

Francisco Gallego Lupiáñez

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α -PARACOMPACT SUBSETS AND WELL-SITUATED SUBSETS

FRANCISCO GALLEGO LUPIAÑEZ, Madrid*)

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INTRODUCTION

In Section 1 we study α -paracompact subsets, defined by C. E. Aull. We obtain some covering properties of α -paracompact subsets which are similar to the properties of paracompact spaces. In particular, we characterize α -paracompact subsets in regular spaces. Moreover, we study the behaviour of α -paracompact subsets under perfect mappings.

In Section 2 we consider R. Telgársky's well-situated subsets. The properties of α -paracompact subsets of Section 1 yield properties of well-situated subsets. Well-situated subsets are related to Tamano's problem (i.e.: to give an intrinsic description of T_2 spaces X such that $X \times Y$ is paracompact for each paracompact T_2 space Y) which remains open.

In Section 3 we solve a problem of Telgársky. We establish that in the realm of T_2 spaces, the class Π^* is perfect.

1. α -PARACOMPACT SUBSETS

C. E. Aull in [1] defined the notion of an α -paracompact subset. A subset E of a topological space X is said to be α -paracompact in X if every covering of E by open subsets of X has a refinement by open subsets of X , locally finite in X , which covers E . We continue in this paper the study of α -paracompact subsets. We omit the proofs in this section.

1.1. Proposition. *Let X be a topological space. Then:*

1) *If X is T_2 , E is an α -paracompact subset in X and F is a closed subset of E , then F is α -paracompact in X .*

2) *If $\{E_j\}_{j \in J}$ is a set of subsets of X , locally finite in X and such that E_j is α -paracompact in X for every $j \in J$ and there exists a locally finite family of open*

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subsets $\{U_j\}_{j \in J}$ of X such that $E_j \subset U_j$ for every $j \in J$, then $\bigcup_{j \in J} E_j$ is α -paracompact in X . In particular, every finite union of α -paracompact subsets is α -paracompact.

1.2. Remark. In Proposition 1.1, point 2), the hypothesis “ $\{U_j\}_{j \in J}$ is a locally finite family” cannot be replaced by the hypothesis “ $\{U_j\}_{j \in J}$ is a locally finite set”. Indeed, in the Niemycki plane $X = \{(x, y) \in \mathbb{R}^2 / y \geq 0\}$ (in which for $y_0 > 0$ the neighbourhoods of (x_0, y_0) are the usual neighbourhoods in the plane relativized with respect to X , while for $y_0 = 0$ the neighbourhoods of $(x_0, 0)$ consist of open circles with center (x_0, y) and radius y with the point $(x_0, 0)$ for each $y > 0$), $\{E_q\}_{q \in \mathbb{Q}}$ where $E_q = \{(q, 0)\}$ is a locally finite set of α -paracompact subsets of X such that $\bigcup_{q \in \mathbb{Q}} E_q$ is not α -paracompact ([1] p. 50), and $\{U_q\}_{q \in \mathbb{Q}}$ where $U_q = X$ for each $q \in \mathbb{Q}$ is a locally finite set such that $E_q \subset U_q$ for each $q \in \mathbb{Q}$.

1.3. Theorem. Let X be a regular space and E a subset of X . The following conditions are equivalent:

- a) E is an α -paracompact subset in X .
- b) 1) Every covering \mathcal{U} of E by open subsets of X has a refinement \mathcal{V} by open subsets of X , σ -locally finite in X , which covers E , and 2) Every open subset U of X such that $E \subset U$ has an open subset V such that $E \subset V \subset \bar{V} \subset U$.
- c) Every covering \mathcal{U} of E by open subsets of X has a refinement $\mathcal{A} = \{A_s\}_{s \in S}$ by arbitrary sets of X , locally finite in X , such that $E \subset (\bigcup_{s \in S} A_s)^0$.
- d) Every covering \mathcal{U} of E by open subsets of X has a refinement $\mathcal{F} = \{F_j\}_{j \in J}$ by closed subsets of X , locally finite in X , such that $E \subset (\bigcup_{j \in J} F_j)^0$.

Remark. Theorem 1.3 implies Corollary 3 and Theorem 4 of [1].

1.4. Proposition. Let X be a regular space and E an α -paracompact subset in X . Then:

- 1) Every covering \mathcal{U} of E by open subsets of X has a refinement by open subsets of X , barycentric in X , which covers E .
- 2) Every covering \mathcal{U} of E by open subsets of X has a star refinement by open subsets of X , which covers E .

1.5. Proposition. Let X be a regular space and E an α -paracompact subset in X . Then for every family $\{F_s\}_{s \in S}$ of subsets of E , locally finite (discrete) in X , there is a family $\{U_s\}_{s \in S}$ of open subsets of X , locally finite (discrete) in X and such that $F_s \subset U_s$ for every $s \in S$.

We pass now to the study of the behaviour of the α -paracompact subsets under perfect mappings.

1.6. Proposition. Let X and X' be topological spaces and $f: X \rightarrow X'$ a perfect mapping. If E' is an α -paracompact subset in X' then $f^{-1}(E')$ is an α -paracompact subset in X .

1.7. Remark. Proposition 1.6 implies that if X and Y are topological spaces, E is an α -paracompact subset in X and Y is compact, then $E \times Y$ is an α -paracompact subset in $X \times Y$.

However, if X and Y are topological spaces, E is an α -paracompact subset in X and F is an α -paracompact subset in Y , $E \times F$ is not necessarily an α -paracompact subset in $X \times Y$. Indeed, \mathcal{Q} is an α -paracompact subset in the Michael line $(\mathbb{R}, \mathcal{T})$ $\mathbb{R} \setminus \mathcal{Q}$ is an α -paracompact subset in $\mathbb{R} \setminus \mathcal{Q}$, but $\mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q})$ is not an α -paracompact subset in $(\mathbb{R}, \mathcal{T}) \times (\mathbb{R} \setminus \mathcal{Q})$. (Since the sets $\mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q})$ and $C = \{(x, x) | x \in \mathbb{R} \setminus \mathcal{Q}\}$ are disjoint closed sets which are not strongly separated, it follows from Theorem 5 in [1] that $\mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q})$ is not an α -paracompact subset.)

1.8. Proposition. Let X and X' be topological spaces, where X is regular, let f be a perfect mapping from X onto X' and E' a subset of X' . If $f^{-1}(E')$ is an α -paracompact subset in X then E' is an α -paracompact subset in X' .

1.9. Remark. In Proposition 1.8 the hypothesis “ f is a mapping onto” cannot be omitted. Indeed, let $(\mathbb{R}, \mathcal{T})$ be the Michael line. Then the mapping $i: \mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q}) \rightarrow (\mathbb{R}, \mathcal{T}) \times (\mathbb{R} \setminus \mathcal{Q})$ is a perfect mapping but is not onto, $\mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q})$ is an α -paracompact subset in $\mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q})$ with the usual topology, and $\mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q})$ is not an α -paracompact subset in $(\mathbb{R}, \mathcal{T}) \times (\mathbb{R} \setminus \mathcal{Q})$ (see 1.7).

1.10. Proposition. Let X and X' be topological spaces and f a perfect and open mapping from X onto X' ; if E is an α -paracompact subset in X then $f(E)$ is an α -paracompact subset in X' .

1.11. Remark. Let X and X' be topological spaces and f a perfect mapping from X onto X' ; if E is an α -paracompact subset in X , $f(E)$ is not necessarily α -paracompact in X' . Indeed, let $(\mathbb{R}, \mathcal{T})$ be the Michael line, $j_1: \mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q}) \rightarrow \mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q}) + (\mathbb{R}, \mathcal{T}) \times (\mathbb{R} \setminus \mathcal{Q})$ and $j_2: (\mathbb{R}, \mathcal{T}) \times (\mathbb{R} \setminus \mathcal{Q}) \rightarrow \mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q}) + (\mathbb{R}, \mathcal{T}) \times (\mathbb{R} \setminus \mathcal{Q})$. Then the mapping onto $f: \mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q}) + (\mathbb{R}, \mathcal{T}) \times (\mathbb{R} \setminus \mathcal{Q}) \rightarrow (\mathbb{R}, \mathcal{T}) \times (\mathbb{R} \setminus \mathcal{Q})$ such that

$$\begin{aligned} f(j_1(x, y)) &= (x, y) \quad \text{if } (x, y) \in \mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q}) \\ f(j_2(x, y)) &= (x, y) \quad \text{if } (x, y) \in (\mathbb{R}, \mathcal{T}) \times (\mathbb{R} \setminus \mathcal{Q}) \end{aligned}$$

is a perfect mapping, $f(j_1(\mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q}))) = \mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q})$, $j_1(\mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q}))$ is an α -paracompact subset in $\mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q}) + (\mathbb{R}, \mathcal{T}) \times (\mathbb{R} \setminus \mathcal{Q})$ and $\mathcal{Q} \times (\mathbb{R} \setminus \mathcal{Q})$ is not an α -paracompact subset in $(\mathbb{R}, \mathcal{T}) \times (\mathbb{R} \setminus \mathcal{Q})$ (1.7).

2. WELL-SITUATED SUBSETS

The concept of a well-situated subset was introduced by R. Telgársky in [4]. Using the notion of an α -paracompact subset, H. W. Martin phrased the definition of a well-situated subset of a space X as follows: a subset E of a space X is said

to be *well-situated* in X if for every paracompact T_2 space Y , $E \times Y$ is an α -paracompact subset in $X \times Y$ ([3]).

If E is a well-situated subset of a space X then E is an α -paracompact subset in X , but \mathcal{Q} is α -paracompact in (\mathbb{R}, T) , the Michael line, and \mathcal{Q} is not a well-situated subset in (\mathbb{R}, T) (cf. 1.7).

From Section 1 we easily obtain the following theorems.

2.1. Proposition. *Let X be a topological space. Then:*

1) *If X is T_2 , E is a well-situated subset in X and F is a closed subset of E , then F is a well-situated subset in X .*

2) *If $\{E_j\}_{j \in J}$ is a set of subsets of X , locally finite in X and such that E_j is a well-situated subset in X for every $j \in J$, and there exists a locally finite family of open subsets $\{U_j\}_{j \in J}$ of X such that $F_j \subset U_j$ for every $j \in J$, then $\bigcup_{j \in J} E_j$ is a well-situated subset in X . In particular, every finite union of well-situated subsets is well-situated.*

2.2. Theorem. *Let X be a regular space and E a subset of X . The following conditions are equivalent:*

a) *E is a well-situated subset in X .*

b) *For every paracompact T_2 space Y : 1) Every covering \mathcal{U} of $E \times Y$ by open subsets of $X \times Y$ has a refinement \mathcal{V} by open subsets of $X \times Y$, σ -locally finite in $X \times Y$, which covers $E \times Y$, and 2) Every open subset U of $X \times Y$ such that $E \times Y \subset U$ has an open subset V such that $E \times Y \subset V \subset \bar{V} \subset U$.*

c) *For every paracompact T_2 space Y , every covering \mathcal{U} of $E \times Y$ by open subsets of $X \times Y$ has a refinement $\mathcal{A} = \{A_s\}_{s \in S}$ by arbitrary sets of $X \times Y$, locally finite in $X \times Y$, such that $E \times Y \subset \left(\bigcup_{s \in S} A_s\right)^0$.*

d) *For every paracompact T_2 space Y , every covering of $E \times Y$ by open subsets of $X \times Y$ has a refinement $\mathcal{F} = \{F_j\}_{j \in J}$ by closed subsets of $X \times Y$, locally finite in $X \times Y$, such that $E \times Y \subset \left(\bigcup_{j \in J} F_j\right)^0$.*

2.3. Proposition. *Let X be a regular space and E a well-situated subset in X . Then:*

1) *For every paracompact T_2 space Y , every covering \mathcal{U} of $E \times Y$ by open subsets of $X \times Y$ has a refinement by open subsets of $X \times Y$, barycentric in $X \times Y$, which covers $E \times Y$.*

2) *For every paracompact T_2 space Y , every covering of $E \times Y$ by open subsets of $X \times Y$ has a star refinement by open subsets of $X \times Y$ which covers $E \times Y$.*

2.4. Proposition. *Let X be a regular space and E a well-situated subset in X . Then for every paracompact T_2 space Y , for every family $\{F_s\}_{s \in S}$ of subsets of $E \times Y$, locally finite (discrete) in $X \times Y$, there is a family $\{U_s\}_{s \in S}$ of open subsets of $X \times Y$, locally finite (discrete) in $X \times Y$ and such that $F_s \subset U_s$ for any $s \in S$.*

2.5. Proposition. *Let X and X' be topological spaces and $f: X \rightarrow X'$ a perfect*

mapping. If E' is a well-situated subset in X' then $f^{-1}(E')$ is a well-situated subset in X .

Proof. For every paracompact T_2 space Y , $f \times 1Y: X \times Y \rightarrow X' \times Y$ is a perfect mapping. Now 1.6 implies that $f^{-1}(E')$ is well-situated.

2.6. Proposition. Let X and X' be T_2 topological spaces where X' is paracompact, let E be a well-situated subset in X and F a closed subset of X' . Then $E \times F$ is an α -paracompact subset in $X \times X'$.

Proof follows from 1.1.

Remark. In [4] R. Telgársky denoted by Π the class of all T_2 spaces X such that $X \times Y$ is paracompact for each paracompact T_2 space Y . Let X and Y be T_2 topological spaces, E a well-situated subset in X and $Y \in \Pi$. Then $E \times Y$ is well-situated in $X \times Y$. (Indeed, for each paracompact T_2 space Z , $Y \times Z$ is paracompact and T_2 , hence $(E \times Y) \times Z$ is α -paracompact in $(X \times Y) \times Z$.)

2.7. Proposition. Let X and X' be topological spaces, where X is regular, let f be a perfect mapping from X onto X' and E' a subset of X' . If $f^{-1}(E')$ is a well-situated subset in X then E' is a well-situated subset in X' .

Proof. For every paracompact T_2 space Y , $f \times 1Y$ is a perfect mapping from $X \times Y$ onto $X' \times Y$. It follows from 1.8 that E' is well-situated.

2.8. Remark. In Proposition 2.7 the hypothesis “ f is a mapping onto” cannot be omitted. In deed, let (\mathbf{R}, T) be the Michael line. The mapping $i: \mathcal{Q} \rightarrow (\mathbf{R}, T)$ is perfect but is not onto, $\mathcal{Q} \in \Pi$ ([4] p. 66) but \mathcal{Q} is not well-situated in (\mathbf{R}, T) (cf. 1.7).

2.9. Proposition. Let X and X' be topological spaces and f a perfect and open mapping from X onto X' : If E is a well-situated subset in X then $f(E)$ is a well-situated subset in X' .

Proof. For every paracompact T_2 space Y , $f \times 1Y$ is a perfect and open mapping from $X \times Y$ onto $X' \times Y$. Now 1.10 implies that $f(E)$ is well-situated.

2.10. Remark. Let X and X' be topological spaces and f a perfect mapping from X onto X' . If E is a well-situated subset in X , $f(E)$ is not necessarily a well-situated subset in X' . (See 1.11).

3. THE CLASS Π^*

In [4] R. Telgársky denoted by Π^* the class of all paracompact T_2 spaces which are well-situated in every paracompact T_2 space in which they are embedded as closed subsets.

R. Telgársky showed that Π^* is a very wide class contained in the class Π , and raised the following questions:

1. Is the class Π^* perfect? ([4], Problem 2.1.)
2. Does the class of all paracompact C -scattered spaces coincide with the class Π^* ? ([4], Problem 2.3.)

In the present paper, we shall give an affirmative answer to question 1.

3.1. Proposition. *Let E and E' be topological spaces where E is T_2 , and let f be a perfect mapping from E onto E' . If $E' \in \Pi^*$ then $E \in \Pi^*$.*

Proof. Let X be a paracompact T_2 space such that E is embedded in X as a closed subset. Let $j_1: X \rightarrow X + E'$, $j_2: E' \rightarrow X + E'$ be the embeddings of the subspaces X and E' in the sum $X + E'$, let $X \cup_f E'$ be the adjunction space determined by X , E' and f and let $q: X + E' \rightarrow X \cup_f E'$ be the natural quotient mapping. As the mapping f is closed, q is a continuous and closed mapping; since $X + E'$ is paracompact and T_2 , $X \cup_f E'$ is paracompact (this follows from the Michael Theorem) and T_2 . Further, $q \circ j_2: E' \rightarrow X \cup_f E'$ is a homeomorphic embedding and $(q \circ j_2)(E')$ is closed in $X \cup_f E'$. Since $E' \in \Pi^*$ and $X \cup_f E'$ is paracompact and T_2 , $(q \circ j_2)(E')$ is a well-situated subset in $X \cup_f E'$.

Let $\hat{f} = q \circ j_1: X \rightarrow X \cup_f E'$. Clearly \hat{f} is a perfect mapping.

$$\begin{array}{ccc} X & \xrightarrow{\hat{f}} & X \cup_f E' \\ \uparrow & & \uparrow q \circ j_2 \\ E & \xrightarrow{f} & E' \end{array}$$

Since $(q \circ j_2)(E')$ is a well-situated subset in $X \cup_f E'$, Proposition 2.5 implies that $\hat{f}^{-1}((q \circ j_2)(E'))$ is a well-situated subset in X . However,

$$\hat{f}^{-1}((q \circ j_2)(E')) = j_1^{-1}(q^{-1}(q(j_2(E')))) = E.$$

Thus E is a well-situated subset in X . Hence $E \in \Pi^*$.

3.2. Proposition. *Let E and E' be topological spaces and f a perfect mapping from E onto E' . If $E \in \Pi^*$ then $E' \in \Pi^*$.*

Proof. Since f is continuous, $G_f = \{(x, f(x)) \in E \times E' \mid x \in E\}$ is homeomorphic to E , hence $G_f \in \Pi^*$.

Since E is T_{3a} and f is a perfect mapping from E onto E' , G_f is a closed subset of $\beta E \times E'$ (see [5], proof of Theorem 3.10).

Let X' be a paracompact T_2 space such that E' is embedded in X' as a closed subset. Then G_f is a closed subset of $\beta E \times X'$ which is paracompact and T_2 . Thus G_f is a well-situated subset in $\beta E \times X'$.

The projection $p_2: \beta E \times X' \rightarrow X'$ is perfect and open, and $p_2(G_f) = E'$. It follows from 2.9 that E' is a well-situated subset in X' .

3.3. Theorem. *In the realm of T_2 spaces, the class Π^* is perfect (i.e., if E and E' are topological spaces where E is T_2 , and f is a perfect mapping from E onto E' then $E \in \Pi^*$ if and only if $E' \in \Pi^*$).*

Proof follows from 3.1 and 3.2.

Added in proofs. The author learned, after writing this paper, that J. D. Wine [in: *Locally paracompact spaces*, *Glasnik Mat.*, 10 (30) (1975), 351–357] has obtained Proposition 1.1.2), and that I. Kovacević [in: *Subsets and paracompactness*, *Zbornik Radova PMF Univ. u Novom Sadu, ser Mat.* 14 (1984), 79–87] has obtained also the implication a) \Rightarrow b) of the Theorem 1.3. The author thanks to Professors J. D. Wine and I. Kovacević for making available their papers to him.

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Author's address: Departamento de Geometria y Topologia, Facultad de Matematicas, Universidad Complutense de Madrid, Spain.