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with terms concentrating on the boundary**

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# Singular limit for a nonlinear parabolic equation with terms concentrating on the boundary \*

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## Abstract

We analyze the asymptotic behavior of the attractors of a parabolic problem when some reaction and potential terms are concentrated in a neighborhood of a portion  $\Gamma$  of the boundary and this neighborhood shrinks to  $\Gamma$  as a parameter  $\varepsilon$  goes to zero.

We prove that the family of attractors is upper continuous at the  $\varepsilon = 0$ .

## 1 Introduction

Let  $\Omega$  be an open bounded smooth set in  $\mathbb{R}^N$  with a  $C^2$  boundary  $\partial\Omega$ . Let  $\Gamma \subset \partial\Omega$  be a smooth subset of the boundary, isolated from the rest of the boundary, that is,  $\text{dist}(\Gamma, \partial\Omega \setminus \Gamma) > 0$ .

Define the strip of width  $\varepsilon$  and base  $\Gamma$  as

$$\omega_\varepsilon = \{x - \sigma\vec{n}(x), x \in \Gamma, \sigma \in [0, \varepsilon]\} \quad (1.1)$$

for sufficiently small  $\varepsilon$ , say  $0 < \varepsilon < \varepsilon_0$ , where  $\vec{n}(x)$  denotes the outward normal vector to  $\Gamma$ . We note that for small  $\varepsilon$ , the set  $\omega_\varepsilon$  is a neighborhood of  $\Gamma$  in  $\overline{\Omega}$ , that collapses to  $\Gamma$  when the parameter  $\varepsilon$  goes to zero.

We are interested in the behavior, for small  $\varepsilon$ , of the solutions of the nonlinear parabolic problem

$$\begin{cases} u_t^\varepsilon - \text{div}(a(x)\nabla u^\varepsilon) &= f(x, u^\varepsilon) + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}g_\varepsilon(x, u^\varepsilon) & \text{in } \Omega \\ a(x)\frac{\partial u^\varepsilon}{\partial \vec{n}} + b(x)u &= 0 & \text{on } \Gamma \\ \mathcal{B}u^\varepsilon &= 0 & \text{on } \partial\Omega \setminus \Gamma \\ u^\varepsilon(0) &= u_0 & \text{in } \Omega \end{cases} \quad (1.2)$$

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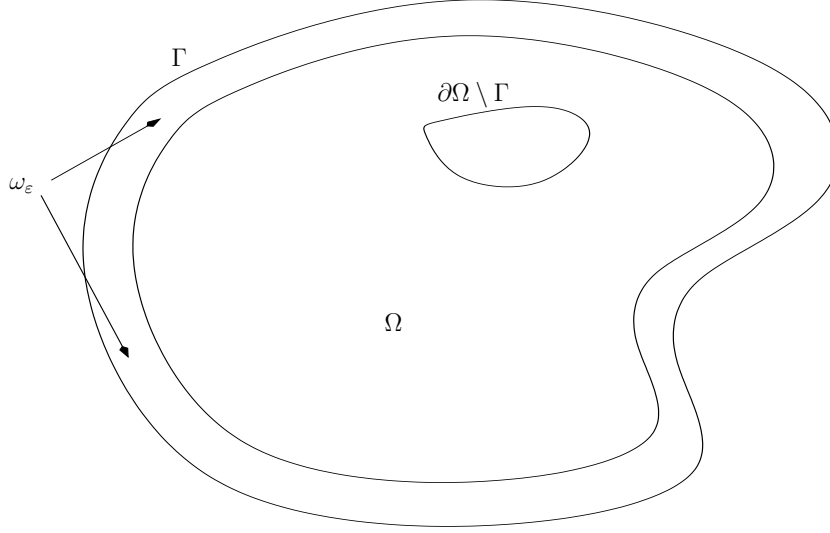


Figure 1: The sets  $\Omega$  and  $\omega_\varepsilon$

where  $a \in C^1(\overline{\Omega})$  with  $a(x) \geq a_0 > 0$  in  $\Omega$  and  $\mathcal{B}$  denotes the boundary operator in  $\partial\Omega \setminus \Gamma$

$$\mathcal{B}u = u, \quad \text{Dirichlet case,} \quad \text{or} \quad \mathcal{B}u = a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u, \quad \text{Robin case,}$$

being  $\vec{n}$  the outward normal vector-field to  $\partial\Omega \setminus \Gamma$  and  $b(x)$  a  $C^1(\partial\Omega)$  function and  $\mathcal{X}_{\omega_\varepsilon}$  denotes the characteristic function of the set  $\omega_\varepsilon$ .

Note that in (1.2) the nonlinear term  $g_\varepsilon(x, u)$  is only effective on the region  $\omega_\varepsilon$  which collapses to  $\Gamma$  as  $\varepsilon \rightarrow 0$ .

We will show in this paper that the “limit problem” for the singularly perturbed problem (1.2) is given by

$$\begin{cases} u_t - \operatorname{div}(a(x)\nabla u) = f(x, u) & \text{in } \Omega \\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u = g_0(x, u) & \text{on } \Gamma \\ \mathcal{B}u = 0 & \text{on } \partial\Omega \setminus \Gamma \\ u(0) = u_0 \end{cases} \quad (1.3)$$

where  $g_0$  is obtained as the limit of the concentrating terms

$$\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} g_\varepsilon(\cdot, u) \rightarrow g_0(\cdot, u)$$

as we now explain. To be more precise, observe that the nonlinear terms in (1.2) may contain zero and first order terms in  $u$ , so they can be written as

$$f(x, u) = h(x) + m(x)u + f_0(x, u) \quad \text{with} \quad f_0(x, 0) = 0, \quad \frac{\partial}{\partial u} f_0(x, 0) = 0 \quad (1.4)$$

and

$$\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} g_\varepsilon(x, u) = \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} \left( h_\varepsilon(x) + V_\varepsilon(x)u + g_\varepsilon^0(x, u) \right) \quad \text{with} \quad g_\varepsilon^0(x, 0) = 0, \quad \frac{\partial}{\partial u} g_\varepsilon^0(x, 0) = 0 \quad (1.5)$$

with certain regularity properties that will be made precise below.

Thus, for small  $\varepsilon$ , the nonhomogeneous terms, the potential functions and the effective reactions are “concentrated” in  $\omega_\varepsilon$ , which collapses to  $\Gamma$ . Note that without loss of generality we can assume that  $g_\varepsilon$  is defined on  $\overline{\Omega} \times \mathbb{R}$ .

Analogously for (1.3) we will assume

$$g_0(x, u) = h_0(x) + V_0(x)u + g_0^0(x, u), \quad x \in \Gamma \quad (1.6)$$

where  $h_0$ ,  $V_0$  and  $g_0^0(x, u)$  are obtained as the limits of the concentrating terms

$$\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} h_\varepsilon \rightarrow h_0, \quad \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} V_\varepsilon \rightarrow V_0,$$

in some sense that we make precise below, while

$$g_\varepsilon^0(x, u) \rightarrow g_0^0(x, u) \quad \text{uniformly in } x \in \Gamma, \text{ for } u \text{ in bounded sets of } \mathbb{R}. \quad (1.7)$$

In order to continue further, we have the following definition.

**Definition 1.1** Consider a family of functions  $J = \{j_\varepsilon\}_\varepsilon$  in  $\Omega$ .

i) The family  $J$  is an “ $L^r$ -concentrated bounded family” near  $\Gamma$  if

$$\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |j_\varepsilon|^r \leq C \quad (1.8)$$

for  $1 \leq r < \infty$ , or

$$\sup_{x \in \omega_\varepsilon} |j_\varepsilon(x)| \leq C \quad (1.9)$$

for the case  $r = \infty$ , and  $C$  a positive constant independent of  $\varepsilon$ .

ii) The family  $J$  is an “ $L^r$ -concentrated convergent family” if it satisfies that for any smooth function  $\varphi$  in  $\overline{\Omega}$ , we have

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\omega_\varepsilon} j_\varepsilon \varphi = \int_\Gamma j_0 \varphi, \quad (1.10)$$

for some  $j_0 \in L^r(\Gamma)$  (or a bounded Radon measure on  $\Gamma$ ,  $j_0 \in \mathcal{M}(\Gamma)$  if  $r = 1$ ). In such a case we write

$$\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} j_\varepsilon \rightarrow j_0 \quad \text{cc} - L^r.$$

iii) The family  $J$  is said to be “ $L^r$ -concentrated (sequentially) compact family” if for any sequence in the family there exist a subsequence (that we still denote the same) and a function  $j_0 \in L^r(\Gamma)$  (or a bounded Radon measure on  $\Gamma$ ,  $j_0 \in \mathcal{M}(\Gamma)$  if  $r = 1$ ) such that for any smooth function  $\varphi$  in  $\overline{\Omega}$ , we have (1.10).

Therefore the results of Lemma 2.2 in [6] can be recast as

**Lemma 1.2** *With the notations above, a “ $L^r$ -concentrated bounded family” is a “ $L^r$ -concentrated (sequentially) compact family”.*

Hence, we will assume that

$$\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} h_\varepsilon \rightarrow h_0, \quad \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} V_\varepsilon \rightarrow V_0, \quad cc - L^r \quad \text{for some } r > N - 1. \quad (1.11)$$

while  $g_\varepsilon^0$  converges to  $g_0^0$  as in (1.7).

Our goal is to prove that under assumptions (1.7) and (1.11), plus some growth and dissipativity conditions on the nonlinear terms, problems (1.2) and (1.3) have globally defined solutions for certain classes of initial data. Moreover, we are going to show that the solutions of both problems have enough compactness so that they are attracted to the global attractors,  $\mathcal{A}_\varepsilon$ ,  $0 \leq \varepsilon \leq \varepsilon_0$  respectively. The global attractor for each problem contains all information about the asymptotic behavior of all solutions.

Furthermore, we are going to show that the asymptotic dynamics of (1.2) and (1.3) are close in the sense that the family of attractors  $\mathcal{A}_\varepsilon$  is upper semicontinuous at  $\varepsilon = 0$ . That is,

$$\text{dist}(\mathcal{A}_\varepsilon, \mathcal{A}_0) := \sup_{u^\varepsilon \in \mathcal{A}_\varepsilon} \inf_{u^0 \in \mathcal{A}_0} \{\|u^\varepsilon - u^0\|\} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0,$$

in a suitable and strong norm which here implies, among others, uniform convergence in  $\overline{\Omega}$  for the functions and convergence of the derivatives in Lebesgue spaces.

Observe that the approach for upper semicontinuity has grounds in, e.g. Section 2.5. in [8]; see also [15] and requires the following ingredients. First, we must prove that all problems have attractors and that they are uniformly bounded with respect to the parameter  $0 \leq \varepsilon \leq \varepsilon_0$ . Then we must prove that the nonlinear semigroups defined by (1.2) converge as  $\varepsilon \rightarrow 0$  to the one defined by (1.3). This in turn, will be obtained from the convergence of solutions for the corresponding linear equations, see [14].

Note that problems with concentrating terms have been considered before. First, linear elliptic problems have been considered in [6] where convergence of solutions and convergence of spectral pairs have been proved. Some related nonlinear problems have been analyzed in [5]. Second, linear parabolic equations have been considered in [14]. All these results are the starting point for the present paper.

The paper is organized as follows. In Section 2 we recall previous results in [14] about linear parabolic equations with concentrated terms. These include results about the setting for the solvability of the linear equations, Theorem 2.2, and results about the convergence of solutions, Theorem 2.3. Section 3 is then devoted to the well posedness of the nonlinear problems (1.2) and (1.3) where, depending on the space for initial data, some growth conditions on the nonlinear terms are imposed, see Theorem 3.2; here the approach is taken from [3]. Also, we impose some sign conditions on the nonlinear terms that imply that the local solutions above are globally defined, see Theorem 3.5. Then, in Section 4 we give some dissipative condition which implies that there are suitable uniform

bounds on the solutions for large times, independent of  $\varepsilon$ . In particular, we obtain the existence of the attractors  $\mathcal{A}_\varepsilon$  and uniform bounds, in strong norms, on them; see Lemma 4.5. Section 5 is somehow independent of (but required for) the rest of the paper and is devoted to analyze how the nonlinear terms  $\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}g_\varepsilon^0(\cdot, u)$  converge to  $g_0^0(\cdot, u)$  under assumption (1.7), when evaluated on families of functions defined in  $\Omega$ ,  $u$ , see Lemma 5.2. With all these, in Section 6 we prove the convergence in finite time of the solutions of (1.2) to the solutions of (1.3), see Lemma 6.1. With this we obtain the upper semicontinuity of attractors in Theorem 6.2. Some earlier and weaker results have been announced in [11].

## 2 Linear problems

In this section we review the functional setting and some results for the linear problems associated to (1.2) and (1.3), namely

$$\begin{cases} u_t^\varepsilon - \operatorname{div}(a(x)\nabla u^\varepsilon) &= m(x)u^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon(x)u^\varepsilon & \text{in } \Omega \\ a(x)\frac{\partial u^\varepsilon}{\partial \bar{n}} + b(x)u^\varepsilon &= m_0(x)u^\varepsilon & \text{on } \Gamma \\ \mathcal{B}u^\varepsilon &= 0 & \text{on } \partial\Omega \setminus \Gamma \\ u^\varepsilon(0) &= u_0 & \text{in } \Omega \end{cases} \quad (2.1)$$

and

$$\begin{cases} u_t - \operatorname{div}(a(x)\nabla u) &= m(x)u & \text{in } \Omega \\ a(x)\frac{\partial u}{\partial \bar{n}} + b(x)u &= (m_0(x) + V_0(x))u & \text{on } \Gamma \\ \mathcal{B}u &= 0 & \text{on } \partial\Omega \setminus \Gamma \\ u(0) &= u_0 & \text{in } \Omega \end{cases} \quad (2.2)$$

with some fixed  $a \in C^1(\bar{\Omega})$ ,  $b \in C^1(\partial\Omega)$  and with  $m \in L^p(\Omega)$ ,  $p > N/2$  and  $m_0, V_0 \in L^r(\Gamma)$ ,  $r > N - 1$  and

$$\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |V_\varepsilon|^r \leq C.$$

The reader is referred to [14] for further details.

For this, denote by  $A_0$  the operator  $A_0u = -\operatorname{div}(a(x)\nabla u)$  with boundary conditions  $a(x)\frac{\partial u}{\partial \bar{n}} + b(x)u = 0$  on  $\Gamma$  and  $\mathcal{B}u = 0$  on  $\partial\Omega \setminus \Gamma$ . Note the coefficients  $a, b$  are  $C^1$ -smooth.

Choosing  $L^q(\Omega)$ , for  $1 < q < \infty$ , as a base space, the unbounded linear operator  $A_0 : D(A_0) \subset L^q(\Omega) \rightarrow L^q(\Omega)$ , with domain  $D(A_0) = H_{bc}^{2,q}(\Omega)$ , consisting of all functions in  $H^{2,q}(\Omega)$  which satisfy all boundary conditions above, generates an analytic semigroup in  $L^q(\Omega)$ , see [2]. Here and below  $H^{s,q}(\Omega)$  denote the Bessel potentials spaces which, for integer  $s$ , coincide with the usual Sobolev spaces.

Using the complex interpolation–extrapolation procedure, one can construct the scale of Banach spaces associated to this operator, which will be denoted  $H_{bc}^{2\alpha,q}(\Omega)$  for  $\alpha \in [-1, 1]$ , which are closed subspaces of  $H^{2\alpha,q}(\Omega)$  incorporating some of the boundary conditions. In particular, we have  $H_{bc}^{0,q}(\Omega) = L^q(\Omega)$ , and

$$H_{bc}^{1,q}(\Omega) = \begin{cases} \{u \in H^{1,q}(\Omega) : u = 0 \text{ in } \partial\Omega \setminus \Gamma\} & \text{for Dirichlet} \\ H^{1,q}(\Omega) & \text{for Robin.} \end{cases}$$

Recall that Bessel spaces have the sharp embeddings

$$H^{s,q}(\Omega) \subset \begin{cases} L^r(\Omega), & s - \frac{N}{q} \geq -\frac{N}{r}, \quad 1 \leq r < \infty, & \text{if } s - \frac{N}{q} < 0 \\ L^r(\Omega), & 1 \leq r < \infty, & \text{if } s - \frac{N}{q} = 0 \\ C^\eta(\bar{\Omega}) & & \text{if } s - \frac{N}{q} > \eta > 0 \end{cases}$$

with continuous embeddings, see [1]. This embeddings are known to be optimal.

Also, if  $\gamma_\Gamma$  denotes the trace operator on  $\Gamma$ , then for  $s > \frac{1}{q}$ ,  $\gamma_\Gamma$  is well defined on  $H^{s,q}(\Omega)$  and

$$H^{s,q}(\Omega) \xrightarrow{\mathfrak{T}} \begin{cases} L^r(\Gamma), & s - \frac{N}{q} \geq -\frac{N-1}{r}, \quad 1 \leq r < \infty, & \text{if } s - \frac{N}{q} < 0 \\ L^r(\Gamma), & 1 \leq r < \infty, & \text{if } s - \frac{N}{q} = 0 \\ C^\eta(\Gamma) & & \text{if } s - \frac{N}{q} > \eta > 0 \end{cases}$$

see [1].

Note that the scale with negative exponents satisfies  $H_{bc}^{-2\alpha,q}(\Omega) = (H_{bc}^{2\alpha,q'}(\Omega))'$ , for  $0 < \alpha < 1$ . Moreover, we have  $H^{-2\alpha,q}(\Omega) = (H^{2\alpha,q'}(\Omega))'$  and  $H^{-2\alpha,q}(\Omega) \hookrightarrow H_{bc}^{-2\alpha,q}(\Omega)$ . See [2] for details.

Note that in this context, the arguments in Lemma 2.1 and 2.2 in [6] prove that

**Lemma 2.1**

i) If  $J = \{j_\varepsilon\}_\varepsilon$  is a “ $L^r$ -concentrated bounded family” near  $\Gamma$ , see (1.8), (1.9), then for any  $s > \frac{1}{\rho'}$  and  $s - \frac{N}{\rho'} \geq -\frac{N-1}{r'}$ ,

$$\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} j_\varepsilon \text{ is bounded in } H^{-s,\rho}(\Omega).$$

ii) If moreover  $J$  is such that such that

$$\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} j_\varepsilon \rightarrow j_0 \quad \text{cc} - L^r$$

then for any  $s > \frac{1}{\rho'}$  and  $s - \frac{N}{\rho'} \geq -\frac{N-1}{r'}$ ,

$$\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} j_\varepsilon \rightarrow j_0 \quad \text{in } H^{-s,\rho}(\Omega).$$

See Lemma 1.2 above.

Also, the operator  $-A_0$  or, more precisely, a suitable realization of it, generates an analytic semigroup,  $S_0(t)$ , in each space of the scale  $H_{bc}^{2\alpha,q}(\Omega)$ ,  $\alpha \in [-1, 1]$ . This semigroup is order preserving and satisfies the smoothing estimates

$$\|S_0(t)u_0\|_{H_{bc}^{2\alpha,q}(\Omega)} \leq \frac{M_{\alpha,\beta} e^{\mu t}}{t^{\alpha-\beta}} \|u_0\|_{H_{bc}^{2\beta,q}(\Omega)}, \quad t > 0, \quad u_0 \in H_{bc}^{2\beta,q}(\Omega)$$

for  $1 \geq \alpha \geq \beta \geq -1$  and some  $\mu \in \mathbb{R}$ . In particular, one has

$$\|S_0(t)u_0\|_{L^\tau(\Omega)} \leq \frac{M_{\rho,\tau} e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{\rho}-\frac{1}{\tau})}} \|u_0\|_{L^\rho(\Omega)}, \quad t > 0, \quad u_0 \in L^\rho(\Omega)$$



for  $1 \leq \rho \leq \tau \leq \infty$ . The reader is referred to [2] and references therein, for further properties of this scale of spaces and semigroups.

In particular, for any  $u_0$  in  $H_{bc}^{2\beta,q}(\Omega)$  or  $L^\rho(\Omega)$ , the function  $u(t; u_0) := S_0(t)u_0$ ,  $t > 0$ , is a classical solution of (2.1) for  $V_\varepsilon = m = m_0 = 0$ .

Now for problems (2.1) and (2.2), the following results have been proved in [14] and will be used in a crucial way in the rest of the paper.

**Theorem 2.2** *Assume that  $m$  lies in a bounded set in  $L^p(\Omega)$ , with  $p > N/2$ ,  $m_0$  lies in a bounded set in  $L^r(\Gamma)$  and also that the family of potentials  $V_\varepsilon$  is a  $L^r$ -concentrated bounded family, for  $r > N - 1$ , that is*

$$\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |V_\varepsilon|^r \leq C, \quad r > N - 1.$$

*Then, for any  $1 < q < \infty$ , the problem (2.1) defines a strongly continuous, order preserving, analytic semigroup,  $S_{m,m_0,\varepsilon}(t)$  in the space  $H_{bc}^{2\gamma,q}(\Omega)$  for any*

$$\gamma \in I(q) := \left(-1 + \frac{1}{2q}, 1 - \frac{1}{2q'}\right).$$

*Moreover the semigroup satisfies the smoothing estimates*

$$\|S_{m,m_0,\varepsilon}(t)u_0\|_{H_{bc}^{2\gamma',q}(\Omega)} \leq \frac{M_{\gamma',\gamma} e^{\mu t}}{t^{\gamma'-\gamma}} \|u_0\|_{H_{bc}^{2\gamma,q}(\Omega)}, \quad t > 0, \quad u_0 \in H_{bc}^{2\gamma,q}(\Omega)$$

*for every  $\gamma, \gamma' \in I(q)$ , with  $\gamma' \geq \gamma$ , for some  $M_{\gamma',\gamma}$  and  $\mu \in \mathbb{R}$  independent of  $m, m_0$  and  $0 < \varepsilon \leq \varepsilon_0$  and  $\gamma, \gamma' \in I(q)$ . In particular, one has*

$$\|S_{m,m_0,\varepsilon}(t)u_0\|_{L^\tau(\Omega)} \leq \frac{M_{\rho,\tau} e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{\rho}-\frac{1}{\tau})}} \|u_0\|_{L^\rho(\Omega)}, \quad t > 0, \quad u_0 \in L^\rho(\Omega)$$

*for  $1 \leq \rho \leq \tau \leq \infty$  with  $M_{\rho,\tau}$  and  $\mu$  independent of  $m, m_0$  and  $0 < \varepsilon \leq \varepsilon_0$ .*

*Finally, for every  $u_0 \in H_{bc}^{2\gamma,q}(\Omega)$ , with  $\gamma \in I(q)$ , the function  $u^\varepsilon(t; u_0) := S_{m,m_0,\varepsilon}(t)u_0$  is in  $C^\nu(\bar{\Omega})$  for any  $0 < \nu < 1$  and is a weak solution of (2.1) in the sense that*

$$\int_{\Omega} u_t^\varepsilon \varphi + \int_{\Omega} a(x) \nabla u^\varepsilon \nabla \varphi + \int_{\Gamma} (b(x) - m_0(x)) u^\varepsilon \varphi = \frac{1}{\varepsilon} \int_{\omega_\varepsilon} V_\varepsilon(x) u^\varepsilon \varphi + \int_{\Omega} m(x) u^\varepsilon \varphi$$

*for all sufficiently smooth  $\varphi$ .*

Note that if  $V_0 \in L^r(\Gamma)$ , for  $r > N - 1$ , with the choice  $V_\varepsilon = 0$  and  $m_0 + V_0$  replacing  $m_0$ , the result above allows to define the semigroup  $S_{m,m_0+V_0}(t)$  such that for every  $u_0 \in H_{bc}^{2\gamma,q}(\Omega)$ , with  $\gamma$  as above, the function  $u(t; u_0) := S_{m,m_0+V_0}(t)u_0$  is a weak solution of (2.2) in the sense that

$$\int_{\Omega} u_t \varphi + \int_{\Omega} a(x) \nabla u \nabla \varphi + \int_{\Gamma} (b(x) - m_0(x)) u \varphi = \int_{\Gamma} V_0(x) u \varphi + \int_{\Omega} m(x) u \varphi$$

for all sufficiently smooth  $\varphi$ . With these notations we have

**Theorem 2.3** Assume that as  $\varepsilon \rightarrow 0$

$$\begin{aligned} m_\varepsilon &\rightarrow m \quad \text{in } L^p(\Omega), \quad p > \frac{N}{2}, \\ m_{0,\varepsilon} &\rightarrow m_0 \quad \text{in } L^r(\Gamma), \quad r > N - 1, \\ \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} V_\varepsilon &\rightarrow V_0, \quad cc - L^r \quad \text{for some } r > N - 1 \end{aligned} \tag{2.3}$$

and for any  $1 < q < \infty$ , consider the semigroups  $S_{m_\varepsilon, m_{0,\varepsilon}, \varepsilon}(t)$  and  $S_{m, m_0 + V_0}(t)$  as above.

Then for every  $\gamma, \gamma' \in I(q) := (-1 + \frac{1}{2q}, 1 - \frac{1}{2q'})$ ,  $\gamma' \geq \gamma$ , and  $T > 0$  there exists  $C(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , such that

$$\|S_{m_\varepsilon, m_{0,\varepsilon}, \varepsilon}(t) - S_{m, m_0 + V_0}(t)\|_{\mathcal{L}(H_{bc}^{2\gamma, q}(\Omega), H_{bc}^{2\gamma', q}(\Omega))} \leq \frac{C(\varepsilon)}{t^{\gamma' - \gamma}}, \quad \text{for all } 0 < t \leq T.$$

In particular, for any  $0 < \nu < 1$  the solutions  $u^\varepsilon(t; u_0) := S_{m_\varepsilon, m_{0,\varepsilon}, \varepsilon}(t)u_0$  of (2.1) converge to solutions  $u(t; u_0) := S_{m, m_0 + V_0}(t)u_0$  of (2.2) in  $C^\nu(\overline{\Omega})$  uniformly on bounded time intervals away from  $t = 0$ .

Finally, about the optimal exponential bound for the semigroups above we have the following

**Proposition 2.4** Assume (2.3) and denote by  $\lambda_1^\varepsilon$  the first eigenvalue of the following eigenvalue problem

$$\begin{cases} -\operatorname{div}(a(x)\nabla\varphi^\varepsilon) = m_\varepsilon(x)\varphi^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}V_\varepsilon(x)\varphi^\varepsilon + \lambda\varphi^\varepsilon & \text{in } \Omega \\ a(x)\frac{\partial\varphi^\varepsilon}{\partial\bar{n}} + b(x)\varphi^\varepsilon = m_{0,\varepsilon}(x)\varphi^\varepsilon & \text{on } \Gamma \\ \mathcal{B}\varphi^\varepsilon = 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases}$$

i) We have that, as  $\varepsilon \rightarrow 0$ ,

$$\lambda_1^\varepsilon \rightarrow \lambda_1^0$$

which is the first eigenvalue of the limit eigenvalue problem

$$\begin{cases} -\operatorname{div}(a(x)\nabla\varphi) = m(x)\varphi + \lambda\varphi & \text{in } \Omega, \\ a(x)\frac{\partial\varphi}{\partial\bar{n}} + b(x)\varphi = (m_0(x) + V_0(x))\varphi & \text{on } \Gamma, \\ \mathcal{B}\varphi = 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases}$$

ii) For sufficiently small  $\varepsilon$  and for any  $-\mu < \lambda_1^0$ , the semigroups  $S_{m_\varepsilon, m_{0,\varepsilon}, \varepsilon}(t)$  and  $S_{m, m_0 + V_0}(t)$  defined above satisfy

$$\begin{aligned} \|S_{m_\varepsilon, m_{0,\varepsilon}, \varepsilon}(t)u_0\|_{H_{bc}^{2\gamma', q}(\Omega)} &\leq \frac{M_{\gamma', \gamma} e^{\mu t}}{t^{\gamma' - \gamma}} \|u_0\|_{H_{bc}^{2\gamma, q}(\Omega)}, \quad t > 0, \quad u_0 \in H_{bc}^{2\gamma, q}(\Omega) \\ \|S_{m, m_0 + V_0}(t)u_0\|_{H_{bc}^{2\gamma', q}(\Omega)} &\leq \frac{M_{\gamma', \gamma} e^{\mu t}}{t^{\gamma' - \gamma}} \|u_0\|_{H_{bc}^{2\gamma, q}(\Omega)}, \quad t > 0, \quad u_0 \in H_{bc}^{2\gamma, q}(\Omega) \end{aligned}$$

for every  $\gamma, \gamma' \in I(q) := (-1 + \frac{1}{2q}, 1 - \frac{1}{2q'})$ , with  $\gamma' \geq \gamma$ , for some  $M_{\gamma', \gamma}$  independent of  $0 < \varepsilon \leq \varepsilon_0$ . In particular,

$$\|S_{m_\varepsilon, m_0, \varepsilon}(t)u_0\|_{L^\tau(\Omega)} \leq \frac{M_{\rho, \tau} e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{\rho} - \frac{1}{\tau})}} \|u_0\|_{L^\rho(\Omega)}, \quad t > 0, \quad u_0 \in L^\rho(\Omega)$$

and

$$\|S_{m, m_0 + V_0}(t)u_0\|_{L^\tau(\Omega)} \leq \frac{M_{\rho, \tau} e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{\rho} - \frac{1}{\tau})}} \|u_0\|_{L^\rho(\Omega)}, \quad t > 0, \quad u_0 \in L^\rho(\Omega)$$

with  $M_{\rho, \tau}$  independent of  $0 < \varepsilon \leq \varepsilon_0$ .

### 3 Well posedness for nonlinear problems

In this section we give some results on the well posedness for both problems (1.2) and (1.3). For these we use the results in [3] adapted to the particularities of problems (1.2) and (1.3) mentioned above. Also note that we will make use of the semigroups described in Section 2 with boundary potential  $m_0 = 0$ .

Hence we consider (1.2) and (1.3) in the space  $X = L^q(\Omega)$  or  $X = H_{bc}^{1,q}(\Omega)$ , for  $1 < q < \infty$ , where

$$H_{bc}^{1,q}(\Omega) = \begin{cases} \{u \in H^{1,q}(\Omega) : u = 0 \text{ in } \partial\Omega \setminus \Gamma\} & \text{for Dirichlet} \\ H^{1,q}(\Omega) & \text{for Robin.} \end{cases}$$

For either choice of  $X$  there exist suitable growth restrictions on the nonlinearities, such that problems (1.2) and (1.3) are locally well posed in  $X$ . For this we consider the following class of nonlinear terms  $\mathcal{N}_X$

**Definition 3.1** *The class  $\mathcal{N}_X$  is formed up with functions  $j(x, u)$  such that*  
*i)  $j(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$  is locally Lipschitz, uniformly on  $x \in \overline{\Omega}$  or  $x \in \Gamma$*   
*ii) If  $X = L^q(\Omega)$ , assume that*

$$|j(x, u) - j(x, v)| \leq c|u - v|(|u|^{\rho-1} + |v|^{\rho-1} + 1), \quad (3.1)$$

*iii) If  $X = H_{bc}^{1,q}(\Omega)$  and*

*a) if  $1 < q < N$ , assume (3.1)*

*b) if  $q = N$  assume that for every  $\eta > 0$ , there exists  $c_\eta > 0$  such that*

$$|j(x, u) - j(x, v)| \leq c_\eta (e^{\eta|u|^{\frac{N}{N-1}}} + e^{\eta|v|^{\frac{N}{N-1}}})|u - v|, \quad (3.2)$$

*c) if  $q > N$ , no further conditions are assumed.*

Then the techniques from [3] applied here give the following result.

**Theorem 3.2** Assume the nonlinear terms  $f(x, u)$ ,  $g_\varepsilon(x, u)$  and  $g_0(x, u)$  satisfy (1.4), (1.5) and (1.6) respectively such that for every fixed  $0 < \varepsilon \leq \varepsilon_0$  we have  $h, \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} h_\varepsilon \in L^\infty(\Omega)$ ,  $m, \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} V_\varepsilon \in L^p(\Omega)$  for some  $p > N/2$ , and for  $\varepsilon = 0$ ,  $h_0 \in L^\infty(\Gamma)$  and  $V_0 \in L^r(\Gamma)$ , for some  $r > N - 1$ .

Also, assume  $X = L^q(\Omega)$  or  $X = H_{bc}^{1,q}(\Omega)$ , with

$$f_0, g_\varepsilon^0, g_0^0 \in \mathcal{N}_X.$$

Moreover assume that either

- i) For (1.2), with fixed  $0 < \varepsilon \leq \varepsilon_0$ ,
- a) if  $X = L^q(\Omega)$  the exponents  $\rho_{f_0}$  and  $\rho_{g_\varepsilon^0}$  in (3.1), are such that

$$\rho_{f_0}, \rho_{g_\varepsilon^0} \leq \rho_\Omega := 1 + \frac{2q}{N},$$

- b) if  $X = H_{bc}^{1,q}(\Omega)$  exponents  $\rho_{f_0}$  and  $\rho_{g_\varepsilon^0}$  in (3.1), are such that

$$\rho_{f_0}, \rho_{g_\varepsilon^0} \leq \rho_\Omega := 1 + \frac{2q}{N - q}.$$

- ii) For (1.3)

- a) if  $X = L^q(\Omega)$  the exponents  $\rho_{f_0}$  and  $\rho_{g_0^0}$  in (3.1), are such that with  $N \geq 2$  (respectively  $N = 1$ )

$$\rho_{f_0} \leq \rho_\Omega := 1 + \frac{2q}{N}, \quad \text{and} \quad \rho_{g_0^0} \leq \rho_\Gamma := 1 + \frac{q}{N}, \quad (\text{respectively, } \rho_{g_0^0} < \rho_\Gamma := 1 + q),$$

- b) if  $X = H_{bc}^{1,q}(\Omega)$  exponents  $\rho_{f_0}$  and  $\rho_{g_0^0}$  in (3.1), are such that

$$\rho_{f_0} \leq \rho_\Omega := 1 + \frac{2q}{N - q} \quad \text{and} \quad \rho_{g_0^0} \leq \rho_\Gamma := 1 + \frac{q}{N - q}.$$

Then for any  $u_0 \in X$  there exists a unique (in certain sense) mild solution  $u(\cdot, u_0) \in C([0, \tau], X)$ , of problems (1.2) or (1.3), respectively, satisfying  $u(0, u_0) = u_0$  in  $X$ . This solution depends continuously on the initial data  $u_0 \in X$ .

**Remark 3.3** Observe that adding a term  $\lambda u$  to both left and right hand sides of (1.2) and (1.3) and, with the notations of Section 2, considering the semigroups  $S_{m,\varepsilon}(t)e^{-\lambda t}$  and  $S_{m,V_0}(t)e^{-\lambda t}$ , which correspond to the case  $m_0 = 0$  in Section 2, the solutions of (1.2) and (1.3) in Theorem 3.2 satisfy the modified variation of constants formula

$$u^\varepsilon(t) = S_{m,\varepsilon}(t - t_0)e^{-\lambda(t-t_0)}u^\varepsilon(t_0) + \int_{t_0}^t S_{m,\varepsilon}(t - s)e^{-\lambda(t-s)}H_\varepsilon(u^\varepsilon(s)) ds \quad (3.3)$$

and

$$u(t) = S_{m,V_0}(t - t_0)e^{-\lambda(t-t_0)}u(t_0) + \int_{t_0}^t S_{m,V_0}(t - s)e^{-\lambda(t-s)}H_0(u(s)) ds \quad (3.4)$$

for  $t \geq t_0 \geq 0$  respectively, where  $\lambda \in \mathbb{R}$  is arbitrary and the nonlinear terms are given by

$$H_\varepsilon(u) = h + f_0(\cdot, u) + \lambda u + \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} h_\varepsilon + \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} g_\varepsilon^0(\cdot, u) \quad (3.5)$$

and

$$H_0(u) = (h + f_0(\cdot, u) + \lambda u)_\Omega + (h_0 + g_0^0(\cdot, u))_\Gamma \quad (3.6)$$

respectively. Note that the latter must be understood in the sense that

$$\langle H_0(u), \varphi \rangle = \int_\Omega (h + f_0(\cdot, u) + \lambda u) \varphi + \int_\Gamma (h_0 + g_0^0(\cdot, u)) \varphi$$

for suitable smooth  $u$  and  $\varphi$ ; see [3].

In order to ensure that the local solutions constructed above are globally defined, following [4], we will assume the following sign conditions on the nonlinear terms

**Sign conditions (S)** Assume in addition that there exist  $C \in L^p(\Omega)$ ,  $0 \leq D \in L^p(\Omega)$  with  $p > \frac{N}{2}$

$$uf(x, u) \leq C(x)u^2 + D(x)|u|, \quad x \in \Omega, \quad u \in \mathbb{R}, \quad (3.7)$$

and either

i) for (1.2), with fixed  $0 < \varepsilon \leq \varepsilon_0$ , there exist  $E_\varepsilon \in L^p(\Omega)$ ,  $0 \leq F_\varepsilon \in L^p(\Omega)$ ,  $p > \frac{N}{2}$  such that

$$ug_\varepsilon(x, u) \leq E_\varepsilon(x)u^2 + F_\varepsilon(x)|u|, \quad x \in \omega_\varepsilon, \quad u \in \mathbb{R}, \quad (3.8)$$

ii) for (1.3), there exist  $E_0 \in L^r(\Gamma)$ ,  $0 \leq F_0 \in L^r(\Gamma)$ ,  $r > N - 1$  such that

$$ug_0(x, u) \leq E_0(x)u^2 + F_0(x)|u|, \quad x \in \Gamma, \quad u \in \mathbb{R}. \quad (3.9)$$

**Remark 3.4** Observe that comparing (1.4) with (3.7), (1.5) with (3.8) and (1.6) with (3.9), we get

$$|h(x)| \leq D(x), \quad |h_\varepsilon(x)| \leq F_\varepsilon(x), \quad |h_0(x)| \leq F_0(x).$$

Then we have, see [4, Theorem 2.2] and also [13, Theorems 2.5 and 2.6].

**Theorem 3.5** Under the sign assumptions (S) above, the local solutions in Theorem 3.2 are defined for all  $t \geq 0$  and each solution is bounded in  $L^\infty(\Omega)$  and in  $X$  on bounded time intervals away from  $t = 0$ .

In particular (1.2) and (1.3) define nonlinear semigroups

$$T_\varepsilon(t)u_0 = u^\varepsilon(t; u_0), \quad 0 \leq \varepsilon \leq \varepsilon_0, \quad u_0 \in X,$$

for either  $X = L^q(\Omega)$  or  $X = H_{bc}^{1,q}(\Omega)$ .

**Proof.**

**Step 1.** We first prove the  $L^\infty(\Omega)$  bounds on the solutions. For fixed  $0 < \varepsilon \leq \varepsilon_0$ , let  $U^\varepsilon(t, |u_0|)$  be the solution of

$$\begin{cases} U_t^\varepsilon - \operatorname{div}(a(x)\nabla U^\varepsilon) = C(x)U^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}E_\varepsilon(x)U^\varepsilon + D(x) + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}F_\varepsilon(x) & \text{in } \Omega, \\ a(x)\frac{\partial U^\varepsilon}{\partial \bar{n}} + b(x)U^\varepsilon = 0 & \text{on } \Gamma, \\ \mathcal{B}U^\varepsilon = 0 & \text{on } \partial\Omega \setminus \Gamma \\ U^\varepsilon(0) = |u_0| & \text{in } \Omega. \end{cases} \quad (3.10)$$

Then, since  $D, F_\varepsilon \geq 0$ , we have that  $U^\varepsilon(t, |u_0|) \geq 0$ . Also, from (3.7) and (3.8) by comparison, we have  $u(t, u_0) \leq U^\varepsilon(t, |u_0|)$ , for as long as  $u(t, u_0)$  exists. Proceeding similarly we obtain that  $u(t, u_0) \geq -U^\varepsilon(t, |u_0|)$  for as long as  $u(t, u_0)$  exists. Consequently

$$|u^\varepsilon(t, u_0)| \leq U^\varepsilon(t, |u_0|), \quad (3.11)$$

for as long as  $u(t, u_0)$  exists.

Now observe that the variations of constants formula for (3.10) gives

$$U^\varepsilon(t) = S_\varepsilon(t)|u_0| + \int_0^t S_\varepsilon(t-s)H_\varepsilon ds$$

where we denote temporarily  $H_\varepsilon = D + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}F_\varepsilon$  which is in  $L^p(\Omega)$  for  $p > N/2$ , by assumption and  $S_\varepsilon(t)$  is the semigroup in Theorem 2.2, for the choice  $m = C$ ,  $m_0 = 0$ ,  $V_\varepsilon = E_\varepsilon$ .

Taking now  $L^\infty(\Omega)$  norms we have that, using the estimates in Theorem 2.2, for all  $0 < t \leq T$ ,

$$\|U^\varepsilon(t)\|_{L^\infty(\Omega)} \leq C_\varepsilon(T)t^{-\frac{N}{2q}}\|u_0\|_{L^q(\Omega)} + C_\varepsilon(T)\int_0^t(t-s)^{-\frac{N}{2p}}\|H_\varepsilon\|_{L^p(\Omega)} ds. \quad (3.12)$$

Now, the right hand side term is bounded for  $t$  in compact intervals bounded away from 0 (the integral term is convergent since  $p > N/2$ ). From here, the  $L^\infty(\Omega)$  bound in  $[\delta, T]$  follows.

On the other hand, for  $\varepsilon = 0$ , let  $U(t, |u_0|)$  be the solution of

$$\begin{cases} U_t - \operatorname{div}(a(x)\nabla U) = C(x)U + D(x) & \text{in } \Omega, \\ a(x)\frac{\partial U}{\partial \bar{n}} + b(x)U = E_0(x)U + F_0(x) & \text{on } \Gamma, \\ \mathcal{B}U = 0 & \text{on } \partial\Omega \setminus \Gamma, \\ U(0) = |u_0| & \text{in } \Omega \end{cases} \quad (3.13)$$

Then, since  $D, F_0 \geq 0$ , we have that  $U(t, |u_0|) \geq 0$ . Also, from (3.7) and (3.9) by comparison, we have  $u(t, u_0) \leq U(t, |u_0|)$ , for as long as  $u(t, u_0)$  exists. Proceeding similarly we obtain that  $u(t, u_0) \geq -U(t, |u_0|)$  for as long as  $u(t, u_0)$  exists. Consequently

$$|u(t, u_0)| \leq U(t, |u_0|), \quad (3.14)$$

for as long as  $u(t, u_0)$  exists.

Now observe that the variations of constants formula for (3.13) gives

$$U(t) = S(t)|u_0| + \int_0^t S(t-s)H_0 ds$$

where we denote temporarily  $H_0 = D_\Omega + (F_0)_\Gamma$ , in the sense that, for smooth enough test functions,

$$\langle H_0, \varphi \rangle = \int_\Omega D\varphi + \int_\Gamma F_0\varphi.$$

Note that here  $S(t)$  is the semigroup in Theorem 2.2 for the choice  $m = C$ ,  $m_0 = E_0$ ,  $V_\varepsilon = 0$ .

Now observe that for the term  $S(t)|u_0| + \int_0^t S(t-s)D ds$  the argument runs as in (3.12). On the other hand, note that as  $F_0 \in L^r(\Gamma)$  for  $r > N - 1$ , then using a test function  $\varphi \in H_{bc}^{2\gamma, r'}(\Omega)$  with  $2\gamma > \frac{1}{r'}$  we obtain that  $F_0 \in H_{bc}^{-2\gamma, r}(\Omega)$ . Hence, using the estimates in Theorem 2.2 we get for  $0 \leq t \leq T$ ,

$$\left\| \int_0^t S(t-s)(F_0)_\Gamma ds \right\|_{H_{bc}^{2\gamma', r}(\Omega)} \leq C(T) \int_0^t (t-s)^{-(\gamma'+\gamma)} \|F_0\|_{H_{bc}^{-2\gamma, r}(\Omega)} ds \quad (3.15)$$

and the right hand side is bounded in  $[0, T]$  provided  $\gamma' + \gamma < 1$ . Since  $r > N - 1$  and taking  $\gamma$  close to  $\frac{1}{2r}$ , the sharp embeddings of Bessel spaces in Section 2 imply that there exist  $\gamma'$  as above such that  $H_{bc}^{2\gamma', r}(\Omega) \subset L^\infty(\Omega)$ .

**Step 2.** Now we prove solutions are global. Using the bounds in Step 1 observe that in the variations of constants formula (3.3) and (3.4) with  $\lambda = 0$ , we have that, for  $t > t_0 > 0$ , (3.5) is bounded in  $\Omega$  while in (3.6) the parts in  $\Omega$  and  $\Gamma$  are both bounded. Therefore, on finite time intervals away from  $t = 0$ , (3.5) is bounded in  $L^\rho(\Omega)$  for any  $1 < \rho < \infty$  while (3.6) is bounded in  $H_{bc}^{-2\gamma, \rho}(\Omega)$  for  $2\gamma > \frac{1}{\rho}$  for any  $1 < \rho < \infty$ .

Then, using the estimates in Theorem 2.2, from (3.3) we get for  $0 < t_0 \leq t \leq T$ ,

$$\|u^\varepsilon(t)\|_{H_{bc}^{2\gamma', \rho}(\Omega)} \leq C_\varepsilon(T)(t-t_0)^{-\gamma'} + C_\varepsilon(T) \int_{t_0}^t (t-s)^{-\gamma'} ds, \quad (3.16)$$

for any  $1 < \rho < \infty$  and  $0 \leq \gamma' < 1$ , while from (3.4) we get

$$\|u(t)\|_{H_{bc}^{2\gamma', \rho}(\Omega)} \leq C_0(T)(t-t_0)^{-\gamma'} + C_0(T) \int_{t_0}^t (t-s)^{-(\gamma'+\gamma)} ds \quad (3.17)$$

provided  $1 < \rho < \infty$  and  $\gamma' + \gamma < 1$ , that is for  $\gamma' < 1 - \gamma < 1 - \frac{1}{2\rho} = \frac{1}{2} + \frac{1}{2\rho}$ .

Hence, we obtain bounds in  $H_{bc}^{2\gamma', \rho}(\Omega)$  on finite time intervals away from zero. In particular we can take  $\rho = q$  and  $\gamma' > 1/2$  and then the solutions are global. ■

### Remark 3.6

*i) Observe that in the proof above if the semigroups  $S_\varepsilon(t)$  and  $S(t)$  decay exponentially then the  $L^\infty(\Omega)$  bounds in (3.12) and (3.15) can be obtained for all  $t > 0$  and uniformly for  $u_0$  such that  $\|u_0\|_{L^q(\Omega)} \leq M$ .*

With this, using (3.3) and (3.4) with  $\lambda$  large enough such that the semigroups  $S_{m,\varepsilon}(t)e^{-\lambda t}$  and  $S_{m,V_0}(t)e^{-\lambda t}$  decay exponentially, the bounds (3.16) and (3.17) can also be obtained for all  $t > 0$  and uniformly for  $u_0$  such that  $\|u_0\|_{L^q(\Omega)} \leq M$ .

In such a case the results in [8] would imply that, for each fixed  $0 \leq \varepsilon \leq \varepsilon_0$ , (1.2) and (1.3) have attractors  $\mathcal{A}_\varepsilon$  and  $\mathcal{A}_0$  respectively in  $X$ .

ii) On the other hand, with the argument above we could obtain bounds independent of  $\varepsilon$  in (3.12) and (3.16) if the constants and the exponential bounds of the semigroups  $S_\varepsilon(t)$  and  $S_{m,\varepsilon}(t)$  are independent of  $\varepsilon$  and if  $\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}F_\varepsilon$  was a bounded family in  $L^p(\Omega)$  for  $p > N/2$ .

The first of this conditions can be guaranteed by Theorem 2.2 and Proposition 2.4 provided

$$\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |E_\varepsilon|^r, \quad \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |V_\varepsilon|^r \leq C,$$

and

$$\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} E_\varepsilon \rightarrow E, \quad \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} V_\varepsilon \rightarrow V_0, \quad cc - L^r \quad \text{for some } r > N - 1.$$

However the second condition would read

$$\frac{1}{\varepsilon^p} \int_{\omega_\varepsilon} |F_\varepsilon|^p \leq C$$

which is far more restrictive than actually needed, since conditions of the type (1.8) are much weaker. Note that the conditions of the type (1.8) can only give uniform bounds in  $L^1(\Omega)$  which are not enough to obtain  $L^\infty(\Omega)$  estimates on the solutions. To see this note that

$$\left| \frac{1}{\varepsilon} \int_{\omega_\varepsilon} F_\varepsilon \right| \leq \frac{1}{\varepsilon} \left( \int_{\omega_\varepsilon} |F_\varepsilon|^r \right)^{1/r} |\omega_\varepsilon|^{1/r'} = \left( \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |F_\varepsilon|^r \right)^{1/r} \left( \frac{|\omega_\varepsilon|}{\varepsilon} \right)^{1/r'} \leq C$$

since  $|\omega_\varepsilon| = O(\varepsilon)$ , while if  $1 \leq p < r$

$$\left\| \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} F_\varepsilon \right\|_{L^p(\Omega)}^p = \frac{1}{\varepsilon^p} \int_{\omega_\varepsilon} |F_\varepsilon|^p \leq \frac{1}{\varepsilon^{p-1}} \left( \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |F_\varepsilon|^r \right)^{p/r} \left( \frac{|\omega_\varepsilon|}{\varepsilon} \right)^{1-p/r}$$

would not be bounded if  $p > 1$ .

In the next section we will address the question of obtaining asymptotic bounds (i.e. for  $t \rightarrow \infty$ ) which are independent of  $\varepsilon$ .

## 4 Existence of attractors and uniform bounds

In this section we give conditions that allow to prove that the nonlinear semigroups defined by problems (1.2) and (1.3) in Theorem 3.5 have global attractors  $\mathcal{A}_\varepsilon$  and  $\mathcal{A}_0$  respectively and to obtain suitable uniform bounds on  $\mathcal{A}_\varepsilon$  independent of  $\varepsilon$ .

For this we will assume that in (3.8)

$$\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |E_\varepsilon|^r \leq C, \quad r > N - 1 \tag{4.1}$$



and

$$\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |F_\varepsilon|^r \leq C, \quad r > N - 1. \quad (4.2)$$

Observe that by Lemma 1.2 we may assume without loss of generality that

$$\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} E_\varepsilon \rightarrow E \quad \text{cc} - L^r, \quad r > N - 1, \quad (4.3)$$

and

$$\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} F_\varepsilon \rightarrow F \quad \text{cc} - L^r, \quad r > N - 1 \quad (4.4)$$

as in Definition 1.1. See also Lemma 2.1.

Therefore we will also assume the following dissipativity condition.

**Dissipative condition (D)** There exists  $\delta > 0$  such that the first eigenvalue,  $\lambda_1$ , of the following problem

$$\begin{cases} -\operatorname{div}(a(x)\nabla\varphi) = C(x)\varphi + \lambda\varphi & \text{in } \Omega \\ a(x)\frac{\partial\varphi}{\partial\tilde{n}} + b(x)\varphi = \tilde{E}(x)\varphi & \text{on } \Gamma \\ \mathcal{B}\varphi = 0 & \text{on } \partial\Omega \setminus \Gamma \end{cases} \quad (4.5)$$

satisfies

$$\lambda_1 > \delta > 0 \quad (4.6)$$

for  $\tilde{E} = E$  as in (4.3) and  $\tilde{E} = E_0$  in (3.9).

**Lemma 4.1** *Assume the sign conditions (3.7), (3.8) and (3.9), the concentrated bounds (4.1), (4.2) and (4.3), (4.4) and the dissipativity condition (4.6).*

*Then there exist a constant  $K_\infty$  and a function  $R_\infty(M, t)$ , for  $M, t > 0$ , independent of  $\varepsilon$  such that for each fixed  $M > 0$ ,  $R_\infty(M, t)$ , is monotonically decreasing and converges to zero, as  $t \rightarrow \infty$  and such that for sufficiently small  $0 \leq \varepsilon \leq \varepsilon_0$ , the global solutions of problems (1.2) and (1.3) in Theorem 3.5, satisfy that for initial data such that  $\|u_0\|_{L^q(\Omega)} \leq M$*

$$\sup_{0 \leq \varepsilon \leq \varepsilon_0} \sup_{\|u_0\|_{L^q(\Omega)} \leq M} \|u^\varepsilon(t, \cdot; u_0)\|_{L^\infty(\Omega)} \leq K_\infty + R_\infty(M, t).$$

*In particular, for any  $M > 0$ ,*

$$\limsup_{t \rightarrow \infty} \sup_{0 \leq \varepsilon \leq \varepsilon_0} \sup_{\|u_0\|_{L^q(\Omega)} \leq M} \|u^\varepsilon(t, \cdot; u_0)\|_{L^\infty(\Omega)} \leq K_\infty.$$

**Proof.** In this proof we keep the notations in the proof of Theorem 3.5. We start with the case  $\varepsilon = 0$ , that is, for problem (1.3) and we follow the argument in Proposition 3.2 in [4]; see also Theorem 3.15 in [13].

Since condition (D) holds for  $\tilde{E} = E_0$  in (3.9), see (4.5), (4.6), consider  $0 \leq \Phi^0(x)$  the unique solution of the following problem

$$\begin{cases} -\operatorname{div}(a(x)\nabla\Phi^0) = C(x)\Phi^0 + D(x) & \text{in } \Omega \\ a(x)\frac{\partial\Phi^0}{\partial\tilde{n}} + b(x)\Phi^0 = E_0(x)\Phi^0 + F_0(x) & \text{on } \Gamma \\ \mathcal{B}\Phi^0 = 0 & \text{on } \partial\Omega \setminus \Gamma \end{cases} \quad (4.7)$$

which is the unique stationary point of (3.13). Thus, since  $C, D \in L^p(\Omega)$  with  $p > N/2$ ,  $E_0, F_0 \in L^r(\Gamma)$  with  $r > N - 1$ , then elliptic regularity implies  $\Phi^0 \in L^\infty(\Omega)$ ; see e.g. [6, 14].

Note that condition (D) implies that  $S(t)$  decays exponentially, that is the estimates in Proposition 2.4 hold for  $\mu = -\delta$ . Then in (3.14) we have

$$U(t, |u_0|) = S(t)(|u_0| - \Phi^0) + \Phi^0.$$

Then estimates in Proposition 2.4 applied to the semigroup  $S(t)$  imply that

$$\|U(t, |u_0|)\|_{L^\infty(\Omega)} \leq \frac{K e^{-\delta t}}{t^{\frac{N}{2q}}} \| |u_0| - \Phi^0 \|_{L^q(\Omega)} + \|\Phi^0\|_{L^\infty(\Omega)}$$

and the result follows for  $\varepsilon = 0$ .

Now for  $0 < \varepsilon \leq \varepsilon_0$ , that is, for problem (1.2), we denote by  $\lambda_1^\varepsilon$  the first eigenvalue of the following eigenvalue problem

$$\begin{cases} -\operatorname{div}(a(x)\nabla\varphi^\varepsilon) &= C(x)\varphi^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}E_\varepsilon(x)\varphi^\varepsilon + \lambda\varphi^\varepsilon & \text{in } \Omega \\ a(x)\frac{\partial\varphi^\varepsilon}{\partial\bar{n}} + b(x)\varphi^\varepsilon &= 0 & \text{on } \Gamma \\ \mathcal{B}\varphi^\varepsilon &= 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases}$$

Since (4.3) holds, by the spectral convergence obtained in [6], we have  $\lambda_1^\varepsilon \rightarrow \lambda_1$  with  $\lambda_1$  the first eigenvalue of the elliptic limit problem (4.5) with  $\tilde{E} = E$ ; see Proposition 2.4. From condition (4.6), we get that for small enough  $\varepsilon_0$  we have  $\lambda_1^\varepsilon > \delta$  for every  $0 < \varepsilon \leq \varepsilon_0$ .

Therefore, since  $C, D \in L^p(\Omega)$  with  $p > N/2$ , and for each fixed  $\varepsilon$  we have  $\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}E_\varepsilon, \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}F_\varepsilon \in L^r(\Omega)$  with  $r > N - 1 \geq N/2$ , there exists a unique solution  $0 \leq \Phi^\varepsilon \in L^\infty(\Omega)$  of the elliptic problem

$$\begin{cases} -\operatorname{div}(a(x)\nabla\Phi^\varepsilon) &= C(x)\Phi^\varepsilon + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}E_\varepsilon(x)\Phi^\varepsilon + D(x) + \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}F_\varepsilon(x) & \text{in } \Omega \\ a(x)\frac{\partial\Phi^\varepsilon}{\partial\bar{n}} + b(x)\Phi^\varepsilon &= 0 & \text{on } \Gamma \\ \mathcal{B}\Phi^\varepsilon &= 0 & \text{on } \partial\Omega \setminus \Gamma \end{cases} \quad (4.8)$$

which is the unique stationary solution of (3.10).

Thus, as before, we have that in (3.11)

$$U^\varepsilon(t, |u_0|) = S_\varepsilon(t)(|u_0| - \Phi^\varepsilon) + \Phi^\varepsilon.$$

Now from condition (D), (4.6) and part ii) in Proposition 2.4, we have that

$$\|U^\varepsilon(t, |u_0|)\|_{L^\infty(\Omega)} \leq \frac{K e^{-\delta t}}{t^{\frac{N}{2q}}} \| |u_0| - \Phi^\varepsilon \|_{L^q(\Omega)} + \|\Phi^\varepsilon\|_{L^\infty(\Omega)}.$$

for some  $K > 0$  independent of  $\varepsilon$ .

Now, since (4.3) and (4.4) hold, the convergence results for elliptic problems in [6], we have that  $\Phi^\varepsilon(x) \rightarrow \Phi(x)$ , as  $\varepsilon \rightarrow 0$ , in  $\mathcal{C}^\beta(\bar{\Omega})$ , for some  $\beta > 0$ , where  $\Phi$  solves

$$\begin{cases} -\operatorname{div}(a(x)\nabla\Phi) &= C(x)\Phi + D(x) & \text{in } \Omega \\ a(x)\frac{\partial\Phi}{\partial\bar{n}} + b(x)\Phi &= E(x)\Phi + F(x) & \text{on } \Gamma \\ \mathcal{B}\Phi &= 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases}$$

In particular,

$$\|\Phi^\varepsilon\|_{L^\infty(\Omega)} \leq C$$

for some  $C$  independent of  $\varepsilon$  and we get the result. ■

**Remark 4.2** *Observe that in the Lemma we have  $R_\infty(M, t) \rightarrow \infty$  as  $t \rightarrow 0^+$ . Also observe that for either  $X = L^q(\Omega)$  or  $X = H_{bc}^{1,q}(\Omega)$  the bounds on the solutions in Lemma 4.1 are uniform for bounded sets in  $L^q(\Omega)$  of initial data.*

In particular, we get

**Corollary 4.3** *With the notations of the Lemma 4.1 and with the functions defined in (4.7) and (4.8), we have for all  $0 \leq \varepsilon \leq \varepsilon_0$*

$$\limsup_{t \rightarrow \infty} |u^\varepsilon(t, x; u_0)| \leq \Phi^\varepsilon(x)$$

*uniformly in  $x \in \bar{\Omega}$  and for any initial data such that  $\|u_0\|_{L^q(\Omega)} \leq M$ .*

*Also, if  $|u_0(x)| \leq \Phi^\varepsilon(x)$  for all  $x \in \Omega$  then*

$$|u^\varepsilon(t, x; u_0)| \leq \Phi^\varepsilon(x)$$

*for all  $0 \leq \varepsilon \leq \varepsilon_0$ ,  $x \in \Omega$  and  $t > 0$ .*

**Remark 4.4** *The uniform bound in  $L^\infty(\Omega)$  on  $\Phi^\varepsilon$  does not follow from the arguments in Theorem 4.5 in [4]. In fact this would require uniform bounds of  $\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}E_\varepsilon, \frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}F_\varepsilon$  in some  $L^p(\Omega)$  with  $p > N/2$ , but a bound like (1.8), for  $1 < r < \infty$ , only gives uniform bounds in  $L^1(\Omega)$ ; see Remark 3.6. Instead the argument above relies on the sharp results in [6] that allow to conclude that the concentrating terms near the boundary actually behave as boundary terms. This explains why (1.8) for  $r > N - 1$  suffices.*

With this and the smoothing effect of the equations we get

**Lemma 4.5** *Under the assumptions in Lemma 4.1 assume moreover that*

$$\sup_{x \in \omega_\varepsilon} |h_\varepsilon(x)| \leq C, \quad \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |V_\varepsilon|^r \leq C, \quad r > N - 1$$

*and  $\{g_\varepsilon^0(x, u)\}_\varepsilon$  is uniformly bounded in  $\bar{\Omega}$  on bounded sets of  $\mathbb{R}$ , i.e. for any  $R > 0$  there exists a positive constant  $C(R)$  independent of  $\varepsilon$  such that*

$$|g_\varepsilon^0(x, u)| \leq C(R), \quad \text{for all } x \in \bar{\Omega}, \quad \text{and } |u| \leq R.$$

*Then, for any  $1 < \rho < \infty$  and  $\gamma' < \frac{1}{2} + \frac{1}{2\rho}$  there exists a constant  $K_{\rho, \gamma'}$  and a function  $R_{\rho, \gamma'}(M, t)$ , for  $M, t > 0$ , independent of  $\varepsilon$  such that for each fixed  $M > 0$ ,  $R_{\rho, \gamma'}(M, t)$ , is monotonically decreasing and converges to zero, as  $t \rightarrow \infty$  and such that for sufficiently*

small  $0 \leq \varepsilon \leq \varepsilon_0$ , the global solutions of problems (1.2) and (1.3) in Theorem 3.5, satisfy that for initial data such that  $\|u_0\|_{L^q(\Omega)} \leq M$

$$\sup_{0 \leq \varepsilon \leq \varepsilon_0} \sup_{\|u_0\|_{L^q(\Omega)} \leq M} \|u^\varepsilon(t, \cdot; u_0)\|_{H_{bc}^{2\gamma', \rho}(\Omega)} \leq K_{\rho, \gamma'} + R_{\rho, \gamma'}(M, t).$$

In particular,

$$\limsup_{t \rightarrow \infty} \sup_{0 \leq \varepsilon \leq \varepsilon_0} \sup_{\|u_0\|_{L^q(\Omega)} \leq M} \|u^\varepsilon(t, \cdot; u_0)\|_{H_{bc}^{2\gamma', \rho}(\Omega)} \leq K_{\rho, \gamma'}.$$

Therefore, the global semigroups defined by problems (1.2) and (1.3) in Theorem 3.5, have global attractors  $\mathcal{A}_\varepsilon$  in  $X$  which satisfy

$$\sup_{0 \leq \varepsilon \leq \varepsilon_0} \sup_{v \in \mathcal{A}_\varepsilon} \|vx1\|_{H_{bc}^{2\gamma', \rho}(\Omega)} \leq K_{\rho, \gamma'}.$$

In particular the attractors are uniformly bounded in  $H_{bc}^{1, \rho}(\Omega)$  and  $C^\nu(\overline{\Omega})$  for any  $1 < \rho < \infty$  and for any  $0 < \nu < 1$  and for every function  $u^\varepsilon \in \mathcal{A}_\varepsilon$  we have, for  $0 \leq \varepsilon \leq \varepsilon_0$

$$|u^\varepsilon(x)| \leq \Phi^\varepsilon(x)$$

for all  $x \in \Omega$ .

**Proof.** We start with  $0 < \varepsilon \leq \varepsilon_0$ . From the assumptions on  $V_\varepsilon$  of this lemma and Proposition 2.4, we can choose  $\lambda$  large enough such that in the variations of constants formula (3.3),  $S_{m, \varepsilon}(t)e^{-\lambda t}$  decays exponentially and independent of  $\varepsilon$ .

Now, using Lemma 4.1, we use the variations of constants formula (3.3), for  $t \geq t_0 > 0$ . Then, using that  $\|u^\varepsilon(s; u_0)\|_{L^\infty(\Omega)} \leq K_\infty + R_\infty(M, t_0)$  for  $s \geq t_0$  and for any initial data such that  $\|u_0\|_{L^q(\Omega)} \leq M$ , in (3.5) we have that  $h, f_0^0(\cdot, u^\varepsilon)$  are uniformly bounded in  $L^\infty(\Omega)$ , while for  $t \geq t_0$

$$\sup_{x \in \omega_\varepsilon} |g_\varepsilon^0(x, u^\varepsilon(t, x))| \leq C.$$

These combined with the assumption on  $h_\varepsilon$  and Lemma 2.1, with  $r = \infty$ , gives that for any  $1 < \rho < \infty$ ,  $2\gamma > \frac{1}{\rho'}$  and  $2\gamma - \frac{N}{\rho'} \geq -N + 1$ , the nonlinear term  $H_\varepsilon(u^\varepsilon(s; u_0))$  in (3.5) is uniformly bounded in  $H_{bc}^{-2\gamma, \rho}(\Omega)$ , for any  $s \geq t_0$  and for any initial data such that  $\|u_0\|_{L^q(\Omega)} \leq M$ .

Thus part ii) in Proposition 2.4, gives that, for  $\gamma' + \gamma < 1$  and some  $\mu > 0$

$$\|u^\varepsilon(t; u_0)\|_{H_{bc}^{2\gamma', \rho}(\Omega)} \leq (K_\infty + R_\infty(M, t_0))Ke^{-\mu(t-t_0)}(t-t_0)^{-\gamma'} + KC \int_{t_0}^t e^{-\mu(t-s)}(t-s)^{-(\gamma'+\gamma)} ds.$$

Note that the second condition on  $\gamma$  above reads  $2\gamma > \frac{1}{\rho'} - \frac{N-1}{\rho}$  and therefore the estimates above hold for any  $2\gamma > \frac{1}{\rho'}$  and  $\gamma' < 1 - \gamma < 1 - \frac{1}{2\rho'} = \frac{1}{2} + \frac{1}{2\rho}$ .

In particular, starting with  $\rho = q$  we get bounds in  $H_{bc}^{2\gamma', q}(\Omega)$  for some  $\gamma' > 1/2$  and then the results in [8] imply the existence of the attractor. The rest is immediate.

For  $\varepsilon = 0$  we use the same argument on the variations of constants formula (3.4) using similar bounds now on the nonlinear term in (3.6). ■

**Remark 4.6** *i) Again in the Lemma we have  $R_{\rho,\gamma'}(M,t) \rightarrow \infty$  as  $t \rightarrow 0^+$ . Also observe that for either  $X = L^q(\Omega)$  or  $X = H_{bc}^{1,q}(\Omega)$  the bounds on the solutions in Lemma 4.1 are uniform for bounded sets in  $L^q(\Omega)$  of initial data.*

*ii) As a consequence of the uniform estimates above, for any  $M > 0$ ,  $t_0 > 0$  and  $\|u_0\|_{L^q(\Omega)} \leq M$  the set*

$$\{T_\varepsilon(t)u_0, t \geq t_0, \|u_0\|_{L^q(\Omega)} \leq M\} = \{T_\varepsilon(s)T_\varepsilon(t_0)u_0, s \geq 0, \|u_0\|_{L^q(\Omega)} \leq M\}$$

*is bounded in  $L^\infty(\Omega)$  and in  $H_{bc}^{2\gamma',\rho}(\Omega)$  for any  $1 < \rho < \infty$  and  $\gamma' < \frac{1}{2} + \frac{1}{2\rho}$ .*

*In particular, once we have fixed such a family of initial data, we can assume that the nonlinear terms are globally Lipschitz and the semigroups  $T_\varepsilon(t)$  and  $T_0(t)$  are defined on  $L^\rho(\Omega)$  for any  $1 < \rho < \infty$ . In particular, the attractors  $\mathcal{A}_\varepsilon$  attract solutions in the norm of  $H_{bc}^{2\gamma',\rho}(\Omega)$  for any  $1 < \rho < \infty$  and  $\gamma' < \frac{1}{2} + \frac{1}{2\rho}$ .*

Now since the nonlinear semigroups  $T_\varepsilon(t)$  and  $T_0(t)$  are order preserving and the estimates above, from Theorem 3.2 in [13], see also [7], we get the existence of extremal equilibria for problems (1.2) and (1.3) which are the caps of the attractors

**Proposition 4.7** *Under the above notations and hypotheses, for each  $0 \leq \varepsilon \leq \varepsilon_0$ , there exists two ordered extremal equilibria  $\varphi_m^\varepsilon \leq \varphi_M^\varepsilon$  such that  $\mathcal{A}_\varepsilon \subset [\varphi_m^\varepsilon, \varphi_M^\varepsilon]$ ,  $\varphi_m^\varepsilon, \varphi_M^\varepsilon \in \mathcal{A}_\varepsilon$  and*

$$\varphi_m^\varepsilon \leq \liminf_{t \rightarrow \infty} u^\varepsilon(t, x; u_0) \leq \limsup_{t \rightarrow \infty} u^\varepsilon(t, x; u_0) \leq \varphi_M^\varepsilon$$

*uniformly in  $x \in \Omega$  and for initial data  $u_0$  such that  $\|u_0\|_{L^q(\Omega)} \leq M$ .*

## 5 Concentrated nonlinear terms

Observe that in [6] we obtained several results that allow to pass to the limit in linear elliptic problems with term that concentrate near the boundary  $\Gamma$ . In the present paper we need to pass to the limit in nonlinear terms, Therefore, in this section, we prove two technical results that will allow to pass to the limit in nonlinear terms which are concentrating near the boundary as  $\varepsilon \rightarrow 0$ . Hence, in a sense, this section is independent of, but needed for, the rest of the paper.

We note that the region  $\omega_\varepsilon$  in (1.1) can be written as  $\omega_\varepsilon = \cup_{0 \leq \delta < \varepsilon} \Gamma_\delta$  where

$$\Gamma_\delta = \{x - \delta \vec{n}(x), x \in \Gamma\}$$

where  $\vec{n}(x)$  denotes the outward normal unit at  $x \in \Gamma$  and  $0 \leq \delta < \varepsilon$ . Note that  $\Gamma_0 = \Gamma$ .

Observe that for sufficiently small  $\varepsilon_0$  and for  $0 < \delta < \varepsilon_0$ , denoting  $\Omega_\delta = \Omega \setminus \overline{\omega_\delta}$ , then we can construct a  $C^2$  diffeomorphism  $\tau_\delta : \overline{\Omega} \rightarrow \overline{\Omega_\delta}$  of the form

$$\tau_\delta(x) = \begin{cases} x & \text{if } \text{dist}(x, \Gamma) \geq \varepsilon_0 \\ z - \psi_\delta(\sigma) \vec{n}(z) & \text{if } x = z - \sigma \vec{n}(z), \sigma \in [0, \varepsilon_0) \end{cases}$$

with an increasing  $C^2$  function  $\psi_\delta : [0, \varepsilon_0] \rightarrow [\delta, \varepsilon_0]$  such that  $\psi_\delta(\varepsilon_0) = \varepsilon_0$ ,  $\psi_\delta(0) = \delta$ . With this construction we also have

$$\|\tau_\delta - I\|_{C^2(\overline{\Omega})} \rightarrow 0, \quad \text{as } \delta \rightarrow 0, \quad (5.1)$$

and also  $\tau_\delta$  is a  $C^2$  diffeomorphism between  $\Gamma$  and  $\Gamma_\delta$  and  $\tau_\delta(x) = x - \delta\vec{n}(x)$  for  $x \in \Gamma$ ; see [5, 6].

In particular, for any  $H$  defined on  $\omega_\varepsilon$  and for  $\varepsilon < \varepsilon_0$ , we have

$$\int_{\omega_\varepsilon} H dx = \int_0^\varepsilon \int_{\Gamma_\delta} H dS_\delta d\delta, \quad (5.2)$$

and

$$\int_{\Gamma_\delta} H dS_\delta = \int_\Gamma H(\tau_\delta(x)) J(\tau_\delta(x)) dS_0(x), \quad (5.3)$$

where  $dS_\delta$  is the surface measure associated to  $\Gamma_\delta$  and  $J(\tau_\delta(x)) := J(x, \delta)$  is the surface Jacobian of the transformation  $\tau_\delta$ . Note that in particular there exists constants  $0 < J_1 \leq J_2$  such that for all  $x \in \Gamma$  and for all  $\delta \in [0, \varepsilon_0]$

$$J_1 \leq J(x, \delta) \leq J_2 \quad \text{and} \quad \|J_\delta - 1\|_{L^\infty(\Gamma)} \rightarrow 0 \quad \text{as } \delta \rightarrow 0. \quad (5.4)$$

Then we have the following result.

**Lemma 5.1** *With the notations above, if  $\varepsilon_0 > 0$  is sufficiently small and  $0 \leq \delta < \varepsilon_0$ , then for any  $1 < q \leq \infty$ , there exists a positive constant  $M$  independent of  $\delta$  such that for every  $\varphi \in H^{1,q}(\Omega)$  we have*

$$\|\varphi(\tau_\delta) - \varphi\|_{L^q(\Gamma)} \leq M\delta^{1-\frac{1}{q}} \|\nabla\varphi\|_{L^q(\omega_\delta)}$$

*In particular, if  $\varphi \in H^{1,r}(\Omega)$  and  $1 < q \leq r$  (or  $q = 1 < r$ ) we have that*

$$\|\varphi(\tau_\delta) - \varphi\|_{L^q(\Gamma)} \leq MC(\Gamma)\delta^{1-\frac{1}{r}} \|\nabla\varphi\|_{L^r(\omega_\delta)}.$$

**Proof.** First, by density we can assume that  $\varphi \in C^1(\overline{\Omega})$ . In such a case, we consider the function  $\phi(t, x) = \varphi(x - t\delta\vec{n}(x))$  with  $t \in [0, 1]$  and  $x \in \Gamma$ . Then, for  $x \in \Gamma$  and  $1 < q < \infty$  we have

$$|\varphi(\tau_\delta(x)) - \varphi(x)|^q = |\phi(1) - \phi(0)|^q = \left| \int_0^1 \phi'(t) dt \right|^q,$$

and

$$\left| \int_0^1 \phi'(t) dt \right|^q \leq \int_0^1 |\nabla\varphi(x - t\delta\vec{n}(x))|^q |\delta\vec{n}(x)|^q dt \leq \delta^q \int_0^1 |\nabla\varphi(x - t\delta\vec{n}(x))|^q dt.$$

Thus, we have that

$$\int_\Gamma |\varphi(\tau_\delta(x)) - \varphi(x)|^q dS_0(x) \leq \delta^q \int_0^1 \int_\Gamma |\nabla\varphi(x - t\delta\vec{n}(x))|^q dS_0(x) dt. \quad (5.5)$$

Therefore, using (5.4) we get from (5.5)

$$\|\varphi(\tau_\delta) - \varphi\|_{L^q(\Gamma)}^q \leq \frac{\delta^q}{J_1^q} \int_0^1 \int_\Gamma |\nabla \varphi(x - t\delta \vec{n}(x))|^q |J(x, t\delta)|^q dS_0(x) dt$$

which, by (5.3), leads to

$$\|\varphi(\tau_\delta) - \varphi\|_{L^q(\Gamma)}^q \leq \frac{\delta^q}{J_1^q} \int_0^1 \int_{\Gamma_{t\delta}} |\nabla \varphi(\tau_{t\delta}(x))|^q dS_{t\delta}(x) dt.$$

Now, the change of variables  $s = t\delta$  shaves off a  $\delta$  and (5.2) gives

$$\|\varphi(\tau_\delta) - \varphi\|_{L^q(\Gamma)}^q \leq \frac{\delta^{q-1}}{J_1^q} \int_0^\delta \int_{\Gamma_s} |\nabla \varphi(\tau_s(x))|^q dS_s(x) ds = \frac{\delta^{q-1}}{J_1^q} \|\nabla \varphi\|_{L^q(\omega_\delta)}^q.$$

The case  $q = \infty$  follows along the same lines as above and the rest follows from Holder's inequality and the fact that  $|\omega_\delta| \leq C(\Gamma)\delta$ . ■

Now, we consider a family of functions

$$g_\varepsilon^0 : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R},$$

for  $0 \leq \varepsilon \leq \varepsilon_0$ , satisfying the following conditions

i)  $\{g_\varepsilon^0(x, u)\}_\varepsilon$  is uniformly bounded in  $\overline{\Omega}$  on bounded sets of  $\mathbb{R}$ , i.e. for any  $R > 0$  there exists a positive constant  $C(R)$  independent of  $\varepsilon$  such that

$$|g_\varepsilon^0(x, u)| \leq C(R), \quad \text{for all } x \in \overline{\Omega}, \quad \text{and } |u| \leq R. \quad (5.6)$$

ii)  $\{g_\varepsilon^0(x, u)\}_\varepsilon$  is uniformly continuous in  $\overline{\Omega}$ , uniformly on bounded sets of  $\mathbb{R}$  and also uniformly Lipschitz on bounded sets of  $\mathbb{R}$ , i.e. for any  $R > 0$  there exists a positive constant  $L(R)$  independent of  $\varepsilon$  such that

$$|g_\varepsilon^0(x, u) - g_\varepsilon^0(x, v)| \leq L(R)|u - v|, \quad \text{for all } x \in \overline{\Omega}, \quad |u| \leq R, \quad |v| \leq R. \quad (5.7)$$

iii)  $g_\varepsilon^0(x, u)$  converges to  $g_0^0(x, u)$  uniformly on  $\Gamma$  and on bounded sets of  $\mathbb{R}$ , i.e. for any  $R > 0$

$$g_\varepsilon^0(x, u) \rightarrow g_0^0(x, u) \quad \text{as } \varepsilon \rightarrow 0, \quad \text{uniformly on } x \in \Gamma \text{ and } |u| \leq R \quad (5.8)$$

Then we have the following result. Note that here  $p$  and  $q$  are not meant to be the same as in previous Sections. Also, the result below applies in the case  $g_\varepsilon^0 = g_0^0$ , that is, when the family does not depend on  $\varepsilon$ .

**Lemma 5.2** *Consider a family of functions*

$$g_\varepsilon^0 : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$$

for  $0 \leq \varepsilon \leq \varepsilon_0$ . Also, consider a family of functions,  $\mathcal{C}$ , in  $\Omega$  such that, for some  $1 < p < \infty$  and  $R > 0$

$$\|v\|_{H^{1,p}(\Omega) \cap L^\infty(\Omega)} \leq R \quad \text{for all } v \in \mathcal{C}. \quad (5.9)$$

i) If  $\{g_\varepsilon^0\}_\varepsilon$  satisfies (5.6), then there exists a positive constant,  $M(R)$ , independent of  $\varepsilon$  such that for every  $1 < q < \infty$  and any  $\varphi \in H^{s,q'}(\Omega)$  with  $s > \frac{1}{q'}$  and every  $v \in \mathcal{C}$  we have

$$\left| \frac{1}{\varepsilon} \int_{\omega_\varepsilon} g_\varepsilon^0(\cdot, v) \varphi \right| \leq M(R) \|\varphi\|_{H^{s,q'}(\Omega)}. \quad (5.10)$$

In particular

$$\sup_{v \in \mathcal{C}} \left\| \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} g_\varepsilon^0(\cdot, v) \right\|_{H^{-s,q}(\Omega)} \leq M(R).$$

ii) If  $\{g_\varepsilon^0\}_\varepsilon$  satisfies (5.6), (5.7) and (5.8), then there exists  $M(\varepsilon, R)$  with  $M(\varepsilon, R) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  such that for every  $\varphi \in H^{1,q'}(\Omega)$  and  $v \in \mathcal{C}$

$$\left| \frac{1}{\varepsilon} \int_{\omega_\varepsilon} g_\varepsilon^0(\cdot, v) \varphi - \int_\Gamma g_0^0(\cdot, v) \varphi \right| \leq M(\varepsilon, R) \|\varphi\|_{H^{1,q'}(\Omega)}, \quad (5.11)$$

provided

$$p \geq \frac{q(N-1)}{N}. \quad (5.12)$$

In particular

$$\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} g_\varepsilon^0(\cdot, v) \rightarrow g_0^0(\cdot, v) \quad \text{in } H^{-1,q}(\Omega), \text{ uniformly in } v \in \mathcal{C}. \quad (5.13)$$

**Proof.** i) First, given  $s > \frac{1}{q'}$  there exists  $r' \geq 1$  such that  $s - \frac{N}{q'} \geq -\frac{N-1}{r'}$ . From Sobolev embeddings we have  $H^{s,q'}(\Omega) \subset L^{r'}(\Gamma)$  and from Lemma 2.1 in [6] we have that, for some constant independent of  $\varepsilon$ ,

$$\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |\varphi|^{r'} \leq C \|\varphi\|_{H^{s,q'}(\Omega)}^{r'}.$$

Now, we consider  $r$  such that  $\frac{1}{r} + \frac{1}{r'} = 1$ . Then (5.6), the  $L^\infty(\Omega)$  bound on  $v \in \mathcal{C}$  and using  $|\omega_\varepsilon| \leq C(\Gamma)\varepsilon$  we get  $\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |g_\varepsilon^0(\cdot, v)|^r \leq C(R)$  with  $C(R)$  independent of  $\varepsilon$ . Hence, for all  $v \in \mathcal{C}$

$$\left| \frac{1}{\varepsilon} \int_{\omega_\varepsilon} g_\varepsilon^0(\cdot, v) \varphi \right| \leq \left[ \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |g_\varepsilon^0(\cdot, v)|^r \right]^{\frac{1}{r}} \left[ \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |\varphi|^{r'} \right]^{\frac{1}{r'}} \leq M(R) \|\varphi\|_{H^{s,q'}(\Omega)}$$

and we get (5.10).

ii) Observe that using (5.2), for all  $v \in \mathcal{C}$

$$\begin{aligned} \left| \frac{1}{\varepsilon} \int_{\omega_\varepsilon} g_\varepsilon^0(\cdot, v) \varphi - \int_\Gamma g_0^0(\cdot, v) \varphi \right| &= \left| \frac{1}{\varepsilon} \int_0^\varepsilon \int_{\Gamma_\delta} g_\varepsilon^0(\cdot, v) \varphi - \int_\Gamma g_0^0(\cdot, v) \varphi \right| \leq \\ &\leq \sup_{\delta \in [0, \varepsilon]} \left| \int_{\Gamma_\delta} g_\varepsilon^0(\cdot, v) \varphi - \int_\Gamma g_0^0(\cdot, v) \varphi \right| = \sup_{\delta \in [0, \varepsilon]} I(\delta) \end{aligned}$$



where, using (5.3),

$$I(\delta) = \left| \int_{\Gamma_\delta} g_\varepsilon^0(\cdot, v)\varphi - \int_\Gamma g_0^0(\cdot, v)\varphi \right| \leq \int_\Gamma |g_\varepsilon^0(\tau_\delta, v(\tau_\delta))\varphi(\tau_\delta)J(\tau_\delta) - g_0^0(\cdot, v)\varphi| dS_0.$$

Adding and subtracting  $g_\varepsilon^0(\tau_\delta, v(\tau_\delta))\varphi J(\tau_\delta)$ ,  $g_\varepsilon^0(\tau_\delta, v(\tau_\delta))\varphi$ ,  $g_\varepsilon^0(\tau_\delta, v)\varphi$  and  $g_\varepsilon^0(\cdot, v)\varphi$  in the expression above, we have that

$$I(\delta) \leq I_1 + I_2 + I_3 + I_4 + I_5$$

with

$$I_1 \equiv \int_\Gamma |g_\varepsilon^0(\tau_\delta, v(\tau_\delta))J(\tau_\delta)[\varphi(\tau_\delta) - \varphi]| dS_0,$$

$$I_2 \equiv \int_\Gamma |g_\varepsilon^0(\tau_\delta, v(\tau_\delta))\|\varphi\||J(\tau_\delta) - 1| dS_0,$$

$$I_3 \equiv \int_\Gamma |g_\varepsilon^0(\tau_\delta, v(\tau_\delta)) - g_\varepsilon^0(\tau_\delta, v)|\|\varphi\| dS_0,$$

$$I_4 \equiv \int_\Gamma |g_\varepsilon^0(\tau_\delta, v) - g_\varepsilon^0(\cdot, v)|\|\varphi\| dS_0,$$

and

$$I_5 \equiv \int_\Gamma |g_\varepsilon^0(\cdot, v) - g_0^0(\cdot, v)|\|\varphi\| dS_0.$$

Then we now prove that there exist  $M_i(\varepsilon, R)$  such that, for every  $v \in \mathcal{C}$ ,  $I_i \leq M_i(\varepsilon, R)\|\varphi\|_{H^1(\Omega)}$  with  $i = 1, 2, 3, 4, 5$  for every  $0 \leq \delta \leq \varepsilon$ , and with  $M_i(\varepsilon, R) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

**Step 1.** Observe that

$$I_1 \leq \|g_\varepsilon^0(\tau_\delta, v(\tau_\delta))\|_{L^\infty(\Gamma)} J_2 \int_\Gamma |\varphi(\tau_\delta) - \varphi|$$

and using (5.6) and Lemma 5.1, we get for every  $0 \leq \delta \leq \varepsilon$  and every  $v \in \mathcal{C}$

$$I_1 \leq M_1(\varepsilon, R)\|\varphi\|_{H^1, q'(\Omega)}$$

with  $M_1(\varepsilon, R) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

**Step 2.** Now we have that,

$$I_2 \leq \int_\Gamma |g_\varepsilon^0(\tau_\delta, v(\tau_\delta))\|\varphi\||J(\tau_\delta) - 1| \leq \|J_\delta - 1\|_{L^\infty(\Gamma)} \|g_\varepsilon^0(\tau_\delta, v(\tau_\delta))\|_{L^\infty(\Gamma)} \|\varphi\|_{L^1(\Gamma)}$$

with  $\|\varphi\|_{L^1(\Gamma)} \leq C(\Gamma, \Omega)\|\varphi\|_{H^1, q'(\Omega)}$ . Then (5.6) and (5.4) implies, for every  $0 \leq \delta \leq \varepsilon$  and  $v \in \mathcal{C}$

$$I_2 \leq M_2(\varepsilon, R)\|\varphi\|_{H^1, q'(\Omega)}$$

with  $M_2(\varepsilon, R) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

**Step 3.** Choose again  $r'$  such that  $H^{1,q'}(\Omega) \subset L^{r'}(\Gamma)$ , i.e.  $r' \leq \frac{q'(N-1)}{N-q'}$ , and then with  $r$  such that  $\frac{1}{r} + \frac{1}{r'} = 1$  we have  $r \geq \frac{q(N-1)}{N}$  and

$$I_3 \leq \|g_\varepsilon^0(\tau_\delta, v(\tau_\delta)) - g_\varepsilon^0(\tau_\delta, v)\|_{L^r(\Gamma)} \|\varphi\|_{L^{r'}(\Gamma)}.$$

with  $\|\varphi\|_{L^{r'}(\Gamma)} \leq C\|\varphi\|_{H^{1,q'}(\Omega)}$ . Then, using (5.7), we get

$$\|g_\varepsilon^0(\tau_\delta, v(\tau_\delta)) - g_\varepsilon^0(\tau_\delta, v)\|_{L^r(\Gamma)} \leq L\|v(\tau_\delta) - v\|_{L^r(\Gamma)}.$$

Thus, Lemma 5.1 gives  $\|v(\tau_\delta) - v\|_{L^r(\Gamma)} \leq M\delta^{1-\frac{1}{q}}\|v\|_{H^{1,r}(\Omega)}$  and, from the assumptions on  $\mathcal{C}$ ,  $\|v\|_{H^{1,r}(\Omega)}$  is bounded provided  $\frac{q(N-1)}{N} \leq r \leq p$ . Note that this condition can be met because of (5.12). Hence, for every  $0 \leq \delta \leq \varepsilon$  and  $v \in \mathcal{C}$  we have that

$$I_3 \leq M_3(\varepsilon, R)\|\varphi\|_{H^{1,q'}(\Omega)}$$

with  $M_3(\varepsilon, R) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

**Step 4.** Now, we have

$$I_4 \leq \|g_\varepsilon^0(\tau_\delta, v) - g_\varepsilon^0(\cdot, v)\|_{L^\infty(\Gamma)} \|\varphi\|_{L^1(\Gamma)},$$

with  $\|\varphi\|_{L^1(\Gamma)} \leq C(\Gamma, \Omega)\|\varphi\|_{H^{1,q'}(\Omega)}$ . Hence, the uniform continuity of  $g_\varepsilon^0$  in the first variable and (5.1) implies

$$I_4 \leq M_4(\varepsilon, R)\|\varphi\|_{H^{1,q'}(\Omega)}$$

for every  $v \in \mathcal{C}$ , with  $M_4(\varepsilon, R) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

**Step 5.** Finally, observe that

$$I_5 \leq \|g_\varepsilon^0(\cdot, v) - g_0^0(\cdot, v)\|_{L^\infty(\Gamma)} \|\varphi\|_{L^1(\Gamma)}$$

with  $\|\varphi\|_{L^1(\Gamma)} \leq C(\Gamma, \Omega)\|\varphi\|_{H^{1,q'}(\Omega)}$ . Then using now (5.8), we have

$$I_5 \leq M_5(\varepsilon, R)\|\varphi\|_{H^{1,q'}(\Omega)}$$

for every  $v \in \mathcal{C}$ , where  $M_5(\varepsilon, R) \rightarrow 0$  if  $\varepsilon \rightarrow 0$ .

Therefore, (5.11) is proved. ■

In particular, we get

**Corollary 5.3** *Assume (5.6), (5.7), (5.8) and*

$$\frac{1}{\varepsilon}\mathcal{X}_{\omega_\varepsilon}h_\varepsilon \rightarrow h_0, \quad cc - L^\infty$$

*and consider the nonlinear terms defined in (3.5) and (3.6). Finally, consider a family  $\mathcal{C}$  as in Lemma 5.2, that is satisfying (5.9). Then we have that for any  $1 < q < \infty$  and  $\frac{1}{q'} < s \leq 1$*

i) There exists  $C > 0$  independent of  $\varepsilon > 0$  such that

$$\sup_{v \in \mathcal{C}} \{ \|H_\varepsilon(v)\|_{H_{bc}^{-s,q}(\Omega)}, \|H_0(v)\|_{H_{bc}^{-s,q}(\Omega)} \} \leq C.$$

ii) If (5.12) holds, that is  $p \geq \frac{q(N-1)}{N}$ , there exists  $M(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  such that

$$\sup_{v \in \mathcal{C}} \|H_\varepsilon(v) - H_0(v)\|_{H_{bc}^{-s,q}(\Omega)} \leq M(\varepsilon).$$

**Proof.** Part i) follows from part i) in Lemma 5.2. On the other hand, part ii) with  $s = -1$  follows from part ii) in Lemma 5.2.

Then, for  $\frac{1}{q'} < s < 1$ , fix  $\frac{1}{q'} < s_0 < s < 1$  and by interpolation we get

$$\|H_\varepsilon(v) - H_0(v)\|_{H_{bc}^{-s,q}(\Omega)} \leq C \|H_\varepsilon(v) - H_0(v)\|_{H_{bc}^{-s_0,q}(\Omega)}^\theta \|H_\varepsilon(v) - H_0(v)\|_{H_{bc}^{-1,q}(\Omega)}^{1-\theta}$$

for some  $0 < \theta < 1$  and a positive constant  $C$  independent of  $\varepsilon$ . By part i) the first term in the right hand side above is bounded uniformly in  $\varepsilon$ , while the second goes to zero, uniformly for  $v \in \mathcal{C}$ , and we conclude. ■

## 6 Upper semicontinuity of attractors

With all the above we can then obtain the convergence of the nonlinear semigroups. Note that although the nonlinear problems (1.2) and (1.3) are set in the space  $X = L^q(\Omega)$  or  $X = H_{bc}^{1,q}(\Omega)$  as in Section 3, depending on the growth of the nonlinear term, the convergence results below always take place in  $H_{bc}^{1,\rho}(\Omega)$  for any  $1 < \rho < \infty$ .

**Lemma 6.1** Fix any  $M > 0$  and  $t_0 > 0$  and consider any initial data such that  $\|u_0\|_{L^q(\Omega)} \leq M$  and denote  $u_\varepsilon = T_\varepsilon(t_0)u_0$ .

Then, for any  $1 < \rho < \infty$  and any  $T > 0$ , there exists a constant  $C(M, T, \varepsilon) \rightarrow 0$  if  $\varepsilon \rightarrow 0$ , such that for  $\varepsilon \in (0, \varepsilon_0)$ ,

$$\|T_\varepsilon(t)u_\varepsilon - T_0(t)u_\varepsilon\|_{H_{bc}^{1,\rho}(\Omega)} \leq C(M, T, \varepsilon) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0, \text{ for } t \in [0, T].$$

In particular

$$\sup_{v_\varepsilon \in \mathcal{A}_\varepsilon} \|T_\varepsilon(t)v_\varepsilon - T_0(t)v_\varepsilon\|_{H_{bc}^{1,\rho}(\Omega)} \leq C(M, T, \varepsilon) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0, \text{ for } t \in [0, T].$$

**Proof.** Denote  $\mathcal{C} = \{T_\varepsilon(t)u_\varepsilon = T_\varepsilon(t + t_0)u_0, 0 \leq t \leq T, \|u_0\|_{L^q(\Omega)} \leq M\}$ . Then by Lemma 4.1 and 4.5 the family  $\mathcal{C}$  is bounded in  $L^\infty(\Omega)$  and in  $H_{bc}^{1,\rho}(\Omega)$  for any  $1 < \rho < \infty$  and the bound depends only on  $M, t_0$  and  $T$ . In particular  $\mathcal{C}$  satisfies the assumption (5.9) in Lemma 5.2 for any  $1 < p < \infty$ . Then (5.13) and Corollary 5.3 hold for any  $1 < q < \infty$ .

From the variation of constants formula (3.3) and (3.4), with  $\lambda = 0$  and  $t_0 = 0$ , and Theorem 2.3, we will get below that for any  $1 < \rho < \infty$

$$\|T_\varepsilon(t)u_\varepsilon - T_0(t)u_\varepsilon\|_{H_{bc}^{1,\rho}(\Omega)} \leq C(T, \varepsilon) + M(T) \int_0^t (t-s)^{-\alpha} \|T_\varepsilon(s)u_\varepsilon - T_0(s)u_\varepsilon\|_{H_{bc}^{1,\rho}(\Omega)} ds \quad (6.1)$$

for  $\frac{1}{\rho'} < s < 1$  and  $\alpha = \frac{1}{2}(s+1) < 1$ , with  $C(T, \varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Hence, applying the singular Gronwall Lemma, Lemma 7.1.1. in [10], to (6.1), we get the result.

We now split the proof of (6.1) in several steps. In effect, from the variation of constants formula (3.3) and (3.4) we have that

$$\begin{aligned} & \|T_\varepsilon(t)u_\varepsilon - T_0(t)u_\varepsilon\|_{H_{bc}^{1,\rho}(\Omega)} \leq \|S_{m,\varepsilon}(t)u_\varepsilon - S_{m,V_0}(t)u_\varepsilon\|_{H_{bc}^{1,\rho}(\Omega)} + \\ & + \int_0^t \|S_{m,\varepsilon}(t-s)H_\varepsilon(T_\varepsilon(s)u_\varepsilon) - S_{m,V_0}(t-s)H_\varepsilon(T_\varepsilon(s)u_\varepsilon)\|_{H_{bc}^{1,\rho}(\Omega)} ds + \\ & + \int_0^t \|S_{m,V_0}(t-s)(H_\varepsilon(T_\varepsilon(s)u_\varepsilon) - H_0(T_\varepsilon(s)u_\varepsilon))\|_{H_{bc}^{1,\rho}(\Omega)} ds + \\ & + \int_0^t \|S_{m,V_0}(t-s)(H_0(T_\varepsilon(s)u_\varepsilon) - H_0(T_\varepsilon(s)u_\varepsilon))\|_{H_{bc}^{1,\rho}(\Omega)} ds = I_1 + I_2 + I_3 + I_4 \end{aligned}$$

**Step 1.-** From Theorem 2.3, we obtain

$$I_1 = \|S_{m,\varepsilon}(t)u_\varepsilon - S_{m,V_0}(t)u_\varepsilon\|_{H_{bc}^{1,\rho}(\Omega)} \leq C(M, T, \varepsilon) \|u_\varepsilon\|_{H_{bc}^{1,\rho}(\Omega)} \leq C(M, T, \varepsilon) K_0$$

with  $C(M, T, \varepsilon) \rightarrow 0$  if  $\varepsilon \rightarrow 0$  and  $K_0$  a positive constant independent of  $\varepsilon$ .

**Step 2.-** Again Theorem 2.3 gives

$$\begin{aligned} I_2 &= \int_0^t \|S_{m,\varepsilon}(t-s)H_\varepsilon(T_\varepsilon(s)u_\varepsilon) - S_{m,V_0}(t-s)H_\varepsilon(T_\varepsilon(s)u_\varepsilon)\|_{H_{bc}^{1,\rho}(\Omega)} ds \leq \\ & \leq C(T, \varepsilon) \int_0^t (t-s)^{-\alpha} \|H_\varepsilon(T_\varepsilon(s)u_\varepsilon)\|_{H_{bc}^{-s,\rho}(\Omega)} ds, \end{aligned}$$

with  $C(T, \varepsilon) \rightarrow 0$ , for  $\frac{1}{\rho'} < s < 1$  and  $\alpha = \frac{1}{2}(s+1) < 1$ .

Now, from part i) in Corollary 5.3 we obtain  $\|H_\varepsilon(T_\varepsilon(s)u_\varepsilon)\|_{H_{bc}^{-s,\rho}(\Omega)} \leq K_1$  for  $s \in [0, T]$  for some positive constant  $K_1$  independent of  $\varepsilon$ . From this

$$I_2 \leq C(M, T, \varepsilon) K_2 T^{1-\alpha}$$

since  $t \leq T$ .

**Step 3.-** From Theorem 2.2 we have

$$\begin{aligned} I_3 &= \int_0^t \|S_{m,V_0}(t-s)(H_\varepsilon(T_\varepsilon(s)u_\varepsilon) - H_0(T_\varepsilon(s)u_\varepsilon))\|_{H_{bc}^{1,\rho}(\Omega)} ds \leq \\ & \leq C(T) \int_0^t (t-s)^{-\alpha} \|H_\varepsilon(T_\varepsilon(s)u_\varepsilon) - H_0(T_\varepsilon(s)u_\varepsilon)\|_{H_{bc}^{-s,\rho}(\Omega)} ds \end{aligned}$$

for  $\frac{1}{\rho'} < s < 1$  and  $\alpha = \frac{1}{2}(s+1) < 1$ .

Using now part ii) in Corollary 5.3, since  $p$  is arbitrary, we obtain that  $\|H_\varepsilon(T_\varepsilon(s)u_\varepsilon) - H_0(T_\varepsilon(s)u_\varepsilon)\|_{H_{bc}^{-s,\rho}(\Omega)} \leq C(M, \varepsilon)$  with  $C(M, \varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  uniformly in  $s \in [0, T]$ . Hence  $I_3 \leq C(M, \varepsilon)K_3T^{1-\alpha}$ , with  $K_3$  a positive constant independent of  $\varepsilon$ .

**Step 4.-** Again Theorem 2.2 gives

$$\begin{aligned} I_4 &= \int_0^t \|S_{m, V_0}(t-s) \left( H_0(T_\varepsilon(s)u_\varepsilon) - H_0(T_0(s)u_\varepsilon) \right)\|_{H_{bc}^{1,\rho}(\Omega)} ds \leq \\ &\leq C(T) \int_0^t (t-s)^{-\alpha} \|H_0(T_\varepsilon(s)u_\varepsilon) - H_0(T_0(s)u_\varepsilon)\|_{H_{bc}^{-s,\rho}(\Omega)} ds \end{aligned}$$

for  $\frac{1}{\rho'} < s < 1$  and  $\alpha = \frac{1}{2}(s+1) < 1$ .

Now, observe that from the assumptions on the nonlinear terms we get that

$$\|H_0(u) - H_0(v)\|_{H_{bc}^{-s,\rho}(\Omega)} \leq L\|u - v\|_{H_{bc}^{1,\rho}(\Omega)}$$

with  $L = L(R)$  if the norm of both  $u$  and  $v$  in  $H_{bc}^{1,p}(\Omega) \cap L^\infty(\Omega)$  is bounded by  $R$  for any  $1 < p < \infty$ . Hence, from the bounds in Lemmas 4.1 and 4.5 we get

$$I_4 \leq C(M, T) \int_0^t (t-s)^{-\alpha} \|T_\varepsilon(s)u_\varepsilon - T_0(s)u_\varepsilon\|_{H_{bc}^{1,\rho}(\Omega)} ds$$

Putting all the estimates above together, we get (6.1) and the proof is complete.

The statement about the attractors follows by the invariance of such sets. ■

We are now in a position to prove the upper semicontinuity of the family of attractors.

**Theorem 6.2** *Under the above assumptions, for any  $1 < \rho < \infty$ , the family of global attractors of (1.2) and (1.3),  $\mathcal{A}_\varepsilon$ , is upper semicontinuous at  $\varepsilon = 0$  in  $H_{bc}^{1,\rho}(\Omega)$ , that is*

$$\text{dist}_{H_{bc}^{1,\rho}(\Omega)}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \rightarrow 0, \text{ if } \varepsilon \rightarrow 0$$

where

$$\text{dist}_{H_{bc}^{1,\rho}(\Omega)}(\mathcal{A}_\varepsilon, \mathcal{A}_0) := \sup_{u^\varepsilon \in \mathcal{A}_\varepsilon} \inf_{u^0 \in \mathcal{A}_0} \{\|u^\varepsilon - u^0\|_{H_{bc}^{1,\rho}(\Omega)}\}$$

**Proof.** First, note that from Lemma 4.5,  $\cup_{0 \leq \varepsilon \leq \varepsilon_0} \mathcal{A}_\varepsilon$ , is a bounded set in  $H_{bc}^{1,\rho}(\Omega)$ . Then, as observed at the end of Section 4 we can always assume that, for any  $1 < \rho < \infty$ , the nonlinear semigroups  $T_\varepsilon(t)$  and  $T_0(t)$  are defined in  $H_{bc}^{1,\rho}(\Omega)$  and the attractors attract in the norm of  $H_{bc}^{1,\rho}(\Omega)$ .

In particular,  $\mathcal{A}_0$  attracts in that norm the set  $\cup_{0 < \varepsilon \leq \varepsilon_0} \mathcal{A}_\varepsilon$ . Hence, given  $\delta > 0$ , there exists  $\tau = \tau(\delta)$  such that  $\text{dist}_{H_{bc}^{1,\rho}}(T_0(\tau)u_\varepsilon, \mathcal{A}_0) \leq \frac{\delta}{2}$  for every  $u_\varepsilon \in \mathcal{A}_\varepsilon$  with  $\varepsilon \in (0, \varepsilon_0)$ .

Next, using that  $\mathcal{A}_\varepsilon$  is an invariant set, given  $v_\varepsilon \in \mathcal{A}_\varepsilon$ , there exists  $u_\varepsilon \in \mathcal{A}_\varepsilon$  such that  $T_\varepsilon(\tau)u_\varepsilon = v_\varepsilon$ . Therefore,

$$\text{dist}_{H_{bc}^{1,\rho}}(v_\varepsilon, \mathcal{A}_0) \leq \|v_\varepsilon - T_0(\tau)u_\varepsilon\|_{H_{bc}^{1,\rho}(\Omega)} + \text{dist}_{H_{bc}^{1,\rho}}(T_0(\tau)u_\varepsilon, \mathcal{A}_0).$$

Then from Lemma 6.1, it is clear if  $\varepsilon$  is small enough we get

$$\|v_\varepsilon - T_0(\tau)u_\varepsilon\|_{H_{bc}^{1,\rho}(\Omega)} = \|T_\varepsilon(\tau)u_\varepsilon - T_0(\tau)u_\varepsilon\|_{H_{bc}^{1,\rho}(\Omega)} \leq \frac{\delta}{2},$$

and we conclude. ■

In particular, we get the upper semicontinuity of equilibria

### Corollary 6.3

i) For every sequence  $\varepsilon_k$  with  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$  and for every sequence of equilibria  $\varphi^{\varepsilon_k} \in \mathcal{A}_{\varepsilon_k}$  there exists a subsequence (that we denote the same) and a equilibrium point  $\varphi^0 \in \mathcal{A}_0$  such that

$$\varphi^{\varepsilon_k} \rightarrow \varphi^0, \quad k \rightarrow \infty \quad \text{in } H_{bc}^{1,\rho}(\Omega)$$

for any  $1 < \rho < \infty$ .

ii) In particular, considering the extremal equilibria in Proposition 4.7, we obtain that

$$\varphi_m^0 \leq \liminf_{\varepsilon \rightarrow 0} \varphi^\varepsilon \leq \limsup_{\varepsilon \rightarrow 0} \varphi^\varepsilon \leq \varphi_M^0$$

### Proof.

i) First, we note that if  $\varepsilon_k \rightarrow 0, k \rightarrow \infty$  and  $\varphi^{\varepsilon_k} \in \mathcal{A}_{\varepsilon_k}$  in  $H_{bc}^{1,\rho}(\Omega)$  then, by Theorem 6.2 we get that  $\varphi^0 \in \mathcal{A}_0$ .

Since  $\varphi^{\varepsilon_k}$  is a stationary solution of (1.2), using Lemma 5.2 and Corollary 5.3 it is easy to obtain that  $\varphi^0$  is a stationary solution of (1.3).

ii) This part is immediate. ■

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