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Covering properties in intuitionistic fuzzy topological spaces

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Abstract

Purpose – D.Çoker constructed the fundamental theory of intuitionistic fuzzy topological spaces. The purpose of this paper is to introduce a new concept of compactness and a definition of paracompactness for intuitionistic fuzzy topological spaces, and obtain several preservation properties.

Design/methodology/approach – Two new covering properties in intuitionistic fuzzy topological spaces are defined and studied.

Findings – Relations on these new properties and covering properties on fuzzy topology in the Chang's sense are obtained.

Research/limitations/implications – Clearly, this paper is devoted to intuitionistic fuzzy topological spaces.

Practical implications – The main applications are in the mathematical field.

Originality/value – The paper shows original results on fuzzy topology.

Keywords Cybernetics, Mathematics, Topology, Fuzzy logic

Paper type Research paper

Introduction

The introduction of "intuitionistic fuzzy sets" is due to Atanassov (1983), and this theory has been developed in many papers (Atanassov, 1986, 1988, 1999). In particular, Çoker and co-workers have constructed the basic concepts of the intuitionistic fuzzy topological spaces, specially fuzzy compactness and fuzzy connectedness, and have obtained many results on it (Çoker, 1996, 1997; Çoker and Demirci, 1995; Çoker and Eş 1995; Eş and Çoker, 1996, 1997). Finally, Lee and Lee (2000) showed that the category of fuzzy topological spaces in the sense of Chang is a bireflective full subcategory of that of intuitionistic fuzzy topological spaces, and Wang and He (2000) showed that every intuitionistic fuzzy set may be regarded as an L-fuzzy set for some appropriate lattice L.

In this paper, we define a new concept of compactness for intuitionistic fuzzy topological spaces and the paracompactness for these spaces.

Fundamental concepts

The fundamental concept of cover is due to Çoker:

Definition 1. (Çoker, 1997) Let (X, τ) be an IFTS. If a family $\mathcal{G} = \{\langle x, \mu_{G_j}, \gamma_{G_j}\rangle | j \in J\}$ of IFOSs in X satisfies the condition:

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$$\bigcup_{j\in J}\langle x, \mu_{G_j}, \gamma_{G_j}\rangle = 1.$$

then it is called a fuzzy open cover of X.

Definition 2. (Çoker, 1997) Let (X, τ) be an IFTS and A an IFS in X. If a family $\mathcal{G} = \{\langle x, \mu_{G_i}, \gamma_{G_i}\rangle | j \in J\}$ of IFOSs in X satisfies the condition:

$$A \subseteq \bigcup_{j\in J}\langle x, \mu_{G_i}, \gamma_{G_i}\rangle,$$

then it is called a fuzzy open cover of A.

Remark. In this paper, we assume that an IFTS (X, τ) is in the sense of Lowen (Eş and Çoker, 1996), i.e. (X, τ) is an IFTS as in Çoker (1997) and each IFS in the form $c_{\alpha,\beta} = \{\langle x, c_\alpha, c_\beta\rangle | x \in X\}$, where $\alpha, \beta \in I$ are arbitrary and $\alpha + \beta \leq 1$, belongs to τ.

Definition 3. If μ is a fuzzy set in X, and c_r is a constant fuzzy set in X, we denote $\mu - c_r$ the fuzzy set such that $(\mu - c_r)(x) = \mu(x) - r$ if $\mu(x) - r \geq 0$ and $(\mu - c_r)(x) = 0$ otherwise, analogously we denote $\mu + c_r$ the fuzzy set such that $(\mu + c_r)(x) = \mu(x) + r$ if $\mu(x) + r \leq 0$ and $(\mu + c_r)(x) = 1$ otherwise. If $A = \langle x, \mu_A, \gamma_A\rangle$ is an IFS on X, we denote $A - r = \langle x, \mu_A - c_r, \gamma_A + c_r\rangle$ for each $r \in (0, 1]$.

Definition 4. If (X, τ) is an IFTS and $A = \langle x, \mu_A, \gamma_A\rangle$ is an IFS of X, we will say that A is strong intuitionistic compact if for each fuzzy open cover $\mathcal{G} = \{G_j\}_{j\in J}$ of A and for each $r \in (0, 1]$, there exist a finite subset $J_0 \subset J$ such that $\{G_j\}_{j\in J_0}$ is fuzzy open cover of $A - r$. We say that (X, τ) is strong intuitionistic compact if for each $h \in [0, 1]$, $h \in [0, 1]$, $\tilde{c}_h \equiv \langle x, c_h, 1 - c_h\rangle$ is an strong intuitionistic compact IFS.

Proposition 1. Let (X, τ) be an IFTS and $\tau_1 = \{\mu_G | G \in \tau\}$, $\tau_2 = \{1 - \gamma_G | G \in \tau\}$ the associate fuzzy topologies on X in the Chang's sense. If (X, τ) is strong intuitionistic compact, then we have that the fuzzy topological spaces (X, τ_1) is fuzzy compact with the definition of (Lowen, 1976).

Proof. Let $h \in [0, 1]$ and let $\mathcal{U} = \{\mu_{G_j} | j \in J\}$ be a family of fuzzy open sets in τ_1 such that:

$$c_h \leq \bigvee_{j\in J}\mu_{G_j}.$$

Then:

$$1 - c_h \geq 1 - \bigvee_{j\in J}\mu_{G_j} = \bigwedge_{j\in J}(1 - \mu_{G_j}) \geq \bigwedge_{j\in J}\gamma_{G_j}$$

i.e. $\mathcal{U}^* = \{\langle x, \mu_{G_j}, \gamma_{G_j} | j \in J\}$ is a fuzzy open cover of $\tilde{c}_h \equiv \langle x, c_h, 1 - c_h\rangle$ and, by the hypothesis, for each $r \in (0, 1]$ there exists a finite subset $J_r \subset J$ such that $\mathcal{V}^* = \{\langle x, \mu_{G_j}, \gamma_{G_j}\rangle | j \in J_r\}$ is a fuzzy open cover of $\tilde{c}_h - r$. In this case, $\mathcal{V} = \{\mu_{G_j} | j \in J_r\}$ is a finite subfamily of $\mathcal{U}$ which covers $c_h - r$. Thus, (X, τ_1) is fuzzy compact in the sense of Lowen. □

Proposition 2. Let (X, τ_0) be a fuzzy topological space in the sense of Lowen and $\tau = \{\langle x, \mu_A, 1 - \mu_A\rangle | A \in \tau_0\}$ the associate IFT on X. If (X, τ_0) is compact with the definition of Lowen, then we have that (X, τ) is a strong intuitionistic compact IFTS.

Proof. For each $h \in [0, 1]$, let $\mathcal{U} = \{\langle x, \mu_j, 1 - \mu_j\rangle | j \in J\}$ be a fuzzy open cover of $\tilde{c}_h \equiv \langle x, c_h, 1 - c_h\rangle$, then:

$$c_h \leq \bigvee_{j\in J} \mu_j,$$

and by the hypothesis, for each $r \in (0, 1]$ there exist a finite subset $J_r \subset J$ such that:

$$c_h - r \leq \bigvee_{j\in J_r} \mu_j.$$

Thus, $\mathscr{V} = \{\langle x, \mu_j, 1-\mu_j\rangle | j \in J_r\} \subset \mathscr{U}$ is a fuzzy open cover of $c_h - r$. □

Proposition 3. Let (X, τ), (Y, s) be IFTSs and $f : X \to Y$ be a fuzzy continuous map. If A is an strong intuitionistic compact IFS in (X, τ) then so is $f(A)$ in (Y, s).

Proof. Let $A = \langle x, \lambda_A, \vartheta_A\rangle$ and $\{\langle y, \mu_{u_j}, \gamma_{u_j}\rangle | j \in J\}$ be a fuzzy open cover of $f(A) = \langle x, f(\lambda_A), 1 - f(1-\vartheta_A)\rangle$. Then we have that $\{\langle x, f^{-1}(\mu_{u_j}), f^{-1}(\gamma_{u_j})\rangle | j \in J\}$ is a fuzzy open cover of A, too. By the hypothesis, for each $r \in (0, 1]$, there exists a finite subset $J_0 \subset J$ such that $\{\langle x, f^{-1}(\mu_{u_j}), f^{-1}(\gamma_{u_j})\rangle | j \in J_0\}$ is a fuzzy open cover of $A - r$. Then:

$$\begin{cases} \lambda_A - c_r \leq \bigvee_{j\in J_0} f^{-1}(\mu_{u_j}) \\ \vartheta_A + c_r \geq \bigwedge_{j\in J_0} f^{-1}(\gamma_{u_j}) \end{cases}$$

and this implies that:

$$\begin{cases} f(\lambda_A) - c_r \leq f(\lambda_A - c_r) \leq \bigvee_{j\in J_0} \mu_{u_j} \\ 1 - f(1-\vartheta_A) + c_r \geq 1 - f(1 - \vartheta_A - c_r) \geq \bigwedge_{j\in J_0} \gamma_{u_j} \end{cases}$$

and:

$$f(A) - r \leq \bigcup_{j\in J_0} \langle y, \mu_{u_j}, \gamma_{u_j}\rangle.$$

□

Corollary. Let (X, τ), (Y, s) be IFTSs and $f : X \to Y$ a fuzzy continuous onto map. If (X, τ) is strong intuitionistic compact, then so is (Y, s).

Proof. Obviously, for each $h \in [0, 1]$ and each constant IFS $\tilde{c}_h$ we have $f(\tilde{c}_h) = \tilde{c}_h$. The IFS c_h of X is strong intuitionistic compact by the hypothesis, then the result is clear. □

Definition 5. Let (X, τ) be an IFTS and $\mathscr{U} = \{\langle x, \mu_{G_j}, \gamma_{G_j}\rangle | j \in J\}$ and $\mathscr{V} = \{\langle x, \mu_{A_i}, \gamma_{A_i}\rangle | i \in I\}$ be two families of IFOSs in X. We will say that $\mathscr{V}$ refines $\mathscr{U}$ (or $\mathscr{V}$ is a refinement of $\mathscr{U}$), if for each $i \in I$ there exists some $j \in J$ such that $\langle x, \mu_{A_i}, \gamma_{A_i}\rangle \subseteq \langle x, \mu_{G_j}, \gamma_{G_j}\rangle$.

Definition 6. Let (X, τ) be an IFTS and $\mathscr{U} = \{\langle x, \mu_{G_j}, \gamma_{G_j}\rangle\}_{j\in J}$ be a family of IFSs in X. We will say that $\mathscr{U}$ is locally finite in an IFS A of X if, for each intuitionistic fuzzy point $p \in A$, there exists an ε-neighbourhood N of p such that $N \cap \langle x, \mu_{G_j}, \gamma_{G_j}\rangle = 0_\sim$ for all $j \in J$ in the complement of a finite subset of J.

Definition 7. If (X, τ) is an IFTS and A is an IFS of X, we will say that A is paracompact if for each fuzzy open cover $\mathscr{U} = \{\langle x, \mu_{G_j}, \mu_{G_j} \langle | j \in J\}$ of A and for each $r \in (0, 1]$, there exists a refinement of u which is locally finite in A and a fuzzy open cover of $A - r$. We will say that X is paracompact if $1_\sim$ is a paracompact IFS.

Remark. An strong intuitionistic compact IFTS is a paracompact IFTS.

Note. There is no problem with these concept and the result of Wang and He, because here we used ε-neighbourhoods (Lupiañez, 2006).

Proposition 4. If (X, τ) is a paracompact IFTS, then c_1 is a^*-paracompact fuzzy set of (X, τ_1).

Proof. The definition of *-paracompact fuzzy set is in (Abd El-Monsef *et al.*, 1992).

Let $\mathscr{U}^* = \{\mu_{G_j}\}_{j \in J}$ be a family of open fuzzy sets such that:

$$c_1 \leq \bigvee_{j \in J} \mu_{G_j}.$$

Then $\mathscr{U}^* = \{\langle x, \mu_{G_j}, 1 - \mu_{G_j} \rangle | j \in J\}$ is a fuzzy open cover of $1_\sim = \langle x, c_1, 1 - c_1 \rangle$ and, by the hypothesis, for each $r \in (0, 1]$ there exists a refinement $\mathscr{V}^* = \{\langle x, v_s, \eta_s \rangle | s \in S\}$ of $\mathscr{U}^*$ which is locally finite in $1_\sim$ and a fuzzy open cover of $1_\sim - r$. Thus, $\mathscr{V} = \{v_s | s \in S\}$ is an open refinement of $\mathscr{U}$, which covers $c_1 - r = c_1 - r$ and is *-locally finite in c_1. Indeed, suppose that there is some fuzzy point $e = z_\lambda$ such that $z_\lambda \leq c_1$ and, for every fuzzy open set μ which is quasi-coincident with e, we have $\mu \wedge v_s \neq 0$ for infinitely many $s \in S$. Now consider the IFP $z(\alpha\ \beta)$, where $\alpha = 1 - \lambda$, $\beta = \lambda$. Hence, there exists an ε-neighbourhood $\langle x, \mu^*, \gamma^* \rangle \in \tau$ of $z(\alpha\ \beta)$ with $\alpha \langle \mu^*(z),\ \beta \rangle \gamma^*(z)$ and $\langle x, \mu^*, \gamma^* \rangle \cap \langle x, v_s, \eta_s \rangle = 0_\sim$ for all $s \in S$ in the complement of a finite subset of S. Since, $\mu^*(z) + \lambda > 1 - \lambda + \lambda = 1$, μ^* is a fuzzy open set in τ_1 which is quasi-coincident with e and $\mu^* \wedge v_s \neq 0$ for infinitely many $s \in S$. In this case $\langle x, \mu^*, \gamma^* \rangle \cap \langle x, v_s, \eta_s \rangle \neq 0$ for infinitely many $s \in S$, which is a contradiction. □

Proposition 5. Let (X, τ_0) be a fuzzy topological space in the sense of Lowen and $\tau = \{\langle x, \mu_A, 1 - \mu_A \rangle | A \in \tau_0\}$ the associate IFT on X. If c_1 is a *-paracompact fuzzy set of (X, τ_0) then (X, τ) is a paracompact IFTS.

Proof. Let $\mathscr{U} = \{\langle x, \mu_j, 1 - \mu_j \rangle | j \in J\}$ be a fuzzy open cover of $1_\sim$, then:

$$c_1 \leq \bigvee_{j \in J} \mu_j,$$

and by the hypothesis, for each $r \in (0, 1]$ there exists an open fuzzy refinement $\{v_s | s \in S\}$ of $\{\mu_j | j \in J\}$ which is *-locally finite in c_1 and:

$$c_1 - r \leq \bigvee_{s \in S} v_s.$$

Thus, $\mathscr{V} = \{\langle x, v_s, 1 - v_s \rangle | s \in S\}$ is a family of IFOSs in X, which refines $\mathscr{U}$, covers $\tilde{c}_1 - r = 1_\sim - r$, and is locally finite in $1_\sim$, indeed.

If there is some IFP $p = z(\alpha\ \beta)$ such that for every ε-neighbourhood $\langle x, \mu, 1 - \mu \rangle \in \tau$ of p, we have $\langle x, \mu, 1 - \mu \rangle \cap \langle x, v_s, 1 - v_s \rangle \neq 0_\sim$ for infinitely many $s \in S$, then $\alpha + \beta \leq 1$, $\alpha < \mu(z)$ and $\beta > 1 - \mu(z)$. Then there exists a fuzzy point $z_\beta \leq c_1$, and we have that for every fuzzy open set μ^* such that μ^* is quasi-coincident with z_β, i.e. $\mu^*(z) + \beta > 1$, thus $\mu^*(z) > 1 - \beta \geq \alpha$ and $1 - \mu^*(z) < \beta$ $\langle x, \mu^*, 1 - \mu^* \rangle \in \tau$ and it is an ε-neighbourhood of p. Thus, $\langle x, \mu^*, 1-\mu^*\rangle \cap \langle x, v_s, 1-v_s\rangle \neq 0_\sim$ for infinitely many $s \in S$ and this implies that $\mu^* \wedge v_s \neq 0_\sim$ for infinitely many $s \in S$ which is contradictory. □

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