A survey on strong reflexivity of abelian topological groups

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Abstract

The Pontryagin duality Theorem for locally compact abelian groups (briefly, LCA-groups) has been the starting point for many different routes of research in Mathematics. From its appearance there was a big interest to obtain a similar result in a context broader than LCA-groups. Kaplan in the 40's proposed -and it remains open- the characterization of all the abelian topological groups for which the canonical mapping into its bidual is a topological isomorphism, assuming that the dual and the bidual carry the compact-open topology. Such groups are called reflexive.

In this survey we deal with results on reflexivity of certain classes of groups, with special emphasis on the class which better reflects the properties of LCA-groups, namely that of strongly reflexive groups. A topological abelian group is said to be strongly reflexive if all its closed subgroups and its Hausdorff quotients as well as the closed subgroups and the Hausdorff quotients of its dual group are reflexive.

Introduction

One instance of spectacular interplay between topology and algebra is Pontryagin-van Kampen duality theorem for locally compact abelian groups. Undoubtedly, it is one of the masterpieces in Mathematics. This explains why the abelian topological groups satisfying the Pontryagin-van Kampen duality, the so called *reflexive groups*, have received considerable attention from the late 40's of the past century.

Locally compact abelian groups (LCA-groups) were initially studied by Pontryagin as the natural class of groups embracing Lie groups. In his remarkable book "Topological groups" (the first English edition from 1939) he already touches the main topics involved in what is commonly understood by "duality theory for abelian groups". Roughly speaking, the duality by him established consists on assigning to an LCA-group another LCA-group called the dual

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group. The good knowledge of categorical language nowadays available permits us to describe Pontryagin's approach as follows. Take first the circle group of the complex plane $\mathbb{T}$, with its natural topology, as dualizing object. Then assign to a group G in the class LCA the group $G^\wedge := CHom(G, \mathbb{T})$ of continuous homomorphisms, and endow it with the compact open topology. This is precisely the dual group of G. After observing that the dual of a compact group is discrete and conversely, he proved that the dual of a group G in LCA is again in LCA.

The celebrated Theorem of Pontryagin and van-Kampen establishes that the natural evaluation mapping from an LCA-group into its bidual is a topological isomorphism (See Theorem 32 of [73]). The contribution of van-Kampen was to withdraw the "separability", a constraint in Pontryagin's first claim. A topological abelian group G is called *reflexive* if the canonical mapping α_G from G into its bidual $G^{\wedge\wedge}$ is a topological isomorphism. Since the "dual groups" $CHom(G, \mathbb{T})$ are abelian and Hausdorff, reflexivity only makes sense within the class of abelian Hausdorff groups.

The first examples of reflexive groups out of the class of LCA groups were found by Kaplan in a very deep paper ([62], 1948) where he established the duality between arbitrary products of abelian topological groups and direct sums of their duals. To this end, he first defined the so called asterisk topology for direct sums of topological groups, which is a group topology made "ad casum", in order to get the mentioned duality. With this instrument at hand he proved that arbitrary products of reflexive groups (in particular of LCA-groups) are reflexive, which stimulated further research in order to find new classes of reflexive groups. As pointed out in [62]: *an as yet unsolved problem is to characterize the class of topological abelian groups for which the Pontryagin duality holds, that is those groups which are the character groups of their character groups.* Several authors have claimed that they had solved this problem: however their proofs either have gaps, or are too complicated to deserve the name of "intrinsic characterization of reflexive groups".

To date many reflexive groups have been found within different classes of topological groups. For instance, in the class of locally convex vector spaces, in the class of free topological groups, in the class of metrizable groups and very recently in the class of precompact groups. (See e.g. [62], [63], [76], [9], [64], [65], [72], [25] [47], [52], [4], [58], [1], [23], [49], [48].)

The simple observation that closed subgroups and Hausdorff quotients of LCA groups are again LCA, and therefore reflexive, leads to a more strict point of view for extending the Pontryajin van-Kampen duality theory: just to consider classes of reflexive groups in which the closed subgroups and the Hausdorff quotients are again reflexive. In a remarkable paper by Brown, Higgins and Morris ([16]) "strong duality" is considered for the first time. A precise definition, after eliminating several non-independent requirements in [16], can be stated as follows.

A topological abelian group G is *strongly reflexive* if the closed subgroups and the Hausdorff quotients of G and of its dual group $G^{\wedge}$ are reflexive ([9]).

Varopoulos in [78] already studied the duality properties of subgroups and quotients of a class of reflexive non locally compact groups. Noble [71] proved that closed subgroups of countable products of LCA groups are reflexive and Leptin [67] showed that this cannot be extended to arbitrary products.

Another sort of reflexivity has been originated by recourse to convergence groups. For a topological abelian group G, define the convergence dual as $CHom(G, \mathbb{T})$ endowed with the continuous convergence structure (instead of the compact-open topology). In general this is no longer a topological group: however, if G is locally compact the convergence dual is exactly the same as the ordinary dual. The duality thus originated by an excursion to convergence groups, may be considered as an extension of Pontryagin duality (see [15], [17], [18], [20], [24], [28]).

Some reflexivity theories for non-abelian groups have been also developed ([32], [46], [29],[14], [54]) but we will not treat on them here.

In this survey we bring together the main results beyond reflexivity known to hold for distinct classes of abelian topological groups. Some of them are very recent and unexpected, for instance those referred to precompact groups. We do not pretend to be exhaustive: a difficult task in a growing field. We have tried to give the flavor of the topic and a good number of references.

1 Preliminaries

All groups considered are abelian, therefore we usually omit this word in the sequel. The symbol $\mathbb{T}$ denotes the multiplicative group of complex numbers with modulus 1, with its natural topology. The set $\mathbb{T}_+ := \{x \in \mathbb{T} : \mathrm{Re} \geq 0\}$ is a particular neighborhood of $1 \in \mathbb{T}$ which plays a pivot role in duality. For a topological group G, $G^{\wedge}$ denotes the group of all continuous homomorphisms from G into $\mathbb{T}$, also called continuous characters. If $G^{\wedge}$ is endowed with the compact-open topology, it is a Hausdorff topological group which is defined to be the *dual group* of G. We shall use the symbol τ_{co} to denote the compact-open topology on $G^{\wedge}$ when a distinction is necessary. Frequently, $G^{\wedge}$ already denotes the dual with the corresponding compact-open topology. If G has sufficiently many continuous characters (that is, $G^{\wedge}$ separates the points of G) then G is said to be *maximally almost periodic* or *MAP*.

The bidual group $G^{\wedge\wedge}$ is $(G^{\wedge})^{\wedge}$ and the canonical evaluation mapping $\alpha_G : G \to G^{\wedge\wedge}$ is defined by $\alpha_G(g)(\kappa) := \kappa(g)$, for all $g \in G$ and $\kappa \in G^{\wedge}$.

Theorem 1.1 *(***Pontryagin, van-Kampen** *1935). If G denotes a locally compact Abelian group, the canonical mapping $\alpha_G : G \to G^{\wedge\wedge}$ is a topological isomorphism.*

Nonreflexive abelian groups occur frequently. A natural easy example is the group of rational numbers $\mathbb{Q}$ endowed with the euclidean topology. Its dual -and therefore its bidual- can be identified algebraically and topologically with the dual of $\mathbb{R}$. It is well known that $\mathbb{R}$ is autodual, thus the non-reflexivity of $\mathbb{Q}$ follows immediately from the reflexivity of $\mathbb{R}$.

A subgroup H of a topological group G is said to be:

- *dually closed* if, for every element x of $G \setminus H$, there is a continuous character φ in $G^\wedge$ such that $\varphi(H) = 1$ and $\varphi(x) \neq 1$.
- *dually embedded* if every continuous character defined on H can be extended to a continuous character on G.
- *h-embedded* if every character defined on H can be extended to a continuous character on G.

Dually closed and dually embedded subgroups already appear in Kaplan's writing, but Noble was the first to call them in this way [71]. On the other hand Tkachenko introduced the h-embedded subgroups in [77].

It is easy to prove that a closed subgroup H of a topological group G is dually closed if and only if the quotient group G/H has sufficiently many continuous characters to separate points.

The *annihilator of* a subgroup $H \subset G$ is defined as the subgroup $H^\perp := \{\varphi \in G^\wedge : \varphi(H) = \{1\}\}$. If L is a subgroup of $G^\wedge$, the *inverse annihilator* is defined by $L^\perp := \{g \in G : \varphi(g) = 1, \forall \varphi \in L\}$. Although the inverse annhilator is frequently denoted by ${}^\perp L$, we shall simply warn the reader if we are taking a direct annhilator of the subgroup L in $G^{\wedge\wedge}$.

Annhilators are the particularizations for subgroups of the more general notion of polars of subsets. Namely, for $A \subset G$ and $B \subset G^\wedge$, the polar of A is $A^\triangleright := \{\varphi \in G^\wedge : \varphi(A) \subset \mathbb{T}_+\}$ and the inverse polar of B is $B^\triangleleft := \{g \in G : \varphi(g) \in \mathbb{T}_+, \forall \varphi \in B\}$. For a topological Abelian group G, it is not difficult to prove that a set $M \subset G^\wedge$ is *equicontinuous* if there exists a neighborhood U of the neutral element in G such that $M \subset U^\triangleright$.

Let $f : G \to E$ be a continuous homomorphism of topological groups. The *dual mapping* $f^\wedge : E^\wedge \to G^\wedge$ defined by $(f^\wedge(\chi))(g) := (\chi \circ f)(g)$ is a continuous homomorphism. If f is onto, then $f^\wedge$ is injective. For a closed subgroup H of a topological group G, denote by $p : G \to G/H$ the canonical projection and by $i : H \to G$ the inclusion. The dual mappings $p^\wedge$ and $i^\wedge$ give rise to the natural continuous homomorphisms $\varphi : (G/H)^\wedge \to H^\perp$ and $\psi : G^\wedge/H^\perp \to H^\wedge$. Observe that if H is dually embedded, ψ is onto. In general φ and ψ are not **topological** isomorphisms: they are under certain conditions that we will study later, and in such case they produce a duality between closed subgroups and Hausdorff quotients of the corresponding dual group.

2 Locally quasi-convex groups

Reflexive groups lie in a wider class of groups, the so called *locally quasi-convex groups.* Vilenkin had the seminal idea to define a sort of convexity for abelian topological groups. Inspired on the Hahn-Banach theorem for locally convex spaces, the following definitions are given in [79]:

Definition 2.1 *A subset A of a topological group G is called* quasi-convex *if for every $g \in G \setminus A$, there is some $\chi \in A^{\triangleright}$ such that $Re\chi(g) < 0$.*

It is easy to prove that for any subset A of a topological group G, $A^{\triangleright\triangleleft}$ is a quasi-convex set. It will be called the *quasi-convex hull* of A since it is the smallest quasi-convex set that contains A. Obviously, A is quasi-convex iff $A^{\triangleright\triangleleft} = A$.

If A is a subgroup of G, A is quasi-convex if and only if A is dually closed.

Remark 1 The definition of a quasi-convex subset A of G relies on the topology of G, since the characters in $A^{\triangleright}$ are required to be continuous. There are "good" subsets A (like the zero neighborhoods of G) for which the continuity of a character ϕ is automatically derived from the fact $\phi(A) \subset \mathbb{T}_+$, but this is not the case in general.

Definition 2.2 *A Hausdorff topological group G is* locally quasi-convex *if it has a basis of zero neighborhoods whose elements are quasi-convex subsets.*

The Hausdorff assumption in the definition of locally quasi-convex groups makes possible the claim that they are MAP groups. Examples of locally quasi-convex groups are provided by any dual group, say $G^\wedge$. In fact, it is easy to prove that the sets $K^{\triangleright}$ where $K \subset G$ is compact, constitute a quasi-convex zero-neighborhood basis for the compact-open topology in $G^\wedge$. Thus, any reflexive group is locally quasi-convex, as it is the dual of its character group.

It is straightforward to prove that any subgroup of a locally quasi-convex group is locally quasi-convex; a Hausdorff quotient of a locally quasi-convex group may not be locally quasi-convex [4, 12.8]. However, in the class of locally compact groups, and in the more general class of nuclear groups, every Hausdorff quotient is locally quasi-convex, see [9, 7.5]. Also quotients of locally quasi-convex groups by compact subgroups are locally quasi-convex [8].

An easy example of a non-reflexive locally quasi-convex topological group is the group of rational numbers $\mathbb{Q}$ with the topology induced by $\mathbb{R}$. As a subgroup of the locally quasi-convex group $\mathbb{R}$, it also has this property and it is non reflexive as mentioned above.

Since topological vector spaces are special instances of abelian topological groups, it is natural to compare quasi-convexity and convexity for subsets of a topological vector space as well as the corresponding local properties. There is a subtle difference between the convex subsets and the quasi-convex subsets of a topological vector space E. To begin with, convexity is a merely algebraic property, while quasi-convexity involves the topology of E. It is amazing that there are finite or countably infinite quasi-convex subsets [66, Chapter 7]. Nevertheless, if $A \subset E$ is a balanced subset and $co(A)$ denotes its convex hull, then $co(A) = A^{\triangleright\triangleleft}$ [17, 6.3.1]. On the other hand, a Hausdorff topological vector space E is locally convex if and only if E is locally quasi-convex as an additive group, [9, 2.4]. Thus, local quasi-convexity is an extension for groups of the notion of local convexity in vector spaces.

Reflexive locally convex spaces constitute a well established topic in Functional Analysis. Smith in 1952 was the first to relate reflexivity in the sense of Pontryagin for a locally convex space with the by now traditional concept of reflexivity in the Functional Analysis sense. As a result she obtained in [76] a new class of Pontryagin reflexive non LCA- groups, from a point of view totally different of that of Kaplan. She was inspired by a paper of Arens of 1947, where the reflexivity of topological vector spaces is treated for the first time, [2]. It is interesting to describe her approach to the question, which we outline in the next paragraph.

For a topological vector space E, denote by E^* the vector space of all continuous linear forms on E. Arens introduced the term "reflexive topology" to denote a topology t on E^* such that the continuous linear forms on (E^*, t) were precisely the "evaluations" at the elements of E, [2]. The current notion of reflexive space is much stronger at present. By the dual of E it is commonly understood E^* endowed with the topology of uniform convergence on the family $\mathcal{B}$ of all the bounded subsets of E. If E_b^* denotes the dual so topologized, then E is said to be *reflexive* if the canonical mapping from E into $(E_b^*)_b^*$ is a topological isomorphism. After proving that $E^\wedge$ and E^* are algebraically isomorphic as groups, Smith points out that, although there is no reason why $(E_b^*)_b^* \approx E$ should imply $(E_c^\wedge)_c^\wedge \approx E$, it happens to be so (here c denotes the compact open topology and $\approx$ topological isomorphism as spaces in the first case and as groups in the second one). Therefore, reflexive locally convex spaces are reflexive as topological groups.

In [76] it is also proved that all Banach spaces are reflexive as topological groups, thus reflexivity in Pontryagin sense is a property strictly weaker than reflexivity in the sense of Functional Analysis. The result is valid for complete metrizable locally convex spaces as well. With the tools of locally quasi-convex groups -which were not available to M. Smith -, the proof is much easier, and the use of a norm is not necessary. In fact, in [9, (15.7)] it is proved that even a complete metrizable locally convex **vector group** is reflexive. Nevertheless, the structure of vector space is essential here, and the result does not have a counterpart for topological groups. As proved in [4, (11.15)], the group $G := L^p_{\mathbb{Z}}([0,1])$ of the almost everywhere integer-valued functions, with $p > 1$ and topology induced by the classical of $L^p([0,1])$, is a locally quasi-convex complete metrizable group which is not reflexive. Later on we will see that for nuclear groups the claim holds even in a stronger sense: that is, a nuclear complete metrizable group is strongly reflexive.

Along the past 20 years there has been intensive research in order to give counterparts for abelian locally quasi-convex groups of theorems known to hold for locally convex spaces. For instance the Theorems of Grothendieck about completeness [22], of Mackey-Arens [31], of Dunford-Pettis [69], of Eberlein-Smulyan [21], and some others. The weak topology $\omega(E, E^*)$ on a topological vector space E has a parallell for topological groups, as we express below after a few considerations on precompact topological groups.

Let $Hom(G, \mathbb{T})$ be the group of all characters on an abstract abelian group G, and let H be a subgroup of $Hom(G, \mathbb{T})$. The weak topology induced by H on G is a precompact group topology which is Hausdorff whenever H separates the points of G, and will be denoted by $\omega(G, H)$. As proved in [31] (and in [36] for separating H), $(G, \omega(G, H))^\wedge = H$. On the other hand, as a consequence of Peter-Weyl theorem, any precompact Hausdorff group G carries the weak topology corresponding to its character group. Thus, precompact Hausdorff groups are simply subgroups of compact Hausdorff groups.

For a topological group G, $\omega(G, G^\wedge)$ is called the *Bohr topology of* G. It is weaker than the original topology of G, and the notation $G^+ := (G, \omega(G, G^\wedge))$ is quite extended in the Literature. As said above, G^+ is precompact and $(G^+)^\wedge = G^\wedge$. In particular, since the additive group of real numbers $\mathbb{R}$ is not precompact, $\mathbb{R}$ and $\mathbb{R}^+$ are not topologically isomorphic groups. If $\mathbb{R}^*$ denotes the vector space of continuous linear forms on $\mathbb{R}$ and $\omega(\mathbb{R}, \mathbb{R}^*)$ the corresponding weak topology on $\mathbb{R}$, which as a vector space topology coincides with the usual topology of $\mathbb{R}$, we obtain that $\omega(\mathbb{R}, \mathbb{R}^\wedge) \neq \omega(\mathbb{R}, \mathbb{R}^*)$. The same happens for any topological vector space E.

3 The canonical mapping α_G

The canonical mapping $\alpha_G : G \to G^{\wedge\wedge}$ is the backbone for the reflexivity of a topological group G. It is the mapping associated to the evaluation $e : G^\wedge \times G \to \mathbb{T}$, defined by $e(\phi, x) = \phi(x)$ ($\phi \in G^\wedge$, $x \in G$), in the following sense: $\alpha_g(x)(\phi) = e(\phi, x)$. If $G^\wedge$ carries the compact-open topology and $G^\wedge \times G$ the product topology, it is well known that the continuity of e implies that of α_G, but the converse does not hold. Observe also that for any locally compact abelian group e is continuous. There is an amazing result which allows us to distinguish the class of LCA-groups in the framework of reflexive groups. Namely, if G is a reflexive group and e is continuous, then G must be locally compact [68]. However one could go an step further to unveal this property: reflexivity is not needed in its full strength. We will introduce below the quasi-convex compactness property and come back to the question.

We first study under very general assumptions on G, when is α_G 1-1, onto, continuous or open.

Proposition 3.1 *Let G be a topological group. The canonical mapping $\alpha_G : G \to G^{\wedge\wedge}$ is an homomorphism such that:*

(1) α_G is injective iff G is MAP.

(2) α_G is continuous if and only if the compact subsets of $G^\wedge$ are equicontinuous.

(3) α_G is k-continuous. In other words, the restriction of α_G to any compact subset $K \subset G$ is continuous. In particular, α_G is sequentially continuous.

(4) If the group G is a k-space, then α_G is continuous.

(5) If G is locally quasi-convex, α_G is relatively open and one to one.

(6) If the compact subsets of G are finite, then α_G is onto.

Proof. The assertion (1) is straightforward, (2) is [4, Proposition 5.10]. Item (3) derives from well known topological results, and (4) is a consequence of (3). In order to prove (5), observe that for a quasi-convex zero-neighborhood V, $\alpha_G(5) = V^{\triangleright\triangleright} \cap \alpha_G(G)$, where $V^{\triangleright}$ is compact in the compact-open topology of $G^\wedge$. Under the assumption (6) the compact-open topology on $G^\wedge$ coincides with the pointwise convergence topology $\omega(G^\wedge, G)$, and the assertion follows. □

A topological group is said to be Raikov-complete (or just complete) if it is complete for its standard two-sided uniformity as a topological group. The conjunction of the notions of k-space and the compact open topology give a straightforward proof of the following:

Fact A. If a topological group G is a k-space, then its dual group $G^\wedge$ is complete.

A topological group G is said to have the the quasi-convex compactness property (briefly, **qcp**) if for every compact $K \subset G$ its quasi-convex hull $K^{\triangleright\triangleleft}$ is again compact. In the framework of locally quasi-convex groups, this property is related with completeness and with the mapping α_G as follows:

Proposition 3.2 *Let G be a locally quasi-convex group. The following assertions hold:*

(1) If G is complete, then G has the qcp. The converse also holds if G is moreover metrizable.

(2) If α_G is onto, G has the qcp. The converse does not hold, even for metrizable groups.

(3) If $e : G^\wedge \times G \to \mathbb{T}$ is continuous and G has the qcp, then G is locally compact.

(4) If α_G is continuous, then $G^\wedge$ has the qcp.

(5) If G is reflexive, both G and $G^\wedge$ have the qcp.

Proof. The proofs are not hard. They can be seen in [17, chapter 6], where the qcp is formally introduced for the first time, the second part of (2) in [52] and in [19]. The converse of (3) holds without the assumption of the qcp. An assertion similar to (3) (without qcp) was known to hold substituting the first factor $G^\wedge$ by the set $C(G, \mathbb{T})$ of continuous functions from G to $\mathbb{T}$. □

We give now some examples which can be quoted later on for distinct properties.

Example 3.1 Let G be the group of rational numbers with the usual euclidean topology. Since G is metrizable non-complete, α_G is continuous, but G does not have the qcp by (1). The evaluation mapping $e : G^\wedge \times G \to \mathbb{T}$ is also continuous, because $G^\wedge$ is precisely $\mathbb{R}$ endowed with the usual topology, and the product $\mathbb{R} \times \mathbb{Q}$ is a k-space by Whitehead's Theorem. This example proves that $G^\wedge$ cannot be substituted by G in Proposition 3.2 (4).

Example 3.2 Let $G := \mathbb{R}^{(\mathbb{N})} \times \mathbb{R}^{\mathbb{N}}$, where $\mathbb{R}^{\mathbb{N}}$ carries the ordinary Tychonoff topology, and $\mathbb{R}^{(\mathbb{N})}$ is the countable direct sum of real lines with the rectangular topology. Then G is a reflexive selfdual topological group: in particular, it is locally quasi-convex and α_G is continuous. It is not a k-space (e.g. [9]), and consequently non-metrizable. As it is the product of the two complete groups, $\mathbb{R}^{\mathbb{N}}$ and its dual $\mathbb{R}^{(\mathbb{N})}$ (Fact A), G is complete.

Example 3.3 Let $G := L^2_{\mathbb{Z}}[0, 1]$ be the group of the almost everywhere integer-valued functions, with the topology induced by the classical norm of $L^2([0, 1])$. This example appears in [4], where the dual is calculated obtaining that $G^\wedge = L^2([0, 1])$. Therefore $G \leq L^2([0, 1])$ is a closed subgroup which has the same dual as the whole group. As said above, it is lqc complete metrizable and nonreflexive. Further it has the qcp by Proposition 3.2 (1).

4 Strong reflexivity and related notions.

So far it is clear that LCA-groups are the best suited for consideration of reflexivity issues. This claim can be supported by the fact that closed subgroups and Hausdorff quotients are also reflexive and further, there is a perfect duality between closed subgroups of an LCA group G (resp. of $G^\wedge$) and Hausdorff quotients of its dual $G^\wedge$ (resp. of G). In the next proposition -whose proof is contained in [9]-, we describe more precisely these properties, and in successive sections we study how much of them is shared by other classes of groups.

Proposition 4.1 *([9, 17.1]) For a topological abelian group G, the following claims -which may or may not hold for G- can be related as we indicate below:*

1) Closed subgroups and Hausdorff quotients of G and of $G^\wedge$ are reflexive.

2) All the closed subgroups of G and of $G^\wedge$ are dually closed and dually embedded.

3) For every pair H and L of closed subgroups of G and of $G^\wedge$ respectively, the natural homomorphisms

$$\varphi : (G/H)^\wedge \to H^\perp, \quad \psi : G^\wedge/H^\perp \to H^\wedge; \quad \varphi' : (G^\wedge/L)^\wedge \to L^\perp, \quad \psi' : G/L^\perp \to L^\wedge.$$

are topological isomorphisms.

Then, 1) implies 2) and 3).

Definition 4.1 *An abelian topological group G is called* strongly reflexive (s.r.) *if every closed subgroup and every Hausdorff quotient of G and of $G^\wedge$ are reflexive.*

Countable products and sums of real lines and circles were the first examples of non locally compact strongly reflexive groups [16]. Banaszczyk extended this result proving that all countable products and sums of LCA groups are strongly reflexive [11], and observed that these examples were included in a larger class of groups, which he defined and studied in [9], calling them nuclear groups. Although we will deal with the class of nuclear groups in Section 6.1, we anticipate that it contains the locally convex nuclear vector spaces and the locally compact Abelian groups, and it is closed under forming products, subgroups and Hausdorff quotients.

Strong reflexivity was obtained in [9] for complete metrizable nuclear groups and in [4, 20.40] for Čech-complete nuclear groups. More recently, in [7] it has been proved that all k_ω and all the locally k_ω nuclear groups are strongly reflexive. A k_ω group is a group whose underlying space is an hemicompact k-space and a locally k_ω group is a group which has an open neighborhood of zero which is a k_ω space. The locally k_ω groups have been introduced in [51] where their interest is sufficiently motivated.

Außenhofer constructed in 2007 [5] a non reflexive quotient of the uncountable product $\mathbb{Z}^{\mathbb{R}}$. With this result she answered in the negative the question posed by Banaszczyk in 1990 if uncountable products of real lines are strongly reflexive.

We do not know any example of strongly reflexive group out of the classes above mentioned. For this reason we bisect strong reflexivity in the two weaker properties, introduced next:

Definition 4.2 *A topological group G will be called*

(i) s-reflexive *if all closed subgroups of G are reflexive;*

(ii) q-reflexive *if all Hausdorff quotients of G are reflexive.*

They are notions stronger than reflexivity as the following example shows:

Example 4.1 The space $L^2[0,1]$ is a reflexive group, but fails to be either s-reflexive or q-reflexive. Indeed, by a theorem of Banaszczyk ([9, (5.3)] or [10]), every infinite-dimensional Banach space has a quotient which does not have non-null characters, witnessing that $L^2[0,1]$ is not a q-reflexive group. On the other hand, $L^2[0,1]$ is neither an s-reflexive group since its closed subgroup $L^2_{\mathbb{Z}}[0,1]$ is not reflexive (Example 3.3).

The following open question arises:

Question 4.1 If a group G is s-reflexive and q-reflexive, is it strongly reflexive.?

Closed subgroups and Hausdorff quotients of strongly reflexive group are strongly reflexive ([9, 17.1]), however even finite products of strongly reflexive groups may be not strongly reflexive (the selfdual group $G := \mathbb{R}^{(\mathbb{N})} \times \mathbb{R}^{\mathbb{N}}$ is not strongly reflexive as proved in [9, 17.7]). In the sequel we study general properties of closed subgroups and Hausdorff quotients and what is lacking in some cases, in order that a reflexive group be strongly reflexive.

Proposition 4.2 *Let G be a reflexive group, H a closed subgroup of G and L a closed subgroup of $G^\wedge$. Then the following facts hold:*

(1) The mapping α_H (respectively, α_L) is relatively open and injective.

(2) The evaluation mapping $\alpha_{G/H}$ is continuous.

(3) If H is dually closed, $\alpha_G(H) = H^{\perp\perp}$.

(4) If H is dually closed and dually embedded, α_H is open and bijective.

(5) A subgroup H is dually closed if and only if $\alpha_{G/H}$ is injective.

(6) $\alpha_{G/H}$ surjective implies $H^\perp$ is dually embedded.

(7) If H is dually closed and $\alpha_{G^\wedge/H^\perp}$ surjective, then H is dually embedded.

(8) If L is dually closed, there exists a closed subgroup H of G such that $H^\perp = L$.

Proof. Item (1) follows from the fact that subgroups of locally quasi-convex groups are locally quasi-convex and Proposition 3.1 (1) and (4). The proof of (2) is straightforward, (3) holds because α_G is surjective. In order to prove (4), consider the following commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\iota} & G \\ \downarrow{\scriptstyle \alpha_H} & & \downarrow{\scriptstyle \alpha_G} \\ H^{\wedge\wedge} & \xrightarrow{\iota^{\wedge\wedge}} & G^{\wedge\wedge} \end{array}$$

Let $\gamma \in H^{\wedge\wedge}$. Since G is reflexive, $\gamma \circ \iota^{\wedge} = \alpha_G(x)$, for some $x \in G$. Then $x \in H$, for otherwise, there would exist $\chi \in G^{\wedge}$ such that $\chi(H) = \{1\}$ and $\chi(x) \neq 1$. Hence $\iota^{\wedge}(\chi) = 1$ but $\gamma \circ \iota^{\wedge}(\chi) = \alpha_G(x)(\chi) = \chi(x) = 1$, which is a contradiction. Item (4) is also proved in [71], (5) is easy and well known, (6) and (7) are in [30, 1.4] and (8) in [9, 14.2]. □

The open subgroups and the compact subgroups of a topological group G characterize reflexivity (or strong reflexivity) of the original group in the following way:

Proposition 4.3 *Let G be a topological group $H \subset G$ an open subgroup, and let $K \subset G$ be a compact subgroup. Then the following assertions hold:*

(1) G is reflexive (strongly reflexive) iff H is reflexive (strongly reflexive).

(2) If G has sufficiently many continuous characters, G is reflexive (strongly reflexive) iff G/K is reflexive (strongly reflexive)

(3) The statements (1) and (2) also hold if "reflexive" is replaced by s-reflexive or by q-reflexive.

Proof. Item (1) is [13, 2.3 and 3.3] and the proof of (2) follows from [13, 2.6, 3.3, 3.4] and [20, 1.4]. The definitions of s-reflexive and q-reflexive are new. The proof of (3) will appear elsewhere. □

Remark 2 Hofmann and Morris in [61] define another strengthening of the concept of reflexivity as follows: a reflexive topological group G has *sufficient duality* if for every closed subgroup H of G the quotient G/H has sufficiently many continuous characters and the quotient $G^{\wedge}/H^{\perp}$ is reflexive. Clearly, a strongly reflexive group has sufficient duality, and in the next proposition we analyze how much of the converse holds.

Proposition 4.4 *If G is a topological group with sufficient duality, all its closed subgroups are dually closed, dually embedded, and have reflexive dual.*

Proof. Let H be a closed subgroup of G. Since G/H is MAP, H is dually closed. By (7) of Proposition 4.2 H is dually embedded, and by [18, (14.8)] there is an isomorphism between $G^{\wedge}/H^{\perp}$ and $H^{\wedge}$. Thus $H^{\wedge}$ is reflexive. □

Proposition 4.5 *If G is a metrizable group with sufficient duality then G is an s-reflexive group.*

Proof. As in the previous proposition, any closed subgroup $H \leq G$ is dually closed and dually embedded. The results of Propositions 4.2 (4) and 3.1 (4) apply to give that α_H is a topological isomorphism. □

5 Metrizable and almost metrizable groups.

Metrizable groups are a distinguished class of groups from the point of view of Pontryagin duality theory. As pointed out in Proposition 3.1 (4), if G is a metrizable group α_G is continuous. If moreover G is locally quasi-convex, then α_G is injective and relatively open. Thus only surjectivity must be worked out in order to have reflexivity in the class of all metrizable locally quasi-convex groups. The following assertion obtained independently in [25] and [4] is a fundamental result for its many consequences:

Fact B. If G is a metrizable group, $G^\wedge$ is a k-space. Therefore, $G^{\wedge\wedge}$ is complete.

Thus, a reflexive metrizable group must be complete. Example 3.3 shows that completeness is not a sufficient condition for the surjectivity of α_G.

On the other hand, metrizability of G implies that $G^\wedge$ is hemicompact, and this in turn implies that $G^{\wedge\wedge}$ is metrizable. Thus, the square of the duality functor applied to the subcategory of metrizable groups $\mathcal{M}$ leads again to $\mathcal{M}$. This also happens in the broader class of almost metrizable groups which we describe next: the bidual of an almost metrizable group is again almost metrizable, although clearly we cannot expect that the dual of an almost metrizable group be hemicompact.

Recall that a topological space X is called *almost metrizable* if every $x \in X$ is contained in a compact subset having a countable neighbourhood basis in X. A topological group G is almost metrizable if and only if it has a compact subgroup K such that G/K is metrizable [75]. Čech-complete groups are instances of almost metrizable groups. As a matter of fact a topological group G is Čech-complete if and only if it has a compact subgroup K such that G/K is metrizable and **complete** [4, (2.21)].

Since many reflexivity properties of metrizable groups can be extended to the almost metrizable groups, we study them together in the next proposition:

Proposition 5.1 *Let G be an almost metrizable topological group. Then:*

(1) G is a k-space, [4, (1.24)].

(2) The dual group $G^\wedge$ is a k-space and $G^{\wedge\wedge}$ is Čech-complete [4, (5.20)], in particular almost metrizable.

(3) The canonical homomorphisms α_G and $\alpha_{G^\wedge}$ are continuous [Proposition 3.1 (4)].

(4) If G is furthermore reflexive, every dually closed and dually embedded subgroup of G is reflexive [Proposition 3.1 (4) and Proposition 4.2 (4)].

(5) If G is reflexive, closed subgroups of G and of $G^\wedge$ are dually embedded and Hausdorff quotients of G and of $G^\wedge$ are locally quasi-convex, then G is strongly reflexive, [30].

Theorem 5.2 *[30] For an almost metrizable topological group, the following assertions are equivalent:*
i) G is strongly reflexive.
ii) Hausdorff quotients of G and of $G^\wedge$ are reflexive. In other words, G and $G^\wedge$ are q-reflexive.

Question 5.1 *[30]* Does the above theorem hold if "q-reflexive" is replaced by "s-reflexive"?.

An important feature of metrizable groups is that dense subgroups determine the dual in the following sense:

Proposition 5.3 *Let G be a metrizable topological group and H a dense subgroup of G. Then the dual homomorphism of the inclusion mapping $j : H \hookrightarrow G$ is a topological isomorphism.*

In other words the restriction mapping from $G^\wedge$ to $H^\wedge$ is a topological isomorphism. This result, proved in [4] and [25], originated intensive research to obtain other classes of groups with the same property. In [34] those groups are called *determined groups* and it is proved under CH that a compact abelian group is determined if and only if it is metrizable. In [55] a new proof is given without CH, and still another proof is provided in [42].

6 The class of nuclear groups

The class of nuclear groups was formally introduced by Banaszczyk in [9]. A source for inspiration was his previous work [12], where he studied the behaviour of closed subgroups and quotients by closed subgroups of nuclear vector spaces. Earlier he had studied similar questions for Banach spaces, and he was aware that, from some point of view, nuclear spaces -rather than Banach spaces- are natural generalization of finite dimensional vector spaces. So he set out to find a class of topological groups embracing nuclear spaces and locally compact abelian groups (as natural generalizations of finite-dimensional vector spaces). This was the origin of the class of nuclear groups: the definition of the latter in [9] is very technical, as could be expected from its virtue of joining together objects of such different classes. A nice survey on nuclear groups is also provided by L. Außenhofer in [6]. The following are outstanding facts concerning the class of nuclear groups:

(Nuc_1) Nuclear groups are locally quasi-convex, [9, 8.5].

(Nuc_2) Products, subgroups and quotients of nuclear groups are again nuclear, [9, 7.5].

(Nuc_3) Every locally compact abelian group is nuclear, [9, 7.10].

(Nuc_4) A nuclear locally convex space is a nuclear group, [9, 7.4]. Furthermore, if a topological vector space E is a nuclear group, then it is a locally convex nuclear space, [9, 8.9].

(Nuc_5) If G is a nuclear group, every $\omega(G, G^\wedge)$-compact subset is compact in the original topology of G, [14].

Nuc_5 gives rise to the following:

Fact C. If G is a nuclear group, then $G^\wedge$ and $(G^+)^\wedge$ coincide not only algebraically but also topologically.

Fact C can be proved for LCA-groups by means of the Glicksberg Theorem. However the class of nuclear groups is extraordinarily bigger. As a consequence we can obtain a family of groups for which the canonical mapping into the bidual is not continuous.

Example 6.1 *If G is a nuclear reflexive nonprecompact group, and $G^+ := (G, \omega(G, G^\wedge))$, then α_{G^+} is non-continuous. In particular, if G is locally compact noncompact α_{G^+} is not continuous.*

Proof. The topologies of G and of G^+ are distinct since G is nonprecompact. By fact C the duals and hence the biduals of G and of G^+ coincide, therefore $\alpha_{G^+} = \alpha_G$ as mappings. Since $\alpha_G : G \to G^{\wedge\wedge}$ is a topological isomorphism, α_{G^+} cannot be a topological isomorphism. By Proposition 3.1 (5) α_{G^+} is open, therefore it is **not** continuous. $\square$

Proposition 6.1 *Let G be a nuclear group and $H \subset G$ a closed subgroup. The following duality results hold:*

(1) The canonical homomorphisms α_G and α_H, and $\alpha_{G/H}$ are injective and relatively open.

(2) Closed subgroups of G are dually closed and dually embedded, [*9*, 8.3] *and* [*9*, 8.6].

(3) If G is moreover complete, α_G is an open isomorphism, [*4*, 21.5].

(4) If G is a complete k-space, it is reflexive and its closed subgroups are also reflexive. Hence nuclear complete k-spaces are s-reflexive.

(5) If G is Čech-complete, its dual group $G^\wedge$ is also nuclear, [*4*, 20.36] *and strongly reflexive, [4, 20.35].*

(6) If G is moreover metrizable, the following equivalences hold:

$$G \text{ is complete } \Leftrightarrow G \text{ is reflexive } \Leftrightarrow G \text{ is strongly reflexive}$$

it is strongly reflexive, [*4*, 20.35].

Proof. Item (1) derives from Nuc_1, Nuc_2, and Proposition 3.1 (5). Item (4) can be obtained from Nuc_1, (3) and and Proposition 3.1 (4). Item (6) is consequence of (4), (5) and Fact B. □

Remark 3 (i) In spite of the good stability properties of the class of nuclear groups, the dual of a nuclear group may not be nuclear. The constraint of (5) in Proposition 6.1 cannot be totally withdrawn.

(ii) Observe that metrizability as well as nuclearity are essential in Proposition 6.1 (6). Examples of noncomplete reflexive P-groups (therefore nuclear) are provided in [38]. On the other hand $L^2_{\mathbb{Z}}[0,1]$ is complete metrizable and nonreflexive (see Example 3.3).

(iii) Example 3.2 provides a reflexive nuclear space which is not strongly reflexive as proved in [9, 17.7]. The proof also shows that it is neither q-reflexive, nor s-reflexive.

7 Precompact groups

An abelian topological group G is precompact if for every neighborhood of zero V there exists a finite subset $F \subset G$ such that $G = F + V$. As already indicated in Section 2, the class of precompact Hausdorff abelian groups can be identified with the class of subgroups of $\mathbb{T}^\alpha$, where α is any cardinal number. Thus, a precompact Hausdorff group is not only locally quasi-convex, it is even nuclear by Nuc_2 and Nuc_3. The topology of a precompact Abelian group G is precisely $\omega(G, G^\wedge)$.

A "sort of reflexivity" can be considered for the class of precompact abelian Hausdorff groups. In fact, taking the pointwise convergence topology in the character groups instead of the compact-open topology, a precompact group G is topologically isomorphic to its bidual, [74]. This has been recently denominated the Comfort-Ross duality.

We turn now to the "standard" reflexivity. It follows from Fact B that a precompact reflexive metrizable group must be compact. Locally compact, non-compact abelian groups endowed with their Bohr topology are examples of precompact nonmetrizable nonreflexive groups (See Example 6.1). Observe that the dual groups of the latter are locally compact.

Since the class of precompact groups is included in that of nuclear groups, the statements of Proposition 6.1 apply for them. Further results specific for this subclass are the following:

Proposition 7.1 *Let G be a precompact group. Then the following assertions hold:*

(1) The equicontinuous subsets in $G^\wedge$ are finite.

(2) The mapping α_G is continuous if and only all the compact subsets of $G^\wedge$ are finite.

(3) If the compact sets of G and of $G^\wedge$ are finite, then G is reflexive.

Proof. In order to prove (1), consider an equicontinuous subset A in $G^\wedge$. Then $A^\triangleleft$ is a neighborhood of zero in G and its closure in the completion $\tilde{G}$ of G is also is a neighborhood of zero in $\tilde{G}$, which we call $\overline{A^\triangleleft}$. The set $(\overline{A^\triangleleft})^\triangleright = (A^\triangleleft)^\triangleright$ is compact in $(\tilde{G})^\wedge$ and thus finite. Item (2) is a corollary of (1) and Proposition 3.1 (2). Item (3) yields from (1), (2) and Proposition 3.1 (5) and (6). □

Question 7.1 *Are there strongly reflexive precompact noncompact groups?*

8 Pseudocompact groups

A Hausdorff topological group G is said to be pseudocompact if it is pseudocompact as a topological space, that is if every continuous real function defined on G is bounded. This property matched with the algebraic structure of the supporting set produces the highly interesting class of pseudocompact groups, which has been intensively studied by Comfort, Dikranjan, Galindo, van Mill, Shakmatov, and some others. The first relevant properties of this class of groups are the following:

(Psc_1) Every pseudocompact group is precompact, [37, 1.1].

(Psc_2) A Hausdorff topological group G is pseudocompact iff it is G_δ-dense in its Stone-Čech compactification βG, [59, Theorem 28].

(Psc_3) The Bohr compactification of a pseudocompact group G coincides with its Stone-Čech compactification βG. Thus, the group operation of G can be continuously extended to βG, [37, Theorem 4.1].

Examples of pseudocompact groups are the Σ-products of uncountable families of compact groups. It can be easily proved that a locally finite family of open sets in a pseudocompact space must be finite and consequently a paracompact pseudocompact space is necessarily compact. In particular, a metrizable pseudocompact group is compact, and if it is infinite its size must be $\geq c$.

The following characterization of pseudocompact groups within the class of precompact groups is provided in [56, (3.4)]. Since it is an important result, we write the proof filling out some details.

Proposition 8.1 *Let G be a precompact group. Then G is pseudocompact if and only if every countable subgroup of $G^\wedge$ is h-embedded in $(G^\wedge, \omega(G^\wedge, G))$.*

Proof. $\Rightarrow$) Assume that G is pseudocompact and consequently its topology is $\omega(G, G^\wedge)$. Let H be a countable subgroup of $G^\wedge$, $H^\perp$ its inverse polar and take the quotient $\frac{G}{H^\perp}$ endowed with the topology $\omega(\frac{G}{H^\perp}, H)$ where H is now identified with a group of characters of $\frac{G}{H^\perp}$. Obviously H separates the points of $\frac{G}{H^\perp}$.

It is straightforward to check that the canonical projection

$$\pi : (G^\wedge, \omega(G^\wedge, G)) \to (\frac{G}{H^\perp}, \omega(\frac{G}{H^\perp}, H))$$

is continuous and therefore the image group is also pseudocompact. The latter is also metrizable, since its dual group H is countable. Thus $(\frac{G}{H^\perp}, \omega(\frac{G}{H^\perp}, H))$ being pseudocompact and metrizable must be compact, and consequently H is discrete. Any character $\phi : H \to \mathbb{T}$ can be considered as the evaluation on a point of $\frac{G}{H^\perp}$, say $g + H^\perp$, or equivalently, there exists $g \in G$ such that $\phi(\psi) = \psi(g)$ for all $\psi \in H$. This means that ϕ can be extended to $\alpha_G(g)$, a continuous character on $(G^\wedge, \omega(G^\wedge, G))$.

$\Leftarrow$) In order to prove that G is pseudocompact, it is enough to see that G is G_δ-dense in its Bohr-compactification bG. Let V denote a G_δ-set in bG containing the neutral element. Then $V \supset \bigcap_{n\in\mathbb{N}} F_n^\triangleright$ where F_n is some finite subset of $G^\wedge$, for each $n \in \mathbb{N}$. Call $H := \langle \bigcup_{n\in\mathbb{N}} F_n \rangle$ the subgroup generated by $\bigcup_{n\in\mathbb{N}} F_n$. Clearly H is countable and $V \supset H^\perp$ (now the annhilator is taken in bG). If p denotes an element in bG, $p_{|H}$ is a character which by the assumption can be extended to a continuous character on $(G^\wedge, \omega(G^\wedge, G))$. Therefore it exists $g \in G$ such that $\alpha_G(g)_{|H} = p$. Thus $(p + V) \cap \alpha_G(G) \neq \emptyset$, which proves that G is G_δ-dense in bG. □

We will use frequently the following assertion proved in $[1, 2.1]$:

Fact D. If a topological group has the property that its countable subgroups are h-embedded, then its compact subsets must be finite.

We enumerate now some properties of pseudocompact groups related to reflexivity.

Proposition 8.2 *Let G be a pseudocompact group. Then the following assertions hold:*

(1) Every $w(G^\wedge, G)$-compact subset of $G^\wedge$ is finite. Consequently, every compact subset of $(G^\wedge, \tau_{co})$ is also finite.

(2) The mapping α_G is continuous, injective and relatively open. Thus, G is reflexive if and only if α_G is surjective.

(3) If the compact sets of G are finite, then the group G is reflexive.

(4) G is a dual group. In fact, G is topologically isomorphic to $(G^\wedge, w(G^\wedge, G))^\wedge$.

(5) If the countable subgroups of G are h-embedded, then $G^\wedge$ is also pseudocompact with countable subgroups h-embedded. Moreover, G is reflexive.

(6) G is topologically isomorphic to a Hausdorff quotient of a reflexive pseudocompact group.

(7) There exist pseudocompact reflexive non strongly reflexive groups.

Proof. The first assertion of (1) follows from Proposition 8.1 and fact D. Another proof is provided in [56, 4.4].

The continuity of α_G in (2) follows from (1) and Proposition 3.1 (2). Since G is locally quasi-convex, α_G is injective and relatively open by Proposition 3.1(5). Thus, the reflexivity of G is reduced to check that α_G is surjective.

Under the assumption of (3), the compact-open topology in $G^\wedge$ coincides with $\omega(G^\wedge, G)$, and this implies that α_G is surjective. Now the reflexivity of G follows from (2).

In order to prove (4) observe first that $(G^\wedge, w(G^\wedge, G))^\wedge$ may be algebraically identified with G for any topological group G. Since $w(G^\wedge, G)$-compact subset of $G^\wedge$ are finite by (1), it follows that the compact-open topology in $(G^\wedge, w(G^\wedge, G))^\wedge$ coincides with $w(G, G^\wedge)$.

The assumption of (5) implies that the compact subsets of G are finite (Fact D), and the dual of G is $(G^\wedge, \omega(G^\wedge, G))$. By (4), $(G^\wedge, \omega(G^\wedge, G))^\wedge = (G, \omega(G, G^\wedge))$, and the "only if" part of Proposition 8.1 implies that $G^\wedge$ is pseudocompact.

The assertion (6) is [1, 4.4]. For the proof of (7) observe that any pseudocompact group G, can be identified with a quotient H/L, where H is a pseudocompact group such that all its countable subgroups are h-embedded and L is a closed pseudocompact subgroup of H [41, 5.5]. By (5) H is reflexive, and the quotient H/L need not be reflexive. This already proves (7), even more, there are reflexive pseudocompact groups which are not q-reflexive. It can be seen further that the closed subgroups of reflexive pseudocompact groups need not inherit reflexivity. In fact, take a non reflexive pseudocompact group G and let H and L be as above. Observe that $L^{\perp}$ is a closed subgroup of $H^{\wedge}$ and $(L^{\perp})^{\wedge} \cong \frac{H^{\wedge\wedge}}{L^{\perp\perp}}$. Since L is dually closed and H is reflexive, $\frac{H^{\wedge\wedge}}{L^{\perp\perp}} \cong H/L \cong G$ and consequently $H^{\wedge}$ is a pseudocompact reflexive group (by (5)) with a non-reflexive closed subgroup $L^{\perp}$. $\square$

Question 8.1 Are there strongly reflexive pseudocompact non-compact groups?. Are there q-reflexive pseudocompact noncompact groups?.

Observe that if the first question above had a positive answer, it could not be witnessed by a pseudocompact group G, with countable subgroups h-embedded. Otherwise, every countable subgroup H of G carries the maximal precompact topology and thus its dual is compact. By Fact D the countable subgroups of G must be closed therefore reflexive, by the strong reflexivity of G. The contradiction now is that any precompact group does not have infinite discrete subgroups.

Remark 4 The question whether there exist pseudocompact s-reflexive groups seems harder since pseudocompactness is not inherited by the closed subgroups. An instance of this fact is provided by the nonreflexive subgroup $L^{\perp}$ that appears in the proof of (7) of Proposition 8.2. It cannot be pseudocompact for otherwise $\alpha_{L^{\perp}}$ should be continuous. But being $L^{\perp}$ dually closed and dually embedded in the reflexive group $H^{\wedge}$, $\alpha_{L^{\perp}}$ is already open and bijective, which contradicts that $L^{\perp}$ is not reflexive.

Remark 5 Compare Propositions 7.1 (3) and 8.2 (3). If a topological group G is precompact and the compact subsets of G and of $G^{\wedge}$ are finite then G is reflexive. Under the stronger assumption that the group is pseudocompact, reflexivity is obtained only requiring that the compact subsets of G are finite.

In Section 9 we will give examples of reflexive pseudocompact groups whose compact subsets are not finite.

On the other hand there are precompact nonpseudompact groups G such that the compact subsets of $(G^{\wedge}, \omega(G^{\wedge}, G))$ are finite. Any non-measurable subgroup of $\mathbb{T}$ is such an example.

Example 8.1 A family of precompact nonpseudocompact groups. Let G denote a nonprecompact nuclear reflexive group, (in particular, G may be a noncompact locally compact group). Then G^+ is a precompact nonpseudocompact group.

Proof. In Example 6.1 it is proved that α_{G^+} is not continuous. By (2) in Proposition 3.1 there is a compact subset $K \subset G^\wedge$ that is not equicontinuous, in particular K must be infinite. Clearly $\omega(G^\wedge, G) \leq \tau_{co}$ implies that K is also $\omega(G^\wedge, G)$-compact which together with (1) of Proposition 8.2 implies that G^+ is not pseudocompact.

9 ω-bounded groups and P-groups

In 2008 Nickolas asked about the existence of nondiscrete reflexive P-groups. A positive answer was provided in [49], a very suggestive paper, where it is also proved that the dual of a P-group is ω-bounded, in particular pseudocompact. Thus, the obtention of a class of reflexive nondiscrete P-groups in the mentioned paper is accompanied with the parallel obtention of a class of pseudocompact, noncompact reflexive groups.

In this section we recall the definitions and collect some properties of the two classes of groups mentioned in the title, which are related by duality. As pointed out in [49], loosely speaking the class of P-groups is "near" to the class of discrete groups, and the same happens with their duals, the class of ω-bounded groups is "near" to that of compact groups.

A topological group G is said to be *ω-bounded* if every countable subset $M \subset G$ is contained in a compact subset of G. Clearly, "countable subset" may be replaced by "countable subgroup" in the definition of ω-bounded group. If G is ω-bounded and separable, then G is compact. The following fact whose proof is straightforward will be often used in the sequel:

Fact E. Every ω-bounded group is pseudocompact, and hence precompact.

We recall that a topological space X is a *P-space* if all of its G_δ-sets are open. An Abelian topological group which is a P-space is called a *P-group.* For general properties on P-spaces and P-groups the reader can consult [3], where a whole section is devoted to them. We only mention here what is needed for our aims, and for this reason the P-groups in the sequel are assumed to be Hausdorff.

A P-group has a basis of neighborhoods of the neutral element consisting of open subgroups and hence, it can be embedded in a product of discrete groups. Consequently, from Nuc_3 and Nuc_2 it follows that the class of P-groups is included in that of nuclear groups and therefore we can freely apply the results about nuclear or locally quasi-convex groups so far stated to this new subclass of groups.

Any topological group (G, τ) gives rise to a P-group. In fact, let $P\tau$ denote the topology generated by the G_δ subsets of τ. It is a group topology, and the pair $(G, P\tau)$ will be called the *P-modification* of (G, τ) (simply, the P-modification of G and PG, if the original topology of G is clear). The P-modification is a mechanism to obtain P-groups. In fact, the first example of a reflexive P-group was given in [49], and it consists on the P-modification of a product of discrete groups. Later on, in [50] the same authors prove that the P-modification of any locally compact group is also reflexive. This in particular implies that for a locally compact abelian group (G, τ) the topology $P\tau$ is not compatible with the duality $(G, G^\wedge)$ in the sense of [31].

We now state the facts about reflexivity known to hold for P-groups.

Proposition 9.1 *Let G be a P-group. Then:*

(1) The mapping α_G is injective and relatively open.

(2) The compact subsets of G are finite, and hence the compact-open topology in $G^\wedge$ coincides with $w(G^\wedge, G)$.

(3) The countable subgroups of G are discrete, thus closed and h-embedded.

(4) The closure of every countable subset of $G^\wedge$ is equicontinuous and therefore compact in the compact-open topology. Consequently, the dual group $G^\wedge$ is ω-bounded.

(5) A countable union of equicontinuous subsets of $G^\wedge$ is equicontinuous.

(6) The evaluation mapping $\alpha_{G^\wedge}$ is continuous.

If G is furthermore reflexive:

(7) Hausdorff quotients of G are reflexive. Therefore G is q-reflexive.

(8) G may not be strongly reflexive.

Proof. Assertion (1) follows from Proposition 3.1 (5).

In order to prove (2) and (3), observe that since G is a P-space its topology is finer than the countable complement topology on G. In the latter the compact sets are finite, and the countable subsets are discrete. Thus, the same properties hold in G, and this already proves (2). Let now H be a countable subgroup of G. It is discrete and consequently it must be closed in every group in which it can be embedded. In particular, H is closed in G, and by Proposition 6.1 (2) every character on H can be continuously extended to G. So assertion (3) is proved.

In order to prove (4) take a countable subset of $G^\wedge$, say $S := \{\psi_n, n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, $\psi_n^{-1}(\mathbb{T}_+)$ is a neighborhood of zero in G, and $V := \bigcap_{n\in\mathbb{N}} \psi_n^{-1}(\mathbb{T}_+)$ is also a neighborhood of zero in the P-group G. Hence $V^\triangleright$ is an equicontinuous subset of $G^\wedge$ which contains S. By Ascoli theorem $V^\triangleright$ is compact in the compact-open topology, which coincides with $w(G^\wedge, G)$ by (2). Thus $\overline{S}$ is equicontinuous and compact, and this also proves the last assertion of (4), in a different way of that given in [49]. The proof of (5) is similar. Pick for each $n \in \mathbb{N}$ an equicontinuous subset $L_n \subset G^\wedge$. If V_n denotes a neighborhood of zero in G such that $L_n \subset V_n^\triangleright$, then $\bigcup_{n\in\mathbb{N}} L_n \subset \bigcup_{n\in\mathbb{N}} V_n^\triangleright \subset (\bigcap_{n\in\mathbb{N}} V_n)^\triangleright$. Since $\bigcap_{n\in\mathbb{N}} V_n$ is also a neighborhood of zero, the assertion (5) follows.

Item (6) derives from the fact that $G^\wedge$ is ω-bounded, in particular pseudocompact and (2) of Proposition 8.2 applies.

Item (7) is proved in [49], and (8) can be derived from a classical example of Leptin in [67], recalled also in [49]. The example consists on a non reflexive group which is a closed subgroup of the P-modification of the product of c copies of the binary group $\{0, 1\}$. Hence, the P-modification itself is a reflexive non strongly reflexive P-group. $\square$

The following example of a non-reflexive ω-bounded group is given in [1, Example 2]:

Example 9.1 Let I be an arbitrary uncountable index set and $n \geq 2$ a natural number. For every point $x \in \mathbb{Z}(n)^I$, let $\operatorname{supp}(x)$ be the set of those $i \in I$ for which $x(i) \neq 0$. Then

$$\Sigma = \{x \in \mathbb{Z}(n)^I : |\operatorname{supp}(x)| \leq \omega\}$$

is a dense ω-bounded subgroup of the compact topological group $\mathbb{Z}(n)^I$ (see [3, Corollary 1.6.34]). It is shown in [27] that the evaluation mapping α_Σ is continuous but not surjective. In fact, $\Sigma^\wedge$ is the direct sum $\mathbb{Z}(n)^{(I)}$ with the discrete topology, and hence the bidual group $\Sigma^{\wedge\wedge}$ is the full product $\mathbb{Z}(n)^I$ with the Tychonoff product topology.

Question 9.1 *Are there nondiscrete strongly reflexive P-groups?.*

Fact E implies that every ω-bounded group G carries the topology $w(G, G^\wedge)$ and it can be identified with the dual of $(G^\wedge, w(G^\wedge, G))$, as stated in Proposition 8.2 (4). We relate the compact-open topology in $G^\wedge$ with the P-modification of $w(G^\wedge, G)$ as follows:

Lemma 9.2 *Let G be an ω-bounded group. Then:*

$$\omega(G^\wedge, G) \leq P\omega(G^\wedge, G) \leq \tau_{co}$$

Moreover, if every compact subset of G is contained in the closure of a countable set (in particular, if compact subsets of G are separable), then $P\omega(G^\wedge, G) = \tau_{co}$.

Proof. The first inequality is obvious, and in order to prove the second one take $V := \bigcap_{n\in\mathbb{N}} F_n^{\triangleright}$, with $F_n \subset G$ finite. Clearly, V is a standard basic neighborhood of zero in $P\omega(G^\wedge, G)$. Since $\bigcup F_n$ is contained in a compact subset $K \subset G$, we get that $V = (\bigcup F_n)^{\triangleright} \supset K^{\triangleright}$ is a zero neighborhood in $(G^\wedge, \tau_{co})$.

For the last assertion, pick a compact subset $K \subset G$. By the assumption, there exists a countable set $S := \{x_n, n \in \mathbb{N}\} \subset G$ such that $K \subset \overline{S}$. Taking polars,

$$\overline{S}^{\triangleright} = S^{\triangleright} = \bigcap_{n\in\mathbb{N}} \{x_n\}^{\triangleright} \subset K^{\triangleright}$$

Consequently $K^{\triangleright}$ is a neighborhood of zero in $P\omega(G^\wedge, G)$, which proves that $\tau_{co} \leq P\omega(G^\wedge, G)$.

□

If the original group is not ω-bounded the last inequality above stated may not hold, as we prove with the next example.

Example 9.2 *If G is a locally compact noncompact topological group, then $P\omega(G^\wedge, G) \not\leq \tau_{co}$ (being τ_{co} the compact-open topology in $G^\wedge$).*

Proof. A neighborhood of zero in $P\omega(G^\wedge, G)$ which is not a neighborhood in τ_{co} can be found as follows. Since G^+ is not pseudocompact (see Example 8.1), it is neither ω-bounded. There exists then a countable set $S = \{x_n, n \in \mathbb{N}\}$ in G such that for every compact subset K of G^+, $S \setminus K \neq \emptyset$. Clearly, $S^{\triangleright}$ is a neighborhood of zero in $P\omega(G^\wedge, G)$. It is not a neighborhood of zero in τ_{co}, for otherwise it would contain the polar of a compact subset K of G, say $K^{\triangleright} \subset S^{\triangleright}$. Now taking inverse polars,

$$K^{\triangleright\triangleleft} \supset S^{\triangleright\triangleleft} \supset S$$

As G is locally compact, it has the qcp (see section 3.2), and $K^{\triangleright\triangleleft}$ is again compact in G. By the Glicksberg Theorem it is also compact in G^+, and this contradicts the assumption on S. □

In [38] it was proved that if G is ω-bounded, $(G^\wedge, \omega(G^\wedge, G))$ and $(G^\wedge, P\omega(G^\wedge, G))$ admit the same continuous characters. More is true:

Proposition 9.3 *Let G be an ω-bounded group. Then the groups $(G^\wedge, \omega(G^\wedge G))^\wedge$ and $(G^\wedge, P\omega(G^\wedge G))^\wedge$ may be algebraically and topologically identified with $(G, \omega(G, G^\wedge))$.*

Proof. The comment preceding this proposition yields that the groups $(G^\wedge, \omega(G^\wedge, G))^\wedge$ and $(G^\wedge, P\omega(G^\wedge, G))^\wedge$ have the same elements. Since $(G^\wedge, \omega(G^\wedge, G))$ is precompact, both of them can be algebraically identified with G.

By 8.2 (1), the $w(G^\wedge, G)$-compact subsets of $G^\wedge$ are finite, and the $Pw(G^\wedge, G)$-compact subsets are finite as well (Proposition 9.1). Therefore the corresponding compact-open topologies in the dual G are equal, and coincide with the pointwise convergence topology $w(G, G^\wedge)$. □

Corollary 9.4 *A topological group G is ω-bounded if and only if it is the dual of a P-group.*

Remark 6 As said in the initial comments of this section, the dual of a P-group is ω-bounded. On the other hand, the dual of an ω-bounded group is a P-group if certain requirements on the compact subsets hold, (see Lemma 9.2). In the reverse sense duality works better, as proved in Corollary 9.4.

We detail now some other facts concerning reflexivity of ω-bounded groups.

Proposition 9.5 *Let G be an ω-bounded group. Then,*

(1) The evaluation mapping α_G is continuous, open and injective. Therefore G is reflexive if and only if α_G is surjective.

(2) G is reflexive provided that every compact subset of G is contained in the closure of a countable set.

(3) If G is reflexive, the closed subgroups of G are reflexive. Hence, every reflexive ω-bounded groups is s-reflexive.

(4) If every compact subset of G is contained in the closure of a countable set then G has sufficient duality.

Proof. The proof of (1) is covered by Fact E and (2) of Proposition 8.2. In order to prove (2), we must only check that α_G is surjective. To that end, take a continuous character $\varphi : G^\wedge \to \mathbb{T}$. There exists then a compact subset K in G such that $\varphi(K^\triangleright) \subset \mathbb{T}_+$. As in Lemma 9.2, pick $S = \{x_n, n \in \mathbb{N}\} \subset G$ such that $K \subset \overline{S}$. Then $S^\triangleright$ is a neighborhood of zero in $Pw(G^\wedge, G)$, and $\varphi(S^\triangleright) \subset \varphi(K^\triangleright) \subset \mathbb{T}_+$ implies that φ is $(G^\wedge, Pw(G^\wedge, G))$-continuous. By Proposition 9.3, $\varphi \in (G^\wedge, w(G^\wedge, G))^\wedge = G$ and hence $\varphi = \alpha_G(g)$ for some $g \in G$.

In order to prove (3) take a closed subgroup H. It is w-bounded, therefore α_H is continuous by (1). By (2) of Proposition 6.1 H is dually closed and dually embedded, and (4) of Proposition 4.2 applies to obtain that α_H is open and bijective. Therefore H is reflexive and the group G is s-reflexive.

Item (4): Since every closed subgroup H of G is dually closed, the continuous characters of G/H separate points. On the other hand, G is reflexive by (2), and this implies that $G^\wedge$ is also reflexive. From Lemma 9.2 we obtain that $G^\wedge$ is a P-group, and (7) of Proposition 9.1 yields that all the Hausdorff quotients of $G^\wedge$ are reflexive. Thus, G has sufficient duality. □

Proposition 9.6 *A reflexive noncompact ω-bounded group need not be strongly reflexive.*

Proof. Take into account that the dual group of a strongly reflexive group is also strongly reflexive. In (8) of Proposition 9.1 an example of a reflexive non strongly reflexive P-group G is presented. Then $G^\wedge$ is an ω-bounded reflexive group, which is not strongly reflexive. □

Question 9.2 Are there noncompact strongly reflexive ω-bounded groups?

10 The algebraic structure of (strongly) reflexive groups.

It is well known that the algebraic structure of a topological group determines properties of a topological nature. In the present section we gather some results in this line which have to do with reflexivity.

The paper [45] faces the following general question:

Question 10.1 Does every infinite abelian group admit a non-discrete reflexive group topology?

Question 10.2 *[41, Q. 25].* Does every pseudocompact group admit a pseudocompact group topology with no infinite compact subsets?

This was partially answered in [48], where it is proved that every pseudocompact abelian group G with $|G| \leq 2^{2^c}$ admits a pseudocompact group topology τ for which all the countable subgroups are h-embedded. By Fact D such a topology has finite compact subsets.

Among the impressive results obtained by Gabriyelyan very recently, we think the following are upmost interesting:

Example 10.1 *[44]* The group $\mathbb{Z}$ admits a non-discrete reflexive group topology.

Theorem 10.1 *[45] Every abelian group G of infinite exponent admits a non-discrete reflexive group topology.*

Corollary 10.2 *The Prüfer group $\mathbb{Z}(p^\infty)$ admits a nondiscrete reflexive group topology.*

Due to the above theorem only the following problem remains open from the general Question 10.1:

Question 10.3 *[45, Problem 2]* Does every infinite Abelian group G of finite exponent (in particular $\mathbb{Z}(p)^{(\mathbb{N})}$ for a prime p) admit a non-discrete reflexive group topology?

Note that for $p = 2, 3$, every locally quasi-convex (hence, every reflexive) topology on $\mathbb{Z}(p)^{(\mathbb{N})}$ has a basis of zero-neighborhoods formed by open subgroups.

Theorem 10.1 suggests now the following:

Question 10.4 Does every abelian group G of infinite exponent admits a non-discrete strongly reflexive group topology?

For the particular case of the groups $\mathbb{Z}$ and $\mathbb{Z}(p^\infty)$ we have partial answers:

Example 10.2 (a) The group $\mathbb{Z}$ admits an s-reflexive and q-reflexive non-discrete group topology. Indeed, let τ be a non-discrete reflexive group topology on $\mathbb{Z}$, whose existence is ensured by Example 10.1. Let us prove that closed subgroups and Hausdorff quotients of $(\mathbb{Z}, \tau)$ are reflexive. Since the proper quotients of $\mathbb{Z}$ are finite, they are reflexive and $(\mathbb{Z}, \tau)$ is q-reflexive. It is also s-reflexive. In fact, take a closed nonnull subgroup H. As it has finite index, it is open and by [22], reflexive.

(b) For every prime number p, the Prüfer group $\mathbb{Z}(p^\infty)$ admits an s-reflexive and q-reflexive non-discrete group topology. Let τ be a non-discrete reflexive group topology on $\mathbb{Z}(p^\infty)$. Pick a proper closed subgroup H of $G = (\mathbb{Z}(p^\infty), \tau)$. Then H is finite, hence reflexive. The quotient G/H is reflexive as well, again by [22]. This proves that $(\mathbb{Z}(p^\infty))$ is s-reflexive and q-reflexive.

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References

[1] S. Ardanza-Trevijano, M. J. Chasco, X. Dominguez and M. Tkachenko, *Precompact non-compact reflexive abelian grups*, Forum Math. Vol 24 (2012) 289–302.

[2] R. Arens, *Duality in linear spaces*, Duke Math J. 14 (1947) 787–794.

[3] A. V. Arhangel'skii and M. G.Tkachenko, *Topological Groups and Related Structures.*, Atlantis Press/World Scientific, Amsterdam-Paris, 2008.

[4] L. Außenhofer, *Contributions to the Duality Theory of Abelian Topological Groups and to the Theory of Nuclear Groups*, Dissertationes Mathematicae, CCCLXXXIV, Warszawa, 1999.

[5] L. Außenhofer, *A duality property of an uncountable product of* $\mathbb{Z}$, Math. Z. 257 no 2 (2007) 231-237.

[6] L. Außenhofer, *A survey on nuclear groups* , Nuclear Groups and Lie Groups, Research and Exposition in Mathematics, Volume 24. (Edited by E. Martín Peinador and J. Núñez García) Heldermann Verlag, 2001

[7] L. Außenhofer, *On the nuclearity of dual groups*, Preprint, Passau.

[8] L. Außenhofer, D. Dikranjan and E. Martín-Peinador, *Locally quasi-convex compatible topologies on a topological group*, Preprint 2010

[9] W. Banaszczyk, *Additive Subgroups of Topological Vector Spaces*, Lecture Notes in Mathematics 1466. Springer-Verlag, Berlin Heidelberg, New York, 1991.

[10] M. Banaszczyk and W. Banaszczyk, *Characterization of nuclear spaces by means of additive subgroups.* Math. Z. 186 (1984), 125-133.

[11] W. Banaszczyk, *Countable products of LCA groups: their closed subgroups, quotients and duality properties*, Colloq. Math. 59 (1990), 52–57.

[12] W. Banaszczyk, *Pontryagin duality for subgroups and quotients of nuclear spaces*, Math. Ann. 273 (1986), 653-664.

[13] W. Banaszczyk, M. J. Chasco and E. Martín-Peinador, *Open subgroups and Pontryagin duality*, Math. Z. (2) 215 (1994) 195–204.

[14] W. Banaszczyk and E. Martín-Peinador, *The Glicksberg Theorem on Weakly Compact Sets for Nuclear Groups*, Ann. N. Y. Acad. Sci., General Topology and Applications, Volume 788, no 1, (1996) pages 34–39.

[15] R. Beattie, and H. P. Butzmann, *Convergence structures and applications to functional analysis*, (Kluwer Academic 2002).

[16] R. Brown, P. J. Higgings and S. A. Morris, *Countable products and sums of lines and circles: their closed subgroups, quotients and duality properties*, Math. Proc. Cambridge Philos. Soc. 78 (1975), 19–32.

[17] M. Bruguera, *Grupos topológicos y grupos de convergencia: estudio de la dualidad de Pontryagin*, Doctoral Dissertation. Barcelona, 1999.

[18] M. Bruguera and M.J. Chasco, *Strong reflexivity of abelian groups*, Czechoslovak Math. J. 51(126) (2001), no. 1, 213–224.

[19] M. Bruguera and E. Martín-Peinador. *Banach-Dieudonné Therem revisited*, J. Aust. Math. Soc. 75 (2003), 69-83.

[20] M. Bruguera and E. Martín-Peinador, *Open subgroups, compact subgroups and Binz-Butzmann reflexivity*, Topology Appl., 72 (1996) 101-111.

[21] M. Bruguera, E. Martín-Peinador and V. Tarieladze, *Eberlein-Smulyan theorem for abelian topological groups*, J. London Math. Soc., 70 (2) (2004) 341- 355

[22] M. Bruguera, M.J. Chasco, E. Martín-Peinador and V. Tarieladze, *Completeness properties of locally quasi-convex groups*, Topology Appl., 111, (2001) 81– 9.

[23] M. Bruguera and M. Tkachenko, *Duality in the class of precompact Abelian groups and the Baire property.* To appear in Journal of Pure and Applied Algebra

[24] H.-P. Butzmann, *Duality theory for convergence groups*, Topology Appl. 111 (2001) 95–104.

[25] M. J. Chasco, *Pontryagin duality for metrizable groups.* Arch. Math. 70 (1998), 22-28.

[26] M.J. Chasco and X. Domínguez, *Topologies on the direct sum of topological abelian groups*, Topology Appl. 133 , no 3 (2003) 209–223.

[27] M. J. Chasco and E. Martín-Peinador, *An approach to duality on Abelian precompact groups*, J. Group Theory 11 (2008), 635–643.

[28] M.J. Chasco and E. Martín-Peinador, *Binz-Butzmann duality versus Pontryagin Duality*, Arch. Math. (Basel) 63 (3) (1994), 264–270.

[29] M.J. Chasco and E. Martín-Peinador, *Pontryagin reflexive groups are not determined by their continuous characters*, Rocky Mountain J. Math. 28 , no 1 (1998) 155–160.

[30] M. J. Chasco, and E. Martín-Peinador, *On strongly reflexive topological groups*, Appl Gen Topol., 2(2) (2001), 219-226.

[31] M. J. Chasco, E. Martín-Peinador and V. Tarieladze, *On Mackey topology for groups*, Studia Math. 132 (3) (1999) 257-284

[32] H. Chu, *Compactification and duality of topological groups*, Trans. Amer. Math. Soc. 123 (1966), 310–324.

[33] W.W. Comfort, S. Hernández, D. Remus and F.J. Trigos-Arrieta, *Open questions on topological groups*, Nuclear Groups and Lie Groups, Research and Exposition in Mathematics, Volume 24. (Edited by E. Martín Peinador and J. Núñez García) Heldermann Verlag, 2001

[34] W. W. Comfort, S. U. Raczkowski and F. J. Trigos-Arrieta, *The dual group of a dense subgroup*, Czechoslovak Math. J. 54 (129) (2004) 509–533.

[35] W.W. Comfort, S.U. Raczkowski and F.J. Trigos-Arrieta, *Making group topologies with, and without, convergent sequences*, Appl. Gen. Topology **7**, (2006), no. 1.

[36] W. W. Comfort and K. A. Ross, *Topologies induced by groups of characters.* Fund. Math. 55 (1964) 283–291.

[37] W. W. Comfort and K. A. Ross, *Pseudocompactness and uniform continuity in topological groups*, Pacific J. Math. 16 (1966) 483–496.

[38] J.M. Díaz Nieto, *On refinements of ω-bounded group topologies.* Preprint 2011.

[39] D. Dikranjan, Iv. Prodanov and L. Stoyanov, *Topological Groups: Characters, Dualities and Minimal Group Topologies,* Pure and Applied Mathematics, vol. **130**, Marcel Dekker Inc., New York-Basel, 1989.

[40] D. Dikanjan, E. Martín-Peinador and V. Tarieladze, *A class of metrizable locally quasi-convex groups which are not Mackey*, arxiv.org/pdf/1012.5713v1

[41] D. Dikranjan and M. Tkachenko, *Sequential completeness of quotient groups*, Bull. Austral. Math. Soc. 61 (2000) 129–151.

[42] D. Dikranjan and D. Shakhmatov, *Selected topics from the structure theory of topological groups*, In Elliott Pearl, editor, Open problems in topology, 389–406. Elsevier, 2007.

[43] D. Dikranjan and D. Shakhmatov, *Quasi-convex density and determining subgroups of compact abelian groups*, J. Math. Anal. Appl. 362 (2010), 42-48

[44] S. Gabriyelyan, *Groups of quasi-invariance and the Pontryagin duality*, Topology Appl. 157, no 18, (2010) 2786–2802.

[45] S. Gabriyelyan, *Reflexive group topologies on Abelian groups*, J. Group Theory 13, no. 6, (2010) 891–901.

[46] J. Galindo, S. Hernández, and T. S. Wu, *Recent results and open questions relating Chu duality and Bohr compactifications of locally compact groups*, Open problems in topology II (E. Pearl, ed.), Elsevier Science. 2007

[47] J. Galindo and S. Hernández, *Pontryajin van-Kampen reflexivity for free abelian topological groups*, Forum Math. 11 (1999) 399-415.

[48] J. Galindo and S. Macario, *Pseudocompact group topologies with no infinite compact subsets*, J. Pure and Appl. Algebra 215 (2011) 655-663.

[49] J. Galindo, L. Recoder-Núñez and M. Tkachenko, *Nondiscrete P-groups can be reflexive*, Topology Appl. 158 (2011) 194-203.

[50] J. Galindo, L. Recoder-Núñez and M. Tkachenko, *Reflexivity of prodiscrete groups*, J. Math. Anal. Appl. 384, no. 2 (2011) 320330.

[51] H. Glöckner, R. Gramlich and T. Hartnick, *Final group topologies, Kac-Moody groups and Pontryagin duality*, Israel J. Math. 177 (2010), 49-101

[52] S. Hernández, *Pontryagin duality for topological abelian groups*, Math. Z. 238, no. 3, (2001) 493–503.

[53] S. Hernández, *Some new results about the Chu topology of discrete groups*, Monats. Math., 149 (2006), 215–232

[54] S. Hernández, *The Bohr topology of discrete non-abelian groups*, J. Lie Theory 18 (2008), no. 3, 733–746,

[55] S. Hernández, S. Macario, F. J. Trigos-Arrieta, *Uncountable products of determined groups need not be determined*, J. Math. Anal. Appl. 348 (2008) 834–842.

[56] S. Hernández and S. Macario, *Dual properties in totally bounded Abelian groups*, Arch. Math. 80 (2003) 271–283.

[57] S. Hernández and J. Trigos, *Group duality with the topology of precompact convergence* J. Math. Anal. Appl, 303 (2005), 274-287.

[58] S. Hernández and V. Uspenskij, *Pontryagin Duality for Spaces of Continuous Functions*, J. Math. Anal. Appl 242, no 2, (2000) 135-144

[59] E. Hewitt, *Rings of real-valued continuous functions I*, Trans. Amer. Math. Soc, 64 (1948) 45-99.

[60] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vol. I: Structure of topological groups. Integration theory, group representations, Die Grundlehren der math. Wissenschaften, Bd. 115, Academic Press, New York and Springer-Verlag, Berlin, 1963.

[61] K. H. Hofmann and S. A. Morris, *The structure of compact groups.* De Gruyter Studies in Mathematics 25, 1998.

[62] S. Kaplan, *Extensions of the Pontryagin duality I: Infinite products*, Duke Math. J. 15 (1948), 649-658.

[63] S. Kaplan, *Extensions of the Pontryagin duality II: Direct and inverse limits*, Duke Math. J. 17 (1950), 419-435.

[64] S. H. Kye, *Pontryagin duality in real linear topological spaces*, Chinese J. of Math., 12 (2) (1984) 129-136.

[65] S.H. Kye, *Several reflexivities in topological vector spaces*, J. Math. Anal. Appl. 139 (1989), no. 2, 477–482.

[66] L. de Leo, *Weak and strong topologies in topological abelian groups*, PhD Thesis, Universidad Complutense de Madrid, July 2008.

[67] H. Leptin, *Zur Dualitätstheorie Projektiver Limites Abelscher Gruppen*, Abh. Math. Sem. Univ. Hamburg 19 (1955) 264-268. MR 16, 899

[68] E. Martín-Peinador, *A reflexive admissible topological group must be locally compact*, Proc. Amer. Math. Soc. 123 no. 11 (1995) 3563–3566.

[69] E. Martín-Peinador and V. Tarieladze, *A property of Dunford-Pettis type in topological groups*, Proc. Amer. Math. Soc., 132, (2004) 1827-1837

[70] P. Nickolas, *Reflexivity of topological groups*, Proc. Amer. Math. Soc. 65 (1977), 137–141

[71] N. Noble, *k-groups and duality* , Trans. Amer. Math. Soc. 151 (1970), 551-561.

[72] V. Pestov, *Free Abelian topological groups and the Pontryagin-Van Kampen duality*, Bull. Austr. Math. Soc., 52, (1995) 297-311

[73] L. Pontrjagin, *Topological groups.* Princeton University Press, Princeton 1946. (Translated from the Russian).

[74] S. U. Raczkowski and J. Trigos. *Duality of totally bounded Abelian groups*, Bol. Soc. Mat. Mexicana, 3 (7) (2001) 1-12.

[75] W. Roelcke, S. Dierolf, *Uniform Structures on topological groups and their quotients*, McGraw-Hill, New York, 1981.

[76] M. F. Smith, *The Pontrjagin duality theorem in linear spaces*, Ann. of Math. 1952, 56 (2), 248-253.

[77] M. G. Tkachenko, *Compactness type properties in topological groups*, Czechoslovak Math. J. 38 (113) (1988), 324-341

[78] N. Th. Varopoulos, *Studies in harmonic analysis*, Proc. Camb. Philos. Soc. 60,(1964) 465-516

[79] N.Ya. Vilenkin, *The theory of characters of topological Abelian groups with boundedness given* , Akad. Nauk SSSR., Izv. Math. 15 (1951), 439-462.

[80] A. Weil, *Sur les espaces à structure uniforme et sur la topologie générale*, Publ. Math. Univ. Strasbourg, Hermann, Paris, 1937.

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