

A NOTE ON THE RANGE OF THE DERIVATIVES OF ANALYTIC APPROXIMATIONS OF UNIFORMLY CONTINUOUS FUNCTIONS ON c_0

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ABSTRACT. A real Banach space X satisfies property (K) (defined in [7]) if there exists a real-valued function on X which is uniformly (real) analytic and separating. We obtain that every uniformly continuous function $f : U \rightarrow \mathbb{R}$, where U is an open subset of a separable Banach space X with property (K) and containing c_0 (thus $X = c_0 \oplus Y$ for some Banach space Y) can be uniformly approximated by (real) analytic functions $g : U \rightarrow \mathbb{R}$ such that $\frac{\partial g}{\partial c_0}(U) \subset \bigcap_{p>0} \ell_p$ (where $\frac{\partial f}{\partial c_0}(U)$ is the set of partial derivatives $\{\frac{\partial f}{\partial x}(x, y) : (x, y) \in U\}$). Similar statements are obtained for uniformly continuous functions $f : U \rightarrow E$ with values in a (finite or infinite dimensional) Banach space E . Some consequences of these results are studied.

1. INTRODUCTION

Several properties related to the size and shape of the set of derivatives of smooth functions between Banach spaces have been studied in [6], [2], [12], [8], [9] and [3] among others. Azagra and Deville [2] constructed C^1 smooth bump functions $f : X \rightarrow \mathbb{R}$, in any Banach space X with a Lipschitzian and C^1 smooth bump function, such that the set of derivatives fills the dual space, that is $f'(X) = X^*$ (a bump function is a function with non-empty and bounded support). Some generalizations of this result are given in [5]. If X has the Radon-Nikodým property it follows from Stegall's variational principle that the cone generated by the set of derivatives of any continuous and Gâteaux differentiable bump function is residual. In turn, P. Hájek [13] proved that every Fréchet differentiable function with locally uniformly continuous derivative from c_0 into a Banach space with non trivial type has locally compact derivative. Azagra and Cepedello [1] proved that every continuous function from ℓ_2 into \mathbb{R}^m can be uniformly approximated by C^∞ smooth functions with no critical points. P. Hájek and M. Johanis [14] stated a related result for separable Banach spaces X with a C^p smooth bump function and containing c_0 : every real-valued continuous function defined on X can be uniformly approximated by C^p smooth functions whose range of derivatives is of first category and avoids a prefixed K_σ subset. Recently, it was proved in [4] that every real valued continuous function defined on a separable Banach space X with separable dual X^* (with a LUR and C^p smooth equivalent norm, respectively) can be approximated by C^1 smooth (C^p smooth, respectively) functions with no critical points.

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Let us focus now on the (real) analytic case. J. Kurzweil [15] proved that every continuous function from ℓ_2 into a Banach space E can be uniformly approximated by analytic functions. R. Fry [11] proved the same assertion for uniformly continuous functions defined on c_0 . Independently, M. Cepedello and P. Hájek [7] proved the same result for c_0 and for the class of separable Banach spaces with property (K). A real separable Banach space X satisfies property (K) if there is a function $d : X \rightarrow \mathbb{R}$ which is uniformly analytic and separating: that is, d is real analytic at every point x with radius of convergence r_x uniformly bounded below by some constant $M > 0$ and there is some α such that the set $\{x \in X : d(x) < \alpha\} \subset B_X$ and is not empty. In this note, we will prove that every uniformly continuous function $f : U \rightarrow \mathbb{R}$, where U is an open subset of a separable Banach space X with property (K) and containing c_0 (thus $X = c_0 \oplus Y$ for some Banach space Y), can be uniformly approximated by (real) analytic functions $g : U \rightarrow \mathbb{R}$ such that $\frac{\partial g}{\partial c_0}(U) \subset \bigcap_{p>0} \ell_p$ (where $\frac{\partial f}{\partial c_0}(U)$ is the set of partial derivatives $\{\frac{\partial f}{\partial x}(x, y) : (x, y) \in U\}$). Our proof is based on the construction of real analytic approximations $g : c_0 \rightarrow \mathbb{R}$ for a uniformly continuous function $f : c_0 \rightarrow \mathbb{R}$ given by M. Cepedello and P. Hájek [7]. We shall prove that, under additional conditions, we can obtain that $g'(c_0) \subset \bigcap_{p>0} \ell_p$. This result allows us to obtain real analytic approximations $h : U \rightarrow \mathbb{R}$ such that $\frac{\partial h}{\partial c_0}(U) \subset \ell_q \setminus \ell_p$ for any pre-fixed $0 < p < q < 1$ (or even $\frac{\partial h}{\partial c_0}(U) \subset \text{span}\{z^*, \bigcap_{p>0} \ell_p\} \setminus \bigcap_{p>0} \ell_p$, for any pre-fixed $z^* \in \ell_1 \setminus \bigcap_{p>0} \ell_p$). A similar estimation of the set of derivatives is given for vectorial functions $f : U \rightarrow E$, where E is a (finite or infinite) dimensional Banach space.

Let us point out that from the results of P. Hájek [13] we can obtain an approximate Morse-Sard theorem, namely the existence of real analytic approximations with no critical points for uniformly continuous functions $f : U \rightarrow E$ with U open subset of X (separable real Banach space with property (K) containing c_0) and E finite dimensional. The results given in this note provide an alternative proof of this assertion, but with the advantage that it gives an additional description of the range of the derivatives.

Finally, let us remark that, whenever X has the property that every uniformly continuous function can be approximated by real analytic functions with no critical points, a real analytic separation result can be applied to any pair of closed subsets at positive distance in X (via 1-codimensional real analytic submanifolds of X). This can be viewed as a real analytic approximation result for closed subsets in X .

2. MAIN RESULTS

Let us first state the following result for c_0 . Then we will state an analogous result for (real) separable Banach spaces containing c_0 with property (K).

Theorem 2.1. *Every real-valued, uniformly continuous function defined on an open subset of c_0 can be uniformly approximated by analytic functions whose set of derivatives is included in $\ell_{1/2}$.*

Proof. From now on, let us follow the notation given in [7]. Let us consider a uniformly continuous function $f : c_0 \rightarrow \mathbb{R}$. Let us fix $\varepsilon := 1$ and take $\rho > 1$ satisfying $|f(x) - f(x')| \leq \frac{1}{4}$ whenever $\|x - x'\|_\infty \leq \frac{1}{\rho}$. We shall denote the complex Banach space c_0 by \hat{c}_0 . Recall that $d(x) := \sum_k x_k^{2k}$ for $x \in c_0$ and the complex

function $d^{\mathbb{C}}(z) = \sum_k z_k^{2k}$ defined in \hat{c}_0 are real-analytic and analytic respectively, with radius of convergence one at every point [16, Example 5.5]. We will consider the real Banach space ℓ_1 , the complex Banach space ℓ_1 (which we shall denote by $\hat{\ell}_1$) and the real quasi-Banach space $\ell_{1/2} \subset \ell_1$ with the quasi-norm $\|\cdot\|_{1/2}$ defined as $\|z\|_{1/2} = \left(\sum_k \sqrt{|z_k|}\right)^2$ for all $z \in \ell_{1/2}$. Notice, that for every $z \in c_0$, we have that $\|(d^{\mathbb{C}})'(z)\|_{1/2} = \left(\sum_k \sqrt{2k|z_k|^{2k-1}}\right)^2 < \infty$ and thus $(d^{\mathbb{C}})'(z) \in \ell_{1/2}$. Let us consider, following the proof of [7],

- (1) the real-analytic function $d_\rho(x) := d(\rho x)$ and the analytic function $d_\rho^{\mathbb{C}}(z) := d^{\mathbb{C}}(\rho z)$ defined in c_0 and \hat{c}_0 respectively, with radius of convergence at every point $\frac{1}{\rho}$. Let us denote by $B_\rho(x, r) = \{y \in c_0 : d_\rho(x - y) < r\}$;
- (2) a dense sequence $\{x_n\}$ in c_0 ,
- (3) a strictly decreasing sequence of real numbers $\{\delta_n\}$ converging to 0 with $0 < \delta_n < \frac{1}{3}$;
- (4) the two locally finite coverings of c_0 formed by the open sets $\{D_n\}$ and $\{\tilde{D}_n\}$ defined as

$$\begin{aligned} D_1 &= B_\rho(x_1, 1), & \tilde{D}_1 &= B_\rho(x_1, 1 + 2\delta_1) \\ D_n &= \cap_{i < n} (B_\rho(x_n, 1) \setminus \overline{B_\rho(x_i, 1 - \delta_n)}), & \text{for } n > 1, \\ \tilde{D}_n &= \cap_{i < n} (B_\rho(x_n, 1 + 2\delta_n) \setminus \overline{B_\rho(x_i, 1 - 3\delta_n)}), & \text{for } n > 1; \end{aligned}$$

- (5) the n -dimensional intervals $\{I_n\}$ defined as

$$\begin{aligned} I_1 &= [-\delta_1, 1 + \delta_1] \\ I_n &= [1 - 2\delta_n, \infty) \times \overset{(n-1)}{\dots} \times [1 - 2\delta_n, \infty) \times [-\delta_n, 1 + \delta_n], & \text{for } n > 1; \end{aligned}$$

- (6) the holomorphic functions defined on \hat{c}_0 as

$$\varphi_n^{\mathbb{C}}(z) := \frac{(|f(x_n)| + 1)}{\nu_n} \int_{I_n} \exp \left[-t_n \sum_{i=1}^n a_i \left(d_\rho^{\mathbb{C}}(z - x_i) - \tau_i \right)^2 \right] d\tau,$$

where $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n$, $\{t_n\}$ and $\{a_n\}$ are sequence of positive numbers and ν_n is the norming factor $\nu_n = \int_{\mathbb{R}^n} \exp \left[-t_n \sum_{i=1}^n a_i \tau_i^2 \right] d\tau = \frac{\pi^{n/2}}{t_n^{n/2} (a_1 \dots a_n)^{1/2}}$.

In order to simplify the notation, let us write the following $\phi_n := \varphi_n^{\mathbb{C}}$ and $\delta := d_\rho^{\mathbb{C}}$. M. Cepedello and P. Hájek proved in [7] that whenever the sequence of positive numbers $\{a_n\}$ is taken small enough so that

$$\sum_{n > m} a_n M_{m,n}^2 < \infty, \quad \text{for every } m \in \mathbb{N},$$

where $M_{m,n} = \sup\{|\delta(z)| : z \in (x_m - x_n) + \frac{1}{4e\rho} B_{\hat{c}_0}\}$, ($B_{\hat{c}_0}$ is the closed unit ball of \hat{c}_0) and the sequence $\{t_n\}$ is taken large enough so that

- (1) $\phi_n(x) > |f(x_n)| + \frac{3}{4}$, whenever $x \in D_n$,
- (2) $|\phi_n(x)| < 2^{-4n} (|f(x_n)| + 1)^{-1}$, whenever $x \in c_0 \setminus \tilde{D}_n$,
- (3) $\sum_n (|f(x_n)| + 1)^2 \exp(-\frac{t_n}{n}) < \infty$,

then the function $\sum_n (|f(x_n)| + 1)\phi_n(z)$ locally uniformly converges in c_0 and the functions $\phi(z) := \sum_n \phi_n(z)$, $\psi(z) := \sum_n f(x_n)\phi_n(z)$ and $h(z) := \frac{\psi(z)}{\phi(z)}$ are well defined and holomorphic on an open subset of \hat{c}_0 containing c_0 . Therefore, the restrictions of the above functions to c_0 are real-analytic on c_0 . Moreover, they prove that $|f(z) - h(z)| < 1$ for every $z \in c_0$, that is h 1-approximates f in c_0 .

An additional condition we set for $\{a_n\}$ is that it is a decreasing sequence (otherwise, we change $\min\{a_1, \dots, a_n\}$ by a_n). Once we fix the sequence $\{a_n\}$, we shall prove in the next that, under additional conditions on $\{t_n\}$, the derivatives $\phi'(x) = \sum_n \phi'_n(x) \in \ell_{1/2}$, $\psi'(x) = \sum_n f(x_n)\phi'_n(x) \in \ell_{1/2}$ and $h'(x) \in \ell_{1/2}$, for every $x \in c_0$.

First, let us compute the derivative of $\phi_n(z)$, for $z \in c_0$. Then,

$$\begin{aligned} \phi'_n(z) = & -\frac{(|f(x_n)| + 1)t_n}{\nu_n} \\ & \times \left(\sum_{j=1}^n 2a_j \delta'(z - x_j) \int_{I_n} (\delta(z - x_j) - \tau_j) \exp \left[-t_n \sum_{i=1}^n a_i (\delta(z - x_i) - \tau_i)^2 \right] d\tau \right). \end{aligned}$$

The derivative $\phi'_n(z)$ is a finite linear combination of $\{\delta'(z - x_j) : j = 1, \dots, n\}$. Since $\{\delta'(z - x_j) : j = 1, \dots, n\} \subset \ell_{1/2}$, we obtain that $\phi'_n(z) \in \ell_{1/2}$. Moreover,

$$\phi'_n(z), \delta'(z) \in \bigcap_{p>0} \ell_p, \text{ for every } z \in c_0.$$

If a sequence of holomorphic functions locally uniformly converges on an open set $\Omega \subset \hat{c}_0$, then the sequence of the derivatives also locally uniformly converges on Ω to the derivative of the limit. Thus, if $\sum_n f(x_n)\phi_n(z)$ and $\sum_n \phi_n(z)$ locally uniformly converge on an open set $\Omega \subset \hat{c}_0$, then $\sum_n f(x_n)\phi'_n(z)$ and $\sum_n \phi'_n(z)$ locally uniformly converge in the space $H(\Omega, \ell_1)$ (holomorphic functions). Nevertheless, we cannot deduce from this fact that $\sum_n f(x_n)\phi'_n(z) \in \ell_{1/2}$ and $\sum_n \phi'_n(z) \in \ell_{1/2}$, whenever $z \in c_0$.

We will find an upper bound for $\|\phi'_n(z)\|_{1/2}$. Let us define, for every $n \in \mathbb{N}$, the sequence of positive numbers $\{r_n\}$ as

$$r_n = \frac{1}{n 2^n (1 + \sum_{j=1}^n \|\delta'(2x_j)\|_{1/2}^{1/2})}.$$

Recall that the quasi-Banach space $\ell_{1/2}$ satisfies the following condition $\|\sum_{k=1}^n v_k\|_{1/2} \leq \left(\sum_{k=1}^n \|v_k\|_{1/2}^{1/2}\right)^2$, for every $n \in \mathbb{N}$ and $v_1, \dots, v_n \in \ell_{1/2}$. Therefore,

$$\begin{aligned} \|\phi'_n(z)\|_{1/2} \leq & \frac{(|f(x_n)| + 1)t_n}{\nu_n} \\ & \times \left(\sum_{j=1}^n \sqrt{2a_j} \|\delta'(z - x_j)\|_{1/2}^{1/2} \left(\int_{I_n} |\delta(z - x_j) - \tau_j| \exp \left[-t_n \sum_{i=1}^n a_i (\delta(z - x_i) - \tau_i)^2 \right] d\tau \right)^{1/2} \right)^2. \end{aligned}$$

Let us fix n and define $A_j(z)$ for $j = 1, \dots, n$ as

$$A_j(z) = \int_{I_n} |\delta(z - x_j) - \tau_j| \exp \left[-t_n \sum_{i=1}^n a_i (\delta(z - x_i) - \tau_i)^2 \right] d\tau.$$

Let us find an upper bound for $A_j(z)$ whenever $z \in c_0 \setminus \tilde{D}_n$. The sequence $\{t_n\}$ will be chosen to be increasing and satisfying $t_n a_n \delta_n^2 \geq 1$, for every $n \in \mathbb{N}$. If $z \in c_0 \setminus \tilde{D}_n$, then either $\delta(z - x_n) \geq 1 + 2\delta_n$ or there is $i < n$ such that $\delta(z - x_n) \leq 1 - 3\delta_n$. Let us study both cases.

(1) Consider A_j for $j < n$. Then $A_j(z)$ is the product of the n integrals

$$A_j(z) = \int_{1-2\delta_n}^{\infty} |\delta(z - x_j) - \tau_j| e^{-t_n a_j (\delta(z - x_j) - \tau_j)^2} d\tau_j \times \int_{-\delta_n}^{1+\delta_n} e^{-t_n a_n (\delta(z - x_n) - \tau_n)^2} d\tau_n \\ \times \prod_{\substack{k < n \\ k \neq j}} \int_{1-2\delta_n}^{\infty} e^{-t_n a_k (\delta(z - x_k) - \tau_k)^2} d\tau_k.$$

(a) Assume that $\delta(z - x_n) \geq 1 + 2\delta_n$. Then, for every $\tau = (\tau_1, \dots, \tau_n) \in I_n$ we have that $\delta(z - x_n) - \tau_n \geq \delta_n$. Then,

$$A_j(z) \leq \frac{1}{t_n a_j} \frac{1}{\sqrt{t_n a_n}} \int_{\sqrt{t_n a_n} \delta_n}^{\infty} e^{-t^2} dt \times \prod_{\substack{k < n \\ k \neq j}} \frac{\sqrt{\pi}}{\sqrt{t_n a_k}} \leq \frac{1}{\pi} \frac{1}{\sqrt{t_n a_j}} \frac{1}{2e^{t_n a_n \delta_n^2}} \nu_n.$$

(b) Now assume that there is $i < n$ such that $\delta(z - x_i) \leq 1 - 3\delta_n$. Then for every $\tau = (\tau_1, \dots, \tau_n) \in I_n$ we have that $\delta(z - x_i) - \tau_i \leq -\delta_n$. In the case $i \neq j$,

$$A_j(z) \leq \frac{1}{t_n a_j} \frac{1}{\sqrt{t_n a_i}} \int_{\sqrt{t_n a_i} \delta_n}^{\infty} e^{-t^2} dt \times \prod_{\substack{k \leq n \\ k \neq i, j}} \frac{\sqrt{\pi}}{\sqrt{t_n a_k}} \leq \frac{1}{\pi} \frac{1}{\sqrt{t_n a_j}} \frac{1}{2e^{t_n a_n \delta_n^2}} \nu_n.$$

In the case $i = j$,

$$A_j(z) \leq \frac{1}{t_n a_j} \int_{\sqrt{t_n a_j} \delta_n}^{\infty} t e^{-t^2} dt \times \prod_{\substack{k \leq n \\ k \neq j}} \frac{\sqrt{\pi}}{\sqrt{t_n a_k}} \leq \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{t_n a_j}} \frac{1}{2e^{t_n a_n \delta_n^2}} \nu_n.$$

(2) Consider A_n . Then,

$$A_n(z) = \int_{-\delta_n}^{1+\delta_n} |\delta(z - x_n) - \tau_n| e^{-t_n a_n (\delta(z - x_n) - \tau_n)^2} d\tau_n \times \prod_{k < n} \int_{1-2\delta_n}^{\infty} e^{-t_n a_k (\delta(z - x_k) - \tau_k)^2} d\tau_k.$$

(a) Assume that $\delta(z - x_n) \geq 1 + 2\delta_n$. Then,

$$A_n(z) \leq \frac{1}{t_n a_n} \int_{\sqrt{t_n a_n} \delta_n}^{\infty} t e^{-t^2} dt \times \prod_{k < n} \frac{\sqrt{\pi}}{\sqrt{t_n a_k}} \leq \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{t_n a_n}} \frac{1}{2e^{t_n a_n \delta_n^2}} \nu_n.$$

(b) Finally assume that there is $i < n$ such that $\delta(z - x_i) \leq 1 - 3\delta_n$. Then

$$A_n(z) \leq \frac{1}{t_n a_n} \frac{1}{\sqrt{t_n a_i}} \int_{\sqrt{t_n a_i} \delta_n}^{\infty} e^{-t^2} d\tau_i \times \prod_{\substack{k < n \\ k \neq i}} \frac{\sqrt{\pi}}{\sqrt{t_n a_k}} \leq \frac{1}{\pi} \frac{1}{\sqrt{t_n a_n}} \frac{1}{2e^{t_n a_n \delta_n^2}} \nu_n.$$

Therefore, for $z \in c_0 \setminus \tilde{D}_n$, we have

$$\begin{aligned} \|\phi'_n(z)\|_{1/2}^{1/2} &\leq \sum_{j=1}^n \sqrt{2a_j} \frac{(|f(x_n)| + 1)^{1/2} t_n^{1/2}}{\nu_n^{1/2}} A_j(z)^{1/2} \|\delta'(z - x_j)\|_{1/2}^{1/2} \\ &\leq \sum_{j=1}^n \frac{a_j^{1/4} t_n^{1/4} (|f(x_n)| + 1)^{1/2}}{\pi^{1/4}} \frac{1}{e^{t_n a_n \delta_n^2/2}} \|\delta'(z - x_j)\|_{1/2}^{1/2}. \end{aligned}$$

It is now clear that we can proceed to select t_n large enough such that

$$\frac{a_1^{1/4} t_n^{1/4} (|f(x_n)| + 1)^{1/2}}{\pi^{1/4}} \frac{1}{e^{t_n a_n \delta_n^2/2}} \leq \frac{r_n}{|f(x_n)|^{1/2} + 1},$$

which implies

$$(|f(x_n)|^{1/2} + 1) \|\phi'_n(z)\|_{1/2}^{1/2} \leq r_n \sum_{j=1}^n \|\delta'(z - x_j)\|_{1/2}^{1/2},$$

whenever $z \in c_0 \setminus \tilde{D}_n$.

Fact 2.2. For every $v, w \in c_0$

$$(2.1) \quad \|\delta'(v + w)\|_{1/2}^{1/2} \leq \|\delta'(2v)\|_{1/2}^{1/2} + \|\delta'(2w)\|_{1/2}^{1/2}.$$

Proof. If $z \in c_0$, then $\delta'(z) = \rho(d^{\mathbb{C}})'(\rho z) = \rho \sum_n 2n(\rho z_n)^{2n-1} e_n$, where $\{e_n\}$ is the canonical basis of ℓ_1 . Then, $\|\delta'(z)\|_{1/2}^{1/2} = \rho^{1/2} \|(d^{\mathbb{C}})'(\rho z)\|_{1/2}^{1/2} = \rho^{1/2} \sum_n \sqrt{2n} |\rho z_n|^{n-1/2}$.

It is enough to check that for every $v, w \in c_0$, $\|(d^{\mathbb{C}})'(v + w)\|_{1/2}^{1/2} \leq \|(d^{\mathbb{C}})'(2v)\|_{1/2}^{1/2} + \|(d^{\mathbb{C}})'(2w)\|_{1/2}^{1/2}$. For $n = 1$, we have that $|v_1 + w_1|^{1/2} \leq (|v_1| + |w_1|)^{1/2} \leq |v_1|^{1/2} + |w_1|^{1/2} \leq |2v_1|^{1/2} + |2w_1|^{1/2}$. Now, the function $\kappa(v) = \sum_{n>1} \sqrt{2n} |v_n|^{n-1/2}$ is convex. Then, $\kappa(v + w) \leq \frac{1}{2}\kappa(2v) + \frac{1}{2}\kappa(2w) \leq \kappa(2v) + \kappa(2w)$ and inequality (2.1) follows easily. \square

From inequality (2.1), we obtain

$$(2.2) \quad \|\delta'(z - x_j)\|_{1/2}^{1/2} \leq \|\delta'(2z)\|_{1/2}^{1/2} + \|\delta'(2x_j)\|_{1/2}^{1/2}, \quad \text{for every } j \in \mathbb{N}.$$

Let us now consider $z \in c_0$. Since $\{\tilde{D}_n\}$ is locally finite, there is $i_0 \in \mathbb{N}$ such that $z \notin \tilde{D}_n$, for $n > i_0$. Thus,

$$\begin{aligned} \sum_{n>i_0} (|f(x_n)|^{1/2} + 1) \|\phi'_n(z)\|_{1/2}^{1/2} &\leq \sum_{n>i_0} r_n \left(\sum_{j=1}^n \|\delta'(z - x_j)\|_{1/2}^{1/2} \right) \\ &\leq \sum_{n>i_0} r_n \left(\sum_{j=1}^n (\|\delta'(2z)\|_{1/2}^{1/2} + \|\delta'(2x_j)\|_{1/2}^{1/2}) \right). \end{aligned}$$

Let us denote $s_n = \sum_{j=1}^n \|\delta'(2x_j)\|_{1/2}^{1/2}$. Then,

$$(2.3) \quad \sum_{n>i_0} (|f(x_n)|^{1/2} + 1) \|\phi'_n(z)\|_{1/2}^{1/2} \leq \left(\sum_{n>i_0} n r_n \right) \|\delta'(2z)\|_{1/2}^{1/2} + \sum_{n>i_0} r_n s_n.$$

Since $\sum_n nr_n < \infty$ and $\sum_n r_n s_n < \infty$, we deduce that

$$\begin{aligned} \left\| \sum_n \phi'_n(z) \right\|_{1/2}^{1/2} &\leq \sum_n \|\phi'_n(z)\|_{1/2}^{1/2} < \infty \quad \text{and} \\ \left\| \sum_n f(x_n) \phi'_n(z) \right\|_{1/2}^{1/2} &\leq \sum_n |f(x_n)|^{1/2} \|\phi'_n(z)\|_{1/2}^{1/2} < \infty, \end{aligned}$$

for every $z \in c_0$. Therefore $h'(z) \in \ell_{1/2}$ for every $z \in c_0$.

Now it is clear that, for every $\varepsilon > 0$, we can ε -approximate f with a real analytic function whose set of derivatives is in $\ell_{1/2}$.

Finally, let us mention that if f is defined in an open subset U of c_0 , the proof is analogous with ϕ , ψ and h defined in an open subset of \hat{c}_0 containing U . \square

Let us consider, for every $0 < p < 1$, the quasi-Banach spaces ℓ_p with the quasi-norm $\|z\|_p = (\sum_k |z_k|^p)^{1/p}$, for $z \in \ell_p$. Recall that $\|\sum_{k=1}^n v_k\|_p^p \leq \sum_{k=1}^n \|v_k\|_p^p$ for every $v_1, \dots, v_n \in \ell_p$. We shall deduce the following corollaries.

Corollary 2.3. *Let us consider real numbers $0 < p < q < 1$. Then, every real-valued, uniformly continuous function defined on an open subset U of c_0 can be uniformly approximated by analytic functions such that the set of derivatives is in $\ell_q \setminus \ell_p$.*

Proof. The proof of Theorem 2.1 works not only for $p = \frac{1}{2}$ but also for every $0 < p < 1$. With the notation of Theorem 2.1, if we select $\{t_n\}$ large enough so that $\sum_n (|f(x_n)|^p + 1) \|\phi'_n(z)\|_p^p < \infty$, and follow the rest of the proof (Fact 2.2 holds for every $p > 0$), we obtain a 1-approximating analytic function $h : U \rightarrow \mathbb{R}$ such that $h'(U) \subset \ell_p$. Next, for $0 < p < q < 1$, we choose $z^* \in \ell_q \setminus \ell_p$ and define a real-analytic function $\omega : \mathbb{R} \rightarrow (-1, 1)$ such that $\omega'(t) > 0$ for every t , for instance $\omega(t) = \frac{2e^t}{1+e^t} - 1$. Then, $(\omega \circ z^*)'(U) \subset \ell_q \setminus \ell_p$ and $(h + \omega \circ z^*)'(U) \subset \ell_q \setminus \ell_p$. Thus $h + \omega \circ z^*$ is a real analytic function which 2-approximates f . \square

Corollary 2.4. (1) *Every real-valued, uniformly continuous function defined on an open subset U of c_0 can be uniformly approximated by analytic functions such that the set of derivatives is in $\cap_{p>0} \ell_p$.*

(2) *Consequently, every real-valued, uniformly continuous function defined on an open subset U of c_0 can be uniformly approximated by analytic functions such that the set of derivatives is in $\text{span}\{z^*, \cap_{p>0} \ell_p\} \setminus \cap_{p>0} \ell_p$ for any pre-fixed $z^* \in \ell_1 \setminus \cap_{p>0} \ell_p$.*

Proof. (1) The proof of Theorem 2.1 works with some modifications: We find an upper bound for $\|\phi'_n(z)\|_{1/n}^{1/n}$ whenever $z \in U \setminus \tilde{D}_n$, such that

$$(2.4) \quad \sum_n (|f(x_n)|^{1/n} + 1) \|\phi'_n(z)\|_{1/n}^{1/n} < \infty,$$

for every $z \in U$. In order to do that (and following the same notation) we consider

$$r_n = \frac{1}{n2^n(1+s_n)(1+\Omega_n)}$$

where $s_n = \sum_{j=1}^n \|\delta'(2x_j)\|_{1/n}^{1/n}$ and $\Omega_n = \sum_k \sqrt[n]{\frac{2k}{2^{2k-1}}}$. The calculations are similar except in the step corresponding to equation (2.3) where we have

$$\sum_{n>i_0} (|f(x_n)|^{1/n} + 1) \|\phi'_n(z)\|_{1/n}^{1/n} \leq \left(\sum_{n>i_0} nr_n \|\delta'(2z)\|_{1/n}^{1/n} \right) + \sum_{n>i_0} r_n s_n.$$

Since $\sum_{n>i_0} r_n s_n < \infty$ we only need to prove that $\sum_{n>i_0} nr_n \|\delta'(2z)\|_{1/n}^{1/n} < \infty$. We need an additional bound for $\|\delta'(2z)\|_{1/n}^{1/n} = \rho^{1/n} \sum_k \sqrt[n]{2k} \sqrt[n]{|\rho 2z_k|^{2k-1}}$. Let us denote by $N_1 = \{k \in \mathbb{N} : |\rho 2z_k| \geq \frac{1}{2}\}$ and $N_2 = \mathbb{N} \setminus N_1$. Then

$$\begin{aligned} \rho^{1/n} \sum_{k \in N_1} \sqrt[n]{2k} \sqrt[n]{|\rho 2z_k|^{2k-1}} &\leq \rho^{1/n} \sum_{k \in N_1} \sqrt[n]{2k} \sqrt[n]{|\rho 4z_k|^{2k-1}} \\ &\leq \rho \sum_{k \in N_1} 2k |\rho 4z_k|^{2k-1} \leq \rho \sum_{k \in \mathbb{N}} 2k |\rho 4z_k|^{2k-1} = \|\delta'(4z)\|_1 \end{aligned}$$

In addition, we have

$$\rho^{1/n} \sum_{k \in N_2} \sqrt[n]{2k} \sqrt[n]{|\rho 2z_k|^{2k-1}} \leq \rho^{1/n} \sum_{k \in N_2} \sqrt[n]{2k} \sqrt[n]{\frac{1}{2^{2k-1}}} \leq \rho \sum_{k \in \mathbb{N}} \sqrt[n]{\frac{2k}{2^{2k-1}}} = \rho \Omega_n.$$

Therefore,

$$\|\delta'(2z)\|_{1/n}^{1/n} \leq \|\delta'(4z)\|_1 + \rho \Omega_n.$$

Now,

$$\begin{aligned} \sum_{n>i_0} nr_n \|\delta'(2z)\|_{1/n}^{1/n} &\leq \sum_{n>i_0} nr_n (\|\delta'(4z)\|_1 + \rho \Omega_n) \\ &= \left(\sum_{n>i_0} nr_n \right) \|\delta'(4z)\|_1 + \rho \sum_{n>i_0} nr_n \Omega_n < \infty, \end{aligned}$$

and (2.4) is proved. From inequality (2.4) we obtain, for every $0 < p < 1$ and $z \in U$, a natural number n_p such that $\frac{1}{n_p} \leq p$ and $\sum_{n \geq n_p} (|f(x_n)|^{1/n} + 1) \|\phi'_n(z)\|_{1/n}^{1/n} < 1$. Since $a^{1/n} \geq a^p$ whenever $0 < a < 1$ and $n \geq n_p$, it follows that

$$\sum_{n \geq n_p} (|f(x_n)|^p + 1) \|\phi'_n(z)\|_{1/n}^p \leq \sum_{n \geq n_p} (|f(x_n)|^{1/n} + 1) \|\phi'_n(z)\|_{1/n}^{1/n} < \infty.$$

Moreover, $\|\cdot\|_p \leq \|\cdot\|_{1/n}$ whenever $p \geq \frac{1}{n}$. Therefore,

$$\sum_{n \geq n_p} (|f(x_n)|^p + 1) \|\phi'_n(z)\|_p^p \leq \sum_{n \geq n_p} (|f(x_n)|^p + 1) \|\phi'_n(z)\|_{1/n}^p < \infty$$

and $h'(z) = \left(\frac{\psi}{\phi}\right)'(z) \in \cap_{p>0} \ell_p$.

Now in order to prove (2), it is clear that $(h + \omega \circ z^*)'(z) \neq 0$ for every $z \in U$, because $z^* \notin \cap_{p>0} \ell_p$ (where ω is the real analytic function considered in the proof of Corollary 2.3). Moreover, $(h + \omega \circ z^*)'(U) \subset \text{span}\{z^*, \cap_{p>0} \ell_p\} \setminus \cap_{p>0} \ell_p$. \square

In the next corollary we extend Theorem 2.1, Corollaries 2.3 and 2.4 to every separable real Banach space X with property (K) and containing c_0 . Recall that, by the Sobczyk theorem $X = c_0 \oplus Y$, for some separable real Banach space Y . Let us consider a differentiable function f defined on an open subset U of X . For every $z = (x, y) \in U$, let us denote by $\frac{\partial f}{\partial c_0}(U)$ the set of partial derivatives $\{\frac{\partial f}{\partial x}(x, y) : (x, y) \in U\}$.

Corollary 2.5. *Let X be a separable real Banach space with property (K) such that X contains a copy of c_0 . Then,*

- (1) *every real-valued, uniformly continuous function f defined on an open subset U of X can be uniformly approximated on U by an analytic function h such that the set $\frac{\partial h}{\partial c_0}(U) \subset \cap_{p>0} \ell_p$.*
- (2) *Consequently, every real-valued, uniformly continuous function f defined on an open subset U of X can be uniformly approximated on U by an analytic function g such that the set $\frac{\partial g}{\partial c_0}(U) \subset \text{span}\{z^*, \cap_{p>0} \ell_p\} \setminus \cap_{p>0} \ell_p$, for any pre-fixed $z^* \in \ell_1 \setminus \cap_{p>0} \ell_p$.*

Proof. We follow the proofs of [7, Theorem 1], Theorem 2.1 and Corollary 2.4 for the space X and the uniformly analytic and separating function D defined below:

- (a) We consider in $X = c_0 \oplus Y$, the function $D(x, y) = d(x) + r(y)$, for $x \in c_0$ and $y \in Y$, where $r : Y \rightarrow \mathbb{R}$ is uniformly analytic and separating in Y and $d(x) = \sum_n x^{2n}$ for $x \in c_0$. Clearly D is uniformly analytic and separating in X .
- (b) We consider the complexified space of X , $X^{\mathbb{C}} = \hat{c}_0 \oplus Y^{\mathbb{C}}$, where $Y^{\mathbb{C}}$ is the complexified space of Y . The function $D^{\mathbb{C}}(x, y) := d^{\mathbb{C}}(x) + r^{\mathbb{C}}(y)$, where $r^{\mathbb{C}}$ is the analytic extension of r to an open subset Ω of $Y^{\mathbb{C}}$ of the form $Y + sU_{Y^{\mathbb{C}}}$ ($U_{Y^{\mathbb{C}}}$ is the open unit ball of $Y^{\mathbb{C}}$) for some $s > 0$.
- (c) Let us define, for every $0 < \rho < 1$, the functions $D_\rho(x, y) = D(\rho x, \rho y)$, for $x \in c_0$ and $y \in Y$.
- (d) $\frac{\partial D_\rho}{\partial x}(x, y) = d'_\rho(x)$ and $\frac{\partial D_\rho^{\mathbb{C}}}{\partial x}(x, y) = (d_\rho^{\mathbb{C}})'(x)$, for $x \in c_0$ and $y \in Y$.

Then, following the notation of Theorem 2.1, we have: (i) $\frac{\partial \phi_n}{\partial x}(x, y) \in \cap_{p>0} \ell_p$, for every $(x, y) \in U$ and $n \in \mathbb{N}$, (ii) $\frac{\partial \phi}{\partial x}(x, y) = \sum_n \frac{\partial \phi_n}{\partial x}(x, y) \in \cap_{p>0} \ell_p$ and $\frac{\partial \psi}{\partial x}(x, y) = \sum_n f(x_n) \frac{\partial \phi_n}{\partial x}(x, y) \in \cap_{p>0} \ell_p$ for every $(x, y) \in U$, where $x_n := (x_n^1, x_n^2) \in \hat{c}_0 \oplus Y^{\mathbb{C}}$ and $\{x_n\}$ is the pre-fixed dense sequence in U , and thus, (iv) the analytic approximations h and $g = h + \omega \circ z^*$ satisfy that $\frac{\partial h}{\partial x}(x, y) \in \cap_{p>0} \ell_p$ and $\frac{\partial g}{\partial x}(x, y) = \frac{\partial(h + \omega \circ z^*)}{\partial x}(x, y) = \frac{\partial h}{\partial x}(x, y) + \omega'(z^*(x))z^* \in \text{span}\{z^*, \cap_{p>0} \ell_p\} \setminus \cap_{p>0} \ell_p$ for every $(x, y) \in U$, if $z^* \notin \cap_{p>0} \ell_p$. \square

Let us now generalize Theorem 2.1, Corollaries 2.3, 2.4 and 2.5 for the case of uniformly continuous functions with values into a (finite or infinite dimensional) real Banach space E .

Corollary 2.6. *Let E be a real Banach, X a separable real Banach space with property (K) so that X contains a copy of c_0 and U an open subset of X . Then,*

- (1) *every uniformly continuous function $f : U \rightarrow E$ can be uniformly approximated by analytic functions $h : U \rightarrow E$ such that $v^* \circ \frac{\partial h}{\partial c_0}(U) \subset \cap_{p>0} \ell_p$ for every $v^* \in E^*$.*

- (2) Let us fix any $z^* \in \ell_1 \setminus \cap_{p>0} \ell_p$ and $v \in E \setminus \{0\}$. Then, every uniformly continuous function $f : U \rightarrow E$ can be uniformly approximated by an analytic function $g : U \rightarrow E$ such that $v^* \circ \frac{\partial g}{\partial c_0}(U) \subset \text{span}\{z^*, \cap_{p>0} \ell_p\} \setminus \cap_{p>0} \ell_p$ for every $v^* \notin [v]^\perp$.
- (3) In addition to (2), if E is finite dimensional, we obtain that the analytic approximation g has no critical points, i.e. $g'(z)$ is surjective for every $z \in U$.

Proof. (1) The proofs of Theorem 2.1 and Corollary 2.5 can be applied to obtain the above assertion. We only have to consider $\|f(x_n)\|_E$ instead of $|f(x_n)|$ and select $\{\phi_n\}$ so that $\sum_n (\|f(x_n)\|_E^{1/n} + 1) \|\frac{\partial \phi_n}{\partial x}(x, y)\|_{1/n}^{1/n} < \infty$ for every $(x, y) \in U$ (we follow the notation of the proof of Corollary 2.5). Then, the analytic function $h(x, y) = \frac{\sum f(x_n) \phi_n(x, y)}{\sum \phi_n(x, y)}$ (which 1-approximates f) satisfies the following property: $v^* \circ \frac{\partial h}{\partial x}(x, y) \in \cap_{p>0} \ell_p$ for every $v^* \in E^*$ and $(x, y) \in U$.

(2) Let us consider $\tilde{v} = \frac{v}{\|v\|_E}$ ($\|\cdot\|_E$ is the norm on E) and define $T : X \rightarrow E$, $T(x, y) = \tilde{v} \cdot w \circ z^*(x)$ for all $(x, y) \in X$, where $w : \mathbb{R} \rightarrow (-1, 1)$ is the analytic function already defined in the proof of Corollary 2.3. On the one hand, $|f(x, y) - g(x, y)| \leq 2$ for every $(x, y) \in U$. On the other hand, $\frac{\partial g}{\partial x}(x, y) = \frac{\partial h}{\partial x}(x, y) + \tilde{v} w'(z^*(x)) z^*$ for every $(x, y) \in U$. This implies $v^* \circ \frac{\partial g}{\partial x}(x, y) = v^* \circ \frac{\partial h}{\partial x}(x, y) + v^*(\tilde{v}) w'(z^*(x)) \cdot z^*$ for every $(x, y) \in U$ and $v^* \in E^*$. Moreover, $v^*(\tilde{v}) w'(z^*(x)) \neq 0$ whenever $v^* \notin [v]^\perp$ and $(x, y) \in U$. Thus, $v^* \circ \frac{\partial g}{\partial x}(x, y) \in \text{span}\{z^*, \cap_{p>0} \ell_p\} \setminus \cap_{p>0} \ell_p$ whenever $v^* \notin [v]^\perp$ and $(x, y) \in U$.

(3) Let us consider a continuous function $f : U \rightarrow \mathbb{R}^n$ with components $f_i : U \rightarrow \mathbb{R}$, $i = 1, \dots, n$. Let us select linear independent elements $z_1^*, \dots, z_n^* \in \ell_1 \setminus \cap_{p>0} \ell_p$ such that $(\text{span}\{z_1^*, \dots, z_n^*\}) \cap (\cap_{p>0} \ell_p) = \{0\}$. By Corollary 2.5, we can uniformly approximate f_i by analytic functions $g_i : U \rightarrow \mathbb{R}$ satisfying the following condition: for every $(x, y) \in U$ there are $\lambda_i \in \mathbb{R} \setminus \{0\}$ and $k_i^* \in \cap_{p>0} \ell_p$ such that $\frac{\partial g_i}{\partial x}(x, y) = \lambda_i z_i^* + k_i^*$. Therefore $\{\frac{\partial g_1}{\partial x}(x, y), \dots, \frac{\partial g_n}{\partial x}(x, y)\}$ are linearly independent (thus $\{g_1'(x, y), \dots, g_n'(x, y)\}$ are l.i. as well) for every $(x, y) \in U$ and the analytic function $g := (g_1, \dots, g_n)$ uniformly approximate f . \square

The existence of real analytic approximations with no critical points for uniformly continuous functions $f : X \rightarrow E$ (X with the conditions of Corollary 2.6 and E finite dimensional) can be obtained from the results of P. Hájek [13]. Statement (3) in Corollary 2.6 provides an alternative proof to this assertion with an additional description of the range of the derivatives. Let us sketch how to deduce the assertion from [13] and for $X = c_0$ (the general case is deduced in an analogous way). If $h : c_0 \rightarrow E$ is a real analytic function approximating f and we consider $h_1, \dots, h_n : c_0 \rightarrow \mathbb{R}$ the n components of h , then by the results given in [13] there is a K_σ set \mathcal{K} (namely a countable union of compact sets) of ℓ_1 , such that $h_1'(c_0) \cup \dots \cup h_n'(c_0) \subset \mathcal{K}$. We may consider that \mathcal{K} is a linear subspace of ℓ_1 ; otherwise we consider the $\text{span}\{\mathcal{K}\}$ which is also a K_σ subset. Let us follow the notation $[a_1, \dots, a_n] = \text{span}\{a_1, \dots, a_n\}$. Therefore, there is $z_1^* \in \ell_1 \setminus \{0\}$ such that $[z_1^*] \cap \mathcal{K} = \{0\}$. Since $[z_1^*] + \mathcal{K}$ is again a K_σ subspace of ℓ_1 , there is $z_2^* \in \ell_1 \setminus \{0\}$ such that $[z_2^*] \cap ([z_1^*] + \mathcal{K}) = \{0\}$ and then $[z_1^*, z_2^*] \cap \mathcal{K} = \{0\}$. Proceeding in this way, we obtain linear independent elements $\{z_i^* : i = 1, \dots, n\}$ in ℓ_1 , such that $[z_1^*, \dots, z_n^*] \cap \mathcal{K} = \{0\}$. Now we proceed as in the proof of Corollary 2.6(3) and define $g_i := h_i + \varepsilon \cdot \omega \circ z_i^*$, $i = 1, \dots, n$ and $\varepsilon > 0$ (w is

the auxiliary real analytic function considered in the proof of Corollary 2.3). Then, $\{g'_1(x), \dots, g'_n(x)\}$ are linearly independent for every $x \in c_0$ and $g := (g_1, \dots, g_n)$ is a real analytic function with no critical points and uniformly approximates f . In addition, it can be obtained that $g'(c_0) \cap \mathcal{M} = \emptyset$ for any pre-fixed K_σ subset $\mathcal{M} \subset L(c_0, E) = \ell_1 \times \dots \times (n) \times \dots \times \ell_1$ (linear continuous function from c_0 into E). In order to do that, we denote by \mathcal{M}_1 the linear span of the projection of \mathcal{M} over the first ℓ_1 . Then, \mathcal{M}_1 is a K_σ subset of ℓ_1 . Following the above arguments, we select elements $z_i^* \in \ell_1$, $i = 1, \dots, n$ such that $[z_1^*, \dots, z_n^*] \cap (\mathcal{K} + \mathcal{M}_1) = \{0\}$. In this case, $g'_1(x) \notin \mathcal{M}_1$ and then $g'(x) \notin \mathcal{M}$ for every $x \in c_0$.

Let us finish this note with a remark on the separation of closed sets and a question on the separable Hilbert space. Recall that, whenever a real Banach space X has the property that every uniformly continuous, real-valued function on X can be uniformly approximated by real-analytic functions with no critical points, one can deduce the following non-linear separation and approximation result. An open subset U of X is said to be real-analytic smooth provided its boundary ∂U is a real-analytic smooth one-codimensional submanifold of X .

Remark 2.7. *Let X be a real and separable Banach space X containing c_0 with property (K).*

- (1) *Let A and B be two disjoint and closed subsets of X such that $\text{dist}(A, B) > 0$. Then, there is a real-analytic function $g : X \rightarrow \mathbb{R}$ with no critical points such that the level set $L = g^{-1}(0)$ is a 1-codimensional analytic submanifold of X that separates A and B in the following sense: Define $U_1 = \{x \in X : g(x) < 0\}$ and $U_2 = \{x \in X : g(x) > 0\}$, then U_1 and U_2 are disjoint real-analytic open subsets of X with common boundary $\partial U_1 = \partial U_2 = L$ such that $A \subset U_1$ and $B \subset U_2$.*
- (2) *Every closed subset of X can be approximated by real-analytic smooth open subsets of X in the following sense: For every closed set $C \subset X$ and every $\varepsilon > 0$, there is a real-analytic open set $U \subset X$ so that $C \subset U \subset C + \varepsilon B_X$, where B_X is the closed unit ball of X .*

Proof. In the first case, let us define $\varepsilon := \text{dist}(A, B) > 0$ and consider the function $f(x) = \text{dist}(x, A)$, which is Lipschitz. We can $\frac{\varepsilon}{4}$ -approximate f by a real-analytic function h with no critical points. Then $g := h - \frac{\varepsilon}{2} < 0$ on A , $g > 0$ on B , and $g^{-1}(0)$ is a real-analytic 1-codimensional submanifold of X separating A and B . In the second case, we consider in (1) the sets $A := C$ and $B := X \setminus (C + \varepsilon U_X)$, where U_X is the open unit ball of X . \square

Question. Recall that it is an open problem whether every real-valued and continuous function on ℓ_2 can be uniformly approximated by analytic functions with no critical points. Let us consider in ℓ_2 a non negative separating polynomial Q with $Q(0) = 0$ and a continuous function $f : \ell_2 \rightarrow \mathbb{R}$. A similar strategy could be applied to ℓ_2 following the proof of Kurzwil [15] to obtain: (1) real-analytic functions $\phi_n : \ell_2 \rightarrow \mathbb{R}$ so that $\phi'_n(x) \in \text{span}\{Q'(x-x_1), \dots, Q'(x-x_n)\}$ for every $x \in \ell_2$, where $\{x_n\}$ is the auxiliary sequence of ℓ_2 considered in the proof, (2) if $\|\cdot\|$ is the Hilbertian norm and $Q(x) = \|x\|^2$, then $\phi'_n(x) \in \text{span}\{x, x_1, \dots, x_n\} \subset \text{span}\{x, c_{00}\}$, whenever the auxiliary sequence $\{x_n\} \subset c_{00}$ (this can be assumed). In that case, we could construct the real analytic function $h(x) = \frac{\sum f(x_n)\phi_n(x)}{\sum \phi_n(x)}$ so that $h'(x) \in \text{span}\{x, \cap_{p>0} \ell_p\}$

for every $x \in \ell_2$ and h 1-approximates f . Thus, it would be enough to find a bounded real-analytic function $T : \ell_2 \rightarrow \mathbb{R}$ satisfying $T'(x) \notin \text{span}\{x, \cap_{p>0} \ell_p\}$ for every $x \in \ell_2$, to obtain a real-analytic function $h + \lambda T$ with no critical points, that 2-approximates f for some $\lambda > 0$. Nevertheless, we do not know the existence of such a function T .

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