s)u_0\|_{L^r(\Omega)} &\leq c(|\Omega|)\|U(t,s)u_0\|_{L^q(\Omega)} \\
&\leq c(|\Omega|)\|U(t,s+1)\|_{\mathcal{L}(L^q(\Omega))}\|U(s+1,s)u_0\|_{L^q(\Omega)} \\
&\leq c(|\Omega|)LM\mathrm{e}^{\delta-\beta}\mathrm{e}^{\beta(t-s)}\|u_0\|_{L^r(\Omega)}.
\end{aligned}$$

Thus,

$$\|U(t,s)\|_{\mathcal{L}(L^r(\Omega))} \leq K_0\mathrm{e}^{\beta(t-s)}$$

for all $t - s > 1$, with K_0 as above.

Finally, for $t - s \leq 1$ and for either case of r, from (2.2), we have

$$\|U(t,s)\|_{\mathcal{L}(L^r(\Omega))} \leq L\mathrm{e}^{\delta}\mathrm{e}^{-\beta(t-s)}\mathrm{e}^{\beta(t-s)} \leq K_1\mathrm{e}^{\beta(t-s)}$$

with $K_1 = \left\{\begin{array}{ll} L\mathrm{e}^{\delta} & \text{if } \beta \geq 0 \\ L\mathrm{e}^{\delta-\beta} & \text{if } \beta < 0 \end{array}\right\} = L\mathrm{e}^{\delta+\beta_-}$, where $\beta_- = \max\{-\beta, 0\} \geq 0$ denotes the negative part of β.

Now we take $K = \max\{K_0, K_1\}$ and the result follows. ■

Note that the constant K in the lemma also depends on q and r but we will not pay attention to this dependence.

Hence we can define

Definition 2.2

i) The exponential type at ∞ *of the evolution operator* $U_C(t,s)$ *is the best exponent in the inequality*

$$\|U(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M\mathrm{e}^{\beta(t-s)} \quad \textit{for all} \quad t > s \geq s_0, \tag{2.4}$$

that is,

$$\beta_0^+(C) = \inf\{\beta \in I\!R, \ \textit{such that } (2.4) \textit{ holds for some } M > 0 \textit{ and } t > s \geq s_0\}$$

for some s_0*.*

ii) The exponential type at $-\infty$ of the evolution operator $U_C(t,s)$ is the best exponent in the inequality

$$\|U(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M e^{\beta(t-s)} \quad \textit{for all} \quad t_0 \geq t > s, \tag{2.5}$$

that is,

$$\beta_0^-(C) = \inf\{\beta \in I\!R, \ \textit{such that } (2.5) \textit{ holds for some } M > 0 \textit{ and } t_0 \geq t > s\}$$

for some t_0.
iii) The exponential type of the evolution operator $U_C(t,s)$ is the best exponent in the inequality

$$\|U(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M e^{\beta(t-s)} \quad \textit{for all} \quad t > s,$$

that is,

$$\beta_0(C) = \max\{\beta_0^-(C), \beta_0^+(C)\}.$$

Now observe that for each β such that (2.4) or (2.5) are satisfied, the optimal constant in these inequalities depends on β, i.e. $M = M(\beta)$. Also, it can be easily shown that $M(\beta)$ is a decreasing function of β. This constant may depend on q and Ω as well, but we will not pay attention to such dependence.

In general it may happen that as β approaches the optimal value, $\beta_0^{\pm}(C)$, the best constant $M(\beta)$ diverges. Hence we have the following

Definition 2.3 *We say $U_C(t,s)$ has "defect $\gamma \geq 0$" at $\pm\infty$, if for $\varepsilon > 0$ we have that the best constant in (2.4) or (2.5) satisfies*

$$M(\beta_0^{\pm}(C) + \varepsilon) \leq D_0(\varepsilon^{-\gamma} + 1)$$

for some constant $D_0 > 0$.

Note that the defect is zero iff the exponential type is attained, that is, if (2.4) or (2.5) hold for $\beta = \beta_0^{\pm}(C)$ respectively. Hence, summarizing the consequences of Lemma 2.1, we have

Corollary 2.4 *With the notations above,*
i) The exponential type of the evolution operator $U_C(t,s)$ at $\pm\infty$, is independent of the $L^q(\Omega)$ space.
ii) The defect $\gamma \geq 0$ of the evolution operator $U_C(t,s)$ at $\pm\infty$, is independent of the $L^q(\Omega)$ space.

Proof. Part i) is clear. For part ii) just note that the constant K in Lemma 2.1 satisfies $K(\beta) = Le^{\delta}\max\{M(\beta)e^{-\beta}c(|\Omega|), e^{\beta_-}\}$, and the result follows taking $\beta = \beta_0^{\pm}(C) + \varepsilon$. ■

Remark 2.5 *Note that if $\beta_0^+(C) < 0$ then for a bounded set of initial data u_0, all solutions of (2.1), $u(t,s;u_0)$, decay exponentially to zero, as $t \to \infty$. We say then that $U_C(t,s)$ is exponentially stable at ∞.*

On the other hand, if $\beta_0^-(C) < 0$ then for a bounded set of initial data u_0, all solutions of (2.1), $u(t,s;u_0)$ decay exponentially to zero, as $s \to -\infty$, that is in the pullback sense. We say then that $U_C(t,s)$ is exponentially stable at $-\infty$ or exponentially pullback stable.

Observe that the concept of exponential type used above is closely related to that of principal spectrum for nonautonomous equations, [6], [7], [5] and references therein. More precisely the principal spectrum, related to the dynamical spectrum or the Sacker and Sell spectrum in finite dimensions, [12], is defined as the set of all possible limits

$$\lim_n \frac{\ln \|U(t_n, s_n)\|_{\mathcal{L}(L^q(\Omega))}}{t_n - s_n}$$

on all sequences such that $t_n - s_n \to \infty$. This set is a closed interval and thus coincides with $[\beta_{inf}, \beta_{sup}]$ where

$$-\infty \leq \beta_{inf} = \liminf \frac{\ln \|U(t_n, s_n)\|_{\mathcal{L}(L^q(\Omega))}}{t_n - s_n}, \quad \beta_{sup} = \limsup \frac{\ln \|U(t_n, s_n)\|_{\mathcal{L}(L^q(\Omega))}}{t_n - s_n}$$

where the liminf and limsup are taken on all sequences such that $t_n - s_n \to \infty$. It is clear then that

$$\beta_0(C) = \beta_{sup},$$

which, is also denoted the principal Lyapunov exponent.

Also note that considering only sequences such that $t_n - s_n \to \infty$ and $s_n \geq s_0$ or $t_0 \geq t_n$ leads, respectively, to the numbers $\beta^{\pm}_{sup}$ and $\beta^{\pm}_{inf}$. Also clearly

$$\beta^{\pm}_0(C) = \beta^{\pm}_{sup}.$$

See Section 3 for further details on the principal spectrum for (2.1).

Now we show that the exponents in (2.4), (2.5) are also related to the smoothing estimates between Lebesgue spaces of the evolution operator, see [9], Lemma 3.2.

Lemma 2.6 *If (2.4) or (2.5) is satisfied with $M = M(\beta)$ then, for $1 \leq q \leq r \leq \infty$, for every $\varepsilon > 0$*

$$\|U_C(t, s)\|_{\mathcal{L}(L^q(\Omega), L^r(\Omega))} \leq M(\varepsilon, \beta) \frac{\mathrm{e}^{(\beta+\varepsilon)(t-s)}}{(t-s)^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}}, \tag{2.6}$$

for $t > s \geq s_0$ or $t_0 \geq t > s$ respectively, with

$$M(\varepsilon, \beta) = K(\beta)\mathrm{e}^{|\beta|} \left\{ \begin{array}{ll} (\frac{\alpha}{\mathrm{e}})^{\alpha} \varepsilon^{-\alpha} & \text{if } 0 < \varepsilon < \varepsilon_0 = \frac{\alpha}{\mathrm{e}} \\ 1 & \text{if } \varepsilon \geq \varepsilon_0 = \frac{\alpha}{\mathrm{e}} \end{array} \right\} \leq K(\beta)\mathrm{e}^{|\beta|} c(\alpha)(\varepsilon^{-\alpha} + 1)$$

where $K(\beta) = L\mathrm{e}^{\delta} \max\{1, M(\beta)\mathrm{e}^{-\beta}\}$, with L, δ as in (2.2), and $\alpha = \frac{N}{2}\left(\frac{1}{q} - \frac{1}{r}\right)$.

Proof. From (2.2) for $t - s \leq 1$,

$$\|U(t, s)\|_{\mathcal{L}(L^q(\Omega), L^r(\Omega))} \leq L\mathrm{e}^{\delta}(t-s)^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}$$

and, for $t - s > 1$, from (2.2) and (2.3)

$$\begin{aligned} \|U(t, s)\|_{\mathcal{L}(L^q(\Omega), L^r(\Omega))} &\leq \|U(t, t-1)\|_{\mathcal{L}(L^q(\Omega), L^r(\Omega))} \|U(t-1, s)\|_{\mathcal{L}(L^q(\Omega))} \\ &\leq LM(\beta)\mathrm{e}^{\delta-\beta}\mathrm{e}^{\beta(t-s)}. \end{aligned}$$

Hence,

$$\|U(t,s)\|_{\mathcal{L}(L^q(\Omega),L^r(\Omega))} \leq \begin{cases} K(\beta)(t-s)^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} & \text{if } t-s\leq 1 \\ K(\beta)\mathrm{e}^{\beta(t-s)} & \text{if } t-s>1 \end{cases} \tag{2.7}$$

for $K(\beta) = L\mathrm{e}^{\delta}\max\{1, M(\beta)\mathrm{e}^{-\beta}\}$.

The right hand side in (2.7) can be bounded above by a right hand side as in (2.6), iff

$$M(\varepsilon,\beta) \geq K(\beta)\mathrm{e}^{-(\beta+\varepsilon)}, \quad \text{and} \quad M(\varepsilon,\beta)\geq K(\beta)\sup_{z\geq 1} h(z)$$

with $\alpha = \frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)$ and $h(z) = z^{\alpha}\mathrm{e}^{-\varepsilon z}$. Note that $\max\{1, \mathrm{e}^{-(\beta+\varepsilon)}\} \leq \mathrm{e}^{|\beta|}$, hence the condition above can be recast as

$$M(\varepsilon,\beta) \geq K(\beta)\mathrm{e}^{|\beta|}\max\{1, \sup_{z\geq 1} h(z)\}.$$

Since, the sup of $h(z)$ for $z \geq 0$ is attained at $z_* = \alpha/\varepsilon$ and $h(z_*) = (\frac{\alpha}{\mathrm{e}})^{\alpha}\varepsilon^{-\alpha}$, comparing with $h(1) = \mathrm{e}^{-\varepsilon}$ we get

$$\sup_{z\geq 1} h(z) = \begin{cases} (\frac{\alpha}{\mathrm{e}})^{\alpha}\varepsilon^{-\alpha} & \text{for } \varepsilon\leq\alpha \\ \mathrm{e}^{-\varepsilon} & \text{for } \varepsilon>\alpha \end{cases}.$$

Now, comparing this sup with 1, the result follows. ■

Next we give an upper bound on the exponential type of an evolution operator $U_C(t,s)$. See the next section for further upper and lower bounds.

Lemma 2.7 *Let $C \in C^{\theta}(I\!R, L^p(\Omega))$ with $0 < \theta \leq 1$ and $p > N/2$. Denote by $\lambda_1(t)$ is the first eigenvalue of the problem*

$$\begin{cases} -\Delta u - C(t,x)u &= \lambda(t)u \quad \text{in} \quad \Omega \\ \mathcal{B}u &= 0 \quad\quad\ \ \text{on} \quad \partial\Omega. \end{cases}$$

Assume there exists s_0 (or t_0 respectively), $\tau > 0$ and $m \in I\!R$, such that for all $t > s \geq s_0$ (or $s < t \leq t_0$ respectively) and $t - s \geq \tau$

$$\frac{1}{t-s}\int_s^t \lambda_1(r)\,dr \geq m.$$

Then the exponential type of evolution operator $U_C(t,s)$ satisfies

$$\beta_0^+(C) \leq -m.$$

($\beta_0^-(C) \leq -m$ respectively).

Proof. As the exponential type is independent of the Lebesgue space, we take $X = L^2(\Omega)$ and for any fixed $t \in I\!R$, the first eigenvalue satisfies

$$\int_{\Omega}\left(|\nabla\varphi|^2 - C(t,x)|\varphi|^2\right)\,dx + I(u,\mathcal{B}) \geq \lambda_1(t)\|\varphi\|^2, \tag{2.8}$$

for all smooth functions φ satisfying $\mathcal{B}u = 0$ on $\partial\Omega$, where we have denoted by $\|\cdot\|$ the norm in $L^2(\Omega)$ and $I(u,\mathcal{B}) = 0$ for Dirichlet or Neumann boundary conditions or $I(u,\mathcal{B}) = \int_{\partial\Omega} b(x)u^2$ for Robin boundary conditions.

Multiplying the first equation in (2.1) by $u(t)$ and integrating in Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \int_\Omega \left(|\nabla u|^2 - C(t,x)|u|^2\right)\, dx + I(u(t),\mathcal{B}) = 0.$$

By (2.8) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + 2\lambda_1(t)\|u(t)\|^2 \le 0$$

and then

$$\|u(t)\|^2 \le \mathrm{e}^{-2\int_s^t \lambda_1(r)\, dr}\|u(s)\|^2.$$

The rest follows from part v) of the Lemma below. ■

Now we prove the Lemma used above, that introduces a class of real functions that will be used several times henceforth.

Lemma 2.8 *Given $m \in I\!R$, define the class $\mathcal{C}^+(m)$ (or $\mathcal{C}^-(m)$ respectively) of real continuous and bounded functions $f(t)$, such that there exists s_0 (or t_0 respectively) and $\tau > 0$, such that for all $t > s \ge s_0$ (or $s < t \le t_0$ respectively) and $t - s \ge \tau$*

$$\frac{1}{t-s}\int_s^t f(r)\, dr \ge m.$$

Then
i) $\mathcal{C}^\pm(m)$ is a nonempty, convex, closed subset of $C_b(I\!R)$.
ii) If $f(t) = m$ for all t, then $f \in \mathcal{C}^\pm(m)$.
iii) If $f \in \mathcal{C}^\pm(m)$ and $g(t)$ is a continuous and bounded function such that $g(t) \ge f(t)$, then $g \in \mathcal{C}^\pm(m)$.
iv) If $f(t)$ is a T-periodic continuous function, set

$$m = \frac{1}{T}\int_0^T f(r)\, dr.$$

Then $f \in \mathcal{C}^\pm(m-\varepsilon)$ for every $\varepsilon > 0$.
v) If $f \in \mathcal{C}^\pm(m)$ then there exists s_0 (or t_0 respectively) and $M = M(f,m)$ such that for all $t > s \ge s_0$ (or $s < t \le t_0$ respectively) we have

$$\mathrm{e}^{-\int_s^t f(r)\, dr} \le M\mathrm{e}^{-m(t-s)}.$$

Even more $M = 1$ if $f \ge 0$ and $m \ge 0$.

Proof. Parts i)–iii) are immediate. For part iv), given $t > s$, using $z = t - s > 0$, we have

$$\frac{1}{t-s}\int_s^t f(r)\, dr = \frac{1}{z}\int_0^z f(r+s)\, dr.$$

Therefore it is enough to prove that for a T-periodic function and for every $\varepsilon > 0$ there exists $\tau = \tau(\varepsilon) > 0$ such that for $t > \tau$ we have

$$F(t) = \frac{1}{t}\int_0^t f(r)\, dr \ge m - \varepsilon,$$

for every $\varepsilon > 0$, with τ independent of all translations of f.

Now note that $F(0) = f(0)$ and $F(jT) = m$ for $j = 1, 2, \ldots$. Then for $t \in [jT, (j+1)T)$, $t = jT + s$, with $0 \le s < T$, using periodicity we have

$$F(t) = \frac{1}{jT+s}\Big(\int_0^{jT} f(r)\,dr + \int_{jT}^{jT+s} f(r)\,dr \Big) = \frac{1}{jT+s}\Big(jmT + \int_0^{s} f(r)\,dr \Big).$$

But for $0 \le s < T$, $\int_0^s f(r)\,dr \ge I_0 := -\int_0^T f_-(r)\,dr$, where $f_-(t) = \max\{-f(t), 0\}$ denotes the negative part of $f(t)$. Also, the same bound hold for any translate of f. Hence,

$$F(t) \ge \frac{1}{(j+1)T}\Big(jmT + I_0 \Big) \to m, \quad \text{as } j \to \infty$$

and the result follows.

Finally, for v), if $f \in \mathcal{C}^{\pm}(m)$ let s_0 (or t_0 respectively) and and $\tau > 0$ as in the definition. Then, clearly, for all $t > s \ge s_0$ (or $s < t \le t_0$ respectively) and $t - s \ge \tau$, we have

$$\mathrm{e}^{-\int_s^t f(r)\,dr} \le \mathrm{e}^{-m(t-s)}.$$

Then it is enough to observe that for any $\gamma \in I\!R$ there exists $M = M(f, \gamma)$ such that for all $t > s \ge s_0$ (or $s < t \le t_0$ respectively) and $t - s \le \tau$, we have

$$\mathrm{e}^{-\int_s^t f(r)\,dr} \le \mathrm{e}^{\|f_-\|_\infty \tau} \le M \mathrm{e}^{\gamma(t-s)}$$

for

$$M(f,\gamma) = \begin{cases} \mathrm{e}^{\|f_-\|_\infty \tau} & \text{if } \gamma \ge 0 \\ \mathrm{e}^{(\|f_-\|_\infty - \gamma)\tau} & \text{if } \gamma < 0 \,. \end{cases}$$

Then we take $\gamma = m$. ■

As we will see below the classes $\mathcal{C}^+(m)$ and $\mathcal{C}^-(m)$ will play an important role in the results of the next sections.

3 Exponential type, principal spectrum and positive solutions

One crucial property of (2.1) that has not been exploited in the previous section is the order preserving property. Thus, in this section we take advantage of this property and relate the exponential type with the behavior of positive solutions.

Lemma 3.1 *i) If $u_0 \ge 0$ then $U_C(t,s)u_0 \ge 0$ and is strictly positive in Ω, for $t > s$. Additionally for every $u_0 \in L^q(\Omega)$ we have $|U_C(t,s)u_0| \le U_C(t,s)|u_0|$.*

In particular, positive functions grow at the maximum rate, i.e. $\beta_0^{\pm}(C)$ is the best exponent in the inequality

$$\|U(t,s)u_0\|_{L^q(\Omega)} \le M \mathrm{e}^{\beta(t-s)} \|u_0\|_{L^q(\Omega)}$$

for all $0 \le u_0 \in L^q(\Omega)$ and for all $t > s \ge s_0$ or $s < t \le t_0$, respectively.
ii) Assume that for all $t > s \ge s_0$ or $s < t \le t_0$, we have

$$C_1(t,x) \le C_2(t,x).$$

Then

$$\beta_0^{\pm}(C_1) \leq \beta_0^{\pm}(C_2)$$

iii) In particular, assume that for either all $t \geq s_0$ or $t \leq t_0$ we have

$$C_0(x) \leq C(t,x) \quad or \quad C(t,x) \leq C_1(x)$$

for every $x \in \Omega$ and for some $C_i \in L^p(\Omega)$ for some $p > N/2$, $i = 1, 2$. Then

$$-\lambda_1(C_0) \leq \beta_0^{\pm}(C) \quad or \quad \beta_0^{\pm}(C) \leq -\lambda_1(C_1),$$

respectively, where $\lambda_1(C_i)$ is the first eigenvalue of the problem

$$\left\{ \begin{array}{rcll} -\Delta u - C_i(x)u &=& \lambda u & in \quad \Omega \\ \mathcal{B}u &=& 0 & on \quad \partial\Omega. \end{array} \right.$$

Proof. For i) we just refer to [10]. Just observe that $U_C(t,s)u_0 \geq 0$ and is strictly positive in Ω, for $t > s$, as a consequence of the maximum principle. This in particular implies, since $U_C(t,s)$ is order preserving, that $|U_C(t,s)u_0| \leq U_C(t,s)|u_0|$ and the rest is easy.

For ii) note that from comparison, we have for $u_0 \geq 0$, $t > s$ and $s \geq s_0$ or $t \leq t_0$

$$U_{C_1}(t,s)u_0 \leq U_{C_2}(t,s)u_0$$

and the result follows from i).

For iii) note that we have again, for $u_0 \geq 0$, $t > s$ and $s \geq s_0$ or $t \leq t_0$

$$U_C(t,s)u_0 \geq U_{C_0}(t,s)u_0 = S_{C_0}(t-s)u_0 \geq 0$$

or

$$U_C(t,s)u_0 \leq U_{C_1}(t,s)u_0 = S_{C_1}(t-s)u_0$$

where $S_{C_i}(t)$ denotes the semigroup associated to the autonomous linear equation

$$\left\{ \begin{array}{rcll} z_t - \Delta z &=& C_i(x)z, & \text{in } \Omega \\ \mathcal{B}z &=& 0 & \text{on } \partial\Omega. \end{array} \right.$$

Then the result follows easily from ii) and (1.4). ■

The next result states that we can obtain some estimates on the norms $\|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))}$ for $t > s \geq s_0$ by observing the forwards behavior of a particular positive solution.

Lemma 3.2
i) If there exists $u_0 \in L^q(\Omega)$, $u_0 > 0$ a.e. in Ω, and a $\beta \in I\!R$, such that

$$\|u(t,s_0;u_0)\|_{L^q(\Omega)} \leq M e^{\beta(t-s_0)}, \qquad for\ all \quad t > s_0 \tag{3.1}$$

then for some positive constant $M_1(s)$ we have

$$\|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M_1(s) e^{\beta(t-s)}, \qquad for\ all \quad t \geq s \geq s_0.$$

ii) If there exists $u_0 \in L^q(\Omega)$ and $\beta' \in I\!R$, $m > 0$, such that

$$m e^{\beta'(t-s_0)} \leq \|u(t,s_0;u_0)\|_{L^q(\Omega)}, \qquad for\ all \quad t > s_0$$

then
$$\|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))} \geq M_0(s)\mathrm{e}^{\beta'(t-s)}, \qquad \textit{for all} \quad t \geq s \geq s_0$$
with $M_0(s) = \frac{m\mathrm{e}^{\beta'(s-s_0)}}{\|u(s,s_0;u_0)\|_{L^q(\Omega)}}$.
iii) Therefore, if
$$m\mathrm{e}^{\beta'(t-s_0)} \leq \|u(t,s_0;u_0)\|_{L^q(\Omega)} \leq M\mathrm{e}^{\beta(t-s_0)}, \qquad \textit{for all} \quad t > s_0$$
with $\beta' \leq \beta$, *then*
$$M_0(s)\mathrm{e}^{\beta'(t-s)} \leq \|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M_1(s)\mathrm{e}^{\beta(t-s)}, \qquad \textit{for all} \quad t \geq s \geq s_0$$
for $M_0(s) = \frac{m}{M}\mathrm{e}^{-(\beta-\beta')(s-s_0)}$ *and some positive constant* $M_1(s)$.

Proof. For i), observe first that we can always assume that (3.1) is satisfied for every $t > s \geq s_0$. For this define $w(s) = \mathrm{e}^{-\beta(s-s_0)}u(s,s_0;u_0)$. Then $\|w(s)\|_{L^q(\Omega)} \leq M$ and
$$\|u(t,s;w(s))\|_{L^q(\Omega)} \leq M\mathrm{e}^{\beta(t-s)}.$$
Also $w(s) \in L^q(\Omega)$, $w(s) > 0$ a.e. in Ω.

Now, fix $s \geq s_0$ and take any $t > s \geq s_0$ and consider an initial data $v_0 \in L^q(\Omega)$, such that there exists $\lambda = \lambda(s, v_0)$ such that $|v_0| \leq \lambda w(s)$. Then
$$\left|\mathrm{e}^{-\beta(t-s)}U_C(t,s)v_0\right| \leq \lambda \mathrm{e}^{-\beta(t-s)}U_C(t,s)w(s).$$
Thus
$$\|\mathrm{e}^{-\beta(t-s)}U_C(t,s)v_0\|_{L^q(\Omega)} \leq \lambda(s,v_0)M, \qquad \text{for any } t > s.$$

Now observe that the set
$$\mathcal{C}(s) = \{v, \ \exists \lambda > 0, \ |v(x)| \leq \lambda w(s,x) \text{ a.e. } x \in \Omega\}$$
is dense in $L^q(\Omega)$ since $w(s) > 0$ a.e. in Ω, see Lemma 3.3 below.

Hence, $T(t,s) = \mathrm{e}^{-\beta(t-s)}U_C(t,s)$, for $t > s \geq s_0$, is pointwise bounded in a dense subset of $L^q(\Omega)$ and hence, the upper bound follows from the Uniform Boundedness Principle.
ii) Just note that $\|u(t,s_0;u_0)\|_{L^q(\Omega)} \geq m\mathrm{e}^{\beta'(s-s_0)}\mathrm{e}^{\beta'(t-s)}$ and also
$$\|u(t,s_0;u_0)\|_{L^q(\Omega)} = \|U_C(t,s)u(s,s_0;u_0)\|_{L^q(\Omega)} \leq \|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))}\|u(s,s_0;u_0)\|_{L^q(\Omega)}.$$
Hence, the lower bound follows and iii) is obvious. ■

Now we prove the Lemma used above.

Lemma 3.3 *Let* $\Omega \subset I\!\!R^N$ *bounded,* $1 \leq q < \infty$ *and* $0 \leq u_0 \in L^q(\Omega)$ *in* Ω. *Then they are equivalent*
i) The set
$$\mathcal{C} = \{v, \ \exists \lambda > 0, \ |v(x)| \leq \lambda u_0(x) \ \textit{a.e. } x \in \Omega\}$$
is dense in $L^q(\Omega)$.
ii) $u_0 > 0$ *a.e. in* Ω.

Proof. i) $\Rightarrow$ ii) Assume $A_0 = \{x \in \Omega,\ u_0(x) = 0\}$ has positive measure. Define $\phi = \mathcal{X}_{A_0} \in L^{q'}(\Omega)$ and then

$$\int_\Omega \phi v = 0 \quad \text{for all} \quad v \in \mathcal{C}.$$

Thus $\mathcal{C}$ is not dense, which is absurd.
ii) $\Rightarrow$ i) Denote $A_\varepsilon = \{x \in \Omega,\ u_0(x) \leq \varepsilon\}$, for $\varepsilon \geq 0$. This is a decreasing family of sets with intersection A_0. Thus their measure converge to zero. Now if $v \in L^q(\Omega)$ then $v_\varepsilon = v\mathcal{X}_{\Omega\setminus A_\varepsilon}$ converges to v in $L^q(\Omega)$. Now, with fixed ε, we truncate v_ε at height $R > 0$, that is, $v_\varepsilon^R = \begin{cases} v_\varepsilon & \text{if } |v_\varepsilon| \leq R, \\ R & \text{if } |v_\varepsilon| \geq R \end{cases}$. Thus $|v_\varepsilon^R| \leq \frac{R}{\varepsilon} u_0$ in Ω, that is $v_\varepsilon^R \in \mathcal{C}$ and $v_\varepsilon^R \to v_\varepsilon$ in $L^q(\Omega)$, as $R \to \infty$. Therefore $\mathcal{C}$ is dense in $L^q(\Omega)$. ■

In order to get constants M_0, M_1 independent of s in Lemma 3.2 above, we will need some additional properties of the solutions. For this we recall the following definition introduced in [11].

Definition 3.4 *A positive function $z(t,\cdot)$ with values in $X = L^q(\Omega)$, $1 \leq q \leq \infty$ or $X = C(\overline{\Omega})$, is non–degenerate (ND) at ∞ (respectively $-\infty$) if there exists $t_0 \in I\!R$ such that z is defined in $[t_0, \infty)$ (respectively $(-\infty, t_0]$) and there exists a $C^1(\overline{\Omega})$ function $\varphi_0(x) > 0$ in Ω, (vanishing on $\partial\Omega$ in case of Dirichlet boundary conditions), such that*

$$z(t,x) \geq \varphi_0(x) \qquad \textit{for all} \quad t \geq t_0$$

(respectively for all $t \leq t_0$).

With this, we have the following improvement of Lemma 3.2.

Lemma 3.5 *If there exists $u_0 \in L^q(\Omega)$, $u_0 > 0$ a.e. in Ω, and a $\beta \in I\!R$, such that*

$$\|u(t, s_0; u_0)\|_{L^q(\Omega)} \leq M e^{\beta(t-s_0)}, \qquad \textit{for all} \quad t > s_0$$

and $e^{-\beta(t-s_0)} u(t, s_0; u_0)$ is nondegenerate, then for all $s \geq s_0$ and for all nontrivial $v_0 \geq 0$, $e^{-\beta(t-s)} u(t, s; v_0)$ is nondegenerate, for $t \geq s+1$. Moreover,

$$M_0 e^{\beta(t-s)} \leq \|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M_1 e^{\beta(t-s)}, \qquad \textit{for all} \quad t \geq s \geq s_0$$

for some M_0, M_1 independent of $t > s \geq s_0$.

In particular the exponential type satisfies

$$\beta_0^+(C) = \beta \quad \textit{and has defect } \gamma = 0 \textit{ at } \infty.$$

Proof. Let $s \geq s_0$ and $v_0 \geq 0$ and observe that for $t > s+1$ we have $u(t,s;v_0) = U_C(t,s)v_0 = U_C(t,s+1)U_C(s+1,s)v_0$ and $w_0 = U_C(s+1,s)v_0 \in C^1_{\mathcal{B}}(\overline{\Omega})$ is positive in Ω. Then there exists $\delta = \delta(s, v_0) > 0$ such that $w_0 \geq \delta u(s+1, s_0; u_0)$ and then

$$u(t,s;v_0) = U_C(t,s+1)w_0 \geq \delta U_C(t,s+1)u(s+1,s_0;u_0) = \delta u(t,s_0;u_0) \geq \delta e^{\beta(t-s)} e^{\beta(s-s_0)} \varphi_0.$$

Thus,

$$e^{-\beta(t-s)} u(t,s;v_0) \geq \delta e^{\beta(s-s_0)} \varphi_0$$

and it is nondegenerate.

On the other hand notice that by assumption we have $w(s) = \mathrm{e}^{-\beta(s-s_0)}u(s, s_0; u_0) \geq \varphi_0$ as in Definition 3.4. In particular, this solution satisfies the assumptions of point iv) Lemma 3.2 with $\beta' = \beta$. Hence, the lower bound on $\|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))}$, with M_0 independent of s, follows.

For the upper bound we proceed as in the proof of point ii) in Lemma 3.2. In fact now for every $t \geq s \geq s_0$ consider an initial data $v_0 \in L^q(\Omega)$, such that there exists $\lambda = \lambda(v_0)$, independent of s, such that $|v_0| \leq \lambda\varphi_0$. Then

$$\left|\mathrm{e}^{-\beta(t-s)}U_C(t,s)v_0\right| \leq \lambda\mathrm{e}^{-\beta(t-s)}U_C(t,s)w(s).$$

Thus

$$\|\mathrm{e}^{-\beta(t-s)}U_C(t,s)v_0\|_{L^q(\Omega)} \leq \lambda(v_0)M, \qquad \text{for any } t > s.$$

Now the set

$$\mathcal{C} = \{v,\ \exists\lambda > 0,\ |v(x)| \leq \lambda\varphi_0(x) \text{ a.e. } x \in \Omega\}$$

is dense in $L^q(\Omega)$, see Lemma 3.3. Hence the result follows again from the Uniform Boundedness Principle on the family of operators $T(t,s) = \mathrm{e}^{-\beta(t-s)}U_C(t,s)$, for $t \geq s \geq s_0$.

The rest follows easily. ■

On the other hand, in order to get some estimates on the norms $\|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))}$ for $s \leq t \leq t_0$, for sufficiently negative t_0, we will rely on the concept of complete trajectory as follows.

Definition 3.6 *A complete solution of (2.1) is a solution defined for all times, in the sense that it is a continuous function $z(t,x)$ with values in $X = L^q(\Omega)$, $1 \leq q \leq \infty$ or $X = C(\overline{\Omega})$, such that for each $s \in \mathbb{R}$ the solution of (2.1) with initial data $u_0(x) = z(x,s)$ is given by $z(t,x)$ for each $t > s$. In other words, for every $s \in \mathbb{R}$ and $t > s$ we have*

$$z(t) = U_C(t,s)z(s).$$

When $C(t,x)$ is smooth, [7], [6], or at least bounded, [5], then (1.1) has a unique (up to multiple) global positive solution, $v_C(t)$. Note that we are unaware of such results for the case of nonsmooth-in-space potentials $C \in C^\theta(\mathbb{R}, L^p(\Omega))$, with $0 < \theta \leq 1$ and some $p > N/2$, considered in this paper.

Thus, if complete positive solutions exist, we have the following

Lemma 3.7 *i) Assume there exists a complete positive solution such that*

$$\|z(t)\|_{L^q(\Omega)} \leq M\mathrm{e}^{\beta t}, \qquad \textit{for all} \quad t \leq t_0$$

and for some $\beta' \geq \beta$

$$\mathrm{e}^{-\beta' t}z(t,x) \quad \textit{is nondegenerate at} \quad -\infty.$$

Then for every $t \leq t_0$ and $s \leq t$ we have

$$M_0(t)\mathrm{e}^{\beta(t-s)} \leq \|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M_1(t)\mathrm{e}^{\beta'(t-s)}, \qquad \textit{for all} \quad s \leq t \leq t_0$$

with $M_0(t) = \frac{\|\varphi_0\|_{L^q(\Omega)}}{M}\mathrm{e}^{(\beta'-\beta)t}$ and $M_1(t) = M_1\mathrm{e}^{-(\beta'-\beta)t}$ for certain positive constant M_1.

ii) In particular, assume that (2.1) has a complete positive solution, $z(t,x)$, such that for $t \le t_0$,

$$\mathrm{e}^{-\beta t} z(t,x) \quad \text{is bounded and nondegenerate at} \quad -\infty.$$

Then for each $s \le t \le t_0$ we have

$$M_0(t_0)\mathrm{e}^{\beta(t-s)} \le \|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))} \le M_1(t_0)\mathrm{e}^{\beta(t-s)}$$

Hence

$$\beta_0^-(C) = \beta, \quad \text{with defect } \gamma = 0 \text{ at } -\infty.$$

Proof. First note that for each $u_0 \in C^1(\overline{\Omega})$, vanishing on $\partial\Omega$, there exists $\lambda = \lambda(u_0)$ such that $|u_0(x)| \le \lambda \varphi_0(x)$ in Ω, where φ_0 is as in Definition 3.4. Then by comparison, we have for each $s \le t \le t_0$

$$|U_C(t,s)u_0(x)| \le \lambda U_C(t,s)\varphi_0(x) \le \lambda U_C(t,s)\mathrm{e}^{-\beta' s} z(s)(x) = \lambda \mathrm{e}^{-\beta' s} z(t,x)$$

and then

$$\|\mathrm{e}^{\beta' s} U_C(t,s)u_0\|_{L^q(\Omega)} \le \lambda \|z(t)\|_{L^q(\Omega)} \le \lambda M \mathrm{e}^{\beta t}.$$

Hence, $\mathrm{e}^{\beta' s} U_C(t,s)$ is pointwise bounded in a dense subset of $L^q(\Omega)$ and hence, from the Uniform Boundedness Principle we get the upper bound on $\mathrm{e}^{\beta' s}\|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))} \le M_1 \mathrm{e}^{\beta t}$, independent of t,s. Now we rewrite this estimate as

$$\mathrm{e}^{-\beta'(t-s)}\|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))} \le M_1 \mathrm{e}^{-(\beta'-\beta)t} = M_1(t)$$

for $s \le t$ and we get the upper bound in the statement.

On the other hand, note that

$$\|z(t)\|_{L^q(\Omega)} = \|U_C(t,s)z(s)\|_{L^q(\Omega)} \le \|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))}\|z(s)\|_{L^q(\Omega)} \le M\mathrm{e}^{\beta s}\|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))}.$$

Thus,

$$\|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))} \ge \frac{\|\varphi_0\|_{L^q(\Omega)}\mathrm{e}^{\beta' t}}{M\mathrm{e}^{\beta s}} = M_0(t)\mathrm{e}^{\beta(t-s)}.$$

Hence, the result follows.

The second part follows using $\beta = \beta'$. ■

As mentioned in the previous section, the exponential type is related to the principal spectrum. On the other hand, as shown above, the exponential type is also related to the behavior of positive solutions. In fact, when $C(t,x)$ is smooth, [7], [6], or at least bounded,[8], [5], then (1.1) has a unique (up to multiple) global positive solution, $v_C(t)$. Moreover, for every $s \in I\!R$ and $t > s$, any solution of (2.1) can be split in a unique way

$$u(t,s;u_0) = \alpha v_C(t) + w(t,s;w_0)$$

with $u_0 = \alpha v_C(s) + w_0$, $\alpha \in I\!R$ and $w(t,s;u_0)$ is a sign changing solution of (2.1).

Furthermore there is exponential separation of non positive solutions in the sense that for any sign changing solution of (2.1), we have, for any $s \in I\!R$ and $t > s$

$$\frac{\|w(t,s;u_0)\|_X}{\|v_C(t)\|_X} \le K\mathrm{e}^{-\sigma(t-s)}\frac{\|w_0\|_X}{\|v_C(s)\|_X}$$

with $X = L^\infty(\Omega)$, for some $K > 0$ and $\sigma > 0$.

From here one easily gets that the exponential type is that of this particular global positive solution, i.e.

$$\beta_0(C) = \limsup \frac{\ln \|v_C(t_n)\|_{L^q(\Omega)} - \ln \|v_C(s_n)\|_{L^q(\Omega)}}{t_n - s_n}$$

where the limsup is taken on all sequences $t_n - s_n \to \infty$. Moreover, if we restrict the sequences above to satisfy $s_n \geq s_0$ or $t_n \leq t_0$ we get the exponential types at $\pm\infty$, $\beta_0^\pm(C)$.

As mentioned above, we are unaware of analogous results for the case of nonsmooth-in-space potentials $C \in C^\theta(I\!R, L^p(\Omega))$, with $0 < \theta \leq 1$ and some $p > N/2$, considered in this paper.

4 Effectively changing the exponential type

In this section our goal is to give sufficient conditions on some perturbations of (2.1) to ensure that exponential type of the resulting evolution operator is actually modified.

Note that these results allows to quantitatively estimate the sizes of the "favorable" and "defavorable" parts allowed in the perturbation term. Also, observe that we can assume without loss of generality that all evolution operators considered below satisfy (2.2) with the same constants L and δ.

Before going further, note that if the perturbation is a multiple of the identity, then

$$U_{C-\alpha}(t,s) = \mathrm{e}^{-\alpha(t-s)} U_C(t,s)$$

and hence

$$\beta_0^\pm(C-\alpha) = \beta_0^\pm(C) - \alpha.$$

In particular we prove the following result which complements and somehow improves Proposition 4.4 in [9].

Theorem 4.1 *Let $\tau_0 \geq -\infty$ and denote $J = (\tau_0, \infty)$. Assume that $U = U_C$ is the evolution operator defined by the solutions of (2.1) as above and satisfies (2.3), that is*

$$\|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M(\beta) \mathrm{e}^{\beta(t-s)} \quad \textit{for all} \quad t > s > \tau_0 \tag{4.1}$$

for some $\beta \in I\!R$ and a constant $M(\beta) > 0$.

Assume that $P \in C^\theta(\overline{J}, L^p(\Omega))$ with $0 < \theta \leq 1$ and some $p > N/2$, is a given time–dependent perturbation of $C(t,x)$. Assume there exists a decomposition

$$P(t,x) = P^1(t,x) - P^2(t,x), \quad P^i \in C^\theta(\overline{J}, L^p(\Omega)), \quad i = 1,2$$

such that for all $x \in \Omega$ and $t > \tau_0$,

$$P^2(t,x) \geq a(t)$$

for some continuous and bounded function such that there exists $\tau > 0$, such that for all $t, s \in J$ with $t - s \geq \tau$

$$\frac{1}{t-s} \int_s^t a(r)\, dr \geq a_0$$

and $a_0 \in I\!R$. Also, assume

$$P^1 \in L^\sigma(J, L^p(\Omega)).$$

Then,

i) If $\sigma = 1$ and $p = \infty$,

$$\|U_{C+P}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M(\beta)\mathrm{e}^{(\beta-a_0)(t-s)}, \quad t \geq s > \tau_0. \tag{4.2}$$

ii) If $1 < \sigma < \infty$ and $p > \frac{N\sigma'}{2}$ then for every $\varepsilon > 0$, there exists $s_0(\varepsilon)$ with $s_0(\varepsilon) \to \infty$ as $\varepsilon \to 0$, such that

$$\|U_{C+P}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq 2M(\beta)\mathrm{e}^{(\beta-a_0+2\varepsilon)(t-s)}, \tag{4.3}$$

for $t \geq s \geq s_0(\varepsilon)$.

If $\tau_0 = -\infty$ then, additionally, for every $\varepsilon > 0$, there exists $t_0(\varepsilon)$ with $t_0(\varepsilon) \to -\infty$ as $\varepsilon \to 0$ such that

$$\|U_{C+P}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq 2M(\beta)\mathrm{e}^{(\beta-a_0+2\varepsilon)(t-s)}, \tag{4.4}$$

for $t_0(\varepsilon) > t > s$.

iii) If $\sigma = \infty$ and $p > \frac{N}{2}$ then for every $\varepsilon > 0$,

$$\|U_{C+P}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M(\beta)c(p,N)\mathrm{e}^{(\beta-a_0+\mu(\varepsilon))(t-s)}, \quad t \geq s > \tau_0 \tag{4.5}$$

where $\mu(\varepsilon) = \varepsilon + (M(\varepsilon,\beta)\Gamma(1-\alpha)\|P^1\|_{L^\infty(J,L^p(\Omega))})^{\frac{1}{1-\alpha}}$ with $M(\varepsilon,\beta)$ as in (2.6), $0 \leq \alpha = \frac{N}{2p} < 1$ and some constant $c(p,N)$.

Proof. We consider solutions of (2.1) in $L^q(\Omega)$, $1 \leq q \leq \infty$ to be chosen below.

First we have, by the variation of constants formula, that for every $u_0 \in L^q(\Omega)$ the solution $u(t,s,u_0) = U_{C+P^1}(t,s)u_0$ satisfies for $t \geq s > \tau_0$,

$$u(t,s;u_0) = U_C(t,s)u_0 + \int_s^t U_C(t,\tau)P^1(\tau)u(\tau,s;u_0)\,d\tau.$$

Using this, we chose q such that $p \geq q'$. Then the term $P^1(\tau)u(\tau,s;u_0)$ can be estimated, using Hölder's inequality, in $L^r(\Omega)$ with $\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$. Thus, denoting $z(t) = \mathrm{e}^{-(\beta+\varepsilon)(t-s)}\|u(t,s,u_0)\|_{L^q(\Omega)}$, and $a(\tau) = M(\varepsilon,\beta)\|P^1(\tau)\|_{L^p(\Omega)}$, with $M(\varepsilon,\beta)$ as in (2.6) (or $M(0,\beta) = M(\beta)$ if $\varepsilon = 0$, that is, for $\sigma = 1$ and $p = \infty$), we get, for $t \geq s > \tau_0$, from (4.1) and (2.6)

$$z(t) \leq M(\beta)\|u_0\|_{L^q(\Omega)} + \int_s^t \frac{a(\tau)}{(t-\tau)^{\frac{N}{2p}}} z(\tau)\,d\tau.$$

Using the singular Gronwall Lemma below, Lemma 4.11 and Corollary 4.12, with $\alpha = \frac{N}{2p} < 1$ and $A = M(\beta)\|u_0\|_{L^p(\Omega)}$ we get,

$$\|u(t,s,u_0)\|_{L^q(\Omega)} \leq M(\beta)\mathrm{e}^{\beta(t-s)}\|u_0\|_{L^q(\Omega)}, \quad t \geq s > \tau_0 \tag{4.6}$$

if $\sigma = 1$ and $p = \infty$ (and $\alpha = 0$). Also, we get

$$\|u(t,s,u_0)\|_{L^q(\Omega)} \leq 2M(\beta)\mathrm{e}^{(\beta+2\varepsilon)(t-s)}\|u_0\|_{L^q(\Omega)}, \tag{4.7}$$

if $1 < \sigma < \infty$ and $p > \frac{N\sigma'}{2}$ for $t \geq s \geq s_0(\varepsilon)$ with $s_0(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Additionally, if $\tau_0 = -\infty$ we get (4.7) for $t_0(\varepsilon) > t > s$ with $t_0(\varepsilon) \to -\infty$ as $\varepsilon \to 0$. Finally

$$\|u(t,s,u_0)\|_{L^q(\Omega)} \leq M(\beta)c(\alpha)\mathrm{e}^{(\beta+\mu(\varepsilon))(t-s)}\|u_0\|_{L^q(\Omega)}, \quad t \geq s > \tau_0 \tag{4.8}$$

where $\mu(\varepsilon) = \varepsilon + (M(\varepsilon,\beta)\Gamma(1-\alpha)\|P^1\|_{L^\infty((s,\infty),L^p(\Omega))})^{\frac{1}{1-\alpha}}$, if $\sigma = \infty$ and $p > \frac{N}{2}$.

Now, we prove that for $t \geq s > \tau_0$, we have

$$\|U_{C+P}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M_0 \mathrm{e}^{-a_0(t-s)} \|U_{C+P^1}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \tag{4.9}$$

for some constant $M_0 = M_0(a, a_0)$.

To see this, note first that if $u_0 \geq 0$ then $U_{C+P}(t,s)u_0 \geq 0$ which implies that $|U_{C+P}(t,s)u_0| \leq U_{C+P}(t,s)|u_0|$. Therefore it is enough to prove the claim for non-negative initial data. In such a case, let $u(t,s;u_0) = U_{C+P}(t,s)u_0 \geq 0$ then, since $P^2(t,x) \geq a(t)$, we have for $t > s$

$$\begin{cases} u_t - \Delta u = C(t,x)u + P^1(t,x)u - P^2(t,x)u \leq C(t,x)u + P^1(t,x)u - a(t)u \\ \mathcal{B}u = 0 \\ u(s) = u_0. \end{cases}$$

Now let $0 \leq v(t,x) = u(t,s;u_0)\mathrm{e}^{\int_s^t a(r)\,dr}$, which satisfies

$$\begin{cases} v_t - \Delta v \leq C(t,x)v + P^1(t,x)v, \quad t > s \\ \mathcal{B}v = 0 \\ v(s) = u_0. \end{cases} \tag{4.10}$$

Hence, see [10], for $t \geq s > \tau_0$

$$0 \leq v(t,x) \leq U_{C+P^1}(t,s)u_0$$

and then (4.9) follows from the assumption on $a(\cdot)$ see v) in Lemma 2.8.

The result then follows from (4.6), (4.7), (4.8) and (4.9). ∎

Observe that we do not assume in Theorem 4.1 above any sign on a_0. However, as we are interested in giving conditions on $P(t,x)$ such that the exponent in (4.2), (4.3), (4.4) and (4.5) is less than β, we have the following results. Note that the results below only make use of the asymptotic properties of the perturbations as $t \to \pm\infty$. Also note that we will use the classes $\mathcal{C}^\pm(m)$ as in Lemma 2.8.

Corollary 4.2 *Assume that $U = U_C$ is the evolution operator defined by the solutions of (2.1) as above and satisfies for some $s_0 \in \mathbb{R}$,*

$$\|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M(\beta)\mathrm{e}^{\beta(t-s)} \quad \textit{for all} \quad t > s > s_0$$

for some $\beta \in \mathbb{R}$ and a constant $M(\beta) > 0$.

Assume that $P \in C^\theta(\mathbb{R}, L^p(\Omega))$ with $0 < \theta \leq 1$ and some $p > N/2$, is a given time–dependent perturbation of C. Assume there exists a decomposition

$$P(t,x) = P^1(t,x) - P^2(t,x), \quad P^i \in C^\theta(\mathbb{R}, L^p(\Omega)), \quad i = 1,2$$

such that for all $x \in \Omega$ and $t \geq s_0$

$$P^2(t,x) \geq a(t) \quad \textit{with} \quad a \in \mathcal{C}^+(a_0).$$

satisfying,

Also, assume that

$$P^1 \in L^\sigma((s_0,\infty), L^p(\Omega))$$

for some σ, p, *such that either* $\sigma = 1$ *and* $p = \infty$, *or* $1 < \sigma < \infty$ *and* $p > \frac{N\sigma'}{2}$, *or* $\sigma = \infty$ *and* $p > N/2$.

Then, for some sufficiently large $t_0^+ > s_0$, *the perturbed evolution operator satisfies*

$$\|U_{C+P}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \le M' \mathrm{e}^{\beta'(t-s)}, \tag{4.11}$$

for $t \ge s \ge t_0^+$, *with*

$$\beta' < \beta,$$

provided that
i) if $\sigma = 1$ *and* $p = \infty$

$$a_0 > 0$$

and in such a case $M' = M(\beta) M_0(a, a_0)$, *for certain constant* $M_0(a, a_0)$, *or*
ii) if $1 < \sigma < \infty$ *and* $p > \frac{N\sigma'}{2}$

$$a_0 > 0$$

and in such a case $M' = 2M(\beta) M_0(a, a_0)$, *or*
iii) if $\sigma = \infty$ *and* $p > \frac{N}{2}$,

$$a_0 > a_0^c(\limsup_{t\to\infty} \|P^1(t)\|_{L^p(\Omega)}) > 0 \tag{4.12}$$

where the continuous functions $a_0^c(s)$ *is given by*

$$a_0^c(s) = \begin{cases} c_0 s, & \text{if } 0 \le s \le s^* \\ c_1 + c_2 s^{\frac{1}{1-\alpha}}, & \text{if } s \ge s^* \end{cases}$$

where $\alpha = \frac{N}{2p} < 1$ *and all positive constants* c_0, c_1, c_2, s^* *depend on* N, p, δ, L *as in (2.2),* β *and* $M(\beta)$. *In such a case* $M' = M(\beta) c(p,N) M_0(a, a_0)$ *with* $M_0(a, a_0)$ *and* $c(p,N)$ *as in Theorem 4.1.*

Proof. First note that, as in (4.9),

$$\|U_{C+P}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \le M_0 \mathrm{e}^{-a_0(t-s)} \|U_{C+P^1}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \tag{4.13}$$

for sufficiently large $t \ge s \ge t_0^+$ and some constant $M_0 = M_0(a, a_0)$, since $a \in \mathcal{C}^+(a_0)$, see v) in Lemma 2.8.

Now, if $\sigma = 1$ and $p = \infty$, we get from i) in Theorem 4.1 we get

$$\|U_{C+P^1}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \le M(\beta) \mathrm{e}^{\beta(t-s)}, \quad t \ge s > t_0^+$$

and, with (4.13), we get case i) above.

If $1 < \sigma < \infty$ and $p > \frac{N\sigma'}{2}$ then from ii) in Theorem 4.1 we have that for every $\varepsilon > 0$, there exists $s_0(\varepsilon)$ with $s_0(\varepsilon) \to \infty$ as $\varepsilon \to 0$, such that

$$\|U_{C+P^1}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \le 2M(\beta) \mathrm{e}^{(\beta+2\varepsilon)(t-s)},$$

for $t \ge s \ge s_0(\varepsilon)$ and, with (4.13), we get case ii).

For $\sigma = \infty$ note that from iii) in Theorem 4.1 and taking $J = (t_0^+, \infty)$ for t_0^+ large enough, we get that for every $\varepsilon > 0$, as in (4.5)

$$\|U_{C+P^1}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M(\beta)c(p,N)\mathrm{e}^{(\beta+\mu(\varepsilon))(t-s)}, \quad t \geq s \geq t_0^+ \tag{4.14}$$

where $\mu(\varepsilon) = \varepsilon + (M(\varepsilon,\beta)\Gamma(1-\alpha)\limsup_{t\to\infty} \|P^1(t)\|_{L^p(\Omega)})^{\frac{1}{1-\alpha}}$ with $M(\varepsilon,\beta)$ as in (2.6) and $\alpha = \frac{N}{2p} < 1$, that is

$$M(\varepsilon,\beta) = K(\beta)\mathrm{e}^{|\beta|}\left\{\begin{array}{ll} (\frac{\alpha}{\mathrm{e}})^{\alpha}\varepsilon^{-\alpha} & \text{if } 0 < \varepsilon < \varepsilon_0 = \frac{\alpha}{\mathrm{e}} \\ 1 & \text{if } \varepsilon \geq \varepsilon_0 = \frac{\alpha}{\mathrm{e}} \end{array}\right\}$$

where $K(\beta) = L\mathrm{e}^{\delta}\max\{1, M(\beta)\mathrm{e}^{-\beta}\}$.

Hence setting $LS(P^1) = \limsup_{t\to\infty} \|P^1(t)\|_{L^p(\Omega)}$, for some constants A_0, A_1 that depend only on N, p, δ, L as in (2.2), β and $M(\beta)$ we have

$$\mu(\varepsilon) = \begin{cases} \varepsilon + A_0 LS(P^1)^{\frac{1}{1-\alpha}}\varepsilon^{\frac{-\alpha}{1-\alpha}} & \text{if } 0 < \varepsilon < \varepsilon_0 \\ \varepsilon + A_1 LS(P^1)^{\frac{1}{1-\alpha}} & \text{if } \varepsilon > \varepsilon_0 \end{cases}.$$

Thus $\mu(0) = \mu(\infty) = \infty$.

But the function $h(\varepsilon) = \varepsilon + A_0 LS(P^1)^{\frac{1}{1-\alpha}}\varepsilon^{\frac{-\alpha}{1-\alpha}}$ has a unique minimum at $\varepsilon_1 = B_0 LS(P^1)$, and $h(\varepsilon_1) = B_1 LS(P^1)$ for some constants B_0, B_1 that depend only on N, p, δ, L as in (2.2), β and $M(\beta)$. Therefore, using (4.14) and (4.13), comparing ε_0 and ε_1, minimizing $\mu(\varepsilon)$ and setting $a_0 > \inf_{\{\varepsilon>0\}} \mu(\varepsilon)$ leads to (4.12).

In all the cases, (4.13) leads to (4.11). ∎

Analogously, for sufficiently negative time, we have

Corollary 4.3 *Assume that $U = U_C$ is the evolution operator defined by the solutions of (2.1) as above and satisfies for some $t_0 \in \mathbb{R}$,*

$$\|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M(\beta)\mathrm{e}^{\beta(t-s)} \quad \textit{for all} \quad t_0 > t > s \tag{4.15}$$

for some $\beta \in \mathbb{R}$ and a constant $M(\beta) > 0$.

Assume that $P \in C^{\theta}(\mathbb{R}, L^p(\Omega))$ with $0 < \theta \leq 1$ and some $p > N/2$, is a given time–dependent perturbation of C. Assume there exists a decomposition

$$P(t,x) = P^1(t,x) - P^2(t,x), \quad P^i \in C^{\theta}(\mathbb{R}, L^p(\Omega)), \quad i = 1,2$$

such that for all $x \in \Omega$ and $t_0 \geq t$

$$P^2(t,x) \geq a(t) \quad \textit{with} \quad a \in \mathcal{C}^-(a_0).$$

Also, assume that

$$P^1 \in L^{\sigma}((-\infty, t_0), L^p(\Omega))$$

for some σ, p, such that either $\sigma = 1$ and $p = \infty$, or $1 < \sigma < \infty$ and $p > \frac{N\sigma'}{2}$, or $\sigma = \infty$ and $p > N/2$.

Then, for some sufficiently negative $t_0^- < t_0$, the perturbed evolution operator satisfies

$$\|U_{C+P}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M'\mathrm{e}^{\beta'(t-s)}, \tag{4.16}$$

for $t_0^- \geq t \geq s$, with

$$\beta' < \beta,$$

provided that
i) if $\sigma = 1$ and $p = \infty$

$$a_0 > 0$$

and in such a case $M' = M(\beta)M_0(a, a_0)$, for certain constant $M_0(a, a_0)$, or
ii) if $1 < \sigma < \infty$ and $p > \frac{N\sigma'}{2}$

$$a_0 > 0$$

and in such a case $M' = 2M(\beta)M_0(a, a_0)$, or
iii) if $\sigma = \infty$ and $p > \frac{N}{2}$

$$a_0 > a_0^c(\limsup_{t \to -\infty} \|P^1(t)\|_{L^p(\Omega)}) > 0 \tag{4.17}$$

where the continuous functions $a_0^c(s)$ is given by

$$a_0^c(s) = \begin{cases} c_0 s, & \text{if } 0 \leq s \leq s^* \\ c_1 + c_2 s^{\frac{1}{1-\alpha}}, & \text{if } s \geq s^* \end{cases}$$

respectively, where $\alpha = \frac{N}{2p} < 1$ and all positive constants c_0, c_1, c_2, s^* depend on N, p, δ, L as in (2.2), β and $M(\beta)$. In such a case $M' = M(\beta)c(p, N)M_0(a, a_0)$ with $M_0(a, a_0)$ and $c(p, N)$ as in Theorem 4.1.

Proof. First, note that (4.13) can be obtained in the same way as in Corollary 4.3 for sufficiently negative t_0^- and $t_0^- \geq t > s$.

Now observe that we can redefine $C(t, x)$ for $t \geq t_0^-$ in such a way that the corresponding evolution operator satisfy (4.15) for all $t > s > -\infty$. Also we can redefine $P(t, x)$ for $t \geq t_0^-$ in such a way that (4.13) holds for all $t > s > -\infty$ and $P^1 \in L^\sigma(I\!R, L^p(\Omega))$. Even more we can always assume that $\|P^1\|_{L^\sigma(I\!R, L^p(\Omega))} \leq (1 + \delta)\|P^1\|_{L^\sigma((-\infty, t_0^-), L^p(\Omega))}$ for any $\delta > 0$, if $1 \leq \sigma < \infty$, or $\|P^1\|_{L^\infty(I\!R, L^p(\Omega))} = \|P^1\|_{L^\infty((-\infty, t_0^-), L^p(\Omega))}$.

Thus, we can apply Theorem 4.1 to get at once (4.16) in cases i) and ii). For case iii) note that indeed by taking t_0^- very negative we get in (4.5),

$$\mu(\varepsilon) = \varepsilon + (M(\varepsilon, \beta)\Gamma(1 - \alpha) \limsup_{t \to -\infty} \|P^1(t)\|_{L^p(\Omega)})^{\frac{1}{1-\alpha}}$$

with $M(\varepsilon, \beta)$ as in (2.6) and $\alpha = \frac{N}{2p} < 1$. Minimizing in ε, as in Corollary 4.2 we get (4.17). ■

Note that the estimates in Theorem 4.1 and Corollaries 4.2 and 4.3 above, give a quantitative estimate on the admissible sizes of the favorable and defavorable parts of the perturbation P^2 and P^1, respectively, for which one can ensure that a given exponent for an evolution operator is effectively modified.

In the previous results we have considered the question of decreasing the given exponent of the evolution operator $U_C(t, s)$ in (4.1). As we shall see, if we pose the same question about the optimal of such exponents, that is, the exponential type of the evolution operator (see Definition 2.2), some times a higher price must be payed, as the optimal constant (which gets involved in the computations of the perturbation) may get worse as one is closer to the exponential type, that is when $\beta = \beta_0(C) + \varepsilon$. This is expressed in the next results where we will assume below that the evolution operator $U_C(t, s)$ has a "defect $\gamma \geq 0$" as in Definition 2.3.

Theorem 4.4 *Assume that $U = U_C$ is the evolution operator defined by the solutions of (2.1) as above and has defect $\gamma \geq 0$ at ∞, as in Definition 2.3.*

Assume that $P \in C^\theta(\mathbb{R}, L^p(\Omega))$ with $0 < \theta \leq 1$ and some $p > N/2$, is a given time–dependent perturbation of $C(t,x)$. Assume there exists a decomposition

$$P(t,x) = P^1(t,x) - P^2(t,x), \quad P^i \in C^\theta(\mathbb{R}, L^p(\Omega)), \quad i = 1,2$$

such that for all $x \in \Omega$ and sufficiently large s_0 and $t \geq s_0$

$$P^2(t,x) \geq a(t) \quad \text{with} \quad a \in \mathcal{C}^+(a_0), \quad a_0 > 0.$$

Then, for some sufficiently large $t_0^+ \in \mathbb{R}$, the perturbed evolution operator satisfies

$$\|U_{C+P}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M' \mathrm{e}^{\beta'(t-s)}, \tag{4.18}$$

for $t \geq s \geq t_0^+$, with

$$\beta' < \beta_0^+(C),$$

provided that either
i) $P^1 \in L^1((s_0,\infty), L^\infty(\Omega))$ or $P^1 \in L^\sigma((s_0,\infty), L^p(\Omega))$, with $1 < \sigma < \infty$ and $p > \frac{N\sigma'}{2}$, and in such a case

$$M' \leq 2D_0((\frac{a_0}{4})^{-\gamma} + 1)$$

where D_0 is as in Definition 2.3, or
ii) if $P^1 \in L^\infty((s_0,\infty), L^p(\Omega))$ with $p > \frac{N}{2}$, and

$$a_0 > A_0^c(\limsup_{t\to\infty} \|P^1(t)\|_{L^p(\Omega)}) > 0 \tag{4.19}$$

where the continuous function $A_0^c(s)$ is given by

$$A_0^c(s) = \begin{cases} c_0 s^{\frac{1}{\gamma+1}}, & \text{if } 0 \leq s \leq s^* \\ c_1 + c_2 s^{\frac{1}{1-\alpha}}, & \text{if } s \geq s^* \end{cases}$$

and in such a case

$$M' \leq \begin{cases} b_0 LS(P^1)^{\frac{-\gamma}{\gamma+1}}, & \text{if } 0 < LS(P^1) \leq s^* \\ b_1, & \text{if } LS(P^1) \geq s^* \end{cases}$$

where $LS(P^1) = \limsup_{t\to\infty} \|P^1(t)\|_{L^p(\Omega)}$ and all positive constants c_0, c_1, b_0, b_1, s^* depend on $N, p, \delta, L, \beta_0^+(C), \gamma$ and D_0 and $\alpha = N/2p < 1$.

In particular, in all the cases above, we have

$$\beta_0^+(C+P) < \beta_0^+(C).$$

Proof. We proceed as in Theorem 4.1 and Corollary 4.2 with $\beta = \beta_0(C) + \varepsilon$ and $M(\beta) \leq D_0(\varepsilon^{-\gamma} + 1)$ to obtain (4.2), (4.3) and (4.5) for $t > s \geq t_0^+$, according to the cases for $P(t,x)$.

Then (4.18) for the cases in i) follows by taking $0 \leq 2\varepsilon < a_0$, e.g. $\varepsilon = \frac{a_0}{4}$. In these cases, from (4.2), (4.3) we have $M' \leq 2M(\beta)$.

For case ii) note that we have in (4.5), $\mu(\varepsilon) = 2\varepsilon + (M(\varepsilon,\beta)\Gamma(1-\alpha)LS(P^1))^{\frac{1}{1-\alpha}}$ with $M(\varepsilon,\beta)$ as in (2.6) and $\alpha = \frac{N}{2p} < 1$. Thus, according to (2.6) and Definition 2.3, we have

$$M(\varepsilon,\beta) \leq \begin{cases} B_0 \varepsilon^{-\gamma-\alpha} & \text{if } 0 < \varepsilon < \varepsilon_0 = \frac{\alpha}{\mathrm{e}} \\ B_1 & \text{if } \varepsilon > \varepsilon_0 \end{cases}.$$

for some constants B_0, B_1 that depend on δ, L as in (2.2) and D_0. Hence

$$\mu(\varepsilon) \leq \begin{cases} 2\varepsilon + A_0 LS(P^1)^{\frac{1}{1-\alpha}} \varepsilon^{\frac{-\gamma-\alpha}{1-\alpha}} & \text{if } 0 < \varepsilon < \varepsilon_0 \\ 2\varepsilon + A_0 LS(P^1)^{\frac{1}{1-\alpha}} & \text{if } \varepsilon > \varepsilon_0 \end{cases}.$$

and $\mu(0) = \mu(\infty) = \infty$, for some constant A_0 that depends on δ, L and D_0.

But the function $h(\varepsilon) = 2\varepsilon + A_0 LS(P^1)^{\frac{1}{1-\alpha}} \varepsilon^{\frac{-\gamma-\alpha}{1-\alpha}}$ has a unique minimum at $\varepsilon_1 = B_2 LS(P^1)^{\frac{1}{\gamma+1}}$, and $h(\varepsilon_1) = B_3 LS(P^1)^{\frac{1}{\gamma+1}}$, for some constants B_2, B_3 that depend on δ, L and D_0. Therefore, setting

$$a_0 > \inf_{\{\varepsilon > 0\}} \mu(\varepsilon) = \begin{cases} \mu(\varepsilon_1) = h(\varepsilon_1) & \text{if } \varepsilon_1 < \varepsilon_0 \\ \mu(\varepsilon_0) = 2\varepsilon_0 + A_0 LS(P^1)^{\frac{1}{1-\alpha}} & \text{if } \varepsilon_1 > \varepsilon_0 \end{cases}$$

leads to (4.19).

Also, in this case, from (4.5), we have $M' \leq M(\beta_0(C) + \varepsilon)c(p, N) \leq M_0(\varepsilon^{-\gamma} + 1)c(p, N)$ and then taking $\varepsilon = \varepsilon_1$ if $\varepsilon_1 < \varepsilon_0$ or $\varepsilon = \varepsilon_0$ if $\varepsilon_1 > \varepsilon_0$, we get the result. ■

Analogously, we have

Theorem 4.5 *Assume that $U = U_C$ is the evolution operator defined by the solutions of (2.1) as above and has defect $\gamma \geq 0$ at $-\infty$, as in Definition 2.3.*

Assume that $P \in C^\theta(I\!R, L^p(\Omega))$ with $0 < \theta \leq 1$ and some $p > N/2$, is a given time–dependent perturbation of $C(t, x)$. Assume there exists a decomposition

$$P(t, x) = P^1(t, x) - P^2(t, x), \quad P^i \in C^\theta(I\!R, L^p(\Omega)), \quad i = 1, 2$$

such that for all $x \in \Omega$ and sufficiently negative t_0 and $t_0 \geq t$

$$P^2(t, x) \geq a(t) \quad \textit{with} \quad a \in \mathcal{C}^-(a_0), \quad a_0 > 0.$$

Then, for some sufficiently negative $t_0^- \in I\!R$, the perturbed evolution operator satisfies

$$\|U_{C+P}(t, s)\|_{\mathcal{L}(L^q(\Omega))} \leq M' \mathrm{e}^{\beta'(t-s)},$$

for $t_0^- \geq t \geq s$, with

$$\beta' < \beta_0^-(C),$$

provided that either
i) $P^1 \in L^1((-\infty, s_0), L^\infty(\Omega))$ or $P^1 \in L^\sigma((-\infty, s_0), L^p(\Omega))$, with $1 < \sigma < \infty$ and $p > \frac{N\sigma'}{2}$, and in such a case

$$M' \leq 2D_0((\frac{a_0}{4})^{-\gamma} + 1)$$

where D_0 is as in Definition 2.3, or
ii) if $P^1 \in L^\infty((-\infty, s_0), L^p(\Omega))$ with $p > \frac{N}{2}$, and

$$a_0 > A_0^c(\limsup_{t \to -\infty} \|P^1(t)\|_{L^p(\Omega)}) > 0$$

where the continuous function $A_0^c(s)$ is given by

$$A_0^c(s) = \begin{cases} c_0 s^{\frac{1}{\gamma+1}}, & \textit{if } 0 \leq s \leq s^* \\ c_1 + c_2 s^{\frac{1}{1-\alpha}}, & \textit{if } s \geq s^* \end{cases}$$

and in such a case

$$M' \leq \begin{cases} b_0 LS(P^1)^{\frac{-\gamma}{\gamma+1}}, & \text{if } 0 < LS(P^1) \leq s^* \\ b_1, & \text{if } LS(P^1) \geq s^* \end{cases}$$

where $LS(P^1) = \limsup_{t\to-\infty} \|P^1(t)\|_{L^p(\Omega)}$ *and all positive constants* c_0, c_1, b_0, b_1, s^* *depend on* N, p, δ, L *as in (2.2),* $\beta_0^-(C), \gamma$ *and* D_0 *and* $\alpha = N/2p < 1$.

In particular, in all the cases above, we have

$$\beta_0^-(C+P) < \beta_0^-(C).$$

Remark 4.6
i) Observe that we get no information on the defect of the perturbed evolution operator.
ii) All the results above are written in terms of decreasing the exponential type. On the other hand, assume that a given time dependent perturbation $P(t,x)$ *is such that the defect of* $U_{C+P}(t,s)$ *at* $\pm\infty$ *is* γ *and that we can decompose* $-P(t,x) = P^1(t,x) - P^2(t,x)$ *such that* $P^i(t,x)$ *satisfy the assumptions in either Theorem 4.4 or 4.5. Then we have, respectively*

$$\beta_0^\pm(C) < \beta_0^\pm(C+P).$$

iii) On the other hand, note that with Theorem 4.1 it is easy to obtain, taking $P^2(t,x) = 0$*, that if*

$$P \in L^\sigma(I\!R, L^p(\Omega)).$$

with $\sigma = 1$ *and* $p = \infty$ *or* $1 < \sigma < \infty$ *and* $p > \frac{N\sigma'}{2}$*, then*

$$\beta_0^\pm(C \pm P) = \beta_0^\pm(C).$$

In other words, perturbations which are not sustained at $\pm\infty$ *do not change the exponential type of an evolution operator.*

Now we illustrate the scope of our results with the following two examples in which we apply Theorems 4.4 or 4.5. The first one allows to improve the conclusions of [11, Propositions 2, ii)], for $t \to -\infty$ and [11, Proposition 4, ii)], for $t \to \infty$. In fact in that reference only convergence to zero of solutions of (2.1) as $t \to \pm\infty$ was obtained, for fixed initial data; here exponential convergence is obtained in operator norm.

Proposition 4.7 *Assume that* $U = U_C$ *is the evolution operator defined by the solutions of (2.1) as above and has defect* $\gamma \geq 0$ *at* $\pm\infty$*, as in Definition 2.3.*

Assume also that $P \in C^\theta(I\!R, L^p(\Omega))$ *with* $0 < \theta \leq 1$ *and some* $p > N/2$*, is a given time–dependent perturbation of* $C(t,x)$ *that for* $t \geq s_0$*, or* $t \leq t_0$*, respectively, satisfies*

$$P(t,x) \leq -\varphi(x), \quad 0 \leq \varphi \in L^p(\Omega),\ p > N/2$$

and assume for $0 < a$ *sufficiently small we have*

$$\mu(\{x \in \Omega,\ 0 \leq \varphi(x) \leq a\}) \leq K a^\nu$$

with $\frac{\nu}{p} > \gamma$ *and* $K > 0$*. Then, we have*

$$\beta_0^\pm(C+P) < \beta_0^\pm(C),$$

respectively.

Proof. Observe first that for any initial data, we have,

$$|U_{C+P}(t,s)u_0| \le U_{C+P}(t,s)|u_0| \le U_{C-\varphi}(t,s)|u_0|$$

and from here

$$\beta_0^{\pm}(C+P) \le \beta_0^{\pm}(C-\varphi).$$

Now take

$$\varphi(x) = \varphi_2^a(x) - \varphi_1^a(x)$$

with

$$\varphi_2^a(x) = \max\{\varphi(x), a\} \ge a > 0$$

and then $0 \le \varphi_1^a(x) \le a$ with

$$\|\varphi_1^a\|_{L^p(\Omega)}^{\frac{1}{\gamma+1}} \le K a^{(1+\frac{\nu}{p})\frac{1}{\gamma+1}}.$$

Then the result follows from Theorem 4.4 or 4.5 respectively, taking $P^1(t,x) = \varphi_1^a(x)$ and $P^2(t,x) = \varphi_2^a(x)$ and $a_0 = a > 0$ small, provided $(1+\frac{\nu}{p})\frac{1}{\gamma+1} > 1$, i.e. $\frac{\nu}{p} > \gamma$. ■

Remark 4.8 *Note that Theorems 4.4 or 4.5 also give that the constant for the perturbed evolution operator is of order* $a^{(1+\frac{\nu}{p})\frac{-\gamma}{\gamma+1}}$.

For example, if φ is a $C^1(\overline{\Omega})$ with nonzero gradient at the points it vanishes, the above is satisfied with $\nu = 1$. More generally, if φ is a $C^\theta(\overline{\Omega})$, with no "flat" parts where it vanishes, then typically $\nu = \frac{1}{\theta}$. Note that in particular, in any case, $\varphi(x) > 0$ a.e. in Ω is required.

The next example is a time dependent variant of the one above. Observe that we remove the sign assumption above on the perturbation and allow very large bad perturbations in small time–wandering sets in Ω. More precisely we have

Proposition 4.9 *With the notations in Proposition 4.7, assume that for $t \ge s_0$ or for $t \le t_0$, respectively, we have for $0 < a(t)$ sufficiently small,*

$$\mu(\{x \in \Omega,\ P(t,x) \ge -a(t)\}) \le K_0 a(t)^{\nu_0}$$

with $\nu_0 > 0$, $K_0 > 0$ and

$$\sup_{\Omega} P(t,x) \le K_1 a(t)^{-\nu_1}$$

with $\nu_1 \ge 0$. Furthermore, assume that if $K_1 = 0$ then $\frac{\nu_0}{p} > \gamma$, while if $K_1 > 0$ then

$$\frac{\nu_0}{p} > \gamma + 1 + \nu_1,$$

where $\gamma \ge 0$ is the defect of the evolution operator $U_C(t,s)$ at $\pm\infty$ respectively.

Then, if $\lim_{t\to\pm\infty} a(t) = a_0 > 0$ is sufficiently small,

$$\beta_0^{\pm}(C+P) < \beta_0^{\pm}(C),$$

respectively.

Proof. We decompose $P = P^1 - P^2$, where $-P^2(t,x) = \min\{P(t,x), -a(t)\}$. Then $P^2(t,x) \geq a(t)$ and

$$\|P^1(t)\|_{L^p(\Omega)}^{\frac{1}{\gamma+1}} \leq (K_1 a(t)^{-\nu_1} + a(t))^{\frac{1}{\gamma+1}} (K_0 a(t)^{\nu_0})^{\frac{1}{p(\gamma+1)}}$$

with $C_1 \geq 0$. Then the result follows from Theorem 4.4 or 4.5, respectively, since in either case for K_1 the leading term in the estimate above has exponent greater than 1 and $a(t)$ is small. Also note that $a \in \mathcal{C}^{\pm}(\frac{a_0}{2})$. ■

Remark 4.10 *Note that Theorems 4.4 or 4.5 also give that the constant for the perturbed evolution operator is of order* $(K_1 a_0^{-\nu_1} + a_0)^{\frac{-\gamma}{\gamma+1}} (K_0 a_0^{\nu_0})^{\frac{-\gamma}{p(\gamma+1)}}$.

We now state the singular Gronwall lemma used above. Note that a very similar result was proved in Lemma 4.5 in [9] and the present one follows from that proof. Here we pay detailed attention to the dependence of the constants involved. As the proof is short we include it for the reader's convenience.

Lemma 4.11 A singular Gronwall lemma
Assume that $a \in L^{\sigma}((\tau_0, \infty)) \cap L^{\infty}_{loc}(\tau_0, \infty)$ *with* $1 \leq \sigma \leq \infty$, $\tau_0 \geq -\infty$ *and that* $z(t) \geq 0$ *is a locally bounded function that for* $t \geq s > \tau_0$ *satisfies*

$$z(t) \leq M z(s) + \int_s^t \frac{a(\tau)}{(t-\tau)^{\alpha}} z(\tau)\, \mathrm{d}\tau \tag{4.20}$$

with $\alpha\sigma' < 1$. *Then*
i) If $\sigma = 1$ *(and* $\alpha = 0$*), we have for* $t \geq s > \tau_0$

$$0 \leq z(t) \leq M z(s).$$

ii) If $1 < \sigma < \infty$ *and* $\alpha\sigma' < 1$, *for every* $\gamma > 0$ *there exists* $s_0 > \tau_0$ *such that if (4.20) holds for* $t > s \geq s_0$ *then*

$$0 \leq z(t) \leq 2M\mathrm{e}^{\gamma(t-s)} z(s), \quad t > s \geq s_0,$$

and $s_0 = s_0(\gamma) \to \infty$ *as* $\gamma \to 0$.
If $\tau_0 = -\infty$, *for every* $\gamma > 0$ *there exists* t_0 *such that if (4.20) holds for* $t_0 \geq t > s$ *then*

$$0 \leq z(t) \leq 2M\mathrm{e}^{\gamma(t-s)} z(s), \quad t_0 \geq t > s,$$

and $t_0 = t_0(\gamma) \to -\infty$ *as* $\gamma \to 0$.
iii) If $\sigma = \infty$ *and* $0 \leq \alpha < 1$ *then we have for* $t \geq s > \tau_0$

$$0 \leq z(t) \leq M(\alpha)\mathrm{e}^{\gamma(t-s)} z(s)$$

with $\gamma = \gamma(a,s,\alpha) = (\|a\|_{L^{\infty}(s,\infty)} \Gamma(1-\alpha))^{1/(1-\alpha)}$ *and* $M(\alpha)$ *depends only on* M *and* α *but not on the function* $a(\cdot)$ *or* s *or* γ *or* τ_0.

Proof. Note that the case $\sigma = 1$, $\alpha = 0$ reduces to the usual Gronwall lemma and then $z(t) \leq M z(s) \mathrm{e}^{\int_s^t a(\tau)\, \mathrm{d}\tau}$ and the result is obvious.

On the other hand the case $\sigma=\infty$ and $0\le\alpha<1$ is a particular case of the singular Gronwall lemma in Henry [3, Lemma 7.1.1, page 188] which gives $\gamma=(\|a\|_{L^\infty(s,\infty)}\Gamma(1-\alpha))^{1/(1-\alpha)}$ and $M(\alpha)=Mc(\alpha)$ for certain constant $c(\alpha)$.

Therefore, we will consider now the case $1<\sigma<\infty$ and $\alpha\sigma'<1$. Note that in this case we can take s_0 large enough such that $\|a\|_{L^\sigma(s_0,\infty)}$ is as small as we want. Also, from (4.20) we get that for $s_0\le s\le t\le s+T$ we have, denoting $w(s,T)=\sup_{s\le\tau\le s+T}z(\tau)$ and using Hölder's inequality

$$\begin{aligned} z(t) &\le Mz(s)+w(s,T)\|a\|_{L^\sigma(s,s+T)}\Big(\int_s^t \frac{1}{(t-\tau)^{\alpha\sigma'}}\,d\tau\Big)^{1/\sigma'} \\ &\le Mz(s)+w(s,T)\delta(s_0,T) \end{aligned}$$

where we have set $\delta(s_0,T)=\|a\|_{L^\sigma(s_0,\infty)}C(\alpha,\sigma')T^{1/\sigma'-\alpha}$, for some constant $C(\alpha,\sigma')$.

Now, given s_0, choose T such that

$$\delta(s_0,T)=\|a\|_{L^\sigma(s_0,\infty)}C(\alpha,\sigma')T^{1/\sigma'-\alpha}=1/2. \tag{4.21}$$

Taking the supremum for $s\le t\le s+T$ we get

$$z(t)\le w(s)\le 2Mz(s)\quad\text{for all}\quad s\le t\le s+T.$$

Writing $s_1=s+T$ and repeating the process and the estimate above we get a sequence $s_n=s+nT$ such that

$$z(t)\le(2M)^n z(s),\quad\text{for all}\quad s+(n-1)T\le t\le s+nT.$$

From here it follows that

$$z(t)\le(2M)^{\frac{t-s}{T}+1}z(s)=2M\mathrm{e}^{\frac{\ln(2M)}{T}(t-s)}z(s),\quad\text{for all}\quad t\ge s\ge s_0.$$

Now given $\gamma>0$ we choose T such that $\gamma=\frac{\ln(2M)}{T}$ and s_0 large enough, such that (4.21) is satisfied and we get the first part of the result. In particular $s_0=s_0(\gamma)\to\infty$ as $\gamma\to 0$.

If $\tau_0=-\infty$, we slightly change the argument above and proceed "backwards". Take t_0 such that $\|a\|_{L^\sigma(-\infty,t_0)}$ is as small as we want. Then from (4.20) we get that for $t-T\le s\le t\le t_0$ we have, denoting $w(t,T)=\sup_{t-T\le\tau\le t}z(\tau)$ and using Hölder's inequality

$$\begin{aligned} z(t) &\le Mz(s)+w(t,T)\|a\|_{L^\sigma(t-T,t)}\Big(\int_s^t \frac{1}{(t-\tau)^{\alpha\sigma'}}\,d\tau\Big)^{1/\sigma'} \\ &\le Mz(s)+w(t,T)\delta(t_0,T) \end{aligned}$$

where we have set $\delta(t_0,T)=\|a\|_{L^\sigma(-\infty,t_0)}C(\alpha,\sigma')T^{1/\sigma'-\alpha}$, for some constant $C(\alpha,\sigma')$.

Now, given t_0, choose T such that

$$\delta(t_0,T)=\|a\|_{L^\sigma(-\infty,t_0)}C(\alpha,\sigma')T^{1/\sigma'-\alpha}=1/2. \tag{4.22}$$

Then we get

$$z(t)\le 2Mz(s)\quad\text{for all}\quad t-T\le s\le t\le t_0.$$

Writing $t_1=t-T$ and repeating the process and the estimate above we get a sequence $t_n=t-nT$ such that

$$z(t)\le(2M)^n z(s),\quad\text{for all}\quad t-nT\le s\le t-(n-1)T.$$

From here it follows that

$$z(t) \le (2M)^{\frac{t-s}{T}+1} z(s) = 2M \mathrm{e}^{\frac{\ln(2M)}{T}(t-s)} z(s), \quad \text{for all} \quad s \le t \le t_0.$$

Now given $\gamma > 0$ we choose T such that $\gamma = \frac{\ln(2M)}{T}$ and s_0 large enough, such that (4.22) is satisfied and we get the result. In particular $t_0 = t_0(\gamma) \to -\infty$ as $\gamma \to 0$. ■

As a consequence we obtain the following corollary that was used before.

Corollary 4.12 *Assume that $a \in L^\sigma((\tau_0, \infty)) \cap L^\infty_{loc}(\tau_0, \infty)$ with $1 \le \sigma \le \infty$, $\tau_0 \ge -\infty$ and that $z(t) \ge 0$ is a locally bounded function that for $t \ge s > \tau_0$ satisfies*

$$z(t) \le A + \int_s^t \frac{a(\tau)}{(t-\tau)^\alpha} z(\tau)\,\mathrm{d}\tau \tag{4.23}$$

with $\alpha\sigma' < 1$. Then
i) If $\sigma = 1$ (and $\alpha = 0$), we have for $t \ge s > \tau_0$

$$0 \le z(t) \le A.$$

ii) If $1 < \sigma < \infty$ and $\alpha\sigma' < 1$, for every $\gamma > 0$ there exists $s_0 > \tau_0$ such that if (4.23) holds for $t > s \ge s_0$ then

$$0 \le z(t) \le 2A\mathrm{e}^{\gamma(t-s)}, \quad t > s \ge s_0,$$

and $s_0 = s_0(\gamma) \to \infty$ as $\gamma \to 0$.

If $\tau_0 = -\infty$, for every $\gamma > 0$ there exists t_0 such that if (4.23) holds for $t_0 \ge t > s$ then

$$0 \le z(t) \le 2A\mathrm{e}^{\gamma(t-s)}, \quad t_0 \ge t > s,$$

and $t_0 = t_0(\gamma) \to -\infty$ as $\gamma \to 0$.
iii) If $\sigma = \infty$ and $0 \le \alpha < 1$ then we have for $t \ge s > \tau_0$

$$0 \le z(t) \le A(\alpha)\mathrm{e}^{\gamma(t-s)}$$

with $\gamma = \gamma(a, s, \alpha) = (\|a\|_{L^\infty(s,\infty)}\Gamma(1-\alpha))^{1/(1-\alpha)}$ and $A(\alpha)$ depends only on A and α but not on the function $a(\cdot)$ or s or γ or τ_0.

Proof. Denote now $Z(t) = A + \int_s^t \frac{a(\tau)}{(t-\tau)^\alpha} z(\tau)\,d\tau$ and note that for every $s < \rho < t$ we have

$$Z(t) = A + \int_s^\rho \frac{a(\tau)}{(t-\tau)^\alpha} z(\tau)\,d\tau + \int_\rho^t \frac{a(\tau)}{(t-\tau)^\alpha} z(\tau)\,d\tau$$

and using that $t > \rho > s$ and $z(\tau) \le Z(\tau)$, we get

$$Z(t) \le Z(\rho) + \int_\rho^t \frac{a(\tau)}{(t-\tau)^\alpha} z(\tau)\,d\tau \le Z(\rho) + \int_\rho^t \frac{a(\tau)}{(t-\tau)^\alpha} Z(\tau)\,d\tau.$$

Therefore $Z(t)$ satisfies (4.20) with $M = 1$ and then Lemma 4.11 applies and we get the result.
■

5 The nonlinear equation

We apply now the previous results in the analysis of the asymptotic behavior of the positive solutions of the nonlinear problem

$$\left\{\begin{array}{rcll} u_t - \Delta u &=& f(t,x,u) & \text{in } \Omega, \quad t > s \\ \mathcal{B}u &=& 0 & \text{on } \partial\Omega \\ u(s) &=& u_0 \geq 0 & \end{array}\right. \tag{5.1}$$

where $f : I\!R \times \Omega \times I\!R \to I\!R$ is suitably smooth and $f(t,x,0) \geq 0$.

In [9] there were given conditions on the nonlinear term $f(t,x,u)$ ensuring the existence of complete positive solutions of (5.1). Also, conditions guaranteeing that positive solutions are nondegenerate at ∞ and/or $-\infty$, in the sense of Definition 3.4, where also given in [11].

Finally it was also shown in [11] that the additional assumption

$$\frac{f(t,x,u)}{u} \quad \text{decreasing for} \quad u \geq 0 \tag{5.2}$$

implies the uniqueness of the complete, positive, bounded and nondegenerate at $-\infty$ solution of (5.1), $\varphi(t,x)$.

Moreover, such solution $\varphi(t,x)$ describes the asymptotic behavior of all positive solutions of (5.1) in a pullback sense, that is, for any bounded set of positive nondegenerate initial data $u(s)$ for $s \leq t_0$ and for any $t \in I\!R$, we have that

$$u(t,s;u(s)) - \varphi(t) \to 0, \quad \text{as } s \to -\infty \quad \text{in} \quad C(\overline{\Omega}). \tag{5.3}$$

Furthermore, $\varphi(t,x)$ also describes the forwards behavior of positive solutions of (5.1), since in fact it was also shown in [11] that for any $s \in I\!R$ and for any two positive solutions of (5.1) for $t > s$, we have,

$$u_1(t,x) - u_0(t,x) \to 0 \quad \text{as } t \to \infty \quad \text{in} \quad C(\overline{\Omega}). \tag{5.4}$$

Note that standard parabolic regularization implies that (5.3) and (5.4) can also be obtained in $C^1(\overline{\Omega})$.

An important particular example considered in [11] are logistic equations, for which

$$f(t,x,u) = m(t,x)u - n(t,x)u^{\rho}, \qquad \rho \geq 2 \tag{5.5}$$

where $m \in C^{\theta}(I\!R, L^p(\Omega))$ for certain $p > N/2$ and $0 < \theta \leq 1$ and $n \geq 0$ is continuous and locally Hölder in t, not identically zero. See [11] for precise conditions on $m(t,x)$, $n(t,x)$ such that the results above apply.

Assuming (5.2), our goal here is to give conditions such that the convergences in (5.3) and (5.4) above are exponential. In fact, we first have

Proposition 5.1
i) Let $u(t,x)$, for $t > s \geq s_0$, be a positive, bounded and nondegenerate at ∞ solution of (5.1). Assume

$$P(t,x) = \frac{\partial}{\partial u} f(t,x,u(t,x)) - \frac{f(t,x,u(t,x))}{u(t,x)} (\leq 0)$$

(which is nonpositive thanks to (5.2)), satisfies the assumption in Theorem 4.4 or Propositions 4.7 or 4.9 with $\gamma = 0$, as $t \to \infty$.

Then the exponential type at ∞ of the linearized equation along $u(t,x)$ i.e. (2.1), with

$$C(t,x) = \frac{\partial}{\partial u} f(t,x,u(t,x)),$$

is negative, i.e.

$$\beta_0^+(C) < 0.$$

In other words, $u(t,x)$ is linearly exponentially stable for (5.1) as $t \to \infty$.

ii) Let $u(t,x)$ be a positive, bounded and nondegenerate at $-\infty$ complete solution of (5.1). Assume we have, that for $t \le t_0$

$$P(t,x) = \frac{\partial}{\partial u} f(t,x,u(t,x)) - \frac{f(t,x,u(t,x))}{u(t,x)} (\le 0)$$

(which is nonpositive thanks to (5.2)) satisfies the assumption in Theorem 4.5 or Propositions 4.7 or 4.9 with $\gamma = 0$ as $t \to \infty$.

Then the exponential type at $-\infty$ of the linearized equation along $u(t,x)$ i.e. (2.1), with

$$C(t,x) = \frac{\partial}{\partial u} f(t,x,u(t,x)),$$

is negative, i.e.

$$\beta_0^-(C) < 0.$$

In other words, $u(t,x)$ is linearly exponentially stable for (5.1) in the pullback sense.

Proof. i) In fact the linearized equation along u, can be written as

$$\eta_t - \Delta\eta = \frac{\partial}{\partial u} f(t,x,u(t,x))\eta = \Big(P(t,x) + C_0(t,x)\Big)\eta$$

with boundary conditions $\mathcal{B}\eta = 0$, with $P(t,x)$ as in the statement and $C_0(t,x) = \frac{f(t,x,u(t,x))}{u(t,x)}$.

But, since $u(t,x)$ is positive bounded and nondegenerate at $+\infty$, from Lemma 3.5, see also Proposition 3 in [11], we have that, for $s \ge s_0$ and some positive constants M_0, M_1, $M_0 \le \|U_{C_0}(t,s)\| \le M_1$ i.e.

$$\beta_0^+(C_0) = 0,$$

and the defect in ∞ is $\gamma = 0$.

From the assumptions on $P(t,x)$ we get

$$\beta_0^+(C_0 + P) < 0$$

and the solutions of the linearized equation decays exponentially as $t \to \infty$.

The case ii) follows along the same lines. ■

Remark 5.2 *Note that in the case of logistic equations as in (5.5), that is,*

$$f(t,x,u) = m(t,x)u - n(t,x)u^\rho, \qquad \rho \geq 2,$$

(5.2) is satisfied and

$$P(t,x) := \frac{\partial}{\partial u} f(t,x,u(t,x)) - \frac{f(t,x,u(t,x))}{u(t,x)} = (1-\rho)n(t,x)u^{\rho-1}(t,x) \leq (1-\rho)n(t,x)\varphi_0^{\rho-1}(x) \leq 0$$

since $u(t,x) \geq \varphi_0(x)$ is non degenerate. Then Proposition 5.1 above applies, provided $P(t,x)$ satisfies the assumptions of either Theorem 4.4 or 4.5, Proposition 4.7, or Proposition 4.9 with $\gamma = 0$.

For example, if $\liminf_{t\to\pm\infty} n(t,x) = N_0(x) \geq 0$ a.e in Ω, does not have flat regions where it vanishes, then Proposition 4.7 applies.

Now we can translate this linear behavior to the nonlinear equation. In particular, we can prove the next result which improves (5.4).

Theorem 5.3 *Assume (5.2) and, for $t \geq s_0$, $0 \leq u(t,x)$ is a nondegenerate at ∞ and bounded solution of (5.1), satisfying the assumptions in Proposition 5.1 i).*

Then any other nonnegative, nontrivial solution of (5.1) $v(t,x)$ is nondegenerate and bounded and, as $t \to \infty$,

$$u(t,x) - v(t,x) \to 0, \quad \textit{exponentially in} \quad C(\overline{\Omega}).$$

Proof. Note that it was already proved in [11] that any other nonnegative, nontrivial solution of (5.1) is nondegenerate at ∞ and bounded. Now consider such a solution $v(t,x)$ and observe that it is enough to prove the result in the cases $v(t,x) \leq u(t,x)$ or $u(t,x) \leq v(t,x)$.

Assume first then that $v(t,x) \leq u(t,x)$. Denote

$$w(t,x) = u(t,x) - v(t,x) \geq 0$$

which satisfies

$$\begin{cases} w_t - \Delta w = f(t,x,u(t,x)) - f(t,x,v(t,x)) = C(t,x)w \\ \mathcal{B}w = 0 \end{cases}$$

where

$$C(t,x) = \frac{\partial}{\partial u} f(t,x,\xi(t,x)), \quad \text{with} \quad v(t,x) \leq \xi(t,x) \leq u(t,x).$$

Hence

$$C(t,x) = C_0(t,x) + P(t,x), \quad C_0(t,x) := \frac{\partial}{\partial u} f(t,x,u(t,x))$$

and from [11], see (5.4), we have

$$P(t,\cdot) \to 0 \quad \text{in} \quad L^\infty(\Omega)$$

as $t \to \infty$.

Since, $u(t,x)$ satisfies the assumptions in Proposition 5.1 above, we then have

$$\beta_0^+(C_0) < 0$$

and then, from Theorem 4.4, with $P^2(t,x)=0$ and $P^1(t,x)=P(t,x)$, we get

$$\beta_0^+(C)<0.$$

Thus $w(t,x)$ converges to zero exponentially as $t\to\infty$.

On the other hand, if $u(t,x)\leq v(t,x)$ then (5.2) implies that

$$C_1(t,x):=\frac{\partial}{\partial u}f(t,x,v(t,x))\leq C_0(t,x):=\frac{\partial}{\partial u}f(t,x,u(t,x))$$

which implies, from Lemma 3.1,

$$\beta_0^+(C_1)\leq\beta_0^+(C_0)<0.$$

Repeating the argument above, interchanging the roles of u and v, we get the result. ■

Finally, concerning the pullback behavior, we have the following result that improves (5.3).

Theorem 5.4 *Assume (5.2) and, for $t\leq t_0$, let $0\leq\varphi(t,x)$ be a positive nondegenerate at $-\infty$ and complete bounded solution of (5.1), satisfying the assumptions in Proposition 5.1 ii).*

Then for any bounded set of nondegenerate positive initial data $u(s)$ for $s\leq t_0$ and for any $t\in I\!R$, we have that

$$u(t,s;u(s))-\varphi(t)\to 0,\quad \textit{exponentially as } s\to-\infty\quad \textit{in}\quad C(\overline{\Omega}).$$

Proof. Take $t_0^-\leq t_0$ to be chosen later and denote

$$w(t,x)=\varphi(t,x)-u(t,s,x;u(s))\quad\text{for}\quad t_0^-\geq t>s$$

which satisfies, for $t_0^-\geq t>s$,

$$\begin{cases} w_t-\Delta w=f(t,x,\varphi(t,x))-f(t,x,u(t,s,x;u(s)))=C^s(t,x)w\\ \mathcal{B}w=0\\ w(s)=\varphi(s)-u(s)\end{cases}$$

where

$$C^s(t,x):=\frac{\partial}{\partial u}f(t,x,\xi^s(t,x)),\quad\text{with}\quad \xi^s(t,x)=\theta u(t,s,x;u(s))+(1-\theta)\varphi(t,x)$$

and $0\leq\theta=\theta(t,s,x)\leq 1$.

Hence

$$C^s(t,x)=C_0(t,x)+Q^s(t,x),\quad\text{with}\quad C_0(t,x):=\frac{\partial}{\partial u}f(t,x,\varphi(t,x))$$

Then from [11], see (5.3), we have that

$$\sup_{s\leq t\leq t_0^-}\|Q^s(t)\|_{L^\infty(\Omega)}\to 0\quad\text{as}\quad t_0^-\to-\infty.$$

Then we can choose t_0^- and extend $Q^s(t,x)$ for $t\leq s$ to have a family $P^s(t,x)$, with $s\leq t_0^-$, such that

$$\sup_{-\infty<t\leq t_0^-}\|P^s(t)\|_{L^\infty(\Omega)}$$

is as small as we want.

Since, $\varphi(t,x)$ satisfies the assumptions in Proposition 5.1 above, we then have

$$\beta_0^-(C_0) < 0$$

and then, from Theorem 4.5, we get, for all $s \le t_0^-$

$$\beta_0^-(D^s) \le \beta_0 < 0,$$

where

$$D^s(t,x) = C_0(t,x) + P^s(t,x), \quad C_0(t,x) = \frac{\partial}{\partial u} f(t,x,\varphi(t,x)).$$

Therefore, for all s and all $t_0^- \ge t \ge \tau$, we have

$$\|U_{D^s}(t,\tau)\|_{\mathcal{L}(L^\infty(\Omega))} \le M e^{\beta_0(t-\tau)}.$$

In particular, if we restrict to $\tau = s$ we get

$$\|\varphi(t) - u(t,s;u(s))\|_{L^\infty(\Omega)} = \|w(t)\|_{L^\infty(\Omega)} = \|U_{D^s}(t,s)w(s)\|_{L^\infty(\Omega)} \le M e^{\beta_0(t-s)} \|w(s)\|_{L^\infty(\Omega)}$$

which goes to zero as $s \to -\infty$, since $\varphi(s) - u(s)$ remains bounded in $L^\infty(\Omega)$. ■

6 Final remarks

Note that all the results in this paper have been worked out for the model problem (1.1). In fact, from the proofs above it is clear that building blocks of our approach are the smoothing estimates between Lebesgue spaces (2.2) and the subsolution argument around (4.10), which basically amounts for the maximum principle to hold. All the remaining estimates are obtained from this and the variations of constants formula. Hence, everything in this paper applies as well for more general linear non–autonomous problems

$$\left\{ \begin{array}{rcll} u_t + A(t)u & = & C(t,x)u & \text{in } \Omega, \quad t > s \\ \mathcal{B}(t)u & = & 0 & \text{on } \partial\Omega \\ u(s) & = & u_0 & \end{array} \right. \tag{6.1}$$

with time dependent elliptic part of the form

$$A(t,D)u = -\sum_{i,j=1}^N a_{ij}(t,x)\partial_i\partial_j u + \sum_{i=1}^N a_i(t,x)\partial_i u + a(t,x)u$$

with suitable smooth coefficients and either Dirichlet boundary conditions or time–dependent boundary conditions of Robin type

$$\mathcal{B}(t)u = \frac{\partial u}{\partial \vec{\eta}} + b(t,x)u,$$

for suitable exterior (oblique) unit vector $\vec{\eta}$, as long as the estimates mentioned above hold true. For example, for smooth coefficients and time–independent boundary conditions see [4], [2] or [1]. In particular, this applies to the problems considered in [5].

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