

Bifurcation and stability of equilibria with asymptotically linear boundary conditions at infinity*

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1 Introduction

In the last decade a lot of attention has been paid to problems with nonlinear boundary conditions. Hence, nowadays the underlying mechanisms for dissipativeness or blow-up of solutions is fairly well understood; see e.g. [7, 4, 6, 18, 19]. Therefore, it is a natural question to analyse the dynamics and bifurcations induced by the nonlinear boundary conditions, and compare its effects with the case of an interior reaction term, which has been more widely studied. In this direction for example, in [5] the existence of patterns for such problems, i.e. stable nontrivial equilibrium, was considered; see also the references therein for some previous and related results. In this work we consider the evolutionary equation of parabolic type

$$\begin{cases} u_t - \Delta u + u = 0, & \text{in } \Omega, \quad t > 0 \\ \frac{\partial u}{\partial n} = \lambda u + g(\lambda, x, u), & \text{on } \partial\Omega, \quad t > 0 \\ u(0, x) = u_0(x), & \text{in } \Omega \end{cases} \quad (1.1)$$

in a bounded and sufficiently smooth domain $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ and analyze the behavior and stability properties of the equilibrium solutions. These equilibria are solutions of the following elliptic problem with nonlinear boundary conditions

$$\begin{cases} -\Delta u + u = 0, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \lambda u + g(\lambda, x, u), & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

Our main goal here is to analyze some possible bifurcations of solutions as the parameter λ is varied and study the stability of such solutions. In particular we are interested in the possibility of

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producing solutions which are large in Ω in a given sense. We are also interested in characterizing the super or subcritical character of such bifurcations.

As it will be show below it is in fact possible to generate such large solutions, which will be obtained from a “bifurcation from infinity” argument, even in the case in which the nonlinear boundary condition is sublinear at infinity. Such solutions will be generated by a resonant mechanism at the boundary.

We will also show that some stability or instability of such solutions can be derived.

Since we will also give conditions to have either subcritical or supercritical bifurcations, we will obtain, as a by product, the analogous to the well known Landesman–Lazer conditions for the existence of equilibria in resonant cases, [15]. Also, a form of the anti-maximum principle will also be derived, [8]. A similar analysis for the case of an interior reaction term was first established in [2].

Now we present our main results in a more precise way. The main hypothesis on the nonlinearity g is the sublinearity with respect to the variable u . Hence we will assume a condition that, roughly speaking, will be of the type

$$|g(\lambda, x, u)| \leq C|u|^\alpha, \text{ as } |u| \rightarrow \infty \text{ for some } \alpha < 1.$$

Observe that we do not exclude the case where α is negative. This condition means that in the boundary condition, the dominant term for $|u|$ large is the linear term λu . In this respect we call this boundary condition asymptotically linear. This includes the case where $g(\lambda, x, u) = g(x)$ and it is well known that problem (1.2) will have a (unique) solution if λ is not an eigenvalue of the problem

$$\begin{cases} -\Delta\Phi + \Phi = 0, & \text{in } \Omega \\ \frac{\partial\Phi}{\partial n} = \sigma\Phi, & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

This eigenvalue problem is known as the Steklov eigenvalue problem and it is well known that (1.3) has a discrete set of eigenvalues $\{\sigma_i\}_{i=1}^\infty$. These numbers will play an essential role in the analysis below. In particular, for $\lambda \notin \{\sigma_i\}_{i=1}^\infty$, we consider the operator T_λ such that $T_\lambda b := v$, where v is the unique solution of

$$\begin{cases} -\Delta v + v = 0, & \text{in } \Omega \\ \frac{\partial v}{\partial n} - \lambda v = b, & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

for a function b given on $\partial\Omega$.

The fact that for compact sets of λ far from the Steklov eigenvalues, the norm of the operator T_λ , in some appropriate spaces, is uniformly bounded, joint with the sublinearity of the function g will allow us to show, by a fixed point argument, the existence of at least a solution of (1.2) for any λ not an Steklov eigenvalue. Moreover, all solutions will be uniformly bounded for λ in compact intervals far from the Steklov eigenvalues, see Theorem 2.7.

On the other hand, when the parameter λ approaches an Steklov eigenvalue, the norm of the operator T_λ diverges to ∞ . This fact is a first hint of the possibility of finding unbounded branches

of solutions and reveals the resonant mechanism at the boundary that produces such large solutions. For instance, when $g \equiv 0$ the structure of the solutions of the problem (1.2) is well known: if λ is not an Steklov eigenvalue, the only solution is the trivial solution and if λ is an Steklov eigenvalue, the whole space of eigenfunctions associated to that eigenvalue are solutions of the elliptic problem which can be regarded as unbounded branches of solutions. For the case where g is sublinear at infinity, we will apply general techniques of bifurcation theory, see [9], [16], [17], and will prove the existence of unbounded branches of solutions whenever the parameter λ approaches an Steklov eigenvalue of odd multiplicity, see Theorem 3.3. Moreover, since the first Steklov eigenvalue is simple, we will show the existence of unbounded branches of solutions bifurcating from the first eigenvalue. The fact that the first Steklov eigenfunction does not change sign will give us extra information that will permit us to analyze this branch of solutions in detail. In particular, we will show the existence of two branches of solutions one consisting of positive solutions and the other negative solutions, see Theorem 3.4.

Once the existence of these bifurcation branches has been established we pay attention to the type of bifurcation (sub or supercritical) occurring. It is clear that a condition on sublinearity of g is not enough to distinguish between the type of bifurcation and to accomplish this we will need to specify the precise asymptotics of the function g at infinity. For instance, if we consider that the function g behaves like $a|u|^\alpha$ as $u \rightarrow +\infty$, we can easily see that the sign of a will determine whether the bifurcation of positive solutions emanating from the first eigenvalue is sub or supercritical. For this, if $0 < u_n \rightarrow \infty$ is a solution of (1.2) for $\lambda_n \rightarrow \sigma_1$, multiplying the equation by the first Steklov eigenfunction $\Phi_1 > 0$ and integrating by parts we obtain,

$$(\sigma_1 - \lambda_n) \int_{\partial\Omega} u_n \Phi_1 d\zeta = \int_{\partial\Omega} g(\lambda_n, x, u_n) \Phi_1 d\zeta,$$

But since $u_n > 0$ and $u_n \rightarrow \infty$, then

$$\int_{\partial\Omega} u_n \Phi_1 d\zeta > 0, \quad \int_{\partial\Omega} g(\lambda_n, x, u_n) \Phi_1 d\zeta \approx a \int_{\partial\Omega} |u_n|^\alpha \Phi_1 d\zeta,$$

and the sign of $\sigma_1 - \lambda_n$ is the same as the sign of a . Hence, if $a > 0$ the bifurcation of positive solutions will be subcritical and if $a < 0$, it will be supercritical, see Theorem 4.3 below for a more general statement.

Moreover, we will also see that typically, when a bifurcation from infinity occurs at the first eigenvalue, the branch of equilibria will be stable when the bifurcation is subcritical and unstable when the bifurcation is supercritical, see Proposition 7.1 and Proposition 7.3 below.

Being able to give conditions to characterize when the bifurcation is sub or supercritical will allow us to address two important issues for this problem.

On one hand we will be able to give Landesman-Lazer type conditions, guaranteeing that the nonlinear resonant problem (that is, when $\lambda = \sigma_i$ for some i) has at least a solution, see [15]. For this, imagine that for a value σ_i we can determine that all possible bifurcations occurring at this value of the parameter are, say, subcritical. This implies that for $\lambda \in (\sigma_i, \sigma_i + \epsilon)$, for some $\epsilon > 0$ small, the solutions of (1.2) will have to be bounded in certain norms, uniformly for

$\lambda \in (\sigma_i, \sigma_i + \epsilon)$. Using elliptic regularity results, will allow us to pass to the limit in a weak sense as $\lambda \rightarrow \sigma_i$ and show that the limit is a solution of the resonant problem, see Theorem 5.1.

On the other hand, we will be able to prove Anti-Maximum Principles for the problem (1.4). In particular, if b is such that $\int_{\partial\Omega} b\Phi_1 > 0$, then the bifurcation of negative solutions occurring at $\lambda = \sigma_1$ is supercritical and this implies that for $\lambda \in (\sigma_1, \sigma_1 + \epsilon)$ the unique solution of (1.4) has to be strictly negative, see Theorem 6.1. These type of results are known as Anti-Maximum Principles and were first proved for elliptic problems of the form $-\Delta u = \lambda m(x)u + h(x)$ with Dirichlet boundary conditions by Clement and Peletier in [8].

This paper is organized as follows. In Section 2 we formulate the problem and show the existence of solutions for all values of the parameter λ different from the Steklov eigenvalues. To accomplish this, we analyse the linear problem (1.4), stating and proving several important regularity results. Then, we formulate the nonlinear problem (1.2) as a fixed point problem in certain function space on the boundary. Finally, the compactness results obtained through the regularity results and the Schaeffer fixed point theorem will show the existence of solutions.

In Section 3 we apply bifurcation results, mainly from [16, 17], to show the existence of unbounded branches of solutions bifurcating from the Steklov eigenvalues, see Theorem 3.3. We pay special attention to the bifurcations emanating from simple eigenvalues, see Theorem 3.4.

In Section 4 we give conditions on the behavior of the nonlinearity g for $|u|$ large that allow us to determine when sub or supercritical bifurcations occur.

In Section 5 we apply the conditions from the previous section to obtain Landesman-Lazer type conditions for the resonant problem.

In Section 6 we state and prove the anti maximum principle for (1.4) mentioned above.

In Section 7 we analyze the stability properties of the solutions bifurcating from the first eigenvalue.

Finally, in Section 8 we consider several important remarks and extensions. We study the conditions that have to be imposed on the nonlinearity g to obtain bifurcations from the trivial solution, instead of bifurcations from infinity. We also consider the case where the boundary condition is of the type $\frac{\partial u}{\partial n} = \lambda m(x)u + g(\lambda, x, u)$ where m is a potential that may change sign on $\partial\Omega$. We also consider the one dimensional case, that is, where the equation (1.2) is posed in $\Omega = (0, 1) \subset \mathbb{R}$.

2 Setting of the problem

In this Section we rewrite equation (1.2) as a fixed point problem in appropriate function spaces and analyze the existence of solutions for all $\lambda \in \mathbb{R}$ except for a discrete set. To accomplish this task we will use Schaeffer's fixed point theorem, see [10].

With respect to the nonlinearity g , we assume the hypothesis

(H1) $g : \mathbb{R} \times \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function (i.e. $g = g(\lambda, x, s)$ is measurable in $x \in \Omega$, and continuous with respect to $(\lambda, s) \in \mathbb{R} \times \mathbb{R}$). Moreover, there exist $h \in L^r(\partial\Omega)$ with $r > N - 1$ and a continuous functions $\Lambda : \mathbb{R} \rightarrow \mathbb{R}^+$, $U : \mathbb{R} \rightarrow \mathbb{R}^+$, satisfying

$$|g(\lambda, x, s)| \leq \Lambda(\lambda)h(x)U(s), \quad \forall(\lambda, x, s) \in \mathbb{R} \times \partial\Omega \times \mathbb{R}. \quad (2.1)$$

Moreover, we assume also the following condition on the function U

$$\mathbf{(H2)} \quad \lim_{|s| \rightarrow \infty} \frac{U(s)}{s} = 0.$$

Observe that the sublinearity of g at infinity is given by condition (H2).

With respect to the linear problem, it is already well known, see [1], that the operator $A = -\Delta + I$, with homogeneous Neumann boundary conditions defines an unbounded operator in $L^p(\Omega)$ for all $p > 1$ with domain $D(A) = \{u \in W^{2,p}(\Omega); \partial u / \partial n = 0 \text{ in } \partial\Omega\}$. Moreover, the operator A has an associated scale of interpolation-extrapolation spaces and, in particular, for each $p > 1$, we have that $A : W^{1,p}(\Omega) \rightarrow W^{-1,p}(\Omega)$ is an isomorphism.

Hence, for any $q \geq 1$, since we have the embedding $L^q(\partial\Omega) \hookrightarrow W^{-1,p}(\Omega)$, continuous for $p = q \frac{N}{N-1}$ and compact if $p < q \frac{N}{N-1}$ we have that for $b \in L^q(\partial\Omega)$, the unique solution of

$$\begin{cases} -\Delta v + v = 0, & \text{in } \Omega \\ \frac{\partial v}{\partial n} = b, & \text{on } \partial\Omega. \end{cases} \quad (2.2)$$

is given by $v = A^{-1}(b) \in W^{1,p}(\Omega)$ and $\|v\|_{W^{1,p}(\Omega)} \leq C\|b\|_{L^q(\partial\Omega)}$. We will denote by $T_0(b) = v$ and $S_0(b) = \gamma T_0(b)$, where γ is the trace operator. The operator S_0 is known as the Neumann-to-Dirichlet operator. Hence, the operator T_0 takes functions defined on $\partial\Omega$ to functions defined in Ω and S_0 takes functions defined on $\partial\Omega$ to functions defined on $\partial\Omega$.

Our first task will be to show that any weak solution $u \in H^1(\Omega)$ of (1.2) lies in $C^\alpha(\bar{\Omega})$. To accomplish this, we will need several regularity results of the associated linear problems. As a matter of fact, as a consequence of the above and using embedding and trace theorems we can easily show the following regularity results,

Lemma 2.1 *If $N \geq 2$ and $b \in L^q(\partial\Omega)$ with $q \geq 1$. Then, the solution $v = T_0 b$ of (2.2) satisfies $v \in W^{1,p}(\Omega)$ for $1 \leq p \leq qN/(N-1)$ with $\|v\|_{W^{1,p}(\Omega)} \leq C\|b\|_{L^q(\partial\Omega)}$.*

In particular, we have

i) If $1 \leq q < N - 1$, then $\gamma v \in L^r(\partial\Omega)$ for all $1 \leq r \leq \frac{(N-1)q}{N-1-q}$ and the map $S_0 : L^q(\partial\Omega) \rightarrow L^r(\partial\Omega)$ is continuous for $1 \leq r \leq \frac{q(N-1)}{N-1-q}$ and compact for $1 \leq r < \frac{q(N-1)}{N-1-q}$.

ii) If $q = N - 1$, then $\gamma v \in L^r(\partial\Omega)$ for all $r \geq 1$ and the map $S_0 : L^q(\partial\Omega) \rightarrow L^r(\partial\Omega)$ is continuous and compact for $1 \leq r < \infty$.

iii) If $q > N - 1$, then $v \in C^\alpha(\bar{\Omega})$ with $\|v\|_{C^\alpha(\bar{\Omega})} \leq C\|b\|_{L^q(\partial\Omega)}$ for some $\alpha \in (0, 1)$, moreover $\gamma v \in C^\alpha(\partial\Omega)$ and the map $S_0 : L^q(\partial\Omega) \rightarrow C^\alpha(\partial\Omega)$ is continuous and compact.

As an immediate corollary, we have the following technical result,

Corollary 2.2 *i)* For any $q \geq 1$ we have that if $b \in L^q(\partial\Omega)$ then $S_0b \in L^{q+\frac{1}{N}}(\partial\Omega)$.

ii) If b satisfies $|b(x)| \leq h(x)w(x)$ where $h \in L^r(\partial\Omega)$ with $r > N - 1$. Then if we define $\delta = \frac{N-1}{N-2} - r' > 0$ we have that if $w \in L^p(\partial\Omega)$ with $\frac{1}{N-1} \leq \frac{1}{p} + \frac{1}{r} \leq 1$ then $S_0b := \gamma v \in L^{p+\delta}(\partial\Omega)$ and $\|S_0b\|_{L^{p+\delta}(\partial\Omega)} \leq C\|w\|_{L^p(\partial\Omega)}$

Proof. *i)* Observe that if $q \geq N - 1$ then from the above Corollary $\gamma v \in L^r(\partial\Omega)$ for all $r \geq 1$. In case $1 \leq q < N - 1$ then $S_0b \in L^r(\partial\Omega)$ for $r \leq \frac{(N-1)q}{N-1-q}$. A simple computation shows that

$$\frac{(N-1)q}{N-1-q} - q \geq \frac{1}{N}, \text{ for } 1 \leq q < N - 1.$$

ii) Notice that $hw \in L^{pr/(p+r)}(\partial\Omega)$ and $\frac{pr}{p+r} \geq 1$ because $\frac{1}{p} + \frac{1}{r} \leq 1$. Hence, by Lemma 2.2 $\gamma v \in L^s(\partial\Omega)$ with $s = \frac{\frac{pr}{p+r}(N-1)}{N-1-\frac{pr}{p+r}}$. If we denote by $y = \frac{pr}{p+r} = \frac{1}{\frac{1}{p} + \frac{1}{r}}$, then $1 \leq y \leq N - 1$, $p = \frac{ry}{r-y}$ and

$$\min_{\frac{1}{N-1} \leq \frac{1}{p} + \frac{1}{r} \leq 1} \left\{ \frac{\frac{pr}{p+r}(N-1)}{N-1-\frac{pr}{p+r}} - p \right\} = \min_{1 \leq y \leq N-1} \left\{ \frac{y(N-1)}{N-1-y} - \frac{ry}{r-y} \right\}$$

But a simple computation shows that this last minimum is attained at $y = 1$. This concludes the proof of the Corollary. \square

These regularity results with a bootstrap argument will allow us to prove the following

Proposition 2.3 *Assume g satisfies (H1) and (H2). Then, for any $R > 0$ if $u \in H^1(\Omega)$ is a solution of (1.2) for some $|\lambda| \leq R$ we have*

$$\|u\|_{C^\alpha(\bar{\Omega})} \leq C(1 + \|u\|_{L^p(\partial\Omega)}) \quad (2.3)$$

for some positive α , where $C = C(R)$ and $p = 2(N-1)/(N-2)$.

Proof. Assume $N \geq 3$, the proof when $N = 2$ is simpler. Observe that the boundary condition satisfied by u is $\frac{\partial u}{\partial n} = \lambda u + g(\lambda, x, u)$ and by hypotheses (H1), (H2) and assuming that $|\lambda| \leq R$, we have $|g(\lambda, x, u)| \leq Ch(x)(1 + |u(x)|)$ for some constant $C = C(R)$. Hence $\frac{\partial u}{\partial n} = b(x)$ with $|b(x)| \leq C(1 + h(x))(1 + |u(x)|)$. Notice also that $1 + h \in L^r(\partial\Omega)$ for some $r > N - 1$.

Now, if $u \in H^1(\Omega)$, then $\gamma u \in L^p(\partial\Omega)$ with $p = 2\frac{N-1}{N-2}$ which satisfies that $\frac{1}{p} + \frac{1}{r} \leq 1$ for any $r > N - 1$. Hence, applying the regularity result of Corollary 2.2 *ii)* we obtain that $\gamma u \in L^{p+\delta}(\partial\Omega)$ and

$$\|u\|_{L^{p+\delta}(\partial\Omega)} \leq C(1 + \|u\|_{L^p(\partial\Omega)})$$

Repeating this regularity argument k times, we get that $\gamma u \in L^{p+k\delta}(\partial\Omega)$ with $\frac{1}{p+k\delta} + \frac{1}{r} = \frac{1}{s} < \frac{1}{N-1}$. Moreover, we will also have

$$\|u\|_{L^{p+k\delta}(\partial\Omega)} \leq C(1 + \|u\|_{L^{p+(k-1)\delta}(\partial\Omega)}) \leq \dots \leq C(1 + \|u\|_{L^p(\partial\Omega)})$$

In particular, $b \in L^s(\partial\Omega)$ for some $s > N - 1$. Hence, Lemma 2.1 iii) implies that $u \in C^\alpha(\bar{\Omega})$ and

$$\|u\|_{C^\alpha(\bar{\Omega})} \leq C\|b\|_{L^s(\partial\Omega)} \leq C(1 + \|u\|_{L^p(\partial\Omega)}).$$

□

Remark 2.4 *The regularity result of the last proposition tells us that looking for solutions of problem (1.2) in $H^1(\Omega)$ is equivalent to looking for solutions in a more regular space like $C^\alpha(\bar{\Omega})$.*

We analyze now the operator S_0 , the Neumann-to-Dirichlet operator. We have the following result,

Lemma 2.5 *The operator $S_0 : L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$ is a linear selfadjoint, positive and compact operator. If we denote by $\{\tau_i\}_{i=1}^\infty$ its eigenvalues, and by $\sigma_i = 1/\tau_i$ we have that for any $\lambda \in \mathbb{R}$, $\lambda \notin \{\sigma_i\}_{i=1}^\infty$, the operator $S_\lambda : L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$ defined by $S_\lambda(g) = \gamma v$ where v is the unique solution of*

$$\begin{cases} -\Delta v + v = 0, & \text{in } \Omega \\ \frac{\partial v}{\partial n} - \lambda v = g, & \text{on } \partial\Omega \end{cases} \quad (2.4)$$

is selfadjoint, continuous and compact. Moreover, the first eigenvalue σ_1 is simple and its eigenfunction Φ_1 can be chosen strictly positive. Also, if $r > N - 1$ then, $S_\lambda : L^r(\partial\Omega) \rightarrow C^0(\partial\Omega)$ is continuous and compact and for any compact set $K \subset \mathbb{R} \setminus \{\sigma_i\}_{i=1}^\infty$ the norm of $S_\lambda : L^r(\partial\Omega) \rightarrow C^0(\partial\Omega)$ is uniformly bounded for $\lambda \in K$. Also, $\|S_\lambda\| \rightarrow \infty$ as $\lambda \rightarrow \sigma_i$ for some i .

Proof. Observe that if $b_1, b_2 \in L^2(\partial\Omega)$ and if v_1, v_2 are the solutions of $-\Delta v_i + v_i = 0$ in Ω , $\frac{\partial v_i}{\partial n} = b_i$, $i = 1, 2$, then by the weak formulation of this problem we have that

$$(S_0(b_1), b_2)_{L^2(\partial\Omega)} = \int_\Omega \nabla v_1 \nabla v_2 + \int_\Omega v_1 v_2 = (b_1, S_0(b_2))_{L^2(\partial\Omega)} \quad (2.5)$$

From (2.5) it follows that S_0 is selfadjoint and positive. That S_0 is compact follows from Lemma 2.1. The fact that the first eigenfunction can be chosen nonnegative follows easily from the Rayleigh quotient for the first eigenvalue. Then, maximum principles imply that the first eigenfunction is actually strictly positive. In turn, this implies that the first eigenvalue is simple.

The rest of the proof follows just by realizing that $S_\lambda = (I - \lambda S_0)^{-1} \circ S_0$ and applying the regularity results of Corollary 2.2. □

It is clear now that we can set a fixed point problem to obtain the solutions of (1.2). As a matter of fact, $u \in H^1(\Omega)$ is a solution of (1.2) if and only if its trace $v = \gamma u$ is a fixed point of

$$v = S_\lambda(g(\lambda, \cdot, v)) \quad (= (I - \lambda S_0)^{-1} \circ S_0(g(\lambda, \cdot, v))) \quad (2.6)$$

Notice also that once v is obtained we recover u by solving $-\Delta u + u = 0$ in Ω with $u = v$ on the boundary.

Concerning the fixed point problem (2.6), we have

Lemma 2.6 *Under hypotheses (H1) and (H2), the map $C^0(\partial\Omega) \ni v \rightarrow g(\lambda, \cdot, v) \in L^r(\partial\Omega)$ is well defined and continuous. Moreover, for each $M > 0$, $\epsilon > 0$, there exists a constant $C = C(\epsilon, M)$ such that*

$$\|g(\lambda, \cdot, v)\|_{L^r(\partial\Omega)} \leq \epsilon \|v\|_{C^0(\partial\Omega)} + C \quad (2.7)$$

for all $v \in C^0(\partial\Omega)$, $|\lambda| \leq M$.

In particular, the map $C^0(\partial\Omega) \ni v \rightarrow S_\lambda(g(\lambda, \cdot, v)) \in C^0(\partial\Omega)$, is continuous and compact for all $\lambda \in \mathbb{R} \setminus \{\sigma_i\}_{i=1}^\infty$.

Proof. That this map is well defined follows from the bounds of g given by (H1). The continuity follows from the continuity of g with respect to the last variable, the bounds of g given by (H1) and the dominated convergence theorem. Statement (2.7) follows from the fact that for each $\epsilon > 0$ we have the inequality $|U(s)| \leq \epsilon s + C$, for some constant $C = C(\epsilon)$, the fact that the function $\Lambda(\lambda)$ is continuous.

The last part of the lemma follows easily. \square

Now we are in a position where we can show the existence of solutions of our original problem (1.2) for all $\lambda \in \mathbb{R} \setminus \{\sigma_i\}_{i=1}^\infty$. We have the following

Theorem 2.7 *If g satisfies (H1) and (H2) then, for all $\lambda \in \mathbb{R} \setminus \{\sigma_i\}_{i=1}^\infty$ there exists at least one solution of problem (1.2). Moreover, for each compact set $K \subset \mathbb{R} \setminus \{\sigma_i\}_{i=1}^\infty$, we have the existence of a constant $C = C(K)$ such that any solution of problem (1.2) is bounded in $C^0(\Omega)$ by C .*

Proof. Consider the compact set $K \subset \mathbb{R} \setminus \{\sigma_i\}_{i=1}^\infty$ and observe that by Lemma 2.5 we have that there exists a constant $C_1 = C_1(K)$ such that the norm of $S_\lambda : L^r(\partial\Omega) \rightarrow C^0(\Omega)$ is bounded by C_1 for all $\lambda \in K$.

We will apply Schaeffer fixed point argument to (2.6), see [10]. For this, consider $\theta \in [0, 1]$ and let v be a fixed point of

$$v = \theta S_\lambda(g(\lambda, \cdot, v)) \quad (2.8)$$

for some $\lambda \in K$. Then $\|v\|_{C^0(\partial\Omega)} \leq C_1 \|g(\lambda, \cdot, v)\|_{L^r(\partial\Omega)}$. But, by (2.7) we get

$$\|v\|_{C^0(\partial\Omega)} \leq C_1(\epsilon \|v\|_{C^0(\partial\Omega)} + C(\epsilon, K))$$

Choosing ϵ small enough such that $1 - C_1\epsilon \geq 1/2$, we get $\|v\|_{C^0(\partial\Omega)} \leq 2C_1C(\epsilon, K)$. Noticing that by Lemma 2.6 we have that $v \rightarrow S_\lambda(g(\lambda, \cdot, v))$ is compact in $C^0(\partial\Omega)$ when $\lambda \notin \{\sigma_i\}_{i=1}^\infty$ and applying Schaeffer fixed point argument, we prove the proposition. \square

3 Unbounded branches of equilibria

From the results of the previous section it is clear that when the value of the parameter λ is away from the Steklov eigenvalues, the solutions of (1.2) are bounded uniformly in λ . On the other hand, since the norm of the operator S_λ blows up to infinity when λ approaches a Steklov eigenvalue, see Lemma 2.5, it is natural to expect the existence of branches of solutions that diverge to infinity in certain norms when the parameter approaches a Steklov eigenvalue. For instance, if we consider the case where $g \equiv 0$, then, for any $\lambda \notin \{\sigma_i\}_{i=1}^\infty$ the unique solution is $u \equiv 0$, while for $\lambda = \sigma_i$ we have that the whole finite dimensional subspace given by the eigenfunctions associated to σ_i are solution. This subspace constitutes an unbounded branch of solutions.

Let us start by analyzing the behavior of the solutions when we know explicitly that the solution blows up in certain norm.

Proposition 3.1 *Assume $\{\lambda_n\}_{n=1}^\infty$ is a convergent sequence of real numbers for which there exist solutions u_n of (1.2) with $\|u_n\|_{L^\infty(\partial\Omega)} \rightarrow \infty$ as $n \rightarrow \infty$. Then necessarily $\lambda_n \rightarrow \sigma_i$ for certain $i \in \mathbb{N}$ and for any subsequence of u_n , there exists another subsequence, that we denote by $u_{n'}$, and an eigenfunction Φ_i associated to σ_i with $\|\Phi_i\|_{L^\infty(\partial\Omega)} = 1$ such that*

$$\frac{u_{n'}}{\|u_{n'}\|_{L^\infty(\partial\Omega)}} \rightarrow \Phi_i, \quad \text{in } C^\beta(\bar{\Omega})$$

for some $\beta > 0$.

Proof. Applying the Hölder estimate given by (2.3) we obtain that if $v_n = u_n / \|u_n\|_{L^\infty(\partial\Omega)}$, we have $\|v_n\|_{C^\alpha(\bar{\Omega})} \leq C$, for some C independent of n . Using the compact embedding $C^\alpha(\bar{\Omega}) \hookrightarrow C^\beta(\bar{\Omega})$ for $0 < \beta < \alpha$, we obtain that for any subsequence of v_n , there exists another subsubsequence, $v_{n'}$ and a function $\Phi \in C^\beta(\bar{\Omega})$ such that $v_{n'} \rightarrow \Phi$ in $C^\beta(\bar{\Omega})$. Therefore, since $\|v_{n'}\|_{L^\infty(\partial\Omega)} = 1$ we get that $\|\Phi\|_{L^\infty(\partial\Omega)} = 1$ and in particular Φ is not identically zero.

The equation satisfied by $v_{n'}$ is

$$\begin{cases} -\Delta v_{n'} + v_{n'} = 0, & \text{in } \Omega \\ \frac{\partial v_{n'}}{\partial n} = \lambda_{n'} v_{n'} + \frac{g(\lambda, x, u_{n'})}{\|u_{n'}\|_{L^\infty(\partial\Omega)}}, & \text{on } \partial\Omega \end{cases}$$

Passing to the limit in the weak formulation of this equation, taking into account that $\frac{g(\lambda, x, u_{n'})}{\|u_{n'}\|_{L^\infty(\partial\Omega)}} \rightarrow 0$ in $L^r(\partial\Omega)$ as $n' \rightarrow \infty$ and that $v_{n'} \rightarrow \Phi$, we get that Φ is a solution of

$$\begin{cases} -\Delta\Phi + \Phi = 0, & \text{in } \Omega \\ \frac{\partial\Phi}{\partial n} = \sigma\Phi, & \text{on } \partial\Omega \end{cases}$$

where $\sigma = \lim_{n' \rightarrow \infty} \lambda_{n'}$. Since $\|\Phi\|_{L^\infty(\partial\Omega)} = 1$, necessarily σ is an Steklov eigenvalue and Φ is an Steklov eigenfunction associated to σ . This proves the Proposition. \square

We immediately have,

Corollary 3.2 *With the same hypotheses of Proposition 3.1 we have*

i) *The whole sequence satisfies $\|u_n\|_{L^p(\partial\Omega)} \rightarrow \infty$ for any $1 \leq p \leq \infty$.*

ii) *If $u_n \geq 0$ for all n , then necessarily $\lambda_n \rightarrow \sigma_1$ and the whole sequence satisfies*

$$\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}} \rightarrow \Phi_1, \quad \text{in } C^\beta(\bar{\Omega}).$$

Proof. i) Since $L^p(\partial\Omega) \hookrightarrow L^1(\partial\Omega)$, it will be enough to show the result for $p = 1$. If this is not the case, then there will exist a subsequence u_n bounded in $L^1(\partial\Omega)$. From Proposition 3.1, we can get another subsequence $u_{n'}$ satisfying $u_{n'}/\|u_{n'}\|_{L^\infty(\partial\Omega)} \rightarrow \Phi_i$ and in particular $\|u_{n'}\|_{L^1(\partial\Omega)}/\|u_{n'}\|_{L^\infty(\partial\Omega)} \rightarrow \|\Phi_i\|_{L^1(\partial\Omega)} > 0$, which implies that $\|u_{n'}\|_{L^1(\partial\Omega)} \rightarrow \infty$, which is a contradiction.

ii) From Proposition 3.1, any possible convergent subsequence of $u_n/\|u_n\|_{L^\infty(\partial\Omega)}$ has to converge to an Steklov eigenfunction Φ_i with $\|\Phi_i\|_{L^\infty(\partial\Omega)} = 1$. Since in this case $u_n \geq 0$, we have that $\Phi_i \geq 0$. But σ_1 is the unique Steklov eigenvalue with a nonnegative eigenfunction Φ_1 , see Lemma 2.5. \square

We will show now that any Steklov eigenvalue σ of odd multiplicity is a bifurcation point from infinity, that is, there exists a sequence λ_n with $\lambda_n \rightarrow \sigma$ and a sequence of solutions u_n of (1.2) for the value λ_n such that $\|u_n\|_{L^\infty(\Omega)} \rightarrow \infty$. **g**

Before stating the result, consider the following notation. We will consider the solutions of (1.2) in $\mathbb{R} \times C(\bar{\Omega})$, where the first coordinate is the value of λ and the second is the function u , which is a solution of (1.2) for this value of λ . In this sense, we will denote the set of solutions by \mathcal{S} . Recall also that we have denoted the Steklov eigenvalues (eigenvalues of problem (1.3)) by $\{\sigma_i\}_{i=1}^\infty$.

We have the following result,

Theorem 3.3 *Consider problem (1.2) and assume that the nonlinearity g satisfies conditions (H1) and (H2). If σ is an Steklov eigenvalue of odd multiplicity, then the set of solutions of (1.2), denoted by \mathcal{S} , possesses an unbounded component \mathcal{D} which meets $(\sigma, \infty) \in \mathbb{R} \times C(\Omega)$.*

Moreover, if $[\lambda_-, \lambda_+] \subset \mathbb{R}$ is an interval such that $[\lambda_-, \lambda_+] \cap \{\sigma_i\}_{i=1}^\infty = \{\sigma\}$ and $\mathcal{M} = [\lambda_-, \lambda_+] \times \{u \in C(\bar{\Omega}) : \|u\|_{C(\bar{\Omega})} \geq 1\}$, then either

(i) $\mathcal{D} \setminus \mathcal{M}$ is bounded in $\mathbb{R} \times C(\bar{\Omega})$ in which case $\mathcal{D} \setminus \mathcal{M}$ meets the set $\{(\lambda, 0), : \lambda \in \mathbb{R}\}$ at $(\lambda_0, 0)$ such that $g(\lambda_0, \cdot, 0) = 0$, or

(ii) $\mathcal{D} \setminus \mathcal{M}$ is unbounded in $\mathbb{R} \times C(\bar{\Omega})$.

If $\mathcal{D} \setminus \mathcal{M}$ is unbounded, and it has a bounded projection on \mathbb{R} , then $\mathcal{D} \setminus \mathcal{M}$ meets $(\tilde{\sigma}, \infty) \in \mathbb{R} \times C(\bar{\Omega})$, with $\sigma \neq \tilde{\sigma} \in \{\sigma_i\}_{i=1}^{\infty}$, i.e. $\mathcal{D} \setminus \mathcal{M}$ meets another bifurcation point from infinity.

Proof. Observe first that the fixed point problem (2.6) can be recast as

$$v = \lambda S_0 v + S_0(g(\lambda, \cdot, v)) \quad (3.1)$$

where S_0 is the Neumann-to-Dirichlet operator, see Lemma 2.5.

We apply now the general techniques from [17] to the fixed point problem (3.1) in the space $C(\partial\Omega)$. Thus, we have to prove that

(A) $S_0(g(\lambda, \cdot, v)) = o(\|v\|)$ at $v = \infty$ uniformly for λ in bounded intervals, and

(B) the map $(\lambda, v) \rightarrow \|v\|^2 S_0(g(\lambda, \cdot, v/\|v\|^2))$ is compact for λ in bounded intervals.

where for simplicity we denote by $\|v\| := \|v\|_{C(\partial\Omega)}$.

(A) For any $v \in C(\partial\Omega)$ we have, from (H1) that $g(\lambda, \cdot, v) \in L^r(\partial\Omega)$. Therefore,

$$\frac{\|S_0(g(\lambda, \cdot, v))\|}{\|v\|} \leq C \frac{\|g(\lambda, \cdot, v)\|_{L^r(\partial\Omega)}}{\|v\|} \leq C(\epsilon + \frac{C_\epsilon}{\|v\|}) \quad (3.2)$$

where we have used Lemma 2.1 for the first inequality and Lemma 2.6 for the second one. From (3.2) we easily get (A).

(B) We have to verify that $H : \mathbb{R} \times C(\partial\Omega) \rightarrow C(\partial\Omega)$ defined by $H(\lambda, v) := \|v\|^2 S_0(g(\lambda, x, v/\|v\|^2))$ is compact. Note first that the image of $\{(\lambda, v) \in [\underline{\lambda}, \bar{\lambda}] \times C(\partial\Omega) : \delta \leq \|v\|_{C(\partial\Omega)} \leq \rho\}$ under H is relatively compact for any $\underline{\lambda} < \bar{\lambda}$ and $0 < \delta \leq \rho < \infty$. This follows from the boundedness of g and the compactness of S_0 . Thus we only need to prove that the image of $[\underline{\lambda}, \bar{\lambda}] \times B_\delta$ under H is relatively compact in $C(\partial\Omega)$ for some $\delta > 0$ small enough, where $B_\delta := \{v \in C(\partial\Omega) : \|v\| \leq \delta\}$. Let us choose $v \in B_\delta$, and define $w = \frac{v}{\|v\|^2}$, which satisfies $\|w\| \geq \frac{1}{\delta}$.

From (2.7) with $\epsilon = 1$, we get

$$\frac{\|g(\lambda, \cdot, w)\|_{L^r(\partial\Omega)}}{\|w\|} \leq C, \quad (3.3)$$

with $C = C(\lambda, \|h\|_{L^r(\partial\Omega)}, \delta)$. Therefore

$$\|v\|^2 \left\| g\left(\lambda, \cdot, \frac{v}{\|v\|^2}\right) \right\|_{L^r(\partial\Omega)} \leq C\|v\| \leq C\delta \quad (3.4)$$

Now, the compactness of $S_0 : L^r(\partial\Omega) \rightarrow C(\partial\Omega)$ given by Lemma 2.1 ends the proof. \square

We analyze now the case where the eigenvalue σ is simple, and in particular the case of the first eigenvalue. We have the following,

Theorem 3.4 *Let σ denote a simple Steklov eigenvalue and Φ a corresponding eigenfunction. Assume g satisfies hypotheses **(H1)** and **(H2)**. Then, the set of solutions of (1.2), possesses two unbounded components \mathcal{D}^+ , \mathcal{D}^- which meet $(\sigma, \infty) \in \mathbb{R} \times C(\bar{\Omega})$, satisfying*

(i) *there exists a neighbourhood \mathcal{O}_1 of (σ, ∞) such that $(\lambda, v) \in \mathcal{D}^+ \cap \mathcal{O}_1$ and $(\lambda, v) \neq (\sigma, \infty)$ implies*

$$v = \alpha\Phi + w \quad \text{where } \alpha > 0, \quad \text{with } \|w\|_{L^\infty(\partial\Omega)} = o(|\alpha|) \text{ at } |\alpha| = \infty$$

(ii) *there exists a neighbourhood \mathcal{O}_2 of (σ, ∞) such that $(\lambda, v) \in \mathcal{D}^- \cap \mathcal{O}_2$ and $(\lambda, v) \neq (\sigma, \infty)$ implies*

$$v = -\alpha\Phi + w \quad \text{where } \alpha > 0, \quad \text{with } \|w\|_{L^\infty(\partial\Omega)} = o(|\alpha|) \text{ at } |\alpha| = \infty$$

Proof. See [17], Corollary 1.8. \square

Note, in particular, that if $\sigma = \sigma_1$ since the first eigenfunction can be chosen positive, this result implies the existence of branches of positive and negative solutions bifurcating from infinity.

4 Sufficient conditions for subcritical and supercritical bifurcations from infinity

In this section we give conditions on the nonlinearity g that allows us to characterize the different bifurcations occurring. Obviously, the type of bifurcation (sub or supercritical) occurring at a bifurcation point will be dictated by the behavior of the nonlinearity g for large values of s . For instance, assume that we have a sequence of solutions u_n for the value of the parameter λ_n and assume that $\lambda_n \rightarrow \sigma_1$, the first Steklov eigenvalue. From Proposition 3.1 we have that the functions $v_n = \frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}}$, maybe after taking a subsequence, converge in $L^\infty(\partial\Omega)$ to Φ_1 or $-\Phi_1$, where Φ_1 is the unique positive eigenfunction of σ_1 with $L^\infty(\partial\Omega)$ -norm one.

As an example, let us consider the case where $v_n \rightarrow \Phi_1$ and assume, for instance, that the function $g(\lambda, x, s)$ behaves for $s \rightarrow +\infty$ and $\lambda \rightarrow \sigma_1$ as

$$g(\lambda, x, s) \approx G(x)s^\alpha.$$

Then, considering equation (1.2) with $\lambda = \lambda_n$, multiplying it by Φ_1 , integrating by parts and using that Φ_1 is an eigenfunction, we get

$$(\sigma_1 - \lambda_n) \int_{\partial\Omega} u_n \Phi_1 = \int_{\partial\Omega} g(\lambda_n, x, u_n) \Phi_1$$

Hence, since $u_n \rightarrow +\infty$ uniformly in $\partial\Omega$ and using the asymptotic expression of g we easily can get that the sign of $\sigma_1 - \lambda_n$ is dictated, for n large enough, by the sign of

$$\int_{\partial\Omega} G(x) \Phi_1^{1+\alpha}$$

In particular, if this last integral is positive the bifurcation is subcritical and if it is negative the bifurcation is supercritical.

With this in mind, we define the following functions, that describe the behavior of g for large values of s , at a given σ . Define, for some α , the following functions

$$\begin{aligned} \underline{G}_+(x) &:= \liminf_{(\lambda,s) \rightarrow (\sigma, +\infty)} \frac{g(\lambda, x, s)}{s^\alpha} & \overline{G}_+(x) &:= \limsup_{(\lambda,s) \rightarrow (\sigma, +\infty)} \frac{g(\lambda, x, s)}{s^\alpha} \\ \underline{G}_-(x) &:= \liminf_{(\lambda,s) \rightarrow (\sigma, -\infty)} \frac{g(\lambda, x, s)}{|s|^\alpha} & \overline{G}_-(x) &:= \limsup_{(\lambda,s) \rightarrow (\sigma, -\infty)} \frac{g(\lambda, x, s)}{|s|^\alpha} \end{aligned} \quad (4.1)$$

Remark 4.1 *i) Observe that in fact G depends on σ and α . If we need to stress this dependence, we will write $\underline{G}_+^{\alpha,\sigma}(x)$, $\overline{G}_+^{\alpha,\sigma}(x)$, $\underline{G}_-^{\alpha,\sigma}(x)$ and $\overline{G}_-^{\alpha,\sigma}(x)$.*

*ii) Observe that if g satisfies **(H2)** and $\alpha \geq 1$ then all the functions defined above are identically zero.*

iii) The way in which the functions defined in (4.1) describe the behavior of the function g for large values of s can be expressed in the following way: for any $\epsilon > 0$ small enough, we have

$$(\underline{G}_+(x) - \epsilon)s^\alpha \leq g(\lambda, x, s) \leq (\overline{G}_+(x) + \epsilon)s^\alpha, \quad s \rightarrow +\infty, \lambda \approx \sigma$$

and similarly for $s \rightarrow -\infty$

In order to establish conditions for sub or super critical bifurcations at the first eigenvalue, we prove first the following important result,

Lemma 4.2 *Assume the nonlinearity g satisfies hypothesis **(H1)** and **(H2)**. Denote by σ_1 the first Steklov eigenvalue and by Φ_1 the first positive eigenfunction with $\|\Phi_1\|_{L^\infty(\partial\Omega)} = 1$. Consider a sequence of solutions u_n for the value of the parameter λ_n such that $\lambda_n \rightarrow \sigma_1$ and $\|u_n\|_{L^\infty(\partial\Omega)} \rightarrow \infty$. Then,*

i) if $u_n > 0$, we have

$$\frac{\int_{\partial\Omega} G_+ \Phi_1^{1+\alpha}}{\int_{\partial\Omega} \Phi_1^2} \leq \liminf_{n \rightarrow \infty} \frac{\sigma_1 - \lambda_n}{\|u_n\|_{L^\infty(\partial\Omega)}^{\alpha-1}} \leq \limsup_{n \rightarrow \infty} \frac{\sigma_1 - \lambda_n}{\|u_n\|_{L^\infty(\partial\Omega)}^{\alpha-1}} \leq \frac{\int_{\partial\Omega} \overline{G_+} \Phi_1^{1+\alpha}}{\int_{\partial\Omega} \Phi_1^2} \quad (4.2)$$

ii) if $u_n < 0$, we have

$$\frac{\int_{\partial\Omega} G_- \Phi_1^{1+\alpha}}{\int_{\partial\Omega} \Phi_1^2} \leq \liminf_{n \rightarrow \infty} \frac{\sigma_1 - \lambda_n}{\|u_n\|_{L^\infty(\partial\Omega)}^{\alpha-1}} \leq \limsup_{n \rightarrow \infty} \frac{\sigma_1 - \lambda_n}{\|u_n\|_{L^\infty(\partial\Omega)}^{\alpha-1}} \leq \frac{\int_{\partial\Omega} \overline{G_-} \Phi_1^{1+\alpha}}{\int_{\partial\Omega} \Phi_1^2} \quad (4.3)$$

Proof. Let us show i). The other case follows a similar proof. So let us consider a family of solutions u_n of (1.2) for $\lambda = \lambda_n$ with $\lambda_n \rightarrow \sigma_1$ and $0 < u_n \rightarrow \infty$. Multiplying equation (1.2) by Φ_1 and integrating by parts, we get

$$(\sigma_1 - \lambda_n) \int_{\partial\Omega} u_n \Phi_1 = \int_{\partial\Omega} g(\lambda_n, x, u_n) \Phi_1 \quad (4.4)$$

But,

$$\int_{\partial\Omega} g(\lambda_n, x, u_n) \Phi_1 = \|u_n\|_{L^\infty(\partial\Omega)}^\alpha \int_{\partial\Omega} \frac{g(\lambda_n, x, u_n)}{u_n^\alpha} \left(\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}} \right)^\alpha \Phi_1$$

But, from Fatou's Lemma,

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \int_{\partial\Omega} \frac{g(\lambda_n, x, u_n)}{u_n^\alpha} \left(\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}} \right)^\alpha \Phi_1 \\ & \geq \int_{\partial\Omega} \liminf_{n \rightarrow \infty} \left[\frac{g(\lambda_n, x, u_n)}{u_n^\alpha} \left(\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}} \right)^\alpha \Phi_1 \right] \\ & \geq \int_{\partial\Omega} \overline{G_+}(x) \Phi_1^{1+\alpha} \end{aligned} \quad (4.5)$$

where we have used the definition of $\overline{G_+}(x)$, that $\Phi_1 > 0$ for all x on $\partial\Omega$ and the fact that $\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}} \rightarrow \Phi_1$ uniformly in $\partial\Omega$, see Corollary 3.2.

Dividing in (4.4) by $\|u_n\|_{L^\infty(\partial\Omega)}$ and passing to the limit we obtain the first inequality of (4.2). The second inequality is trivial and the third is obtained in a similar manner as the first one. \square

Now, with respect to bifurcations from the first eigenvalue we can prove,

Theorem 4.3 (Bifurcation from the first eigenvalue) *Assume the nonlinearity g satisfies hypothesis (H1) and (H2). Denote by σ_1 the first Steklov eigenvalue and by Φ_1 the first positive eigenfunction with $\|\Phi_1\|_{L^\infty(\partial\Omega)} = 1$. Then,*

i) (Subcritical bifurcations). *Assume there exists an $\alpha < 1$ such that $\overline{G_+} = \overline{G_+}^{\alpha, \sigma_1} \in L^1(\partial\Omega)$ (respectively $\overline{G_-} = \overline{G_-}^{\alpha, \sigma_1} \in L^1(\partial\Omega)$). Then, if*

$$\int_{\partial\Omega} \overline{G_+} \Phi_1^{1+\alpha} > 0 \quad \left(\text{respectively} \quad \int_{\partial\Omega} \overline{G_-} \Phi_1^{1+\alpha} < 0 \right) \quad (4.6)$$

the bifurcation from infinity of positive (resp. negative) solutions at $\lambda = \sigma_1$ is subcritical, i.e. $\lambda < \sigma_1$ for every positive (resp. negative) solution (λ, v) of (1.2) with $(\lambda, \|v\|)$ in a neighbourhood of (σ_1, ∞) .

ii) (Supercritical bifurcations). Assume there exists an $\alpha < 1$ such that $\overline{G_+} = \overline{G_+^{\alpha, \sigma_1}} \in L^1(\partial\Omega)$ (respectively $\underline{G_-} = \underline{G_-^{\alpha, \sigma_1}} \in L^1(\partial\Omega)$). Then, if

$$\int_{\partial\Omega} \overline{G_+} \Phi_1^{1+\alpha} < 0 \quad \left(\text{respectively} \quad \int_{\partial\Omega} \underline{G_-} \Phi_1^{1+\alpha} > 0 \right) \quad (4.7)$$

the bifurcation from infinity of positive (resp. negative) solutions at $\lambda = \sigma_1$ is supercritical, i.e. $\lambda > \sigma_1$ for every positive (resp. negative) solution (λ, v) of (1.2) with $(\lambda, \|v\|)$ in a neighbourhood of (σ_1, ∞) .

Proof. The proof of this Theorem follows directly from Lemma 4.2. Observe that conditions (4.6) and (4.7) impose a definite sign of $\sigma_1 - \lambda_n$ in (4.2) and (4.3). \square

As an example of this result we have

Corollary 4.4 *i) Assume the nonlinearity satisfies $g(\lambda, x, s) \approx a|s|^\alpha$ as $s \rightarrow +\infty$ for some $\alpha < 1$. Then, if $a > 0$ all bifurcations of positive solutions are subcritical, while if $a < 0$ all bifurcations of positive solutions are supercritical.*

ii) Assume the nonlinearity satisfies $g(\lambda, x, s) \approx a|s|^\alpha$ as $s \rightarrow -\infty$ for some $\alpha < 1$. Then, if $a > 0$ all bifurcations of negative solutions are supercritical, while if $a < 0$ all bifurcations of negative solutions are subcritical.

We consider now the general case, that is, u_n are solutions of (1.2) for a sequence λ_n with $\lambda_n \rightarrow \sigma$ and $\|u_n\|_{L^\infty(\partial\Omega)} \rightarrow \infty$. Then, from Proposition 3.1 we have that λ is an eigenvalue and, up to a subsequence, $u_n/\|u_n\|_{L^\infty(\partial\Omega)} \rightarrow \Phi$ uniformly for some eigenfunction Φ associated to the eigenvalue σ and with $\|\Phi\|_{L^\infty(\partial\Omega)} = 1$.

We have the following

Theorem 4.5 (Bifurcation from a general eigenvalue) *Assume the nonlinearity g satisfies hypothesis (H1) and (H2). Let σ be an Steklov eigenvalue for which a bifurcation from infinity of (1.2) occurs at $\lambda = \sigma$. Then,*

i) (Subcritical bifurcation). *Assume that for some $-1 \leq \alpha < 1$ and for this value of σ we have that $\underline{G_+}(x), \overline{G_-}(x) \in L^1(\partial\Omega)$. Then, if for any eigenfunction Φ associated to the eigenvalue σ , we have*

$$\int_{\partial\Omega} \underline{G_+}(x) |\Phi^+|^{1+\alpha} > \int_{\partial\Omega} \overline{G_-}(x) |\Phi^-|^{1+\alpha}, \quad (4.8)$$

the bifurcation from infinity of solutions at $\lambda = \sigma$ is subcritical, i.e. $\lambda < \sigma$ for every solution (λ, v) of (1.2) with $(\lambda, \|v\|)$ in a neighbourhood of (σ, ∞)

ii) (Supercritical bifurcation). Assume that for some $-1 \leq \alpha < 1$ and for this value of σ , we have that $\overline{G_+}(x), \underline{G_-}(x) \in L^1(\partial\Omega)$. Then, if for any eigenfunction Φ associated to the eigenvalue σ , we have

$$\int_{\partial\Omega} \overline{G_+}(x) |\Phi^+|^{1+\alpha} < \int_{\partial\Omega} \underline{G_-}(x) |\Phi^-|^{1+\alpha}, \quad (4.9)$$

the bifurcation from infinity of solutions at $\lambda = \sigma$ is supercritical, i.e. $\lambda > \sigma$ for every solution (λ, v) of (1.2) with $(\lambda, \|v\|)$ in a neighbourhood of (σ, ∞)

Proof. We will show the first case. The supercritical case is proved in a similar way.

As in the proof of Theorem 4.3, we need to study the sign of

$$\int_{\partial\Omega} g(\lambda_n, x, u_n) \Phi.$$

But, if we denote by $\partial\Omega^+ = \{x \in \partial\Omega : \Phi(x) > 0\}$ and by $\partial\Omega^- = \{x \in \partial\Omega : \Phi(x) < 0\}$, we have

$$\begin{aligned} \int_{\partial\Omega} g(\lambda, x, u) \Phi &= \int_{\partial\Omega^+} g(\lambda, x, u) \Phi^+ - \int_{\partial\Omega^-} g(\lambda, x, u) |\Phi^-| \\ &= \|u\|^\alpha \int_{\partial\Omega^+} \frac{g(\lambda, x, u)}{(1+|u|)^\alpha} \Phi^+ \left(\frac{1}{\|u\|} + \frac{|u|}{\|u\|} \right)^\alpha \\ &\quad - \|u\|^\alpha \int_{\partial\Omega^-} \frac{g(\lambda, x, u)}{(1+|u|)^\alpha} |\Phi^-| \left(\frac{1}{\|u\|} + \frac{|u|}{\|u\|} \right)^\alpha. \end{aligned} \quad (4.10)$$

Observe that, for any $\alpha \geq -1$,

$$\Phi^+ \left(\frac{1}{\|u_n\|} + \frac{|u_n|}{\|u_n\|} \right)^\alpha \rightarrow |\Phi^+|^{1+\alpha} \quad \text{in } C(\partial\Omega^+) \quad \text{as } n \rightarrow \infty. \quad (4.11)$$

Now, passing to the limit in (4.10), using (4.11), hypothesis (4.8) and the Fatou Lemma we conclude the proof. \square

5 The resonant case

We are concerned now with the resonant problem, that is,

$$\begin{cases} -\Delta u + u = 0, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \sigma u + g(x, u), & \text{on } \partial\Omega \end{cases} \quad (5.1)$$

where σ is an Steklov eigenvalue of (1.3). We are interested in giving conditions guaranteeing the existence of solutions in this case. As a matter of fact, we will see that if all possible bifurcations of the problem

$$\begin{cases} -\Delta u + u = 0, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \lambda u + g(x, u), & \text{on } \partial\Omega \end{cases} \quad (5.2)$$

with $\lambda \in \mathbb{R}$, $\lambda \approx \sigma$ are either subcritical or supercritical, then the resonant problem necessarily has at least a solution.

Theorem 5.1 *Assume that every possible bifurcation from infinity at $\lambda = \sigma$ of problem (5.2) is subcritical, that is, condition (4.8) holds, or every possible bifurcation from infinity at $\lambda = \sigma$ of problem (5.2) is supercritical, that is, condition (4.9) holds. Then the resonant problem (5.1) has at least one solution.*

Remark 5.2 *Conditions (4.8) and (4.9) are known as Landesman-Lazer type conditions.*

Proof. Observe first that from Theorem 2.7, for $\epsilon > 0$ small enough, we have that problem (5.2) has at least one solution for all $\lambda \in (\sigma - \epsilon, \sigma + \epsilon) \setminus \{\sigma\}$. If, for instance, we assume that all possible bifurcations occurring at $\lambda = \sigma$ are subcritical, then necessarily there exists a constant M such that for any $\lambda \in (\sigma, \sigma + \epsilon)$ all possible solutions of (5.2) satisfy $\|u\|_{L^\infty(\partial\Omega)} \leq M$. This allows us to take a sequence of $\lambda_n \rightarrow \sigma$ and solutions u_n of (5.2) with $\|u_n\|_{L^\infty(\partial\Omega)} \leq M$. Using the compactness given by elliptic regularity results applied to (5.2) and passing to the limit, we obtain a solution of (5.1). \square

6 The Anti-Maximum Principle for the Steklov problem

Let us consider the nonhomogeneous linear Steklov problem (6.1)

$$\begin{cases} -\Delta u + u = 0, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \lambda u + g(x), & \text{on } \partial\Omega \end{cases} \quad (6.1)$$

and let us show an Anti-maximum principle for this problem, see [8], [2] for the case where the nonlinear term is in Ω . As usual, we denote by σ_1 the first Steklov eigenvalue and by Φ_1 its positive eigenfunction.

Theorem 6.1 *For every $g \in L^r(\partial\Omega)$ with $r > N - 1$, there exists $\epsilon = \epsilon(g)$ such that*

1. *If $\int_{\partial\Omega} g\Phi_1 > 0$ then every solution (λ, u) of (6.1) satisfies the following*
 - (a) $u > 0$ if $\sigma_1 - \epsilon < \lambda < \sigma_1$,
 - (b) $u < 0$ if $\sigma_1 < \lambda < \sigma_1 + \epsilon$.
2. *If $\int_{\partial\Omega} g\Phi_1 = 0$ then every solution (λ, u) of (6.1) with $\lambda \neq \sigma_1$ changes sign on $\partial\Omega$ and consequently in Ω .*

Proof. Assume $\int_{\partial\Omega} g\Phi_1 > 0$. The Fredholm Alternative states that the linear problem (6.1) does not have solution if $\lambda = \sigma_1$ and has a unique solution if $\lambda \notin \sigma(S)$. Moreover from Theorem 3.3, $\lambda = \sigma_1$ is a bifurcation point from infinity and from Theorem 4.3, the bifurcation from infinity of positive solutions is subcritical, i.e. there exists an $\epsilon = \epsilon(g)$ such that for all (λ, u) solving (6.1) with $\lambda \rightarrow \sigma_1$, $\|u\| \approx \infty$ and $u > 0$ then $\sigma_1 - \epsilon < \lambda < \sigma_1$.

Moreover, by the same theorem, the bifurcation from infinity of negative solutions is supercritical, i.e. there exists an $\epsilon = \epsilon(g)$ such that for all (λ, u) solving (6.1) with $\lambda \rightarrow \sigma_1$, $\|u\| \approx \infty$ and $u < 0$ then $\sigma_1 < \lambda < \sigma_1 + \epsilon$.

Assume now that $\int_{\partial\Omega} g\Phi_1 = 0$. Multiplying equation (6.1) with $\lambda \neq \sigma_1$, by Φ_1 and integrating by parts, we obtain that $\int_{\partial\Omega} u\Phi_1 = 0$. Since $\Phi_1 > 0$, u has to change sign in $\partial\Omega$ and the proof is concluded. \square

7 Stability analysis

We analyse in this section the stability properties of the branches of solutions of (1.2) found in the previous section. We will regard these solutions as equilibrium points of the following parabolic evolutionary problem with nonlinear boundary condition,

$$\begin{cases} u_t - \Delta u + u = 0, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \lambda u + g(\lambda, x, u), & \text{on } \partial\Omega, \\ u(0, x) = u_0(x), & \text{in } \Omega. \end{cases} \quad (7.1)$$

and will analyse their stability in relation to this problem.

We will also assume that the nonlinearity g , besides conditions **(H1)** and **(H2)** satisfies a locally Lipschitz condition in the variable u . By assuming this, we guarantee that for a given initial condition $u_0 \in C(\bar{\Omega})$ there exists a unique solution $u \in C([0, T], C(\bar{\Omega}))$ of problem (7.1) and that the solutions depend continuously on the initial data, see for instance [3].

From condition **(H2)** we easily get that

$$g(\lambda, x, u)u \leq \epsilon|h(x)|u^2 + D_\epsilon|h(x)||u|$$

on bounded intervals of λ .

Hence, comparison arguments, see for instance [4], show that $|u(t, x)| \leq U(t, x)$ where u is the solution of (7.1) and U is the solution of the following linear problem

$$\begin{cases} U_t - \Delta U + U = 0, & \text{in } \Omega \\ \frac{\partial U}{\partial n} = (\lambda + \epsilon|h(x)|)U + D_\epsilon|h(x)|, & \text{on } \partial\Omega, \\ U(0, x) = |u_0(x)|, & \text{in } \Omega. \end{cases} \quad (7.2)$$

With this comparison we obtain the following information:

1) Since problem (7.2) is linear and $h \in L^r(\partial\Omega)$ with $r > N - 1$, we get that the solutions of (7.2) are in $C(\bar{\Omega})$ and they are globally defined in time. This gives us estimates on the solution $u(t, x)$ of (7.1) which in turn imply that the solutions of (7.1) are global in time. Hence, for each $u_0 \in C(\bar{\Omega})$ we have a unique solution $u \in C([0, \infty), C(\bar{\Omega}))$.

2) If we consider a fixed $\lambda < \sigma_1$, then for ϵ small enough, we have the existence of a unique $\varphi_\epsilon \in C(\bar{\Omega})$, solution of the following elliptic problem

$$\begin{cases} -\Delta\varphi + \varphi = 0, & \text{in } \Omega \\ \frac{\partial\varphi}{\partial n} = (\lambda + \epsilon|h(x)|)\varphi + D_\epsilon|h(x)|, & \text{on } \partial\Omega. \end{cases} \quad (7.3)$$

To see this, we apply Lax-Milgram theorem to the following bilinear form in $H^1(\Omega)$

$$a_\epsilon(u, v) = \int_{\Omega} (\nabla u \nabla v + uv) - \int_{\partial\Omega} (\lambda + \epsilon|h(x)|)uv$$

Observe that since $\lambda < \sigma_1$, the bilinear form above with $\epsilon = 0$ is coercive. Now since, $h \in L^r(\partial\Omega)$ and $r > N - 1$, for ϵ small enough we can show, via Sobolev embeddings and trace theorems, that a_ϵ is also coercive and we obtain the existence and uniqueness of a weak solution. Using regularity results we get that the solution $\varphi_\epsilon \in C(\bar{\Omega})$, since $r > N - 1$.

3) Now, the solution U of (7.2) is given by $U(t, x) = z(t, x) + \varphi_\epsilon(x)$ where $z(t, x)$ is the solution of

$$\begin{cases} z_t - \Delta z + z = 0, & \text{in } \Omega \\ \frac{\partial z}{\partial n} = (\lambda + \epsilon|h(x)|)z, & \text{on } \partial\Omega, \\ z(0, x) = |u_0(x)| - \varphi_\epsilon, & \text{in } \Omega. \end{cases} \quad (7.4)$$

But the coercitivity of the bilinear form a_ϵ and the smoothing properties of the solutions of (7.4) implies that

$$\|z(t, \cdot)\|_{C(\bar{\Omega})} \leq M_\epsilon e^{-\gamma_\epsilon t} \| |u_0| - \varphi \|_{C(\bar{\Omega})}$$

for some $M_\epsilon, \gamma_\epsilon > 0$. Hence, the solution u of (7.1) satisfies

$$\|u(t, \cdot)\|_{C(\bar{\Omega})} \leq \|U(t, \cdot)\|_{C(\bar{\Omega})} \leq M_\epsilon e^{-\gamma_\epsilon t} \| |u_0| - \varphi \|_{C(\bar{\Omega})} + \|\varphi_\epsilon\|_{C(\bar{\Omega})} \quad (7.5)$$

and also

$$\limsup_{t \rightarrow +\infty} |u(t, x)| \leq \varphi_\epsilon(x), \quad \text{a.e. } x \in \Omega \quad (7.6)$$

Estimate (7.5) imply that for $\lambda < \sigma_1$ the evolution of any initial condition for (7.1) is contained in a bounded set. Hence, this problem has an attractor, see [12]. Moreover, all the globally defined and bounded solutions are contained in the attractor. In particular, all the equilibria, conections

between equilibria, etc. are contained in the attractor. Estimate (7.6) tells us that any point in the attractor is bounded pointwise by φ_ϵ . In particular all equilibria are bounded by φ_ϵ .

With respect to the stability of the equilibria bifurcating from infinity at the first eigenvalue σ_1 , when we have a subcritical bifurcation, we have the following

Proposition 7.1 *Assume we are in the conditions of Theorem 4.3. Then,*

i) if the bifurcation of positive solutions (resp. negative solutions) at the first eigenvalue $\lambda = \sigma_1$ is subcritical, then there exists a $\delta > 0$ small enough such that for $\sigma_1 - \delta < \lambda < \sigma_1$, the largest positive (resp. smallest negative) solution bifurcating from infinity is “globally asymptotically stable from above” (resp. from below). That is, if $u_\lambda > 0$ (resp. $u_\lambda < 0$) is this solution then, for all initial condition $w_0 > u_\lambda$, (resp. $w_0 < u_\lambda$), the solution $u(t, x, w_0)$ of (7.1) with this initial condition satisfies $\lim_{t \rightarrow \infty} u(t, x, w_0) = u_\lambda$ uniformly in $x \in \bar{\Omega}$, for $\sigma_1 - \delta < \lambda < \sigma_1$.

ii) if in (4.1) we have $\underline{G}_+ \geq \epsilon$ (resp. $\overline{G}_- \leq -\epsilon$) for some $\epsilon > 0$, then the bifurcation of positive (resp. negative) solutions at $\lambda = \sigma_1$ is subcritical. Moreover there exists a $\beta_0 > 0$ large enough such that if \tilde{u}_λ is the smallest positive (resp. largest negative) solution satisfying $\tilde{u}_\lambda \geq \beta_0$ (resp. $\tilde{u}_\lambda \leq -\beta_0$) then there exists a $\delta > 0$ such that the equilibrium \tilde{u}_λ is asymptotically stable from below (resp. above) for $\sigma_1 - \delta < \lambda < \sigma_1$.

In particular, if for some λ in this range, we have a unique positive (resp. negative) equilibrium, that is $\tilde{u}_\lambda = u_\lambda$, then this equilibrium is asymptotically stable.

Proof. In order to prove this result we analyze the solution of (7.1) with initial condition $u_0 = \beta\Phi_1$, for $\beta \in \mathbb{R}$, where Φ_1 is the positive eigenfunction with $\|\Phi_1\|_{L^\infty(\partial\Omega)} = 1$, associated to the first Steklov eigenvalue. Hence, if we denote this solution by $u(t)$, multiplying the equation (7.1) by a positive test function $\chi \in C^\infty(\mathbb{R}^N)$ and integrating by parts, we get

$$\frac{d}{dt} \int_{\Omega} u(t)\chi = - \int_{\Omega} (\nabla u(t)\nabla\chi + u(t)\chi) + \int_{\partial\Omega} \lambda u(t)\chi + g(\lambda, \cdot, u(t))\chi$$

Evaluating this expression at $t = 0$, we get

$$\left. \frac{d}{dt} \int_{\Omega} u(t)\chi \right|_{t=0} = -\beta \int_{\Omega} (\nabla\Phi_1\nabla\chi + \Phi_1\chi) + \int_{\partial\Omega} \lambda\beta\Phi_1\chi + g(\lambda, \cdot, \beta\Phi_1)\chi$$

and taking into account that Φ_1 is the first Steklov eigenfunction, we get

$$\int_{\Omega} (\nabla\Phi_1\nabla\chi + \Phi_1\chi) = \sigma_1 \int_{\partial\Omega} \Phi_1\chi$$

which implies

$$\left. \frac{d}{dt} \int_{\Omega} u(t)\chi \right|_{t=0} = \int_{\partial\Omega} \left(\lambda - \sigma_1 + \frac{g(\lambda, \cdot, \beta\Phi_1)}{\beta\Phi_1} \right) \beta\phi_1\chi \quad (7.7)$$

This is the basic equality to prove the result.

i) Consider the case where we have a family of positive solutions, bifurcating from infinity and the bifurcation is subcritical. For fixed λ , denote by u_λ the largest positive solution.

We know from Proposition 3.1 and Corollary 3.2 that $\frac{u_\lambda}{\|u_\lambda\|_{L^\infty(\bar{\Omega})}} \rightarrow \Phi_1$.

For a fixed λ with $-\delta < \lambda - \sigma_1 < 0$, let β_λ be large enough such that $\beta_\lambda \Phi_1 > u_\lambda$ and

$$\left| \frac{g(\lambda, x, \beta_\lambda \Phi_1(x))}{\beta_\lambda \Phi_1(x)} \right| \leq \frac{1}{2} |\lambda - \sigma_1|$$

This can be accomplished by condition **(H2)** and using that $\inf_{x \in \partial\Omega} \Phi_1(x) > 0$. Hence, for $\beta \geq \beta_\lambda$ and $\chi > 0$, we get

$$\frac{d}{dt} \int_{\Omega} u(t) \chi \Big|_{t=0} \leq \frac{1}{2} \int_{\partial\Omega} (\lambda - \sigma_1) \beta \phi_1 \chi < 0 \quad (7.8)$$

Since $\chi > 0$ is arbitrary, this implies that the solution starting at $\beta \Phi_1$ for $\beta \geq \beta_\lambda$ is initially decreasing, that is, there exists a small t_0 such that $u(t, x, \beta \Phi_1) \leq \beta \Phi_1$ for $0 < t < t_0$. Since the flow generated by (7.1) is monotone, then we easily get that $u(t, x, \beta \Phi_1) \leq u(s, x, \beta \Phi_1) \leq \beta \Phi_1$ for all $0 < s \leq t$. Moreover, since we have chosen $\beta_\lambda \Phi_1 > u_\lambda$ and u_λ is an equilibria, we get $u_\lambda \leq u(t, x, \beta \Phi_1) \leq \beta \Phi_1$ for all $t > 0$. Now, since the solution $u(t, x, \beta \Phi_1)$ is monotone decreasing in time and bounded below, and u_λ is the largest positive equilibrium solution, then, for each $\beta > \beta_\lambda$ necessarily $u(t, x, \beta \Phi_1) \rightarrow u_\lambda$ as $t \rightarrow \infty$ uniformly in $x \in \bar{\Omega}$.

Hence, for any initial condition $w_0 \in C(\bar{\Omega})$ with $w_0 > u_\lambda$, if we consider $\beta > \beta_\lambda$ such that $u_\lambda \leq w_0 \leq \beta \Phi_1$, by monotonicity of the flow we get that $u_\lambda \leq \limsup_{t \rightarrow \infty} u(t, \cdot, w_0) \leq \lim_{t \rightarrow \infty} u(t, \cdot, \beta \Phi_1) = u_\lambda$, which proves the result.

ii) If $\underline{G}_+ \geq \epsilon$ for some $\epsilon > 0$, then, we know from Theorem 4.3 that the bifurcation of positive solutions is subcritical.

Choose a $\beta_0 > 0$ large enough and $\delta > 0$ small enough such that, from (4.1), we get

$$\frac{g(\lambda, x, \beta_0 \Phi_1(x))}{(\beta_0 \Phi_1(x))^\alpha} \geq \epsilon/2, \quad \sigma_1 - \delta < \lambda < \sigma_1, \quad x \in \partial\Omega$$

This implies that, for this β_0 fixed, we have

$$\frac{g(\lambda, x, \beta_0 \Phi_1(x))}{\beta_0 \Phi_1(x)} \geq \frac{\epsilon}{2(\beta_0 \Phi_1(x))^{1-\alpha}} \geq \bar{\epsilon}, \quad \sigma_1 - \delta < \lambda < \sigma_1, \quad x \in \partial\Omega$$

where

$$\bar{\epsilon} = \inf \left\{ \frac{\epsilon}{2(\beta_0 \Phi_1(x))^{1-\alpha}} : x \in \partial\Omega \right\}$$

Assuming that $\delta \leq \bar{\epsilon}/2$ (if this is not the case we choose $\delta = \bar{\epsilon}/2$) we get from (7.7) with initial condition $\beta_0 \Phi_1$

$$\left. \frac{d}{dt} \int_{\Omega} u(t)\chi \right|_{t=0} \geq \frac{\epsilon}{4} \int_{\partial\Omega} \beta_0 \phi_1 \chi > 0 \quad (7.9)$$

wich implies as in i) that the solution starting at $\beta_0 \Phi_1$ is non decreasing. Now, with similar monotonicity arguments as in i) we prove that the solution of (7.1) with initial condition w_0 with $\beta_0 \Phi_1 \leq w_0 \leq u_\lambda$ has to converge to \tilde{u}_λ .

The case $\overline{G}_- < -\epsilon$ is totally similar. \square

Remark 7.2 A condition that garantees that for a fixed λ there exists a unique large enough positive (resp. negative) solution is to assume that the function $s \rightarrow \frac{g(\lambda, x, s)}{s}$ is strictly monotone for $s > 0$ (resp. $s < 0$) large enough and a.e. $x \in \partial\Omega$. To see this, assume that u_λ and \tilde{u}_λ are two positive solutions with $u_\lambda(x), \tilde{u}_\lambda(x) \geq \beta$ and such that $s \rightarrow \frac{g(\lambda, x, s)}{s}$ is strictly monotone for $s \geq \beta$. Observe that without loss of generality we can assume that $\tilde{u}_\lambda < u_\lambda$. Then, u_λ is the solution of

$$\begin{cases} -\Delta u_\lambda + u_\lambda = 0, & \text{in } \Omega \\ \frac{\partial u_\lambda}{\partial n} = \left(\lambda + \frac{g(\lambda, x, u_\lambda)}{u_\lambda} \right) u_\lambda, & \text{on } \partial\Omega. \end{cases}$$

that is u_λ is an eigenfunction associated to the eigenvalue $\mu = 0$, of the following eigenvalue problem

$$\begin{cases} -\Delta \phi + \phi = \mu \phi, & \text{in } \Omega \\ \frac{\partial \phi}{\partial n} = \left(\lambda + \frac{g(\lambda, x, u_\lambda)}{u_\lambda} \right) \phi, & \text{on } \partial\Omega. \end{cases} \quad (7.10)$$

and since $u_\lambda > 0$ then 0 is the principal eigenfunction.

Similarly we could argue that $\phi = \tilde{u}_\lambda > 0$ is the principal eigenfunction associated to the principal eigenvalue 0 of the following problem

$$\begin{cases} -\Delta \phi + \phi = \mu \phi, & \text{in } \Omega \\ \frac{\partial \phi}{\partial n} = \left(\lambda + \frac{g(\lambda, x, \tilde{u}_\lambda)}{\tilde{u}_\lambda} \right) \phi, & \text{on } \partial\Omega. \end{cases} \quad (7.11)$$

But, since $\tilde{u}_\lambda < u_\lambda$, by monotonicity of $s \rightarrow \frac{g(\lambda, x, s)}{s}$ we cannot have that $\mu = 0$ is the first eigenvalue of both problems (7.10) and (7.11).

When the bifurcation at the first eigenvalue is supercritical we have,

Proposition 7.3 Assume the function g is differentiable with respect to the last variable and consider the functions $\underline{G}_+, \overline{G}_+, \underline{G}_-, \overline{G}_-$ as defined in (4.1) for some $\alpha < 1$ and for $\sigma = \sigma_1$, the first Steklov eigenvalue. Hence, if we have

$$\liminf_{(\lambda, s) \rightarrow (\sigma_1, +\infty)} \frac{g_u(\lambda, x, s)}{s^{\alpha-1}} \geq \alpha \overline{G}_+, \quad (\text{resp. } \limsup_{(\lambda, s) \rightarrow (\sigma_1, -\infty)} \frac{g_u(\lambda, x, s)}{s^{\alpha-1}} \leq \alpha \underline{G}_-) \quad (7.12)$$

and if condition (4.7) holds, that is, $\int_{\partial\Omega} \overline{G}_+(x)\Phi_1^{1+\alpha} < 0$, (resp. $\int_{\partial\Omega} \underline{G}_-(x)\Phi_1^{1+\alpha} > 0$), then, the bifurcation of positive (resp. negative) solutions at the first eigenvalue is supercritical and any positive (resp. negative) equilibrium solution bifurcating from infinity is unstable.

Proof. We only consider the case of bifurcation of positive solutions. The proof for negative solutions is similar.

Condition (4.7) guarantees that there exists a supercritical bifurcation of positive solutions from infinity at the first eigenvalue σ_1 . Let us by u_λ a positive solution bifurcating from infinity. The eigenvalue problem associated to the linearization around u_λ is given by

$$\begin{cases} -\Delta w + w = \mu w, & \text{in } \Omega \\ \frac{\partial w}{\partial n} = \lambda w + g_u(\lambda, x, u_\lambda)w, & \text{on } \partial\Omega. \end{cases} \quad (7.13)$$

We will show that the first eigenvalue, $\mu_1 = \mu_1(\lambda) < 0$ for $\lambda > \sigma_1$ close enough to σ_1 . This eigenvalue is given by

$$\begin{aligned} \mu_1 &= \min_{\phi \in H^1(\Omega)} \frac{\int_{\Omega} |\nabla \phi|^2 + |\phi|^2 - \int_{\partial\Omega} \lambda |\phi|^2 + g_u(\lambda, x, u_\lambda) |\phi|^2}{\int_{\Omega} |\phi|^2} \\ &\leq \frac{\int_{\Omega} |\nabla \Phi_1|^2 + |\Phi_1|^2 - \int_{\partial\Omega} \lambda |\Phi_1|^2 + g_u(\lambda, x, u_\lambda) |\Phi_1|^2}{\int_{\Omega} |\Phi_1|^2} \\ &= \frac{(\sigma_1 - \lambda) \int_{\partial\Omega} |\Phi_1|^2 - \int_{\partial\Omega} g_u(\lambda, x, u_\lambda) |\Phi_1|^2}{\int_{\Omega} |\Phi_1|^2} \end{aligned} \quad (7.14)$$

where we have used that Φ_1 is the first Steklov eigenfunction, associated to the eigenvalue σ_1 .

But observe that from Lemma 4.2, we have

$$\limsup_{\lambda \rightarrow \sigma_1} \frac{\sigma_1 - \lambda}{\|u_\lambda\|_{L^\infty(\partial\Omega)}^{\alpha-1}} \leq \frac{\int_{\partial\Omega} \overline{G}_+ \Phi_1^{1+\alpha}}{\int_{\partial\Omega} \Phi_1^2} \quad (7.15)$$

On the other hand, from (7.12) and Corollary 3.2, we have that

$$\liminf_{\lambda \rightarrow \sigma_1} \int_{\partial\Omega} \frac{g_u(\lambda, x, u_\lambda)}{u_\lambda^{\alpha-1}} |\Phi_1|^2 \geq \int_{\partial\Omega} \alpha \overline{G}_+(x) \Phi_1^{1+\alpha} \quad (7.16)$$

Plugging expressions (7.15) and (7.16) in (7.14), we get,

$$\limsup_{\lambda \rightarrow \sigma_1} \frac{\mu_1(\lambda)}{\|u_\lambda\|_{L^\infty(\partial\Omega)}^{\alpha-1}} \leq \frac{(1 - \alpha) \int_{\partial\Omega} \overline{G}_+(x) \Phi_1^{1+\alpha}}{\int_{\Omega} \Phi_1^2}.$$

Now, since by hypothesis, condition (4.7) holds and $\alpha < 1$, we obtain that $\mu_1 < 0$ for λ close enough to σ_1 and the equilibrium is unstable. \square

8 Remarks and extensions

We consider in this section several important remarks and extensions of the problem we are dealing with. These comments go in three directions.

First, in subsection 8.1, we will consider the case where bifurcations from the trivial solution may occur. For this, we will need to assume that the nonlinearity g is $g(\lambda, x, u) = o(u)$ as $u \rightarrow 0$.

Second, in subsection 8.2, we will consider the case where the nonlinear boundary conditions incorporate a potential with a possible non definite sign, that is, the boundary conditions reads,

$$\frac{\partial u}{\partial n} = \lambda m(x)u + g(\lambda, x, u)$$

Finally, in subsection 8.3, we analyse the simpler, but important and instructive, case where $N = 1$.

8.1 Bifurcation from the trivial solution

We consider problem (1.2) and assume that the nonlinearity g satisfies condition **(H1)** but, instead of specifying the behavior of g for large values of u , we consider the behavior of g for small values of u . That is, we assume

$$\text{(H3)} \quad \lim_{|s| \rightarrow 0} \frac{U(s)}{s} = 0.$$

We have the following result,

Theorem 8.1 *Consider problem (1.2) and assume that the nonlinearity g satisfies conditions **(H1)** and **(H3)**. If σ is an Steklov eigenvalue of odd multiplicity, then the set of solutions of (1.2) possesses a component emanating from the bifurcation point $(\sigma, 0) \in \mathbb{R} \times C(\bar{\Omega})$. Moreover, this component, either it is bounded in $\mathbb{R} \times C(\bar{\Omega})$, in which case it meets another bifurcation point from zero (that is, another point $(\sigma', 0)$ for another Steklov eigenvalue σ'), or it is unbounded.*

Proof. The proof of this result follows the general results on bifurcations from the trivial solution given in [16]. See also [2] for similar results when the nonlinearity is in the interior. \square

Remark 8.2 *Observe that it is possible to have nonlinearities where both situations, the one from Theorem 8.1 and from Theorem 3.3, hold. This is the case, for instance, where the nonlinearity $g(\lambda, x, u)$ is $o(u)$ at $u \rightarrow 0$ and at $u \rightarrow \infty$. In this situation, both Theorems apply and if σ is an Steklov eigenvalue of odd multiplicity (for instance the first one) then both bifurcations, from zero and from infinity occurs at this value of the parameter.*

8.2 Potential on the boundary

We study now the case where the nonlinear elliptic problem contains a potential $m(x)$ in the boundary condition,

$$\begin{cases} -\Delta u + u = 0, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \lambda m(x)u + g(\lambda, x, u), & \text{on } \partial\Omega. \end{cases} \quad (8.1)$$

For simplicity we may assume $m \in L^\infty(\partial\Omega)$ and we will consider the important case where the potential changes sign on $\partial\Omega$.

The role played in the whole analysis of the previous sections by the eigenvalues $\{\sigma_i\}_{i=1}^\infty$, of problem (1.3) are played now by the eigenvalues of the following problem

$$\begin{cases} -\Delta\Phi + \Phi = 0, & \text{in } \Omega \\ \frac{\partial\Phi}{\partial n} = \sigma m(x)\Phi, & \text{on } \partial\Omega. \end{cases} \quad (8.2)$$

We will still denote these values as Steklov eigenvalues. Hence, σ is an Steklov eigenvalue, if problem (8.2) has nontrivial solutions. Moreover, the multiplicity of σ is the number of linearly independent solutions of (8.2). Alternatively, $\sigma \in \mathbb{R}$ is an Steklov eigenvalue if and only if $\mu = 0$ is an eigenvalue of the following eigenvalue problem

$$\begin{cases} -\Delta\Phi + \Phi = \mu\Phi, & \text{in } \Omega \\ \frac{\partial\Phi}{\partial n} = \sigma m(x)\Phi, & \text{on } \partial\Omega. \end{cases} \quad (8.3)$$

and the multiplicity of σ as an Steklov eigenvalue of (8.2) is the same as the multiplicity of the eigenvalue $\mu = 0$ of (8.3).

In terms of the structure of the Steklov eigenvalues we have the following result.

Proposition 8.3 *Let the potential $m \in L^\infty(\partial\Omega)$, with $\Omega \subset \mathbb{R}^N$, $N \geq 2$ and let $\alpha > 0$. Then,*

i) If $m \geq \alpha > 0$ in a subset $\Gamma_+ \subset \partial\Omega$ with $(N-1)$ -dimensional measure $|\Gamma_+|_{N-1} > 0$, then there exists a sequence of Steklov eigenvalues $\{\sigma_i^+\}_{i=1}^\infty$ with $0 < \sigma_1^+ < \sigma_2^+ \leq \dots$ with the property that $\sigma_i^+ \rightarrow +\infty$ as $i \rightarrow +\infty$ and these are all the positive Steklov eigenvalues. Moreover, σ_1^+ is simple and the eigenfunction corresponding to the eigenvalue σ_1^+ does not change sign in $\bar{\Omega}$.

ii) If $m \leq -\alpha < 0$ in $\Gamma_- \subset \partial\Omega$ with $|\Gamma_-|_{N-1} > 0$, then there exists a sequence of Steklov eigenvalues $\{\sigma_i^-\}_{i=1}^\infty$ with $0 > \sigma_1^- > \sigma_2^- \geq \dots$ with the property that $\sigma_i^- \rightarrow -\infty$ as $i \rightarrow +\infty$

and these are all the negative Steklov eigenvalues. Moreover, σ_1^- is simple and the eigenfunction corresponding to the eigenvalue σ_1^- does not change sign in $\bar{\Omega}$.

Proof. We will give an sketch of the proof. The reader may complete the details, since the arguments are similar as for the case of potentials in Ω , see [13, 11].

It is enough to show i) since ii) is obtained from i) by noticing that $\lambda m(x) = (-\lambda)(-m(x))$.

i) Consider, for each fixed $\sigma \in \mathbb{R}$, the eigenvalues $\{\mu_k(\sigma)\}_{k=1}^\infty$ of problem (8.3)

Notice that for fixed $\sigma \in \mathbb{R}$, we have that the sequence $\{\mu_k(\sigma)\}_{k=1}^\infty$ corresponds to the eigenvalues of $-\Delta + I$ with the Robin boundary condition $\frac{\partial u}{\partial n} = \sigma m u$. Hence $\mu_k(\sigma) \rightarrow +\infty$ as $k \rightarrow \infty$. In particular, if $\sigma = 0$ we recover the Neumann eigenvalues of $-\Delta + I$ and we know that $1 = \mu_1(0) < \mu_2(0) \leq \dots \leq \mu_k(0) \rightarrow +\infty$ as $k \rightarrow \infty$. For fixed k we can consider the dependence of μ_k with respect to σ . These curves are continuous in σ , see [14]. Moreover, using the min-max characterization of the eigenvalues, we can easily see that for $\sigma \geq 0$, we have $\tau_k(\sigma) \leq \mu_k(\sigma)$, where $\tau_k(\sigma)$ are the eigenvalues of

$$\begin{cases} -\Delta\Phi + \Phi = \tau\Phi, & \text{in } \Omega \\ \frac{\partial\Phi}{\partial n} = \sigma m^+(x)\Phi, & \text{on } \partial\Omega. \end{cases} \quad (8.4)$$

Using again the min-max characterization of the eigenvalues and the fact that $m^+ \geq 0$ we can easily see that for $\sigma > 0$ the curves $\sigma \rightarrow \tau_k(\sigma)$ are non increasing. Moreover, from the fact that $m \geq \alpha$ in Γ^+ , it can be seen that both curves $\tau_k(\sigma), \mu_k(\sigma) \rightarrow -\infty$ as $\sigma \rightarrow +\infty$. The structure of these curves as $\sigma \rightarrow \infty$ and the characterization of the Steklov eigenvalues as the values $\sigma \geq 0$ for which some of these curves passes through zero, easily prove the result. \square

All the results of the previous sections can be easily adapted to the problem (8.1). In particular, the operator S_λ from Lemma 2.5, which appear in the fixed point problem (2.6), is obtained with the trace of the solution of the following problem

$$\begin{cases} -\Delta u + u = 0, & \text{in } \Omega \\ \frac{\partial u}{\partial n} - \lambda m(x)u = g, & \text{on } \partial\Omega \end{cases} \quad (8.5)$$

and the fixed point problem (3.1) should be rewritten now as $v = \lambda S_0(mv) + S_0(g(\lambda, \cdot, v))$, where S_0 is as in Lemma 2.5.

The existence of bifurcations from infinity at an Steklov eigenvalue σ_i^+ or σ_i^- , of odd multiplicity follows the same line of proof.

The characterization of the type of bifurcation (sub or super critical) when the parameter λ crosses one of the eigenvalues $\sigma_i^+ > 0$ for some $i = 1, 2, \dots$ is the same as in the case $m \equiv 1$, that is, Theorem 4.3 and Theorem 4.5 apply directly to this case. For instance, if $\int_{\partial\Omega} \underline{G}_+^{\alpha, \sigma_1^+}(x) \Phi_{1,+}^{1+\alpha} > 0$ then the bifurcation of positive solutions at $\lambda = \sigma_1^+ > 0$ is subcritical. If the parameter λ crosses $\sigma_i^- < 0$, then the characterizations are exactly the opposite, that is, for

instance if $\int_{\partial\Omega} G_+^{\alpha, \sigma_1^-}(x) \Phi_{1,-}^{1+\alpha} > 0$ then the bifurcation of positive solutions at $\lambda = \sigma_1^- < 0$ is supercritical. That the characterizations are reversed can be easily seen since if we want to analyse the behavior of (8.1) for $\lambda < 0$ is the same as analyzing the same problem for $\tau = -\lambda > 0$ for the potential $n = -m$, since $\lambda m = (-\lambda)(-m) = \tau n$.

In this same spirit, and for the case where the potential changes sign, for which we have two principal eigenvalues, $\sigma_1^- < 0 < \sigma_1^+$, with strictly positive eigenfunctions $\Phi_{1,-}$, $\Phi_{1,+}$ respectively, the Anti-Maximum principle with a potential will be as follows.

Theorem 8.4 *For every $g \in L^r(\partial\Omega)$ with $r > N - 1$, there exists $\epsilon = \epsilon(g)$ such that*

1. *If $\int_{\partial\Omega} g\Phi_{1,+} > 0$ (resp. $\int_{\partial\Omega} g\Phi_{1,-} > 0$) then every solution (λ, u) of (8.5) satisfies*
 - (a) *$u > 0$ if $0 < \sigma_1^+ - \epsilon < \lambda < \sigma_1^+$, (resp. $u < 0$ if $\sigma_1^- - \epsilon < \lambda < \sigma_1^- < 0$)*
 - (b) *$u < 0$ if $\sigma_1^+ < \lambda < \sigma_1^+ + \epsilon$, (resp. $u > 0$ if $\sigma_1^- < \lambda < \sigma_1^- + \epsilon < 0$),*
2. *If $\int_{\partial\Omega} g\Phi_1 = 0$ then every solution (λ, u) of (6.1) with $\lambda \neq \sigma_1$ changes its sign on $\partial\Omega$ and consequently in Ω .*

8.3 The case $N = 1$

So far we have been treating the case where the equation is N -dimensional with $N \geq 2$. We give now some ideas on how to treat the one dimensional case. We will see that the bifurcation problem is a two parameter non linear problem that can be treated using finite dimensional techniques.

Observe that if we consider equation (1.2) (in a similar maner we could argue for equation (8.1)), in the one dimensional domain $\Omega = (0, 1)$, we can rewrite it as

$$\begin{cases} -u_{xx} + u = 0, & \text{in } (0, 1) \\ -u_x(0) = \lambda u + g_0(\lambda, u(0)), \\ u_x(1) = \lambda u + g_1(\lambda, u(1)), \end{cases} \quad (8.6)$$

But in this case, the differential equation can be explicitly solved in terms of two constants a and b . The general solution is $u(x) = ae^x + be^{-x}$. Plugging this expression into the boundary conditions, we get the following two equations, which are the equivalent to equation (2.6)

$$\begin{aligned} -a + b &= \lambda(a + b) + g_0(\lambda, a + b), & (x = 0) \\ ae - be^{-1} &= \lambda(ae + be^{-1}) + g_1(\lambda, ae + be^{-1}), & (x = 1) \end{aligned}$$

Observe that in this case we only have two Steklov eigenvalues, which are given by the values σ for which the following matrix has zero determinant:

$$\begin{pmatrix} -(1 + \sigma) & (1 - \sigma) \\ (1 - \sigma)e & -(1 + \sigma)e^{-1} \end{pmatrix}.$$

These two values are given by

$$\sigma_1 = \frac{e - 1}{e + 1} < \sigma_2 = \frac{1}{\sigma_1} = \frac{e + 1}{e - 1}$$

The eigenfunction Φ_1 and Φ_2 for this problem are given by

$$\Phi_1(x) = \frac{e^x + e^{1-x}}{1 + e}, \quad \Phi_2(x) = \frac{e^x - e^{1-x}}{1 - e}$$

Observe that $\Phi_1(0) = \Phi_1(1) = 1$ and $\Phi_2(0) = 1 = -\Phi_2(1)$.

For any $\lambda \neq \sigma_1, \sigma_2$, the function $u = ae^x + be^{-x}$ is a solution if (a, b) satisfy

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -(1 + \lambda) & (1 + \lambda) \\ (1 - \lambda)e & -(1 + \lambda)e^{-1} \end{pmatrix}^{-1} \begin{pmatrix} g_0(\lambda, a + b) \\ g_1(\lambda, ae + be^{-1}) \end{pmatrix}$$

The sublinearity of g_0 and g_1 as $u \rightarrow \infty$ allows to apply fixed point arguments in \mathbb{R}^2 guaranteeing the existence of at least one solution for any $\lambda \neq \sigma_1, \sigma_2$. Moreover, the fact that both eigenvalues are simple, guarantee that under a sublinearity condition on g as $u \rightarrow \infty$, we have bifurcation curves from infinity.

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