

# A NOTE ABOUT THE SHAPE OF ATTRACTORS OF DISCRETE SEMIDYNAMICAL SYSTEMS

MANUEL A. MORÓN AND FRANCISCO R. RUIZ DEL PORTAL

(Communicated by Alexander N. Dranishnikov)

**ABSTRACT.** We state in a short way a result that improves one of the main theorems in a paper of M. Gobbino concerning the topological properties that the phase space induces in an attractor of a discrete dynamical system.

This note is motivated by a paper of Gobbino ([1]) where it is proved that, with some restrictions, the inclusion of an attractor into the phase space induces isomorphisms between the Čech-Alexander-Spanier cohomology groups for both continuous and discrete dynamical systems in metric spaces. Here we deal just with the discrete case. The aim of this paper is to give a short proof of a stronger result using shape theory. In fact, shape theory allows us to work with more general spaces, without paying attention to the not very easy to handle notion of the attractor to be tautly embedded (see [4]) in the phase space, and to obtain the invariance not only of the Čech-Alexander-Spanier cohomology groups but the Čech cohomology and homology groups (and the corresponding pro-groups that contain much more information). Moreover, in presence of a rest point, the inclusion of an attractor into the phase space is a pointed shape equivalence and also induces isomorphisms between the shape groups and the homotopy pro-groups.

The reader can find information about shape theory and its invariants in the book of S. Mardesić and J. Segal, [3].

Given a map  $f : X \rightarrow X$  and subsets  $A, B \subset X$ , we say that  $A$  attracts  $B$  if for any neighborhood  $U$  of  $A$  in  $X$  there is  $n_0 \in \mathbb{N}$  such that  $f^n(B) \subset U$  for any  $n \geq n_0$ .

Given sets  $C \subset D$ ,  $j_{C,D}$  denotes the inclusion of  $C$  into  $D$ .

**Theorem 1.** *Let  $X$  be a paracompact Hausdorff space. Let  $f : X \rightarrow f(X) \subset X$  be a homeomorphism and let  $K \subset X$  be a compact invariant set. Assume that there exists  $B \subset X$  such that  $K \subset B$  and  $K$  attracts  $B$ . If the inclusions  $j_{B,X} : B \rightarrow X$  and  $j_{f(X),X} : f(X) \rightarrow X$  are homotopy equivalences, then the inclusion  $j_{K,X} : K \rightarrow X$  is a shape equivalence. As a consequence, the inclusion  $j_{K,X}$  induces isomorphisms in the Čech cohomology and homology groups and pro-groups.*

*Proof.* It is enough to show that the map  $j_{K,X}^* : [X, P] \rightarrow [K, P]$  defined as  $j_{K,X}^*([\alpha]) = [\alpha \circ j_{K,X}]$  is a bijection for any CW-complex  $P$  (here  $[\alpha]$  denotes

---

Received by the editors February 24, 2005.

2000 *Mathematics Subject Classification.* Primary 54H20.

*Key words and phrases.* Attractor, discrete (semi)dynamical systems, shape theory.

The authors were supported by MCyT, BFM 2003-0825.

the homotopy class of a map  $\alpha$  and  $[C, D]$  represents the set of homotopy classes of maps  $C \rightarrow D$ .

Let  $[\alpha] \in [K, P]$ . Since  $K$  is compact,  $\alpha(K)$  lies in a finite subcomplex of  $P$ . Using that  $X$  is normal there exists an extension  $\alpha' : U \rightarrow P$  of  $\alpha$  to an open neighborhood  $U$  of  $K$ . Let  $m \in \mathbb{N}$  such that  $f^m(B) \subset U$ .

Since  $j_{f(X), X} : f(X) \rightarrow X$  is a homotopy equivalence and  $f$  is a homeomorphism onto its image, the inclusions  $j_{f^r(X), f^{r-1}(X)}$  are homotopy equivalences for any  $r > 0$ .

Now, we can write  $j_{f^m(B), X} \circ (f^m|_B)$  as the following composition of homotopy equivalences:

$$j_{f^m(B), X} \circ (f^m|_B) : B \xrightarrow{j_{B, X}} X \xrightarrow{f^m} f^m(X) \xrightarrow{j_{f^m(X), X}} X.$$

Therefore,  $j_{f^m(B), X}$  is also a homotopy equivalence. Let  $[g] : X \rightarrow f^m(B)$  be its homotopy inverse.

Consider

$$\beta : X \xrightarrow{g} f^m(B) \xrightarrow{j_{f^m(B), U}} U \xrightarrow{\alpha'} P.$$

It is easy to check that  $j_{K, X}^*([\beta]) = [\alpha]$ . Then,  $j_{K, X}^*$  is onto.

In order to show that  $j_{K, X}^*$  is injective, assume that  $[\alpha_1], [\alpha_2] \in [X, P]$  are such that  $[\alpha_1 \circ j_{K, X}] = [\alpha_2 \circ j_{K, X}]$ . Then, using again the assumption that  $K$  is compact and since  $X \times [0, 1]$  is normal, there is an open neighborhood  $V$  of  $K$  such that  $[\alpha_1 \circ j_{V, X}] = [\alpha_2 \circ j_{V, X}]$ . Let  $n \in \mathbb{N}$  such that  $f^n(B) \subset V$ . Then,  $[\alpha_1 \circ j_{f^n(B), X}] = [\alpha_2 \circ j_{f^n(B), X}]$ . Since  $j_{f^n(B), X}$  is a homotopy equivalence, we have that  $[\alpha_1] = [\alpha_2]$ .  $\square$

Gobbino and Sardella, in [2], gave a nice example of a non-connected attractor in a connected phase space. However, under the hypotheses of Theorem 1, we have the following corollary.

**Corollary 1.** *Under the conditions of Theorem 1,  $K$  is connected iff  $X$  is connected.*

**Corollary 2.** *Under the conditions of Theorem 1, if  $K$  and  $X$  are ANRs the inclusion  $j_{K, X}$  is a homotopy equivalence. Then,  $\mathcal{X}(K) = \mathcal{X}(X)$  (where  $\mathcal{X}(Y)$  denotes the Euler characteristic of  $Y$ ).*

**Corollary 3.** *Let  $X$  be a paracompact Hausdorff space. Let  $f : (X, *) \rightarrow (f(X), *) \subset X$  be a pointed homeomorphism and let  $K \subset X$  be a compact invariant set containing  $*$ . Assume that there exists  $B \subset X$  such that  $K \subset B$  and  $K$  attracts  $B$ . If the inclusions  $j_{B, X} : (B, *) \rightarrow (X, *)$  and  $j_{f(X), X} : (f(X), *) \rightarrow (X, *)$  are pointed homotopy equivalences, then the inclusion  $j_{K, X} : (K, *) \rightarrow (X, *)$  is a pointed shape equivalence. As a consequence, the inclusion  $j_{K, X}$  induces isomorphisms between the shape groups, the homotopy pro-groups and the Čech cohomology and homology groups and pro-groups.*

## REFERENCES

- [1] M. Gobbino, *Topological properties of attractors for dynamical systems*, Topology 40, 2 (2001), 279-298. MR1808221 (2002f:37027)
- [2] M. Gobbino, M. Sardella, *On the connectedness of attractors for dynamical systems*, Journal of Diff. Equations 133 (1997) 1-14. MR1426754 (98e:58113)

SHAPE OF ATTRACTORS

- [3] S. Mardesić, J. Segal, *Shape theory*, North-Holland, Amsterdam, 1982. MR0676973 (84b:55020)
- [4] E. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966. MR0210112 (35:1007)

DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA, FACULTAD DE CIENCIAS MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE DE MADRID, MADRID, 28040, SPAIN  
*E-mail address:* MA.Moron@mat.ucm.es

DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA, FACULTAD DE CIENCIAS MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE DE MADRID, MADRID, 28040, SPAIN  
*E-mail address:* R.Portal@mat.ucm.es