Approximation of functions and their derivatives by analytic maps on certain Banach spaces

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Abstract. Let X be a separable Banach space which admits a separating polynomial. Let $f : X \to \mathbb{R}$ be bounded, Lipschitz, and C^1 with uniformly continuous derivative. Then for each $\varepsilon > 0$, there exists an analytic function $g : X \to \mathbb{R}$ with $|g - f| < \varepsilon$ and $\|g' - f'\| < \varepsilon$.

1. Introduction

The problem of approximating a smooth function and its derivatives by a function of higher order smoothness on a Banach space X has been investigated by several authors, although the number of such results is limited. When X is finite dimensional excellent results are known, and in fact Whitney in his classical paper [**W**] provides essentially a complete answer by showing: for every C^k function $f : \mathbb{R}^n \to \mathbb{R}^m$ and every continuous $\varepsilon : \mathbb{R}^n \to (0, +\infty)$ there exists a real analytic function g such that $\|D^j g(x) - D^j f(x)\| \leq \varepsilon(x)$ for all $x \in \mathbb{R}^n$ and $j = 1, ..., k$. This is the so-called C^k fine approximation of f.

The first results for X infinite dimensional concern the smooth, non-analytic case, and are due to Moulis [**M**]. She proves, in particular, a C^1 fine approximation theorem; namely, that for $X = c_0$ or l_p with $1 < p < \infty$, and Y an arbitrary Banach space, given a C^1 map $f : X \to Y$, and a continuous function $\varepsilon : X \to (0, \infty)$, there exists a C^k smooth map $g : X \to Y$ (where the optimal value of $k \geq 1$ depends on the choice of X) such that $|g(x) - f(x)| < \varepsilon(x)$ and $\|g'(x) - f'(x)\| < \varepsilon(x)$. This result was later extended in [**AFGJL**] to the case where X has an unconditional basis and admits a Lipschitz, C^k smooth bump function. Further work along this line can be found in [**HJ**], where it is shown that for certain range spaces Y, one

2000 *Mathematics Subject Classification.* Primary 46B20.

Key words and phrases. analytic approximation, separable Banach space, separating polynomial.

The second named author partly supported by NSERC (Canada).

can relax the conditions on X in [**AFGJL**] and, for example, take X to be merely separable.

It is important to note that all the results mentioned above require, in a very essential way, a theorem concerning the approximation of Lipschitz functions f by more regular, Lipschitz functions g, where the Lipschitz constant of g is fixedly proportional to the Lipschitz constant of f, regardless of the precision in the approximation. In fact, all the methods to date on C^k fine approximation rely on such results. In [**M**] and [**AFGJL**] this is achieved by reducing the problem to the finite dimensional case using the unconditional basis, but otherwise without this reduction traditional methods of smooth approximation, such as smooth partitions of unity, do not work in addressing this problem. A new approach was found in [**F1**], and further developed in [**AFM**], [**F2**], [**AFK2**], and [**HJ**]. The technique from [**F1**] has been called the method of *sup-partitions of unity* in [**HJ**].

Concerning C^k fine approximation by analytic functions for X infinite dimensional, nothing is known. In view of the remarks in the preceding paragraph, it would appear that first one needs the ability to approximate Lipschitz functions by Lipschitz, analytic functions with good control over the Lipschitz constant. That is, one requires a kind of analytic sup-partition of unity. Only very recently has this been possible with the work of [**AFK1**], where it is proven that if X is separable and admits a separating polynomial, then for every Lipschitz function $f : X \to \mathbb{R}$ and $\varepsilon > 0$ there exists a Lipschitz, analytic function $g : X \to \mathbb{R}$ with $|f - g| < \varepsilon$ and $\mathrm{Lip}(g) \leq C\mathrm{Lip}(f)$, where the constant $C > 1$ depends only on X (for a precursor to this work see [**FK**]). Using this, we are able in this note to give the first results on the C^1 fine analytic approximation problem in infinite dimensions. We remark that this work is new even for X a separable Hilbert space. We establish,

THEOREM 1. *Let X be a separable Banach space which admits a separating polynomial. Let $f : X \to \mathbb{R}$ be bounded and Lipschitz, with uniformly continuous derivative, and $\varepsilon > 0$. Then there exists an analytic function $g : X \to \mathbb{R}$ such that $|f - g| < \varepsilon$ and $\|f' - g'\| < \varepsilon$.*

Our notation is standard, with X denoting a Banach space, and an open ball with centre x and radius r denoted $B_r(x)$. If $\{f_j\}_j$ is a sequence of Lipschitz functions defined on X, then we will at times say this family is *equi-Lipschitz* if there is a common Lipschitz constant for all j. A *homogeneous polynomial of degree* n is a map, $P : X \to \mathbb{R}$, of the form $P(x) = A(x, x, ..., x)$, where $A : X^n \to \mathbb{R}$ is $n-$multilinear and continuous. For $n = 0$ we take P to be constant. A *polynomial of degree* n is a sum $\sum_{i=0}^{n} P_i(x)$, where $i \geq 1$ the P_i are i-homogeneous polynomials.

Let X be a Banach space, and $G \subset X$ an open subset. A function $f : G \to \mathbb{R}$ is called *analytic* if for every $x \in G$, there are a neighbourhood N_x, and homogeneous polynomials $P_n^x : X \to \mathbb{R}$ of degree n, such that

$$f(x+h) = \sum_{n \geq 0} P_n^x(h) \text{ provided } x+h \in N_x.$$

Further information on polynomials may be found, for example, in [**SS**].

For a Banach space X, we define its (Taylor) complexification $\widetilde{X} = X \bigoplus iX$ with norm

$$\|x+iy\|_{\widetilde{X}} = \sup_{0 \leq \theta \leq 2\pi} \|\cos\theta\, x - \sin\theta\, y\|_X = \sup_{T \in X^*, \|T\| \leq 1} \sqrt{T(x)^2 + T(y)^2}.$$

If $L : E \to F$ is a continuous linear mapping between two real Banach spaces then there is a unique continuous linear extension $\widetilde{L} : \widetilde{E} \to \widetilde{F}$ of L (defined by $\widetilde{L}(x+iy) = L(x) + iL(y)$) such that $\|\widetilde{L}\| = \|L\|$. For a continuous k-homogeneous polynomial $P : E \to \mathbb{R}$ there is also a unique continuous k-homogeneous polynomial $\widetilde{P} : \widetilde{E} \to \mathbb{C}$ such that $\widetilde{P} = P$ on $E \subset \widetilde{E}$ (but the norm of P is not generally preserved: one has that $\|\widetilde{P}\| \leq 2^{k-1}\|P\|$). It follows that if $q(x)$ is a continuous polynomial on X, there is a unique continuous polynomial $\widetilde{q}(z) = \widetilde{q}(x+iy)$ on $\widetilde{X}$ where for $y = 0$ we have $\widetilde{q} = q$. For more information on complexifications (and polynomials) we recommend [**MST**]. In the sequel, all extensions of functions from X to $\widetilde{X}$, as well as subsets of $\widetilde{X}$, will be embellished with a tilde.

2. Main Results

To prove Theorem 1, we start with a lemma which is a variation of [**AFK1**, Lemma 3], where here we have made three changes: added part $(4')$; included constants M_n for the estimate in (5); and relaxed the condition that $r \geq 1$ to $r > 0$. To obtain $(4')$, we replace the function b_n in the proof of [**AFK1**, Lemma 3] with a C^1 version; the change in (5) is easily handled; and requiring merely $r > 0$ means that certain constants will depend on r, but as we shall apply the lemma with r fixed throughout, this causes no problem.

First we need some definitions and notation. If X posseses an n^{th} order separating polynomial, then it admits a $2n$-homogeneous polynomial q such that

$$\|x\|^{2n} \leq q(x) \leq A\|x\|^{2n}, \tag{2.1}$$

for some $A > 1$ (see e.g., [**AFK1**]). In [**AFK1**, Lemma 2] it is proved that the function $Q(x) = (q(x)+1)^{1/2n} - 1$ satisfies:

(1) Q is (real) analytic on X,
(2) Q is Lipschitz on X, where we can take $\mathrm{Lip}(Q) > 1$,
(3) $\inf\{Q(x) : \|x\| \geq 1\} > 0 = Q(0)$,

(4) $Q(x) < 4\rho \Rightarrow \|x\| < 8\rho$ when $\rho \geq 1$; otherwise $Q(x) < 4\rho \Rightarrow \|x\| < \delta(\rho) \equiv \left((1+4\rho)^{2n} - 1\right)^{1/2n}$, this latter implication simply using (2.1) and the definition of Q.

(5) there exists $\delta > 0$ such that Q extends to a Lipschitz, holomorphic map $\widetilde{Q}$ on the uniform strip $X \subset W_\delta \subset \widetilde{X}$ given by,

$$W_\delta = \left\{ x + z : x \in X,\ z \in \widetilde{X},\ \|z\|_{\widetilde{X}} < \delta \right\}.$$

We use the notion of a *Q-body*, which for $\rho > 0$ is defined by

$$D_Q(x, \rho) = \{ y \in X : Q(y - x) < \rho \}.$$

Let $\|\cdot\|_{c_0}$ be an equivalent analytic norm on c_0, with $\|x\|_\infty \leq \|x\|_{c_0} \leq A_1 \|x\|_\infty$ for all $x \in c_0$, and some $A_1 > 1$ (see e.g., [**FPWZ**], and also [**AFK1**], [**FK**] where it is referred to as the Preiss norm).

For the remainder of the proof, we fix a dense sequence $\{x_n\}_{n=1}^\infty$ in X.

LEMMA 1. *Let $\widetilde{V} = W_\delta$ be an open strip around X in $\widetilde{X}$ in which the function $\widetilde{Q}$ given above is defined. Given $\varepsilon \in (0,1)$, $r > 0$, and a covering $\{D_Q(x_n, r)\}_{n=1}^\infty$ of X, there exists a sequence of holomorphic functions $\widetilde{\varphi}_n = \widetilde{\varphi}_{n,r,\varepsilon} : \widetilde{V} \to \mathbb{C}$, whose restrictions to X we denote by $\varphi_n = \varphi_{n,r,\varepsilon}$, with the following properties:*

1: *The collection $\{\varphi_{n,r,\varepsilon} : X \to [0,2] \mid n \in \mathbb{N}\}$ is equi-Lipschitz on X, with Lipschitz constant of the form $L_\varphi = L_1 Lip(Q)/r > 1$ (where $L_1 \geq 1$ is independent of ε and n),*

2: $0 \leq \varphi_{n,r,\varepsilon}(x) \leq 1 + \varepsilon$ *for all* $x \in X$.

3: *For each $x \in X$ there exists $m = m_{x,r} \in \mathbb{N}$ (independent of ε) with $\varphi_{m,r,\varepsilon}(x) > 1/2$.*

4: $0 \leq \varphi_{n,r,\varepsilon}(x) \leq \varepsilon$ *for all* $x \in X \setminus D_Q(x_n, 5r)$.

4′: $\left\|\varphi'_{n,r,\varepsilon}(x)\right\| \leq \varepsilon$ *for all* $x \in X \setminus D_Q(x_n, 5r)$.

5: *For each $x \in X$ there exist $\delta_{x,r} > 0$, $a_{x,r} > 0$, and $n_{x,r} \in \mathbb{N}$ (all independent of ε) such that*

$$|\widetilde{\varphi}_{n,r,\varepsilon}(x+z)| < \frac{1}{M_n n! a_{x,r}^n} \quad \text{for } n > n_{x,r},\ \ z \in \widetilde{X} \text{ with } \|z\|_{\widetilde{X}} < \delta_{x,r},$$

where $M_n = e^{2C^2\kappa}(1 + \|x_n\|)$, and the $\kappa = \kappa(r) > 1$ and $C > 1$ are constants that will be specified in the proof of Theorem 1.

6: *For each $x \in X$, there exists $\delta_{x,r} > 0$ (independent of ε) and $n_{x,\varepsilon,r} \in \mathbb{N}$ such that for $\|z\|_{\widetilde{X}} < \delta_{x,r}$ and $n > n_{x,\varepsilon,r}$ we have $|\widetilde{\varphi}_{n,r,\varepsilon}(x+z)| < \varepsilon$.*

7: *For each $x \in X$, there exists $\delta_{x,\varepsilon,r}$ such that*

$$|\widetilde{\varphi}_{n,r,\varepsilon}(x+z)| \leq 1 + 2\varepsilon \quad \text{for } n \in \mathbb{N}, \ \text{and } z \in \widetilde{X} \text{ with } \|z\|_{\widetilde{X}} \leq \delta_{x,\varepsilon,r}.$$

Proof. We largely follow the proof of [**AFK1**, Lemma 3], with the few noted changes. As the proof in [**AFK1**] is rather long and technical, we here indicate only the key constructions, referring the reader to the above cited paper for full details. Note that because r is fixed throughout, for ease of notation, we shall often suppress dependences on r. Define subsets $A_{1,r} = \{y_1 \in \mathbb{R} : -1 \leq y_1 \leq 4r\}$, and, for $n \geq 2$,

$$\begin{aligned} A_{n,r} &= \{y = \{y_j\}_{j=1}^n \in \ell_\infty^n : -1 - r \leq y_n \leq 4r,\ 2r \leq y_j \\ &\leq M_{n,r} + 2r \text{ for } 1 \leq j \leq n-1\}, \end{aligned}$$

$$\begin{aligned} A'_{n,r} &= \{y = \{y_j\}_{j=1}^n \in \ell_\infty^n : -1 \leq y_n \leq 3r,\ 3r \leq y_j \\ &\leq M_{n,r} + r \text{ for } 1 \leq j \leq n-1\}, \end{aligned}$$

$$\text{where } M_{n,r} = \sup\left\{Q\left(x - x_j\right) : x \in D_Q(x_n, 4r),\ 1 \leq j \leq n\right\}.$$

Let $\mu \in C^\infty\left(\mathbb{R}, [0, 1+\varepsilon]\right)$ be Lipschitz such that $\mu\left(t\right) = 0$ iff $t \geq 1$, and $\mu\left(t\right) = 1 + \varepsilon$ iff $t \leq 1/2$. Let $b^n \in C^\infty\left(\mathbb{R}, [0,1]\right)$ be Lipschitz such that $b^n\left(t\right) = 1$ iff $t \notin \left(2r, M_{n,r} + 2r\right)$, and $b^n\left(t\right) = 0$ iff $t \in \left[3r, M_{n,r} + r\right]$. Observe that we may choose the b^n to have common Lipschitz constant, dependent on r, but independent of n. Let $\widehat{b} \in C^\infty\left(\mathbb{R}, [0,1]\right)$ be Lipschitz such that $\widehat{b}\left(t\right) = 1$ iff $t \notin \left(-1-r, 4r\right)$, and $\widehat{b}\left(t\right) = 0$ iff $t \in \left[-1, 3r\right]$. Now define a Lipschitz, C^∞ smooth map $b_n : c_{00} \subset c_0 \to [0,1]$ by $b_n\left(y_1, ..., y_n\right) = \mu\left(\left\|\left(b^n\left(y_1\right), ..., b^n\left(y_{n-1}\right), \widehat{b}\left(y_n\right)\right)\right\|_{c_0}\right)$. Then support$(b_n) = \overline{A}_n$, and $b_n = 1+\varepsilon$ on A'_n. Moreover, b_n is Lipschitz with constant of the form L_1/r, where $L_1 \geq 1$ is independent of n.

Now one defines, on $\mathbb{R}^n$, the map

$$h_n\left(x\right) = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-k \sum_{j=1}^n 2^{-j}(x_j - y_j)^2} dy,$$

$$T_n = \int_{\mathbb{R}^n} e^{-k \sum_{j=1}^n 2^{-j} {y_j}^2} dy.$$

Because $b_n = b_{n,\varepsilon}$ has compact support, is bounded, Lipschitz, and C^1, one can choose $k = k_{n,\varepsilon} > 0$ sufficiently large that

$$|b_n(x) - h_n(x)| \leq \varepsilon/2 \quad \text{for all} \quad x \in \mathbb{R}^n, \tag{2.2}$$

and

$$|b'_n(x) - h'_n(x)| \leq \varepsilon/2 \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{2.3}$$

Next one defines (real) analytic maps $\varphi_n : X \to \mathbb{R}$ by,

$$\varphi_n\left(x\right) = h_n\left(Q\left(x - x_1\right), ..., Q\left(x - x_n\right)\right) = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n\left(y\right) e^{-k_n \sum_{j=1}^n 2^{-j}\left(Q(x-x_j) - y_j\right)^2}.$$

It is more or less standard to show that $\mathrm{Lip}(\varphi_n) \leq \frac{L_1}{r}\ \mathrm{Lip}(Q)$. We can extend the maps $\varphi_{n,r,\varepsilon}$ to complex valued maps defined on W_{Q} (see above) calling them $\widetilde{\varphi}_n$. Namely (where $x \in X$, $z \in \widetilde{X}$),

$$\widetilde{\varphi}_n\left(x+z\right) = \frac{1}{T_n}\int_{\mathbb{R}^n} b_n\left(y\right) e^{-k_n \sum_{j=1}^{n} 2^{-j}\left(\widetilde{Q}(x-x_j+z)-y_j\right)^2} dy$$

Note that the $\widetilde{\varphi}_n$ are well defined (as the b_n have compact supports) and are holomorphic where $\widetilde{Q}$ is (namely on $\widetilde{W_\delta}$).

To see (4) and (4′), note that if $Q\left(x-x_n\right) \geq 5r$, then there is a neighbourhood N of x for which $y \in N$ implies that the point

$$\widehat{x} = \left(Q\left(y-x_1\right), ..., Q\left(y-x_n\right)\right) \in \mathbb{R}^n \backslash A_n,$$

implying $b_n\left(\widehat{x}\right) = 0$ and $b_n'\left(\widehat{x}\right) = 0$. Hence, by (2.2) and (2.3), we have, $\left|\varphi_n\left(x\right)\right| < \varepsilon/2$ and $\left\|\varphi_n'\left(x\right)\right\| < \varepsilon/2$.

The remaining parts are handled as in [**AFK1**], noting that for (5) we choose κ_n larger if need be to ensure the stated estimate involving the M_n. □

We return now to the proof of the theorem. Let $\varepsilon > 0$ be given and choose ε' satisfying

$$0 < \varepsilon' < \min\{\frac{1}{8}, 1/(132C_0A_1^2L_1Lip(Q)), 1/(10A_1r)\},$$

where L_1 is as in part (1) of the preceding lemma, where is defined immediately below, and where C_0 is a constant, only depending on X, which will be fixed later on (see page 9 below). Because f is bounded, we may suppose that $1 \leq f \leq 2$. As f' is uniformly continuous on X, we can find a fixed $\rho > 0$ so that for all n, $x \in B_\rho\left(x_n\right)$ implies $\left\|f'\left(x_n\right) - f'\left(x\right)\right\| < \varepsilon'$. Now, considering property (4) of Q, and noting that $\delta\left(r\right) \to 0^+$ as $r \to 0^+$, we can choose $r \in (0,1)$ (independent of n) so that $D_Q\left(x_n, 5r\right) \subset B_\rho\left(x_n\right)$ for all n. It will be convenient to write $D_n \equiv D_Q\left(x_n, 5r\right)$. This r shall be fixed for the remainder of the proof.

Next let $\overline{\nu} \in C^\infty\left(\mathbb{R}, [0,1]\right)$ be Lipschitz such that $\overline{\nu}\left(t\right) = 1$ iff $\left|t\right| \leq 5r$, and $\overline{\nu}\left(t\right) = 0$ iff $\left|t\right| \geq \frac{11}{2}r$. Put $L = \mathrm{Lip}(f)$. Fix a sequence of functions $\{\varphi_{n,r,\varepsilon_1}\}_{n=1}^\infty$ with respect to the covering $\{D_Q\left(x_n, r\right)\}_{n=1}^\infty$ of X as given by Lemma 1, where r is fixed as above and the ε of the Lemma is chosen to be

$$\varepsilon_1 := \min\left\{\varepsilon' r/3C_0 L\mathrm{Lip}\left(\overline{\nu}\right), \varepsilon' r/25 L\mathrm{Lip}\left(\overline{\nu}\right)\right\},$$

where

We write $\varphi_{n,r,\varepsilon_1}$ as φ_n for ease of notation, and, as in Lemma 1 (1), $L_\varphi = \mathrm{Lip}(\varphi_n) = L_1\mathrm{Lip}(Q)/r \geq 1$, which we recall is independent of n. Often we will subsume dependence on ε_1 as dependence on ε' and L.

Put $\Delta\left(t\right) = \left(\left(\left|t\right|+1\right)^{2n} - 1\right)^{1/2n} \geq 0$. Now via convolution integrals between $\overline{\nu}$ and Gaussian kernels, we can find Lipschitz, analytic functions ν, with $\mathrm{Lip}(\nu) = \mathrm{Lip}(\overline{\nu})$, and which C^1-fine approximate $\overline{\nu}$ in the following sense,

$$\begin{aligned} &\left|\nu\left(t\right)-\overline{\nu}\left(t\right)\right| < \frac{\varepsilon' r/2LL_{\varphi}}{1+\Delta\left(t\right)}, \\ (2.4)\qquad & \\ &\left|\nu'\left(t\right)-\overline{\nu}'\left(t\right)\right| < \frac{\varepsilon' r/2LL_{\varphi}}{1+\Delta\left(t\right)}. \end{aligned}$$

Indeed, we can take ν to be of the form,

$$\nu\left(t\right) = \frac{1}{a}\int_{\mathbb{R}} \overline{\nu}\left(s\right) e^{-\kappa(t-s)^2} ds,$$

$$a = \int_{\mathbb{R}} e^{-\kappa s^2} ds,$$

where $\kappa > 1$ is chosen sufficiently large and is independent of t (although it does depend on $\max\{\Delta\left(t\right) : t \in \ \mathrm{supp}\left(\overline{\nu}\right)\} < \infty$). This is possible because $\overline{\nu}$ is C^{∞} with compact support, and the function $t \to \frac{\varepsilon'/2L}{1+\Delta(t)}$ is strictly positive, continuous and decreases slowly enough with respect to $e^{-\kappa t^2}$ (namely, $\lim_{t\to\infty} \Delta(t)/e^{\kappa t^2} = 0$). Moreover, since $\overline{\nu}$ has compact support, ν has a holomorphic extension,

$$\widetilde{\nu}\left(z\right) = \frac{1}{a}\int_{\mathbb{R}} \overline{\nu}\left(s\right) e^{-\kappa(z-s)^2} ds,$$

to $\mathbb{C}$. Next observe that for $t, s \in \mathbb{R}$ and $z \in \mathbb{C}$ with $|z| \leq \eta$, we have,

$$\begin{aligned} \mathrm{Re}\left(t+z-s\right)^2 &= \left(t-s\right)^2 + 2\left(t-s\right)\mathrm{Re}\, z + \mathrm{Re}\left(z^2\right) \\ &= \left(t-s+\mathrm{Re}\, z\right)^2 - \left(\mathrm{Re}\, z\right)^2 + \mathrm{Re}\left(z^2\right) \\ &\geq \left(t-s+\mathrm{Re}\, z\right)^2 - 2\eta^2. \end{aligned}$$

Therefore when $|z| < \eta$ we get,

$$
\begin{aligned}
|\widetilde{\nu}(t+z)| &= \frac{1}{a}\left|\int_{\mathbb{R}} \overline{\nu}(s)\, e^{-\kappa(t+z-s)^2} ds\right| \\
&\leq \frac{1}{a}\int_{\mathbb{R}} e^{-\kappa \operatorname{Re}(t+z-s)^2} ds \\
&\leq \frac{1}{a}\int_{\mathbb{R}} e^{-\kappa(t-s+\operatorname{Re} z)^2-2\eta^2} ds \qquad (2.5) \\
&= \frac{e^{2\kappa\eta^2}}{a}\int_{\mathbb{R}} e^{-\kappa(t+\operatorname{Re} z-s)^2} ds \\
&= e^{2\kappa\eta^2},
\end{aligned}
$$

where we have used a variable change to obtain the last line. Now define Lipschitz, analytic functions $\nu_n : X \to \mathbb{R}$ by,

$$\nu_n(x) = \nu(Q(x-x_n)).$$

Clearly ν_n has the holomorphic extension $\widetilde{\nu}_n(z) = \widetilde{\nu}\left(\widetilde{Q}(z-x_n)\right)$. It will be convenient to put $\overline{\nu}_n(x) = \overline{\nu}(Q(x-x_n))$. Observe that, writing $\widehat{D}_n = D_Q(x_n, 6r)$,

$$|\nu_n(x)| < \frac{\varepsilon' r/2LL_\varphi}{1+\Delta(Q(x-x_n))}, \quad \text{for } x \notin \widehat{D}_n, \tag{2.6}$$

and

$$\left|\nu_n(x)'\right| < \frac{\operatorname{Lip}(Q)\,\varepsilon' r/2LL_\varphi}{1+\Delta(Q(x-x_n))}, \quad \text{for } x \notin \widehat{D}_n. \tag{2.7}$$

Note that

$$
\begin{aligned}
\frac{\varepsilon' r/2LL_\varphi}{1+\Delta(Q(x-x_n))} &= \frac{\varepsilon' r/2LL_\varphi}{1+q(x-x_n)^{1/2n}} \qquad (2.8)\\
&\leq \frac{\varepsilon' r/2LL_\varphi}{1+\|x-x_n\|}.
\end{aligned}
$$

Now we estimate $|\widetilde{\nu}_n(x+z)| = \left|\widetilde{\nu}\left(\widetilde{Q}(x-x_n+z)\right)\right|$, for $\|z\|_{\widetilde{X}} < \eta$. From [**AFK1**, Lemma 2], we can write

$$\widetilde{Q}(x-x_n+z) = Q(x-x_n) + Z_n,$$

where $Z_n \in \mathbb{C}$ with $|Z_n| \le C\,\|z\|_{\widetilde{X}}$, for some constant $C > 1$. Then from the calculation (2.5) we get, for $\|z\|_{\widetilde{X}} < \eta$,

$$\begin{aligned}
|\widetilde{\nu}_n (x+z)| &= \left|\widetilde{\nu}\left(\widetilde{Q}(x-x_n+z)\right)\right| \\
&= |\widetilde{\nu}(Q(x-x_n)+Z_n)| \qquad (2.9)\\
&\le e^{2C^2\kappa\eta^2}.
\end{aligned}$$

It is also worthwhile to note that $\nu(t) < 1+\varepsilon'$ for all t.

Let $T_n(x) = f'(x_n)(x-x_n) + f(x_n)$ be the first order Taylor polynomial of f at x_n. Note that $\|T_n'(x)\| = \|f'(x_n)\| \le L$. Observe that $T_n - f$ is Lipschitz on $B_\rho(x_n)$, with $\mathrm{Lip}(T_n - f) \le \left\|(T_n-f)'\right\| = \|f'(x_n) - f'(x)\| \le \varepsilon'$ on $B_\rho(x_n)$. It follows that $T_n - f$ is Lipschitz on $D_n \subset B_\rho(x_n)$ with constant not greater than ε'. Denote by $\overline{T_n - f}$ a bounded and Lipschitz extension of $(T_n - f)\,|_{D_n}$ to all of X, having the same bound and Lipschitz constant. For example, one can take, temporarily writing $h = (T_n - f)\,|_{D_n}$,

$$\left(\overline{T_n - f}\right)(x) = \max\{-\|h\|_\infty, \min\{\|h\|_\infty, \inf_{y\in D_n}\{h(y) + \mathrm{Lip}(h)\|x-y\|\}\}\}.$$

Write $\epsilon_n(x) = \left(\overline{T_n - f}\right)(x)$. We now apply [**AFK1**, Proposition 3] to $\epsilon_n(x)$, along with the standard 'scaling argument' that appears at the very end of the proof of [**AFK1**, Theorem 1], to obtain the following: there exists a constant $C_0 > 1$, depending only on X, a neighbourhood $X \subset \widetilde{W} \subset \widetilde{X}$, where $\widetilde{W} = \widetilde{W}_{\varepsilon',r}$ depends only on ε' and r (the dependence on L_φ written as a dependence on r), and an analytic map $\delta_n : X \to \mathbb{R}$ such that

1. $|\epsilon_n(x) - \delta_n(x)| < \varepsilon' r/L_\varphi$ for all $x \in X$,
2. $\mathrm{Lip}(\delta_n) \le C_0\,\mathrm{Lip}(\epsilon_n) \le C_0\varepsilon'$,
3. the map δ_n extends to a holomorphic map $\widetilde{\delta}_n$ on $\widetilde{W}$ (where in particular, $\widetilde{W}$ is independent of n),
4. $|\widetilde{\delta}_n(x+iy) - \delta_n(x)| \le M_\Delta$ for all $x+iy \in \widetilde{W}$, where M_Δ depends on ε' and is independent of n.

Now we define analytic functions on X by,

$$\psi_n(x) = (T_n(x)\,\nu_n(x) - \delta_n(x))\,\varphi_n(x).$$

Observe that from property (3) of δ_n and Lemma 1, ψ_n extends to a holomorphic map $\widetilde{\psi}_n(z) = \left(\widetilde{T}_n(z)\,\widetilde{\nu}_n(z) - \widetilde{\delta}_n(z)\right)\widetilde{\varphi}_n(z)$, where

$$\widetilde{T}_n(z) = \widetilde{T}_n(x+iy) = \widetilde{f'(x_n)}\,(x+iy-x_n) + f(x_n)$$

($\widetilde{f'(x_n)}$ being the canonical extension of $f'(x_n)$ to all of $\widetilde{X}$), on a neighbourhood $X \subset \widetilde{W} \subset \widetilde{X}$, where $\widetilde{W}$ is independent of n.

Let us define the function $g : X \to \mathbb{R}$ by,

$$g(x) = \frac{\|\{\psi_n(x)\}_{n=1}^{\infty}\|_{c_0}}{\|\{\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0}}$$

We next show that g is analytic. Since the norm $\|\cdot\|_{c_0}$ is real analytic on $c_0 \setminus \{0\}$, it is sufficient to check that the mappings $x \mapsto \{\varphi_n(x)\}_{n=1}^{\infty}$ and $x \mapsto \{\psi_n(x)\}_{n=1}^{\infty}$ are real analytic from X into c_0 and do not take the value $0 \in c_0$. Using Lemma 1 it is easy to show that the function $x \mapsto \{\varphi_n(x)\}_{n=1}^{\infty}$ has such properties (see [**AFK1**, Lemma 4]).

As for the function $x \mapsto \{\psi_n(x)\}_{n=1}^{\infty}$, let us first show that it does not take the value 0. In fact we show that for each $x \in X$ there exists an n so that the number $(T_n(x)\,\nu_n(x) - \delta_n(x))\,\varphi_n(x)$ is bounded above 1/4. Indeed, for each $x \in X$, there is a minimal $n = n_x$ with $x \in D_Q(x_{n_x}, 3r)$, and via the proof of [**AFK1**, Lemma 3 (3)], for such n_x we have $\varphi_{n_x}(x) \geq 1/2$. Note also that $D_Q(x_{n_x}, 3r) \subset D_Q(x_{n_x}, 5r) = D_{n_x}$, and $\epsilon_{n_x}(x) = T_{n_x}(x) - f(x)$ on D_{n_x}. So, from this and property (1) of δ_n, we have $|T_{n_x}(x)\,\overline{\nu}_{n_x}(x) - f(x) - \delta_{n_x}(x)| = |\epsilon_{n_x}(x) - \delta_{n_x}(x)| \leq \varepsilon'$. Now to replace $\overline{\nu}_{n_x}$ with ν_{n_x}, we observe by (2.4) and (2.8),

$$\begin{aligned}
|T_n(x)\,\nu_n(x) - T_n(x)\,\overline{\nu}_n(x)| &= |T_n(x)|\,|\nu_n(x) - \overline{\nu}_n(x)| \\
&\leq (L\,\|x - x_n\| + |f(x_n)|)\ \frac{\varepsilon' r/2LL_\varphi}{1 + \|x - x_n\|} \\
&\leq (L\,\|x - x_n\| + 2)\ \frac{\varepsilon' r/2LL_\varphi}{1 + \|x - x_n\|} \\
&\leq \varepsilon' r/2L_\varphi + \varepsilon' r/LL_\varphi \\
&\leq 3\varepsilon' r/L_\varphi \leq 3\varepsilon'.
\end{aligned} \tag{2.10}$$

Therefore, these estimates give, $|T_{n_x}(x)\,\nu_{n_x}(x) - f(x) - \delta_{n_x}(x)| \leq 4\varepsilon'$, and because $f \geq 1$, we have our desired bound

$$\begin{aligned}
|T_{n_x}(x)\,\nu_{n_x}(x) - \delta_{n_x}(x)|\,\varphi_{n_x}(x) &\geq |T_{n_x}(x)\,\nu_{n_x}(x) - \delta_{n_x}(x)|\,(1/2) \\
&\geq \left(f(x) - 4\varepsilon'\right)(1/2) > 1/4.
\end{aligned}$$

We remark that it follows from this that for any x,

$$\left\|\{\psi_n(x)\}_n\right\|_{c_0} \geq \left\|\{\psi_n(x)\}_n\right\|_{\infty} = \left\|\{(T_n(x)-\delta_n(x))\,\varphi_n(x)\}_n\right\|_{\infty} \geq 1/4. \tag{2.11}$$

Next, to show that the function $x \mapsto \{\psi_n(x)\}_{n=1}^{\infty}$ is real analytic from X into c_0, we shall require that for each x, there exists n_x and $\delta_x \in (0,1)$ so that for $n \geq n_x$ and $\|z\|_{\widetilde{X}} < \delta_x$, we have

$$\frac{\left|\widetilde{T}_n(x+z)\,\widetilde{\nu}_n(x+z) - \widetilde{\delta}_n(x+z)\right|}{M_n} \leq M_x, \tag{2.12}$$

where M_x depends on x, but is independent of n.

Recalling that $\widetilde{T}_n(w) = \widetilde{f'(x_n)}\,(x - x_n + w) + f(x_n)$, and using (2.9) and property (1) and (4) of δ_n, where we may suppose that $x + z \in \widetilde{W}$ when $\|z\|_{\widetilde{X}} < \delta_x < 1$, we obtain,

$$\begin{aligned}
&\left|\widetilde{T}_n(x+z)\,\widetilde{\nu}_n(x+z) - \widetilde{\delta}_n(x+z)\right| \\
&\leq \left|\widetilde{T}_n(x+z)\right|\left|\widetilde{\nu}_n(x+z)\right| + \left|\widetilde{\delta}_n(x+z) - \delta_n(x)\right| + \delta_n(x) \\
&\leq \left|\widetilde{f'(x_n)}\,(x - x_n + z) + f(x_n)\right| e^{2C^2\kappa\delta_x^2} + M_\Delta + \varepsilon' \\
&< \left(L\left(\|x - x_n\| + \|z\|_{\widetilde{X}}\right) + |f(x_n)|\right) e^{2C^2\kappa\delta_x^2} + 2M_\Delta \\
&< \left(L\left(\|x - x_n\| + 1\right) + 2\right) e^{2C^2\kappa} + 2M_\Delta \\
&\leq \left(3L\left(\|x - x_n\| + 1\right)\right) e^{2C^2\kappa} + 2M_\Delta
\end{aligned}$$

Now recalling that $M_n = e^{2C^2\kappa}\left(1 + \|x_n\|\right)$, we see that

$$\begin{aligned}
&\frac{3L\left(\|x - x_n\| + 1\right) e^{2C^2\kappa}}{M_n} \\
&\leq \frac{3L\left(\|x\| + \|x_n\| + 1\right)}{1 + \|x_n\|} \\
&\leq 3L\left(1 + \|x\|\right).
\end{aligned}$$

Putting $M_x = 2M_\Delta + 3L\left(1 + \|x\|\right)$, we have established (2.12).

Now, to show the analyticity of $\{\psi_n(x)\}_{n=1}^{\infty}$, we first note that property (5) of Lemma 1 together with (2.12) yield

$$|\widetilde{\psi}_n(x+z)| = |\widetilde{T}_n(x+z)\widetilde{\nu}_n(x+z) - \widetilde{\delta}_n(x+z)|\,|\widetilde{\varphi}_n(x+z)| \leq \frac{M_x}{n!a_{x,r}^n}$$

whenever $n \geq n_x$ and $\|z\|_{\widetilde{X}} < \delta_x$.

Because the numerical series $\sum_{n=1}^{\infty} M_x/n!a_{x,r}^n$ is convergent, we then have that the series of functions $\sum_{n=1}^{\infty} |\widetilde{\psi}_n(x+z)|$ is uniformly convergent on the ball $B_{\widetilde{X}}(0,\delta_x)$, which clearly implies that the series

$$\sum_{n=1}^{\infty} \widetilde{\psi}_n(z)e_n = \{\widetilde{\psi}_n(z)\}_{n=1}^{\infty}$$

is uniformly convergent for $z \in B_{\widetilde{X}}(x,\delta_x)$. Then it is clear that $\{\widetilde{\psi}_n(z)\}_{n=1}^{\infty}$, being a series of holomorphic mappings which converges uniformly on the ball $B_{\widetilde{X}}(x,\delta_x)$, is a holomorphic mapping on this ball. Since $x \in X$ is arbitrary, this shows that $x \mapsto \{\psi_n(x)\}_{n=1}^{\infty}$ is real analytic.

Now we move on to our final estimates; $|g-f|$ and $\|g'-f'\|$. Fix $x \in X$, and put $\mathcal{N} = \mathcal{N}_x = \{n : x \in D_n\}$. Now we have (using $f \geq 1 > 0$), that

$$\begin{aligned}
|g(x) - f(x)| &= \left| \frac{\|\{\psi_n(x)\}_{n=1}^{\infty}\|_{c_0}}{\|\{\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0}} - f(x) \right| \\
&= \left| \frac{\|\{\psi_n(x)\}_{n=1}^{\infty}\|_{c_0}}{\|\{\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0}} - \frac{\|\{f(x)\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0}}{\|\{\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0}} \right| \\
&= \frac{1}{\|\{\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0}} \|\{(T_n(x)\nu_n(x) - f(x) - \delta_n(x))\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0} \\
&\leq 2\|\{(T_n(x)\nu_n(x) - f(x) - \delta_n(x))\varphi_n(x)\}_{n=1}^{\infty}\|_{c_0},
\end{aligned}$$

the last line by Lemma 1 (3). We proceed in cases.

Case 1: For $n \in \mathcal{N}$, we have $\epsilon_n(x) = T_n(x) - f(x) = T_n(x)\overline{\nu}_n(x) - f(x)$, and so by property (1) of δ_n and Lemma 1 (7), we obtain the estimate, $|T_n(x)\overline{\nu}_n(x) - f(x) - \delta_n(x)|\varphi_n(x) \leq (\varepsilon' r/L_\varphi)3$. Then using (2.10), we have

$$|T_n(x)\nu_n(x) - f(x) - \delta_n(x)|\varphi_n(x) \leq 6r\varepsilon'/L_\varphi. \tag{2.13}$$

Case 2: For $n \notin \mathcal{N}$, recall $\varphi_n(x) \leq \varepsilon_1 \leq \varepsilon' r/25L$.

Now, for n such that $x \in \widehat{D}_n$, we have $Q(x - x_n) < 6r$, and so $\|x - x_n\| \leq (q(x - x_n))^{1/2n} < \left((6r + 1)^{2n} - 1\right)^{1/2n} < 7$, as $r < 1$. Hence, for such n we have,

$$|T_n(x)\nu_n(x)| \leq (L\|x - x_n\| + |f(x_n)|)\nu_n(x) \leq (7L + 2)\left(1 + \varepsilon'\right) \leq 18L. \tag{2.14}$$

On the other hand, for n such that $x \notin \widehat{D}_n$, by (2.6) we obtain,

$$\begin{aligned}|T_n(x)\nu_n(x)| &\leq (L\|x - x_n\| + |f(x_n)|)\frac{\varepsilon' r/2LL_\varphi}{1 + \|x - x_n\|}\\ &\leq (L\|x - x_n\| + 2)\frac{\varepsilon'/2L}{1 + \|x - x_n\|}\\ &\leq \varepsilon'/2 + \varepsilon'/L \leq 2\varepsilon'.\end{aligned}$$

In any event, for all n we have,

$$|T_n(x)\nu_n(x)| \leq 18L. \tag{2.15}$$

Therefore, for $n \notin \mathcal{N}$, using again property (1) of δ_n, we have,

$$\begin{aligned}|T_n(x)\nu_n(x) - f(x) - \delta_n(x)|\varphi_n(x) &\leq (|T_n(x)\nu_n(x)| + |f(x)| + \delta_n(x))\varphi_n(x)\\ &\leq \left(18L + 2 + 2\varepsilon'\right)\left(\varepsilon' r/25L\right) \leq \varepsilon' r.\end{aligned} \tag{2.16}$$

It follows that, $|g(x) - f(x)| \leq 10A_1\varepsilon' r < \varepsilon$.

We now establish some derivative estimates. Fix x and consider the expression

$$(T_n(x)\nu_n(x))' = T_n'(x)\nu_n(x) + T_n(x)\nu_n'(x).$$

From an estimate analogous to (2.15), using (2.4) and (2.7), we have that for all n, $\|T_n(x)\nu_n'(x)\| \leq 9L\mathrm{Lip}(Q)\mathrm{Lip}(\nu)$. Also, $\|T_n'(x)\nu_n(x)\| \leq L(1 + \varepsilon') \leq 2L$. Hence,

$$\left\|(T_n(x)\nu_n(x))'\right\| \leq 2L + 9L\mathrm{Lip}(Q)\mathrm{Lip}(\nu) \leq 11L\mathrm{Lip}(Q)\mathrm{Lip}(\nu).$$

Using this, and property (2) of δ_n, we have,

$$\mathrm{Lip}(T_n\nu_n - f - \delta_n) \leq 11L\mathrm{Lip}(Q)\mathrm{Lip}(\nu) + L + C_0\varepsilon' \leq 13C_0L\mathrm{Lip}(Q)\mathrm{Lip}(\nu). \tag{2.17}$$

Next, for $x \in D_n$, $\overline{\nu}_n(x) = 1$, and again by property (2) of δ_n,

$$\mathrm{Lip}((T_n\overline{\nu}_n - f - \delta_n)|_{D_n}) = \mathrm{Lip}((T_n - f - \delta_n)|_{D_n}) \leq \varepsilon' + C_0\varepsilon' \leq 2C_0\varepsilon'. \tag{2.18}$$

Next we compute, using (2.4) ,

$$\left\|\left(T_n(x)\left(\nu_n-\overline{\nu}_n\right)\right)'\right\| = \left\|T_n'(x)\right\|\left|\nu_n(x)-\overline{\nu}_n(x)\right| + \left|T_n(x)\right|\left\|\nu_n'(x)-\overline{\nu}_n'(x)\right\|$$

$$\leq L\frac{\varepsilon' r/2LL_\varphi}{1+\|x-x_n\|} + \left(L\|x-x_n\|+2\right)\frac{\mathrm{Lip}(Q)\,\varepsilon' r/2LL_\varphi}{1+\|x-x_n\|}$$

$$\leq \varepsilon' r/2 + \mathrm{Lip}(Q)\ \varepsilon' r/2 + \mathrm{Lip}(Q)\ \varepsilon' r/L$$

$$\leq 2\mathrm{Lip}(Q)\,\varepsilon'.$$

It follows from this and (2.18) that,

$$\mathrm{Lip}\left(\left(T_n\nu_n - f - \delta_n\right)|_{D_n}\right) \leq 2C_0\varepsilon' + 2\mathrm{Lip}(Q)\,\varepsilon' r \leq 4C_0\mathrm{Lip}(Q)\,\varepsilon'. \tag{2.19}$$

Finally we turn to $\|g'(x)-f'(x)\|$ with the help of the above estimates. Again fix $x\in X$, and we obtain,

$$\left\|g'(x)-f'(x)\right\|$$

$$=\left(\frac{\left\|\left\{\left(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right)\varphi_n(x)\right\}_{n=1}^\infty\right\|_{c_0}}{\left\|\left\{\varphi_n(x)\right\}_{n=1}^\infty\right\|_{c_0}}\right)'$$

$$=\frac{1}{\left\|\left\{\varphi_n(x)\right\}_{n=1}^\infty\right\|_{c_0}^2}\ \times$$

$$\Big(\left\|\left\{\varphi_n(x)\right\}_{n=1}^\infty\right\|_{c_0}\left\|\left\{\left(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right)\varphi_n(x)\right\}_{n=1}^\infty\right\|_{c_0}'$$

$$-\left\|\left\{\varphi_n(x)\right\}_{n=1}^\infty\right\|_{c_0}'\left\|\left\{\left(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right)\varphi_n(x)\right\}_{n=1}^\infty\right\|_{c_0}\Big)$$

Let us first consider

$$\left(\left(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right)\varphi_n(x)\right)'$$

$$=\left(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right)'\varphi_n(x)+\left(T_n(x)\,\nu_n(x)-f(x)-\delta_n(x)\right)\varphi_n'(x)\,.$$

For the first term, and $n\in\mathcal{N}$, we have, using property (2) of Lemma 1 and (2.19) ,

$$\begin{aligned}\left\|\left(T_n\left(x\right)\nu_n\left(x\right)-f\left(x\right)-\delta_n(x)\right)'\varphi_n\left(x\right)\right\| &\le \left\|\left(T_n\left(x\right)\nu_n\left(x\right)-f\left(x\right)-\delta_n(x)\right)'\right\|\varphi_n\left(x\right)\\ &\le 4C_0\mathrm{Lip}\left(Q\right)\varepsilon'\left(1+\varepsilon_1\right)\\ &\le 8C_0\mathrm{Lip}\left(Q\right)\varepsilon'.\end{aligned}$$

For $n\notin\mathcal{N}$, using Lemma 1 $(4')$ and (2.17), we obtain,

$$\begin{aligned}\left\|\left(T_n\left(x\right)\nu_n\left(x\right)-f\left(x\right)-\delta_n(x)\right)'\varphi_n\left(x\right)\right\| &\le \left\|\left(T_n\left(x\right)\nu_n\left(x\right)-f\left(x\right)-\delta_n(x)\right)'\right\|\varphi_n\left(x\right)\\ &\le 13C_0L\mathrm{Lip}\left(Q\right)\mathrm{Lip}\left(\nu\right)\left(\varepsilon'/6C_0L\mathrm{Lip}\left(\nu\right)\right)\\ &\le 3\mathrm{Lip}\left(Q\right)\varepsilon'.\end{aligned}$$

In any event, $\left\|\left(T_n\left(x\right)\nu_n\left(x\right)-f\left(x\right)-\delta_n(x)\right)'\varphi_n\left(x\right)\right\|_{c_0}\le 8C_0A_1\mathrm{Lip}(Q)\,\varepsilon'$.

Next we consider the second term, $\left(T_n\left(x\right)\nu_n\left(x\right)-f\left(x\right)-\delta_n(x)\right)\varphi_n'\left(x\right)$. For $n\in\mathcal{N}$, from the estimate giving (2.13), we have,

$$\begin{aligned}\left\|\left(T_n\left(x\right)\nu_n\left(x\right)-f\left(x\right)-\delta_n(x)\right)\varphi_n'\left(x\right)\right\| &\le \left|T_n\left(x\right)\nu_n\left(x\right)-f\left(x\right)-\delta_n(x)\right|\left\|\varphi_n'\left(x\right)\right\|\\ &\le 6\varepsilon' r/L_\varphi\left(L_\varphi\right)\le 6\varepsilon'.\end{aligned}$$

For $n\notin\mathcal{N}$, just as in (2.16) we obtain,

$$\begin{aligned}\left\|\left(T_n\left(x\right)\nu_n\left(x\right)-f\left(x\right)-\delta_n(x)\right)\varphi_n'\left(x\right)\right\| &\le \left|T_n\left(x\right)\nu_n\left(x\right)-f\left(x\right)-\delta_n(x)\right|\left\|\varphi_n'\left(x\right)\right\|\\ &\le \varepsilon'.\end{aligned}$$

Hence, altogether we see that,

$$\begin{aligned}\left\|\left(\left(T_n\left(x\right)\nu_n\left(x\right)-f\left(x\right)-\delta_n(x)\right)\varphi_n\left(x\right)\right)'\right\|_{c_0} &\le 8C_0A_1\mathrm{Lip}\left(Q\right)\varepsilon'+6A_1\varepsilon'\\ &\le 14C_0A_1\mathrm{Lip}\left(Q\right)\varepsilon'.\end{aligned}$$

Lastly, we examine $\left\|\left\{\varphi_n\left(x\right)\right\}_{n=1}^{\infty}\right\|_{c_0}'\left\|\left\{\left(T_n\left(x\right)\nu_n\left(x\right)-f\left(x\right)-\delta_n(x)\right)\varphi_n\left(x\right)\right\}_{n=1}^{\infty}\right\|_{c_0}$. Recall our estimate of $|f-g|$ found $\left\|\left\{\left(T_n\left(x\right)\nu_n\left(x\right)-f\left(x\right)-\delta_n(x)\right)\varphi_n\left(x\right)\right\}_{n=1}^{\infty}\right\|_{c_0}\le 3A_1\varepsilon' r$. Therefore we have,

$$\left\|\{\varphi_n(x)\}_{n=1}^{\infty}\right\|_{c_0}' \left\|\{(T_n(x)\,\nu_n(x) - f(x) - \delta_n(x))\,\varphi_n(x)\}_{n=1}^{\infty}\right\|_{c_0}$$

$$\leq A_1 L_{\varphi} \left\|\{(T_n(x)\,\nu_n(x) - f(x) - \delta_n(x))\,\varphi_n(x)\}_{n=1}^{\infty}\right\|_{\infty}$$

$$\leq A_1 \frac{L_1 \mathrm{Lip}(Q)}{r}\left(5A_1\varepsilon' r\right) = 5A_1^2 L_1 \mathrm{Lip}(Q)\,\varepsilon'$$

Finally, because $\left\|\{\varphi_n(x)\}_{n=1}^{\infty}\right\|_{c_0}^2 \geq \left\|\{\varphi_n(x)\}_{n=1}^{\infty}\right\|_{\infty}^2 \geq 1/4$ as noted above, putting all the above estimates together yields,

$$\left\|g'(x) - f'(x)\right\| \leq \frac{(2A_1)\,14C_0 A_1 \mathrm{Lip}(Q)\,\varepsilon' + 5A_1^2 L_1 \mathrm{Lip}(Q)\,\varepsilon'}{1/4}$$

$$\leq \left(132 C_0 A_1^2 L_1 \mathrm{Lip}(Q)\right)\varepsilon' < \varepsilon. \quad \square$$

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