PROJECTIVE NORMALITY AND SYZYGIES OF ALGEBRAIC SURFACES AND CALABI-YAU THREEFOLDS

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Abstract. In this article we develop techniques to compute Koszul cohomology groups for a wide class of varieties. As an application of this we prove results on projective normality and syzygies for surfaces and for Calabi-Yau threefolds. From more general results we obtain in particular the following:

a) Mukai's conjecture (and stronger variants of it) regarding projective normality and normal presentation for surfaces with Kodaira dimension 0,
b) results on projective normality for pluricanonical models of surfaces of general type (recovering and strengthening results by Ciliberto; cf. [Ci]) and generalizations of them to higher syzygies, and
c) results on very ampleness (of line bundles), projective normality and higher syzygies for Calabi-Yau threefolds.

Introduction

In this article we develop techniques to compute Koszul cohomology groups for several classes of varieties. Koszul cohomology is important because of its relation to Hodge Theory and to the computation of syzygies of projective variety. In this article we focus on the latter application. The topic of syzygies is particularly interesting as it deals with the interplay between algebra and geometry: the algebra coming from the equations defining the variety and the geometry arising from which line bundles live on the variety. In the last century Castelnuovo showed that a curve of degree greater than $2g$ has a normal homogeneous coordinate ring (g denotes the genus of the curve). He also proved that if the degree was greater than $2g + 1$, then the ideal of the curve was generated by quadratic equations. This result was rediscovered later by many people, among others Fujita, St. Donat, Mumford, Green, etc. Recently Mark Green threw new light on this connection between algebra and geometry by generalizing the study of homogeneous coordinate rings and ideals to the study of free resolutions. He linked the behavior of graded Betti numbers of the resolution of the homogeneous coordinate ring to the cohomology groups of certain vector bundles on the variety (for a particularly nice introduction of the subject see [L]; for the precise statement used in this article see Theorem 1.2). In particular he generalized Castelnuovo result proving that if the degree of

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the curve is greater than $2g+p$, then the resolution is in addition linear until the pth stage. This is the so-called property N_p. The connection between algebra and geometry is better seen in the case of the canonical curve. Here there are classical results by Nöther and Petri on projective normality and normal presentation for canonical curves. The geometric part of the statements is reflected by the Clifford index, which basically tell us how special the most special line bundle that a curve possesses is, and Green's conjecture says that the shape of the canonical resolution, precisely the Betti numbers that are zero, is determined by the Clifford index of the curve.

The landscape in the case of surfaces and higher dimensional varieties is largely uncharted. Among other things one would like to find higher dimensional analogues of the results known for curves. Mukai observed that Green's theorem can be interpreted as follows : any line bundle L on a curve C which is at least as positive as $K_C \otimes A^{\otimes p+3}$ satisfies N_p, where K_C is the canonical bundle of C and A an ample line bundle on C. A way to generalize this result to higher dimensions is therefore to consider adjoint linear series. Mukai also conjectured the following: Let S be a surface and let L be a line bundle on S equal to $K_S \otimes A^{\otimes n}$, where A is ample. If $n \geq p+4$, then L satisfies property N_p. This conjecture is not proven even for $p=0$. In fact it was not known until recently that $K_S \otimes A^{\otimes n}$ was very ample if $n \geq 4$. This was proven by Reider (see [R]). Some progress has been made in this direction by Butler for ruled varieties (see [Bu]; see also [H1], [H2]), Ein and Lazarsfeld (see [EL]) who study syzygies involving adjoint linear series with A very ample and Kempf for abelian varieties (see [K]).

In the case of surfaces and higher dimensional varieties the techniques needed to compute Koszul cohomology groups are scarce – as Green put it, there are more reasons to compute them than ways of computing them. This is especially so if the adjoint linear series involves base point free or ample line bundles. On the other hand one would also like to have some kind of a unifying principle to gather several different results for syzygies of surfaces proved so far. To do this we have formulated the following meta-principle:

0.1. *If L is the product of $(p+1)$ ample and base-point-free line bundles satisfying certain conditions, then L satisfies the condition N_p.*

Almost all known results on syzygies for an algebraic surface (and several results on curves) fit into 0.1. For example, the normal presentation of line bundles of degree greater than $2g+1$ on curves, by Castelnuovo and others (see [GP1]), the result of Kempf mentioned above (see Remark 4.6), and the result by Ein and Lazarsfeld. In our earlier work (see [GP1], [GP2]) we also prove a number of results which fit into the meta-principle. For surfaces with geometric genus 0 we prove that the $p+1$-th power of an ample, free and nonspecial line bundle satisfy property N_p ([GP2], Theorem 2.2). A generalization of this result is Theorem 1.3 which unifies among others the one previously mentioned, Theorem 5.5 for surfaces of general type and Corollary 6.2 for Calabi-Yau threefolds. In [GP1] and [GP2] we prove finer versions of the meta-principle for elliptic ruled surfaces ([GP1], Theorem 4.2, [GP2] and Theorem 6.1, the former implying Mukai's conjecture for elliptic ruled surfaces and $p=1$). In [GP3] we prove stronger versions of the meta-principle for K3 surfaces, and results in the spirit of 0.1 for Fano varieties. The goal of the present article is to show that 0.1 holds for a wider range of varieties: all surfaces of Kodaira dimension 0, pluricanonical models of surfaces of general type (Theorem

5.6, for the square of the canonical bundle is base-point-free, and Theorem 5.7). We also prove results for Calabi-Yau threefolds along the line of 0.1. For surfaces with Kodaira dimension 0 we show precisely the following:

Let S be a minimal surface with Kodaira dimension 0 *and let $B_1, \dots, B_n$ be numerically equivalent, ample and base-point-free line bundles. Then $B_1 \otimes \cdots \otimes B_n$ satisfy the property N_p if $n \geq p+1$ and $p \geq 1$ and if, when X is an Enriques surface, the sectional genus of B_i is greater than or equal to* 2 *and when S is a K3 surface, B_i is non hyperelliptic of sectional genus greater than or equal to* 4 *(in this case we obtain a stronger result imposing extra conditions on B_i, see [GP3]).*

The proof of this result can be found, for Enriques surfaces in Section 2, for abelian and bielliptic surfaces in Section 4 and for K3 surfaces in [GP3]. This general result implies in particular (see Lemma 2.13) the following result regarding adjoint linear series:

Let S be a minimal surface with Kodaira dimension 0*, let B be an ample and base-point-free line bundle, and let A be an ample line bundle. Then the bundle $K_S \otimes B^{\otimes n}$ satisfies property N_p if $n \geq p+1$ and $p \geq 1$ and the bundle $K_S \otimes A^{\otimes m}$ satisfies property N_p if $m \geq 2p+2$ and $p \geq 1$*

The above mentioned results imply Mukai's conjecture for surfaces of Kodaira dimension 0 and $p = 0, 1$, in fact lowering Mukai's bound by one in this case. We also prove stronger variants of Mukai's conjecture (see Corollaries 2.6 and 4.5).

Our Koszul cohomology computations point out relevant geometric properties of the varieties under consideration. Let us be a little more explicit about it. For a long part of the article we deal with varieties X with numerically trivial canonical bundle. Two situations occur: either $H^1(\mathcal{O}_X) = 0$ or $H^1(\mathcal{O}_X) \neq 0$. In the former case (comprising among others K3 surfaces, Enriques surfaces and Calabi-Yau threefolds) we give more or less uniform proofs using, short to say, induction on the dimension of the variety. This allows us to use eventually semistability results and results on surjectivity of multiplication maps of vector bundles on curves, like [Bu], Proposition 2.2 and [P2], Corollary 4. These results which might seem at first glance unmotivated find therefore a meaningful application. In the case when $H^1(\mathcal{O}_X) \neq 0$ (comprising among others abelian surfaces and bielliptic surfaces) we are also able to give homogeneous proofs using the fact that $\mathrm{Pic}^0(\mathcal{O}_X)$ (the group of line bundles homologous to 0 modulo linear equivalence) is, for our purposes, "large enough" (in fact, not discrete in this case).

The article is organized as follows. There are threads of thought which are used repeatedly throughout the article. These arguments are displayed in detail in the first sections (especially Sections 1 and 2). Then, in subsequent proofs, they are sometimes told in a more concise manner (in occasions they are just sketched). In the proofs of the vanishings leading to results on higher syzygies we use induction on the dimension. The process is explained in the proof of Theorem 1.3. Also in Section 1 it is shown how the arguments using Castelnuovo-Mumford regularity are employed.

Another fruitful strategy is, as we have said before, what we could call "induction on the dimension". This is used for varieties with irregularity zero, and it is better explained when first used in this article, that is, in Section 2, which deals with normal generation, normal presentation and syzygies of Enriques surfaces.

In Section 3 we deal with the property of Koszul for coordinate rings. Precisely we prove (Theorem 3.5) that those line bundles which according to Theorem 2.11 satisfy property N_1 have a Koszul coordinate ring. We also show that whenever a line bundle on the variety under consideration (in this article) is normally presented then it embeds the variety with Koszul homogeneous coordinate ring. This gives further evidence to the following (to paraphrase Arnold): Any homogeneous coordinate ring which has a serious reason for being quadratically presented is Koszul.

Section 4 is about the irregular minimal surfaces of Kodaira dimension 0, i.e., abelian and bielliptic. Our results (in particular Corollary 4.4) on abelian surfaces in particular imply Kempf's result. As mentioned above we also find sufficient conditions for the coordinate ring to be Koszul (Theorem 4.7).

In Section 5 using the techniques and methods developed in the previous sections we prove the surjectivity of multiplication maps of certain vector bundles on pluricanonical models of surfaces. This shows the vanishing of certain Koszul cohomology groups which in particular imply results on projective normality, normal presentation and higher syzygies for pluricanonical models. Our results recover results of Ciliberto on projective normality, in particular, we show the following, which is a question posed by Bombieri (in [Bo]):

Let X be a surface of general type such that $p_g \geq 2$ or $K_X^2 \geq 5$. If $n \geq 5$, then the image of X by $|K_X^{\otimes n}|$ is projectively normal

We improve the results of [Ci] in the case of regular surfaces (Theorem 5.4), and we generalize it to higher syzygies (and proving the Koszul property) as mentioned before.

Finally Section 6 is devoted to Calabi-Yau threefolds. We prove results on very ampleness and projective normality. These results are similar in spirit to the well known results of St. Donat for K3 surfaces and Lefschetz for abelian varieties. Precisely, we find sharp conditions on $h^0(B)$ of an ample and base-point-free line bundle B for $B^{\otimes 2}$ and $B^{\otimes 3}$ to satisfy property N_0 (Theorems 6.3, 6.4, 6.6 and 6.7) and for $B^{\otimes 3}$ to satisfy property N_1 (Theorem 6.8). Then we generalize these results to study the syzygies of embeddings induced by higher powers of B (Theorem 6.8).

Convention. Throughout this article we work over an algebraically closed field of characteristic 0. For us surface will be always mean minimal and smooth algebraic surface. We will denote numerical equivalence of line bundles by $\equiv$.

Definition. Let X be a projective variety and let L be a very ample line bundle on X. We say that L is normally generated or that satisfies the property N_0, if $|L|$ embeds X as a projectively normal variety. We say that L is normally presented or that L satisfies the property N_1 if L satisfies property N_0 and, in addition, the homogeneous ideal of the image of X by $|L|$ is generated by quadratic equations. We say that L satisfies the property N_p for $p > 1$, if L satisfies property N_1 and the free resolution of the homogeneous ideal of X embedded by $|L|$ is linear until the pth-stage.

1. A general result on syzygies of algebraic varieties

As we mentioned in the introduction, Green interpreted the Betti numbers of the minimal free resolution of the coordinate ring of an embedded projective variety in terms of Koszul cohomology. Concretely, let X be a projective variety, and let F be a globally generated vector bundle on X. We define the bundle M_F as follows:

$$(1.1) \qquad 0 \to M_F \to H^0(F) \otimes \mathcal{O}_X \to F \to 0$$

If L is an ample line bundle on X and all its positive powers are nonspecial one has the following characterization of the property N_p:

Theorem 1.2. *Let L be an ample line bundle on a variety X. If $H^1(\bigwedge^{p'+1} M_L \otimes L^{\otimes s})$ vanishes for all $0 \leq p' \leq p$, then L satisfies the property N_p. If in addition $H^1(L^{\otimes r}) = 0$, for all $r \geq 1$, then the above is a necessary and sufficient condition for L to satisfy property N_p.*

We will obtain our results on syzygies using the previous lemma. For the proof of it we refer to [EL], Section 1. Recall that we are working over an algebraically closed field of characteristic 0, thus in our proofs we will check the vanishings of $H^1(M_L^{\otimes p'+1} \otimes L^{\otimes s})$ rather than see directly the vanishings of $H^1(\bigwedge^{p'+1} M_L \otimes L^{\otimes s})$.

The purpose of this section is to prove a general result about Koszul cohomology and, by the above lemma, about syzygies of varieties of arbitrary dimension.

Theorem 1.3. *Let X be a projective variety. Let B be a base-point-free line bundle on X with regularity r. If $n \geq \max(r+p-2, p)$, $p \geq 1$ and $m \geq \max(r,1)$, then*

$$H^i(M_{B^{\otimes m}}^{\otimes p+1} \otimes B^{\otimes n+2-i}) = 0 \textit{ for all } i \geq 1 \ .$$

In particular, $H^1(M_{B^{\otimes m}}^{\otimes p+1} \otimes B^{\otimes n+1}) = 0$ and if B is ample and $n \geq \max(r+p-2, r, p)$, then $B^{\otimes n+1}$ satisfies the property N_p.

To prove the theorem we will need the following

Lemma 1.4. *Let X be and B be as in Theorem 1.3. If $n \geq r-1$ and $m \geq 1$, then*

$$H^1(M_{B^{\otimes m}} \otimes B^{\otimes n+1}) = 0 \ .$$

In particular, if B is ample, then $B^{\otimes n+1}$ satisfies the property N_0.

Proof. Since $n+1 \geq r$, $H^1(B^{\otimes n+1}) = 0$. Thus, tensoring the sequence (1.1) relative to $B^{\otimes m}$ with $B^{\otimes n+1}$ and taking global sections one sees that it is enough to check that the multiplication map

$$H^0(B^{\otimes m}) \otimes H^0(B^{\otimes n+1}) \longrightarrow H^0(B^{\otimes m+n+1})$$

surjects. To see that, we use the following useful observation:

Observation 1.4.1. Let E and $L_1, \dots, L_r$ be coherent sheaves on a variety X. Consider the map $H^0(E) \otimes H^0(L_1 \otimes \cdots \otimes L_r) \xrightarrow{\psi} H^0(E \otimes L_1 \otimes \cdots \otimes L_r)$ and the maps

$$H^0(E) \otimes H^0(L_1) \xrightarrow{\alpha_1} H^0(E \otimes L_1),$$
$$H^0(E \otimes L_1) \otimes H^0(L_2) \xrightarrow{\alpha_2} H^0(E \otimes L_1 \otimes L_2),$$
$$\dots,$$
$$H^0(E \otimes L_1 \otimes \cdots \otimes L_{r-1}) \otimes H^0(L_r) \xrightarrow{\alpha_r} H^0(E \otimes L_1 \otimes \cdots \otimes L_r) \ .$$

If $\alpha_1, \dots, \alpha_r$ are surjective then ψ is also surjective.

In our case, we set $L_i = B$ and $E = B^{\otimes n+1}$, and to see that the maps α_i are surjective we use the following generalization by Mumford of a lemma of Castelnuovo (see [Mu]; note that the assumption of ampleness is unnecessary):

(1.4.2). Let L be a base-point-free line bundle on a variety X and let $\mathcal{F}$ be a coherent sheaf on X. If $\mathrm{H}^i(\mathcal{F} \otimes L^{-i}) = 0$ for all $i \geq 1$, then the multiplication map

$$\mathrm{H}^0(\mathcal{F} \otimes L^{\otimes i}) \ \otimes \mathrm{H}^0(L) \to \mathrm{H}^0(\mathcal{F} \otimes L^{\otimes i+1})$$

is surjective for all $i \geq 0$.

Finally, the vanishings required according to (1.4.2) follow from our assumption on regularity. □

(1.5) *Proof of Theorem 1.3.* The proof is by induction on p. We prove the result for $p = 1$. First we show that

$$H^1(M^{\otimes 2}_{B^{\otimes m}} \otimes B^{\otimes n+1}) = 0 \text{ for all } m \geq r, 1 \text{ and all } n \geq r-1, 1 \ .$$

We will use (1.4.2) and Observation 1.4.1 to prove this statement. Observe that tensoring the sequence (1.1) with $M_{B^{\otimes m}} \otimes B^{\otimes n+1}$ and taking global sections yields the following long exact sequence:

$$\begin{aligned} &H^0(M_{B^{\otimes m}} \otimes B^{\otimes n+1}) \otimes H^0(B^{\otimes m}) \xrightarrow{\gamma} H^0(M_{B^{\otimes m}} \otimes B^{\otimes m+n+1}) \\ &\longrightarrow H^1(M^{\otimes 2}_{B^{\otimes m}} \otimes B^{\otimes n+1}) \longrightarrow H^1(M_{B^{\otimes m}} \otimes B^{\otimes n+1}) \otimes H^0(B^{\otimes m}). \end{aligned}$$

The last term in the above sequence is zero by Lemma 1.4 . Thus it is enough to prove that γ surjects. By Observation 1.4.1 it is enough to show that the multiplication map

$$H^0(M_{B^{\otimes m}} \otimes B^{\otimes n+1}) \otimes H^0(B) \longrightarrow H^0(M_{B^{\otimes m}} \otimes B^{\otimes n+2})$$

surjects for all $m \geq r, 1$ and all $n \geq r-1, 1$. Since B is base-point-free, by (1.4.2) we need to check the vanishings $H^i(M_{B^{\otimes m}} \otimes B^{\otimes n+1-i}) = 0$ for all $i \geq 1$, l $m \geq r, 1$ and $n \geq r-1, 1$. For $i \geq 2$, we tensor the sequence (1.1) corresponding to $B^{\otimes m}$ with $B^{\otimes n+1-i}$ and take global sections. The vanishings then follow from our assumption on the regularity of B. Since $m \geq r$ and $n \geq r-1$ it follows in particular that $H^1(B^{\otimes m}) = H^1(B^{\otimes n}) = 0$, hence the vanishing required for $i = 1$ is equivalent to the vanishing of $H^1(M_{B^{\otimes n}} \otimes B^{\otimes m})$, which follows in turn from Lemma 1.4.

The vanishings of $H^i(M^{\otimes 2}_{B^{\otimes m}} \otimes B^{\otimes n+2-i})$ for all $m \geq 1$, all $i \geq 2$ and all $n \geq r-1$ follow from (1.1), Lemma 1.4, and the assumption on regularity.

Let us now assume that the desired vanishings occur for $p-1$. We therefore have:

$$\begin{aligned} H^i(M^{\otimes p}_{B^{\otimes m}} \otimes B^{\otimes n+2-i}) = 0 \text{ for all } n \geq \ \max(p+r-3, p-1), \\ \text{all } m \geq \ \max(r,1) \text{ and all } i \geq 1 \ . \end{aligned}$$

We first prove the desired vanishing for p and $i = 1$. By tensoring the sequence (1.1) with $M^{\otimes p}_{B^{\otimes m}} \otimes B^{\otimes n+1}$ and taking global sections one sees that the desired vanishing can be obtained by showing the surjectivity of the multiplication map δ sitting in the long exact sequence

$$\begin{aligned} &H^0(M^{\otimes p}_{B^{\otimes m}} \otimes B^{\otimes n+1}) \otimes H^0(B^{\otimes m}) \xrightarrow{\delta} H^0(M^{\otimes p}_{B^{\otimes m}} \otimes B^{\otimes m+n+1}) \\ &\longrightarrow H^1(M^{\otimes p+1}_{B^{\otimes m}} \otimes B^{\otimes n+1}) \longrightarrow H^1(M^{\otimes p}_{B^{\otimes m}} \otimes B^{\otimes n+1}) \otimes H^0(B^{\otimes m}) \end{aligned}$$

The last term is zero by induction assumption. In order to prove the surjectivity of δ we use Observation 1.4.1. By Observation 1.4.1 it suffices to show the surjectivity of the map
$H^0(M_{B^{\otimes m}}^{\otimes p} \otimes B^{\otimes n+1}) \otimes H^0(B) \xrightarrow{\epsilon} H^0(M_{B^{\otimes m}}^{\otimes p} \otimes B^{\otimes n+2})$ for all $n \geq p+r-2, p$ and all $m \geq r, 1$.
To prove the surjectivity of ϵ we use (1.4.2). According to it, it suffices that the groups $H^i(M_{B^{\otimes m}}^{\otimes p} \otimes B^{\otimes n+1-i})$ vanish, which follows by induction.

Finally, to show that $H^i(M_{B^{\otimes m}}^{\otimes p+1} \otimes B^{\otimes n+2-i}) = 0$, for all $i \geq 2$ we consider again sequence (1.1) associated to $B^{\otimes m}$, tensor it with $M_{B^{\otimes m}}^{\otimes p} \otimes B^{\otimes n+2-i}$ and take global sections. Then the vanishings follow again from induction hypothesis.

The fact that $B^{\otimes n+1}$ satisfies the property N_p follows from the vanishing of $H^1(M_{B^{\otimes n+1}}^{\otimes p'} \otimes B^{\otimes s(n+1)})$ for all $1 \leq p' \leq p$ and all $s \geq 1$, from Lemma 1.4 and from Theorem 1.2. □

The theorem just proven, which might seem at first glance somehow vague, holds however the power to unify several results for different kinds of varieties: it yields information about pluricanonical embeddings of surfaces of general type (Theorem 5.4) as well as about varieties of arbitrary dimension and canonical divisor numerically trivial (Theorem 6.1). An example of the latter are Calabi-Yau n-folds. Theorem 1.3 also implies a result for surfaces with $p_g = 0$ (among them elliptic ruled surfaces, Enriques surfaces and bielliptic surfaces):

[GP2], Theorem 2.2. *Let X be a surface with $p_g = 0$. Let B be a nonspecial, ample, and base-point-free line bundle. Then $B^{\otimes p+1}$ satisfies the property N_p for all $p \geq 1$.*

Therefore Theorem 1.3 and its corollaries are a good starting point for our study of syzygies of varieties. However, if one focuses on the particular examples and uses the specific geometry of the varieties in question, one can expect to obtain sharper and more complete results. Precisely this was done for elliptic ruled surfaces in [GP2] and is done for Enriques surfaces in Section 2, for bielliptic surfaces in Section 4, for surfaces of general type in Section 5 and for Calabi-Yau threefolds in Section 6.

2. Syzygies of Enriques surfaces

In Section 1 we proved a general theorem, Theorem 1.3, which unifies a number of results for different kinds of varieties. In this section we focus on Enriques surfaces. The geometric genus of an Enriques surface is 0 and, in characteristic 0, a globally generated line bundle over an Enriques surface has null higher cohomology, hence it is 2-regular. Therefore the starting point of our study of syzygies of Enriques surfaces is the following theorem, corollary of Theorem 1.3, which fits indeed in 0.1:

Theorem 2.1, (cf. [GP2], Corollary 2.7.1). *Let X be an Enriques surface. Let B be a base-point-free line bundle. Then the image of X by $|B^{\otimes p+1}|$ satisfies property N_p, for all $p \geq 1$. If in addition B is ample then $B^{\otimes p+1}$ is very ample and satisfies the property N_p, for all $p \geq 1$.*

Our intention now is to study a more general class of line bundles (namely, tensor products of $p+1$ different base-point-free line bundles), and in particular, adjoint line bundles. For that we need to follow a different approach: roughly, we are going to use “induction on the dimension”. This approach will unfold throughout this section and the machinery developed along the way will be used for other results of this article, concretely in Sections 3, 5 and 6. We now resume with a result about normal generation:

Theorem 2.2. *Let X be an Enriques surface. Let B_1, B_1', B_2 and B_2' be ample and base-point-free line bundles on X, such that $B_1 \equiv B_1'$, $B_2 \equiv B_2'$ and, either $B_1 \cdot B_2 \geq 4$, $B_1^2 \geq 6$, and $B_2^2 \geq 6$ or $B_1 \cdot B_2 \geq 5$. Let $L = B_1^{\otimes r} \otimes B_2^{\otimes s}$ and $L' = B_1'^{\otimes k} \otimes B_2'^{\otimes l}$. If $r, s, k \geq 1$, and $l \geq 0$, then the map $H^0(L) \otimes H^0(L') \xrightarrow{\alpha} H^0(L \otimes L')$ surjects and $H^1(M_L \otimes L') = H^1(M_L' \otimes L) = 0$. In particular, L is very ample and satisfies property N_0.*

Before we go on with the proof of Theorem 2.2, we isolate for convenience three ingredients of the argument, which will be used in many other instances. The first is an observation on the relation between the surjectivity of multiplication maps, and the surjectivity of its restrictions to divisors. The other two are a result due to Butler and another one due to Pareschi, about the surjectivity of multiplication maps of vector bundles on curves.

Observation 2.3. *Let X be a regular variety (i.e, a variety such that $H^1(\mathcal{O}_X) = 0$). Let E be a vector bundle on X, let C be a divisor such that $L = \mathcal{O}_X(C)$ is globally generated line bundle and $H^1(E \otimes L^{-1}) = 0$. If the multiplication map $H^0(E \otimes \mathcal{O}_C) \otimes H^0(L \otimes \mathcal{O}_C) \to H^0(E \otimes L \otimes \mathcal{O}_C)$ surjects, then the map $H^0(E) \otimes H^0(L) \to H^0(E \otimes L)$ also surjects.*

Proof. We construct the following commutative diagram:

$$\begin{array}{ccccc} H^0(E) \otimes H^0(\mathcal{O}_X) & \hookrightarrow & H^0(E) \otimes H^0(L) & \twoheadrightarrow & H^0(E) \otimes H^0(L \otimes \mathcal{O}_C) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(E) & \hookrightarrow & H^0(E \otimes L) & \twoheadrightarrow & H^0(E \otimes L \otimes \mathcal{O}_C)\ . \end{array}$$

The surjectivity of the left hand side vertical map is obvious. The surjectivity of the right hand side vertical map follows by hypothesis. The exactness of the top horizontal sequence follows from the fact that X is regular. The claim is the surjectivity of the middle vertical map. ☐

Proposition 2.4 ([Bu], Proposition 2.2).. *Let E and F be semistable vector bundles over a curve C such that E is generated by its global sections. If*

1. *$\mu(F) > 2g$, and*
2. *$\mu(F) > 2g +$ rank$(E)(2g - \mu(E)) - 2h^1(E)$,*

then the multiplication map $H^0(E) \otimes H^0(F) \to H^0(E \otimes F)$ surjects.

Proposition 2.5 ([P2], Corollary 4.). *Let N and L be two base-point-free line bundles on C such that:*

(a) *at least one of them is very ample;*
(b) *$h^0(N), h^0(L) \geq 3$ and*
(c) *deg$N +$ deg $L \geq$ max $(3g - 3, 4g + 1 - 2h^1(N) - 2h^1(L) - \text{Cliff}(C))$.*

Then the multiplication map

$$H^0(L) \otimes H^0(N) \to H^0(L \otimes N)$$

is surjective.

(2.6) *Proof of Theorem 2.2.* Note first that, since we are working over a field of characteristic 0, any base-point-free line bundle on X has null higher cohomology. If we twist the sequences (1.1) relative to L and L' by L' and L respectively and take global sections, we see at once that $H^1(M_L \otimes L') = H^1(M'_L \otimes L)$ and equal to the cokernel of

$$H^0(L) \otimes H^0(L') \xrightarrow{\alpha} H^0(L \otimes L') \ .$$

To see that α indeed surjects, we use Observation 1.4.1. According to it we want to check that several (possibly more than one) multiplication maps surject. We check here the first one; the surjectivity of the rest can be seen in the same way. The map in question is

$$H^0(L) \otimes H^0(B'_1) \xrightarrow{\beta} H^0(L \otimes B'_1) \ .$$

To see the surjectivity of β, we consider a smooth irreducible curve C in $|B'_1|$ (such curve exists by Bertini's Theorem because B'_1 is ample and base-point-free) and use Observation 2.3. It is therefore enough to check that

$$H^0(L \otimes \mathcal{O}_C) \otimes H^0(B'_1 \otimes \mathcal{O}_C) \xrightarrow{\gamma} H^0(L \otimes B'_1 \otimes \mathcal{O}_C)$$

surjects. For that, if $B_1 \cdot B_2 \geq 5$, we may apply Proposition 2.4. Indeed, the line bundle B'_1 is globally generated, and by adjunction $\mu(L) = \deg L \geq 2g(C) + 3 > 2g(C) + 2$. If $B_1 \cdot B_2 = 4$ and $B_1^2 \geq 6$, then $g(C) \geq 4$ and, since C is irreducible, it follows that it is non-hyperelliptic (cf. [CD], Proposition 4.5.1). Then the surjectivity of γ follows from Proposition 2.5. □

As a corollary of Theorem 2.2 we prove a stronger version of the conjecture of Mukai, in the case of Enriques surfaces and for the property N_0. To see that we use the following

Lemma 2.7. *Let A_1 and A_2 be two ample divisors on a surface X with Kodaira dimension 0. Then $A_1 \otimes A_2$ is base-point-free.*

Proof. Since $K_X \equiv 0$, $(A_1 \otimes A_2)^2 \geq 5$. By hypothesis $A_1 \otimes A_2$ is ample. If $A_1 \otimes A_2$ were not base-point-free, it would follow from Reider's theorem that there would exist an effective divisor E such that one of the following holds:

(a) $(A_1 \otimes A_2) \cdot E = 0$ and $E^2 = -1$ or
(b) $(A_1 \otimes A_2) \cdot E = 1$ and $E^2 = 0$.

None of the two possibilities can occur since, A_i being ample, $A_i \cdot E \geq 1$. □

Corollary 2.8. *Let X be an Enriques surface and $A_1, \ldots, A_n$ ample line bundles on X. Let $L = K_X \otimes A_1 \otimes \cdots \otimes A_n$. If $n \geq 4$, then L satisfies property N_0.*

Proof. By Lemma 2.7, $K_X \otimes A_1 \otimes A_2$ and $A_3 \otimes \cdots \otimes A_n$ are base-point-free line bundles. There are furthermore ample, and, by adjunction, $(K \otimes A_1 \otimes A_2)^2 \geq 6$, $(A_3 \otimes \cdots \otimes A_n)^2 \geq 6$ and $(K \otimes A_1 \otimes A_2) \cdot (A_3 \otimes \cdots \otimes A_n) \geq 4$. Then the result follows from Theorem 2.2. □

We now generalize these results to higher syzygies. To do so, we need another two lemmas. In the case in which $\mathfrak{q}$ is a curve C, the former allows us to pass from a multiplication map involving non-semistable bundles (note that $M_F \otimes \mathcal{O}_C$ if $H^1(L \otimes \mathcal{O}(-C)) = 0$) to a multiplication map involving semistable bundles, This situation is of course easier to handle. The latter dealing with positivity and semistability of bundles on curves. They will not only be used for the arguments in the remaining of this section but also in Section 3, 5 and 6.

Lemma 2.9. *Let X be a projective variety, let q be a nonnegative integer and let F_i be a base-point-free line bundle on X for all $1 \leq i \leq q$. Let Q be an effective line bundle on X and let $\mathfrak{q}$ be a reduced and irreducible member of $|Q|$. Let R be a line bundle and G a sheaf on X such that*

1. $H^1(F_i \otimes Q^*) = 0$
2. $H^0(M_{(F_{i_1} \otimes \mathcal{O}_\mathfrak{q})} \otimes \cdots \otimes M_{(F_{i_{q'}} \otimes \mathcal{O}_\mathfrak{q})} \otimes R \otimes \mathcal{O}_\mathfrak{q}) \otimes H^0(G) \to$
   $\to H^0(M_{(F_{i_1} \otimes \mathcal{O}_\mathfrak{q})} \otimes \cdots \otimes M_{(F_{i_{q'}} \otimes \mathcal{O}_\mathfrak{q})} \otimes R \otimes G \otimes \mathcal{O}_\mathfrak{q})$ *surjects for all* $0 \leq q' \leq q$.

Then, for all $0 \leq q'' \leq q$ and any subset $\{j_k\} \subseteq \{i\}$ with $\#\{j_k\} = q''$ and for all $0 \leq k' \leq q''$,

$$H^0(M_{F_{j_1}} \otimes \cdots \otimes M_{F_{j_{k'}}} \otimes M_{(F_{j_{k'+1}} \otimes \mathcal{O}_\mathfrak{q})} \otimes \cdots \otimes M_{(F_{j_{q''}} \otimes \mathcal{O}_\mathfrak{q})} \otimes R \otimes \mathcal{O}_\mathfrak{q}) \otimes H^0(G) \to$$
$$H^0(M_{F_{j_1}} \otimes \cdots \otimes M_{F_{j_{k'}}} \otimes M_{(F_{j_{k'+1}} \otimes \mathcal{O}_\mathfrak{q})} \otimes \cdots \otimes M_{(F_{j_{q''}} \otimes \mathcal{O}_\mathfrak{q})} \otimes G \otimes R \otimes \mathcal{O}_\mathfrak{q})$$

surjects.

Proof. We prove the result by induction on q''. For $q'' = 0$ the corresponding statement is just Condition 2 when $q = 0$. Assume that the result is true for $q'' - 1$. In order to prove the result for q'' we will use induction on k'. If $k' = 0$, the statement is again just Condition 2. Assume that the result is true for $k' - 1$. Now for any F globally generated vector bundle and for any effective divisor $\mathfrak{q}$ such that $H^1(F \otimes Q^*) = 0$, for $Q = \mathcal{O}(\mathfrak{q})$, we have this commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathrm{H}^0(F \otimes Q^*) \otimes \mathcal{O}_\mathfrak{q} & \to & \mathrm{H}^0(F \otimes Q^*) \otimes \mathcal{O}_\mathfrak{q} & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_F \otimes \mathcal{O}_\mathfrak{q} & \to & \mathrm{H}^0(F) \otimes \mathcal{O}_\mathfrak{q} & \to & F \otimes \mathcal{O}_\mathfrak{q} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_{(F \otimes \mathcal{O}_\mathfrak{q})} & \to & \mathrm{H}^0(F \otimes \mathcal{O}_\mathfrak{q}) \otimes \mathcal{O}_\mathfrak{q} & \to & F \otimes \mathcal{O}_\mathfrak{q} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We are interested in the left hand side vertical exact sequence:

2.9.1 $$0 \to \mathrm{H}^0(F \otimes Q^*) \otimes \mathcal{O}_\mathfrak{q} \to M_F \otimes \mathcal{O}_\mathfrak{q} \to M_{(F \otimes \mathcal{O}_\mathfrak{q})}$$

By Condition 1, F can be taken to be $F_{j_{k'}}$. Tensoring 2.9.1 by

$$M_{F_{j_1}} \otimes \cdots \otimes M_{F_{j_{k'-1}}} \otimes M_{(F_{j_{k'+1}} \otimes \mathcal{O}_\mathfrak{q})} \otimes \cdots \otimes M_{(F_{j_{q''}} \otimes \mathcal{O}_\mathfrak{q})} \otimes R \otimes \mathcal{O}_\mathfrak{q} \ ,$$

taking global sections and tensoring by $\mathrm{H}^0(G)$ we obtain this commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & A \otimes \mathrm{H}^0(G) & \to & B \otimes \mathrm{H}^0(G) & \to & C \otimes \mathrm{H}^0(G) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & A' & & B' & & C' & &
\end{array}$$

where $A = \mathrm{H}^0(F_{j_{k'}} \otimes Q^*) \otimes \mathrm{H}^0(\bigotimes_{r=1}^{k'-1} M_{F_{j_r}} \otimes \bigotimes_{r=k'+1}^{q''} M_{(F_{j_r}\otimes\mathcal{O}_\mathfrak{q})} \otimes R \otimes \mathcal{O}_\mathfrak{q})$, $B = \mathrm{H}^0(\bigotimes_{r=1}^{k'} M_{F_{j_r}} \otimes \bigotimes_{r=k'+1}^{q''} M_{(F_{j_r}\otimes\mathcal{O}_\mathfrak{q})} \otimes R \otimes \mathcal{O}_\mathfrak{q})$, $C = \mathrm{H}^0(\bigotimes_{r=1}^{k'-1} M_{F_{j_r}} \otimes \bigotimes_{r=k'}^{q''} M_{(F_{j_r}\otimes\mathcal{O}_\mathfrak{q})} \otimes R \otimes \mathcal{O}_\mathfrak{q})$, $A' = \mathrm{H}^0(F_{j_{k'}} \otimes Q^*) \otimes \mathrm{H}^0(\bigotimes_{r=1}^{k'-1} M_{F_{j_r}} \otimes \bigotimes_{r=k'+1}^{q''} M_{(F_{j_r}\otimes\mathcal{O}_\mathfrak{q})} \otimes R \otimes G \otimes \mathcal{O}_\mathfrak{q})$, $B' = \mathrm{H}^0(\bigotimes_{r=1}^{k'} M_{F_{j_r}} \otimes \bigotimes_{r=k'+1}^{q''} M_{(F_{j_r}\otimes\mathcal{O}_\mathfrak{q})} \otimes R \otimes G \otimes \mathcal{O}_\mathfrak{q})$ and $C' = \mathrm{H}^0(\bigotimes_{r=1}^{k'-1} M_{F_{j_r}} \otimes \bigotimes_{r=k'}^{q''} M_{(F_{j_r}\otimes\mathcal{O}_\mathfrak{q})} \otimes R \otimes G \otimes \mathcal{O}_\mathfrak{q})$. The top horizontal exact sequence is certainly surjective: this follows from chasing the diagram after having taken cohomology. The left hand side vertical sequence surjects by the induction hypothesis on q'' and the right hand side exact sequence surjects by induction on k' (we have assumed the result to be true for $q''-1$ and $k'-1$). Therefore we obtain the surjectivity of the vertical sequence sitting in the middle of the commutative diagram. □

Lemma 2.10. *Let E be a semistable vector bundle with $\mu(E) > 2g$ and F a vector bundle on a curve C of genus g.*

(1) *If $\mu(F) \geq 2g+4$, then $\mu(M_E \otimes F) > 2g+2$.*
(2) *If $\mu(F) \geq 2g+2$, then $\mu(M_E \otimes F) > 2g$.*

Moreover, if F is in addition semistable, then $M_E \otimes F$ is semistable.

Proof. Since E is semistable and $\mu(E) > 2g$, E is globally generated and $h^1(E) = 0$, hence the vector bundle M_E is defined and has slope

$$\mu(M_E) = \frac{-\mu(E)}{\mu(E) - g} \ .$$

Then, for (1), $\mu(M_E \otimes F) \geq \frac{-\mu(E)}{\mu(E)-g} + 2g + 4$. Thus if $\frac{-\mu(E)}{\mu(E)-g} + 2g + 4 > 2g + 2$ we are done, but that inequality is equivalent to $\mu(E) > 2g$. The proof of (2) is analogous. Now, if F is semistable by [Bu], Theorem 1.12 and [Mi], Corollary 3.7, $M_E \otimes F$ is also semistable. □

Theorem 2.11. *Let X be an Enriques surface. Let B_1, B_1', B_2 and B_2' be two ample and base-point-free divisors such that $B_1 \equiv B_1'$, $B_2 \equiv B_2'$ and $B_1 \cdot B_2 \geq 6$. Let $L = B_1^{\otimes s} \otimes B_2^{\otimes r}$ and $L' = {B_1'}^{\otimes k} \otimes {B_2'}^{\otimes l}$. If $k, l, r, s \geq 1$, then $H^1(M_L^{\otimes 2} \otimes L') = 0$. In particular, L' satisfies property N_1.*

Proof. The cohomology group $H^1(M_L^{\otimes 2} \otimes L') = 0$ sits in the long exact sequence

$$H^0(L) \otimes H^0(M_L \otimes L') \xrightarrow{\alpha} H^0(M_L \otimes L \otimes L')$$
$$\longrightarrow H^1(M_L^{\otimes 2} \otimes L') \longrightarrow H^0(L) \otimes H^1(M_L \otimes L') \ ,$$

obtained by tensoring (1.1) relative to L with $M_L \otimes L'$ and taking global sections. The last term is zero by Theorem 2.2, thus it is enough to prove that α is surjective. To show the surjectivity of α we use Observation 1.4.1. According to it we need to check the surjectivity of several maps. Here we will only show the surjectivity of the first of them, since the rest are analogous:

$$H^0(B_1) \otimes H^0(M_L \otimes L') \xrightarrow{\beta} H^0(M_L \otimes L' \otimes B_1) \ .$$

Let C be a smooth member of $|B_1|$. From Theorem 2.2 it follows that $H^1(M_L \otimes L' \otimes B_1^*) = 0$, therefore we may apply Observation 2.3 to reduce the question of surjectivity of β to the surjectivity of the following multiplication map on C:

$$H^0(B_1 \otimes \mathcal{O}_C) \otimes H^0(M_L \otimes L' \otimes \mathcal{O}_C) \longrightarrow H^0(M_L \otimes L' \otimes B_1 \otimes \mathcal{O}_C)$$

By Lemma 2.9 it is enough to check that the following multiplication maps on C are surjective:

$$H^0(B_1 \otimes \mathcal{O}_C) \otimes H^0(L' \otimes \mathcal{O}_C) \longrightarrow H^0(B_1 \otimes L' \otimes \mathcal{O}_C)$$
$$H^0(B_1 \otimes \mathcal{O}_C) \otimes H^0(M_{L\otimes\mathcal{O}_C} \otimes L' \otimes \mathcal{O}_C) \xrightarrow{\gamma} H^0(M_{L\otimes\mathcal{O}_C} \otimes L' \otimes B_1 \otimes \mathcal{O}_C)\ .$$

The surjectivity of the first map was already seen within the course of proving Theorem 2.2. For γ, we use Proposition 2.4. Since $\deg(L \otimes \mathcal{O}_C)$ and $\deg(L' \otimes \mathcal{O}_C)$ are both greater than or equal to $2g+4$, it follows from Lemma 2.10 that the bundle $M_{L\otimes\mathcal{O}_C} \otimes L' \otimes \mathcal{O}_C$ is semistable with slope strictly bigger than $2g+2$. Then it follows from Proposition 2.4 that γ is surjective and we are done. Now, since L' is ample, it follows from the vanishing of $H^1(M_{L'}^{\otimes 2} \otimes L'^{\otimes s})$ for all $s \geq 1$, Theorem 2.2 and Theorem 1.2, that L' satisfies property N_1. □

We obtain the following corollary, which proves Mukai's Conjecture (and when considering powers of the same ample bundle, improves his bound), regarding property N_1 for Enriques surfaces.

Corollary 2.12. *Let X be an Enriques surface. Let A, $A_1, \dots, A_n$ be ample line bundles. Then the line bundles $K_X \otimes A^{\otimes m}$ and $K_X \otimes A_1 \otimes \cdots \otimes A_n$ satisfy property N_1 if $m \geq 4$ and $n \geq 5$ respectively.*

Proof. For the former case, let $B_1 = K_X \otimes A^{\otimes 2}$ and $B_2 = A^{\otimes m-2}$. For the latter, let $B_1 = K_X \otimes A_1 \otimes A_2 \otimes A_3$ and $B_2 = A_4 \otimes \cdots \otimes A_n$. In both cases, B_1 and B_2 are ample, and base-point-free by Lemma 2.7. Furthermore, $B_1 \cdot B_2 \geq 6$, consequently the result follows from Theorem 2.11. □

To finish this section we show a result for higher syzygies of adjoint bundles. Before that we state a useful lemma dealing with the numerical nature of the property of base-point-freeness.

Lemma 2.13. *Let X be a surface such that $K_X \equiv 0$ (i.e., X is any minimal surface with $\kappa = 0$) and let B be an ample and base-point-free line bundle such that $B^2 \geq 5$. If $B' \equiv B$, then B' is ample and base-point-free.*

Proof. The line bundle B' is ample because ampleness is a numerical condition and has self-intersection greater than or equal to 5. If B' has base points, by Reider's theorem there is an effective divisor E such that:

(a) $B' \cdot E = 0$ and $E^2 = -1$ or
(b) $B' \cdot E = 1$ and $E^2 = 0$.

The former cannot happen because B' is ample. We will also rule out (b). The divisor E must be irreducible and reduced because B' is ample and $B' \cdot E = 1$. On the other hand, the arithmetic genus of E is greater than or equal to 1. Now $B \cdot E = B' \cdot E = 1$ so $h^0(B \otimes \mathcal{O}_E) \leq 1$. Since B is base-point-free, E should be a smooth rational curve and this is a contradiction. □

Theorem 2.14. *Let X be an Enriques surface. Let B be an ample and base-point-free line bundle such that $B^2 \geq 6$ and let N, N' be line bundles numerically equivalent to 0 (i.e., they are either trivial or equal to K_X). Let $L = B^{\otimes p+1+l} \otimes N$, $L' = B^{\otimes p+1+k} \otimes N'$ for $p \geq 1$. Then $H^1(M_L^{\otimes p+1} \otimes L')$ vanishes for all $k, l \geq 0$. In particular L satisfies property N_p.*

Proof. Since $B^2 \geq 6$, by Lemma 2.13 the line bundle $B \otimes N$ is also ample and base-point-free. The proof is by induction. The result is true for $p = 1$ by Theorem

2.11. We assume now the result to be true for $p-1$. In particular we have $H^1(M_L^{\otimes p} \otimes L') = 0$. Tensoring the sequence (1.1) with $M_L^{\otimes p} \otimes L'$ and taking global sections yields therefore the following long exact sequence

$$H^0(L) \otimes H^0(M_L^{\otimes p} \otimes L') \xrightarrow{\alpha} H^0(M_L^{\otimes p} \otimes L \otimes L') \to H^1(M_L^{\otimes p+1} \otimes L') \to 0 \ ,$$

thus it is enough to prove that the multiplication map α is surjective. Then by Observation 1.4.1 it is enough to see the surjectivity of

$$H^0(B') \otimes H^0(M_L^{\otimes p} \otimes B^{\otimes p+1+k} \otimes N') \xrightarrow{\beta} H^0(M_L^{\otimes p} \otimes B^{\otimes p+1+k} \otimes B' \otimes N') \ ,$$

where B' is either B or $B \otimes N$. Now to complete the proof one can argue in two ways. One of them is using (1.4.2). The path to follow is shown in the proof of Theorem 1.3 but we outline here the steps to be taken. The first cohomology vanishing required,

$$H^1(M_L^{\otimes p} \otimes B^{\otimes p+1+k} \otimes N' \otimes B'^*) \qquad (2.14.1)$$

follows directly by induction. For the second cohomology vanishing one may observe that, after iteratively chasing the cohomology sequence, it follows by induction, from Theorem 2.2 and Kodaira vanishing Theorem. The other way to argue is as for the surjectivity of β in the proof of Theorem 2.11: one uses and apply Lemma 2.9 to reduce the problem to checking the surjectivity of multiplication maps on a curve.

Finally since L is ample, Theorem 1.2 implies that L satisfies N_p. □

Corollary 2.15. *Let X be an Enriques surface, let A be an ample line bundle and B an ample and base-point-free line bundle on X. If $m \geq p+1$, then $K_X \otimes B^{\otimes m}$ satisfies property N_p. If $n \geq 2p+2$, then $K_X \otimes A^{\otimes n}$ satisfies property N_p.*

Proof. The first statement is a straight forward consequence of the theorem. By Lemma 2.7, the line bundle $A^{\otimes 2}$ is base-point-free, so if n is even the second statement follows from the first. If n is odd the result follows from a slight variation of the argument in the proof of Theorem 2.14: we break up $K_X \otimes A^{\otimes n}$ as tensor product of $n-1$ copies of $B = A^{\otimes 2}$ and $B' = A^{\otimes 3}$, which is base-point-free by Lemma 2.7. When applying Observation 1.4.1 we take the last map among the α_i to be precisely the map involving B'. The reader can easily verify that the vanishings needed in order to apply (1.4.2) follow by induction or, eventually, by Kodaira vanishing Theorem. □

3. Koszul rings of Enriques surfaces

We have devoted Section 2 to the study of syzygies of embeddings of Enriques surfaces. We show in particular a result, Theorem 2.11, about normal presentation of line bundles which were the tensor product of two base-point-free line bundles. Recall that the normal presentation property means that the homogeneous ideal of the (projectively normal) variety is generated by forms of degree 2. As already pointed out in the introduction, an interesting algebraic property that many normally presented rings have is the Koszul property. There exist many significant examples: canonical rings of curves (cf. [FV], [PP]), rings of curves of degree greater than or equal to $2g+2$ (cf. [Bu], [GP1]), elliptic ruled surfaces (cf. [GP1], Theorem 5.8) and those line bundles on Enriques surfaces which are normally presented according to Theorem 2.1 (cf. [GP1], Corollary 5.7). This section provides yet one more case in favor of this philosophy: we will show in Theorem 3.5 that those line bundles on an Enriques surface which are normally presented according to Theorem 2.11 also satisfy the Koszul property. Moreover, in the course of proving the result, it can be seen how the property N_1 is one of the first conditions required for the ring to be Koszul.

To begin we recall some notation and some basic definitions: given a line bundle L on a variety X, we set $R(L) = \bigoplus_{n=0}^{\infty} \mathrm{H}^0(X, L^{\otimes n})$.

Definition 3.1. *Let $R = \mathbf{k} \oplus R_1 \oplus R_2 \oplus \dots$ be a graded ring and $\mathbf{k}$ a field. R is a Koszul ring iff $Tor_i^R(\mathbf{k}, \mathbf{k})$ has pure degree i for all i.*

We recall now a cohomological interpretation, due to Lazarsfeld, of the Koszul property for a coordinate ring $R(L)$. Let L be a globally generated line bundle on a variety X. We will denote $M^{0,L} := L$ and $M^{1,L} := M_L \otimes L = M_{M^{0,L}} \otimes L$. If $M^{1,L}$ is globally generated, we denote $M^{2,L} := M_{M^{1,L}} \otimes L$. We repeat the process and define inductively $M^{h,L} := M_{M^{h-1,L}} \otimes L$, if $M^{h-1,L}$ is globally generated. Now we are ready to state the following slightly modified version of [P1], Lemma 1:

Lemma 3.2. *Let X be a projective variety over an algebraic closed field $\mathbf{k}$. Let L be an ample and base-point-free line bundle on X. Then $R(L)$ is Koszul iff $M^{h,L}$ exists, is globally generated and $H^0(M^{h,L}) \otimes H^0(L^{\otimes s+1}) \to H^0(M^{h,L} \otimes L^{\otimes s+1})$ is surjective for all $h \geq 0$, $s \geq 0$. If, in addition, $H^1(L^{\otimes s+1}) = 0$ for every $s \geq 0$, then $R(L)$ is Koszul iff $H^1(M^{(h),L} \otimes L^{\otimes s}) = 0$ for every $h \geq 0$ and every $s \geq 0$.*

The proof of Theorem 3.5 will follow the same strategy of Section 2, i.e., we will translate the problem in terms of a question about vector bundles over a suitable curve C of X. For that purpose we need now a way to relate $M^{(h),L}$ to $M^{(h),L\otimes\mathcal{O}_C}$. We carry this out link by link:

Definition 3.3. *Let X be a variety, let L be a line bundle on X and let $\mathfrak{b}$ be a (smooth) effective divisor on X. Assume that $M^{h',L}$ is defined for all $h \geq h' \geq 0$ (i.e., inductively, $M^{h'-1,L}$ is defined and globally generated). We then define, for all $0 \leq h' \leq h$, $M_{h',\mathfrak{b}}^{h',L} = M^{h',L} \otimes \mathcal{O}_{\mathfrak{b}}$. Then $M_{h',\mathfrak{b}}^{h',L}$ is globally generated and we define $M_{h',\mathfrak{b}}^{h'+1,L} = M_{M_{h',\mathfrak{b}}^{h',L}} \otimes L$. If $M_{h',\mathfrak{b}}^{h'+1,L}$ is again globally generated we define $M_{h',\mathfrak{b}}^{h'+2,L} = M_{M_{h',\mathfrak{b}}^{h'+1,L}} \otimes L$ and so on.*

Lemma 3.4. *Let X be a variety, let $\mathfrak{b}$ be a (smooth) effective divisor on X and let $B = \mathcal{O}(\mathfrak{b})$. Let L be a base-point-free line bundle on X such that $M^{h',L}$ is globally generated and $H^1(M^{h',L} \otimes B^*) = 0$ for all $0 \leq h' \leq h-1$, $H^1(L \otimes \mathcal{O}_{\mathfrak{b}}) = 0$, and $L \otimes \mathcal{O}_{\mathfrak{b}}$ is Koszul. Then,*

(1) *$M_{h',\mathfrak{b}}^{h,L}$ is globally generated for all $0 \leq h' \leq h$.*
(2) *$H^1(M_{h',\mathfrak{b}}^{h,L}) = 0$ for all $0 \leq h' \leq h$.*
(3) *$0 \to H^0(M^{h'-1,L} \otimes B^*) \otimes M_{0,\mathfrak{b}}^{h-h',L} \to M_{h',\mathfrak{b}}^{h,L} \to M_{h'-1,\mathfrak{b}}^{h,L} \to 0$, for all $1 \leq h' \leq h$.*

Proof. The proof is by induction on h. If $h = 0$, the result is part of the hypotheses. If $h = 1$, the exact sequence in (3) is 2.9.1 when we set $E = L$ and twisted by L. Let us write $L_{\mathfrak{b}} = L \otimes \mathcal{O}_{\mathfrak{b}}$. Since $H^1(L_{\mathfrak{b}}) = 0$ and $L_{\mathfrak{b}}$ is Koszul, $H^1(M^{1,L}_{0,\mathfrak{b}}) = 0$, therefore using (3) we obtain indeed that $H^1(M^{1,L}_{1,\mathfrak{b}}) = 0$. The bundle $M^{1,L}_{0,\mathfrak{b}}$ is globally generated because $L_{\mathfrak{b}}$ is Koszul. Finally the fact that $M^{1,L}_{1,\mathfrak{b}}$ is globally generated follows again from (3): we have the following exact commutative diagram

$$\begin{array}{ccccc} H^0(L\otimes B^*)\otimes H^0(L_{\mathfrak{b}})\otimes \mathcal{O}_{\mathfrak{b}} & \hookrightarrow & H^0(M_L\otimes L_{\mathfrak{b}})\otimes \mathcal{O}_{\mathfrak{b}} & \twoheadrightarrow & H^0(M_{L_{\mathfrak{b}}}\otimes L_{\mathfrak{b}})\otimes \mathcal{O}_{\mathfrak{b}} \\ \downarrow & & \downarrow & & \downarrow \\ H^0(L\otimes B^*)\otimes L_{\mathfrak{b}} & \hookrightarrow & M_L\otimes L_{\mathfrak{b}} & \twoheadrightarrow & M_{L_{\mathfrak{b}}}\otimes L_{\mathfrak{b}} \end{array}$$

in which the vertical side arrows are surjective because $L_{\mathfrak{b}}$ and $M_{L_{\mathfrak{b}}} \otimes L_{\mathfrak{b}} = M^{1,L}_{0,\mathfrak{b}}$ are both globally generated. Let us now assume the result to be true for $h-1$ and prove it for h. We again prove (3) first. If $h = h'$, again (3) is nothing but 2.9.1, setting $F = M^{h-1,L}$ (which we know by induction hypothesis to be globally generated) and twisted by L. If $h > h'$, by induction on h we have the sequence

$$0 \to \mathrm{H}^0(M^{h'-1,L} \otimes B^*) \otimes M^{h-h'-1,L}_{0,\mathfrak{b}} \to M^{h-1,L}_{h',\mathfrak{b}} \to M^{h-1,L}_{h'-1,\mathfrak{b}} \to 0 \ .$$

Call $V = H^0(M^{h'-1,L} \otimes B^*)$. Taking global sections, we build this exact commutative diagram:

$$\begin{array}{ccccc} V\otimes H^0(M^{h-h'-1,L}_{0,\mathfrak{b}})\otimes \mathcal{O}_{\mathfrak{b}} & \hookrightarrow & H^0(M^{h-1,L}_{h',\mathfrak{b}})\otimes \mathcal{O}_{\mathfrak{b}} & \twoheadrightarrow & H^0(M^{h-1,L}_{h'-1,\mathfrak{b}})\otimes \mathcal{O}_{\mathfrak{b}} \\ \downarrow & & \downarrow & & \downarrow \\ V\otimes M^{h-h'-1,L}_{0,\mathfrak{b}} & \hookrightarrow & M^{h-1,L}_{h',\mathfrak{b}} & \twoheadrightarrow & M^{h-1,L}_{h'-1,\mathfrak{b}} \end{array}$$

The top horizontal sequence is exact at the right because $H^1(M^{h-h'-1,L}_{0,\mathfrak{b}}) = 0$, by induction hypothesis. The vertical arrows are surjective because the vector bundles involved are globally generated by induction hypothesis on h. The short exact sequence of kernels is then, after tensoring by $L_{\mathfrak{b}}$, the sequence wanted for (3). To prove (2), we use induction on h'. If $h' = 0$ both (1) and (2) follow from the fact that $L_{\mathfrak{b}}$ is Koszul and $H^1(L_{\mathfrak{b}}) = 0$. Now assume that (1) and (2) hold for $h'-1$. Condition (2) is a straight forward consequence of already proven (3) and induction hypothesis on both h and h'. For (1) we use induction on both h and h' and (3) just proven. If $h = 0$ the surjectivity just follows from the fact that $L_{\mathfrak{b}}$ is Koszul, hence normally generated. If $h' = 0$ the surjectivity just follows from the fact that $L_{\mathfrak{b}}$ is Koszul. Assume now that the claim holds for $h'-1$. The surjectivity of the map for h' follows then by chasing the commutative diagram of multiplication maps, built upon (3), having in account the vanishing of $H^1(M^{h-h',L}_{0,\mathfrak{b}})$, which follows from (2), and the surjectivity of the vertical side maps, which follows from induction hypothesis on h and h'. Then the fact that $L_{\mathfrak{b}}$ is ample implies the global generation of $M^{h,L}_{h',\mathfrak{b}}$ as wished. □

We are now ready to prove the main theorem of this section:

Theorem 3.5. *Let X be an Enriques surface. Let B_1 and B_2 be ample and base-point-free line bundles, such that $B_1 \cdot B_2 \geq 6$. If $L = B_1 \otimes B_2$, then $R(L)$ is Koszul.*

Proof. According to Lemma 3.2 we need to show that $M^{h,L}$ is globally generated and that

$$H^0(M^{h,L}) \otimes H^0(L^{\otimes s}) \xrightarrow{\alpha} H^0(M^{h,L} \otimes L^{\otimes s})$$

surjects for all $h \geq 0$ and $s \geq 1$. To better carry out the argument, is convenient to also prove $H^1(M^{h,L} \otimes B_1^*) = H^1(M^{h,L} \otimes B_2^*) = 0$. The proof is by induction on h. If $h = 0$ the result is the projective normality of $L = B_1 \otimes B_2$, which follows from Theorem 2.2, and Kodaira vanishing. Now assume the result for $h - 1$. Since L is ample, the surjectivity of α implies the global generation of $M^{h,L}$, hence we can assume that $M^{h',L}$ is globally generated for all $0 \leq h' < h$ and we need only to prove that α surjects and that $H^1(M^{h,L} \otimes B_1^*) = H^1(M^{h,L} \otimes B_2^*) = 0$. We start proving the former and in the course of the proof we will also obtain the desired vanishings. According to Observation 1.4.1 we are done if we prove that certain collection of multiplication maps surject. We prove the surjectivity of the first of them, which is

$$H^0(M^{h,L}) \otimes H^0(B_1) \xrightarrow{\beta} H^0(M^{h,L} \otimes B_1) \ .$$

The argument to prove the surjectivity of the rest is analogous. We prove it using again induction on h. We proved the statement for $h = 0$ in the course of proving the projective normality of L in Theorem 2.2. Assume the statement to be true for $h - 1$ (we may also assume the surjectivity of the map β for $h - 1$ if we substitute in the formula B_1 by B_2, since the roles of B_1 and B_2 are interchangeable. Consider the sequence

$$\begin{gathered} H^0(M^{h-1,L}) \otimes H^0(B_2) \xrightarrow{\gamma} H^0(M^{h-1,L} \otimes B_2) \\ \to H^1(M^{h,L} \otimes B_1^*) \to H^1(M^{h-1,L}) \otimes H^0(B_2) \ . \end{gathered}$$

The multiplication map γ is surjective by induction hypothesis. The group $H^1(M^{h-1,L})$ vanishes also by induction hypothesis, therefore $H^1(M^{h,L} \otimes B_1^*) = 0$. On the other hand $H^1(\mathcal{O}_X) = 0$, so in order to see the surjectivity of β it is enough to check the surjectivity of

$$H^0(M^{h,L}_{h,\mathfrak{b}_1}) \otimes H^0(B_1 \otimes \mathcal{O}_{\mathfrak{b}_1}) \xrightarrow{\delta} H^0(M^{h,L}_{h,\mathfrak{b}_1} \otimes B_1) \ ,$$

where $\mathfrak{b}_1$ is a smooth irreducible curve in $|B_1|$. To see the surjectivity of δ we will use Lemma 3.4 inductively on h'. More precisely we want to prove that

$$H^0(M^{h,L}_{h',\mathfrak{b}_1}) \otimes H^0(B_1 \otimes \mathcal{O}_{\mathfrak{b}_1}) \to H^0(M^{h,L}_{h,\mathfrak{b}_1} \otimes B_1)$$

surjects for all $0 \leq h' \leq h$. If $h' = 0$, $M^{h,L}_{0,\mathfrak{b}_1}$ is semistable with slope strictly bigger than $2g + 2$ by Lemma 2.10, hence by Proposition 2.4 the multiplication map in question is surjective. Now assume the statement to be true for $h' - 1$. We take global sections in the sequence in part (3) of the statement of Lemma 3.4 and tensor with $U = H^0(B_1 \otimes \mathcal{O}_{\mathfrak{b}_1})$ to obtain the following exact commutative diagram,

$$\begin{array}{ccccc} W \otimes H^0(M^{h-h',L}_{0,\mathfrak{b}_1}) \otimes U & \hookrightarrow & H^0(M^{h,L}_{h',\mathfrak{b}_1}) \otimes U & \twoheadrightarrow & H^0(M^{h,L}_{h'-1,\mathfrak{b}_1}) \otimes U \\ \downarrow & & \downarrow & & \downarrow \\ W \otimes H^0(M^{h-h',L}_{0,\mathfrak{b}_1} \otimes B_1) & \hookrightarrow & H^0(M^{h,L}_{h',\mathfrak{b}_1} \otimes B_1) & \twoheadrightarrow & H^0(M^{h,L}_{h'-1,\mathfrak{b}_1} \otimes B_1) \ , \end{array}$$

where $W = H^0(M^{h'-1,L} \otimes B_1)$. The surjectivity of the left hand side vertical map and the exactness at the right of the top horizontal sequence follow both from Proposition 2.4 and Lemma 2.10. The surjectivity of the right hand side vertical map follows by the induction hypothesis on h'. $\square$

4. Abelian and bielliptic surfaces

In this section we deal with the remaining classes of surfaces with Kodaira dimension 0, namely, those with nonzero irregularity. For the techniques employed we return to those used in the arguments of Section 1. The main theorem we will prove is

Theorem 4.1. *Let X be an abelian or a bielliptic surface. Let B be an ample and base-point-free line bundle with $B^2 \geq 5$ and let N be a numerically trivial line bundle on X. Let $L_1 \equiv B^{\otimes l_1+1}$ and $L_2 \equiv B^{\otimes l_2+1}$. If $l_1, l_2 \geq p \geq 1$, then*

$$H^1(M_{L_1}^{\otimes p+1} \otimes L_2) = H^1(M_{L_1} \otimes L_2) = 0 \ .$$

In particular, if $n \geq p \geq 1$, then $B^{n+1} \otimes N$ satisfies the property N_p.

Before we prove Theorem 4.1 we need the following

Lemma 4.2. *Let X be a surface with $\kappa = 0$. Let B be an ample and base-point-free line bundle. Let $L_1 = B_1^1 \otimes \cdots \otimes B_{l_1}^1$, where $B_i^1 \equiv B$ are base-point-free line bundles and $l_1 \geq 1$ and $L_2 = B_1^2 \otimes \cdots \otimes B_{l_2}^2$, where $B_j^2 \equiv B$ and $l_2 \geq 1$. If either*

(1) *l_1 or l_2 are greater than or equal to 3 or,*
(2) *$l_1 = 2$, $l_2 = 1$ or 2 and $H^2(L_1 \otimes (B_1^2)^{-2}) = 0$ or,*
(3) *X is abelian or bielliptic surface, $B^2 \geq 5$, and $l_1 = l_2 = 2$,*

then $H^1(M_{L_1} \otimes L_2) = 0$.

Proof. In cases (1) and (2) the result follows from iteratively applying (1.4.2) using Observation 1.4.1 and Kodaira vanishing. In case (3), let us write L_1 as $B^{\otimes 2} \otimes E_1$ with $E_1 \equiv 0$. We can find $E \equiv 0$ with $E^{\otimes 2} \neq E_1 \otimes K^*$, because not all elements in $\text{Pic}^0(X)$ (the group of all numerically trivial line bundles up to linear equivalence) have order 2. Then, by Lemma 2.13, we can assume that $B_1^2 = B \otimes E$. Then $H^2(L_1 \otimes (B_1^2)^{-2}) = H^0(K \otimes E^{\otimes 2} \otimes E_1^*) = 0$, which follows from our choice of E, and we are in case (2). □

4.3. *Proof of Theorem 4.1.* The vanishing of $H^1(M_{L_1} \otimes L_2)$ is a consequence of Lemma 4.2. The proof of the vanishings of $H^1(M_{L_1}^{\otimes p+1} \otimes L_2)$ is by induction. As usual the key step is the first: $p = 1$. We need to prove that $H^1(M_{R_1}^{\otimes 2} \otimes R_2) = 0$ if $R_1 \equiv B^{\otimes r_1}$ and $R_2 \equiv B^{\otimes r_2}$ and $r_1, r_2 \geq 2$. Using the sequence (1.1) we obtain

$$H^0(M_{R_1} \otimes R_2) \otimes H^0(R_1) \xrightarrow{\alpha} H^0(M_{R_1} \otimes R_1 \otimes R_2)$$
$$\to H^1(M_{R_1}^{\otimes 2} \otimes R_2) \to H^1(M_{R_1} \otimes R_2) \otimes H^0(R_1) \ .$$

The group $H^1(M_{R_1} \otimes R_2)$ vanishes by Lemma 4.2. Therefore the sought vanishing is equivalent to the surjectivity of α. If $r_1 \geq 3$, let $B_1^1 = B$. If $r_1 = 2$, let $R_1 = B^{\otimes 2} \otimes E_1$. Analogously, if $r_2 = 2$, let $R_2 = B^{\otimes 2} \otimes E_2$. We may now assume if $r_1 \geq 2$, by Lemma 2.13, that $B_1^1 = B \otimes E$ with $E \equiv 0$ but $E^{\otimes 2} \neq K^* \otimes E_2$ and, if in addition $r_2 = 2$, that $E^{\otimes 2} \neq K \otimes E_2^{\otimes 2} \otimes E_1^*$ also. We can always find such an E if not all elements in $\text{Pic}^0(X)$ have order 2, 4 or 6. This is the case for abelian and bielliptic surfaces, which possess numerically trivial line bundles of infinite order. Then, to see that α surjects, by (1.4.2) and Observation 1.4.1 it suffices to check that

$$(4.3.1) \qquad H^1(M_{R_1} \otimes R_2 \otimes (B_1^1)^*) = 0$$
$$(4.3.2) \qquad H^2(M_{R_1} \otimes R_2 \otimes (B_1^1)^{-2}) = 0$$
$$(4.3.3) \qquad H^1(M_{R_1} \otimes R_2' \otimes B^{\otimes \gamma}) = 0 \text{ for all } 0 \leq \gamma \text{ and } R_2' \equiv R_2$$
$$(4.3.4) \qquad H^2(M_{R_1} \otimes R"_2 \otimes B^{\otimes \gamma - 1}) = 0 \text{ for all } 0 \leq \gamma \text{ and } R"_2 \equiv R_2 \ .$$

The vanishing (4.3.4) follows from (1.1) and Kodaira vanishing Theorem. The vanishing (4.3.2) follows from (1.1), Kodaira vanishing Theorem and the way in which we have chosen E. The vanishing in (4.3.3) follows from Lemma 4.2. Finally, (4.3.1) follows from Lemma 4.2 once we see that if $r_1 = r_2 = 2$, $H^2(R_1 \otimes R_2^{-2} \otimes (B_1^1)^{\otimes 2}) = H^2(E_1 \otimes E^{\otimes 2} \otimes E_2^{-2}) = 0$, which follows from the way in which we have chosen E.

Assume the result true for $p-1$ and $p > 1$. We have the following sequence:

$$H^0(M_{R_1}^{\otimes p} \otimes R_2) \otimes H^0(R_1) \xrightarrow{\beta} H^0(M_{R_1}^{\otimes p} \otimes R_1 \otimes R_2)$$
$$\to H^1(M_{R_1}^{\otimes p+1} \otimes R_2) \to H^1(M_{R_1}^{\otimes p} \otimes R_2) \otimes H^0(R_1) \ .$$

The last term is zero by induction hypothesis, so the desired vanishing is equivalent to the surjectivity of β. This follows from Observation 1.4.1 and (1.4.2). In fact, the required vanishings follow by induction, Kodaira vanishing Theorem and Lemma 4.2.

For the last conclusion of the theorem, note that

$$H^1(M_L^{\otimes p'+1} \otimes L^{\otimes s}) = 0 \text{ for all } p \geq p' \geq 0 \text{ and all } s \geq 1 \ .$$

Then, by Theorem 1.2, L satisfies property N_p.

Either as straight forward consequence of Theorem 4.1 or from the same ideas we have been using we obtain results for adjoint linear series:

Corollary 4.4. *Let X be an abelian or bielliptic surface. Let B be an ample and base-point-free line bundle such that $B^2 \geq 5$. Then $K \otimes B^{\otimes n}$ satisfies property N_p if $n \geq p+1$, $p \geq 1$.*

Corollary 4.4 implies Mukai's conjecture for abelian and bielliptic surfaces and $p = 0$, and for $p = 1$ (in the latter case, our result improves Mukai's bound):

Corollary 4.5. *Let X be an abelian or a bielliptic surface. Let A be an ample line bundle and $L = K_X \otimes A^{\otimes n}$. If $n \geq 2p+2$ and $p \geq 1$, then L satisfies property N_p. In particular, if $n \geq 4$, L satisfies property N_1.*

Proof. $A^{\otimes 2}$ is base-point-free by Lemma 2.7 (for abelian surfaces this also follows from Lefschetz's Theorem) and since $K \equiv 0$, $A^2 \geq 2$ and $(A^{\otimes 2})^2 \geq 8$. Then, if n is even the result is a straight forward consequence of Corollary 4.4. If n is odd the situation is the same as that of Corollary 2.15 and we proceed analogously. □

Remark 4.6. If X is an abelian surface the above result was proven by Kempf (cf. [K]). However, the results proven in this chapter are stronger: for instance, since on an abelian surface a polarization of type $(2,1)$ is base-point-free, Corollary 4.4

is not implied by Kempf's result, which only deals with line bundles of type (n, n) with $n \geq 2p+2$.

To end this section we carry out a study analogous to the one realized for Enriques surfaces in Section 3: the following theorem proves in particular that the line bundles satisfying property N_1 according to Theorem 4.1 have also a Koszul coordinate ring.

Theorem 4.7. *Let X be an abelian or a bielliptic surface. Let B_1 and B_2 be numerically equivalent ample and base-point-free line bundles with self-intersection bigger than or equal to 5. If $L = B_1 \otimes B_2$, then $R(L)$ is Koszul. In particular L satisfies property N_1.*

In order to prove the theorem we use the following result which is basically a reformulation of [GP1], Theorem 5.4 for the case of surfaces with $\kappa = 0$:

Lemma 4.8. *Let X be a surface with $\kappa = 0$, let B_1 and B_2 be two ample and base-point-free line bundles. If $H^2(B_1 \otimes B_2^*) = H^2(B_2 \otimes B_1^*) = 0$, then the following properties are satisfied for all $h \geq 0$:*

1) $M^{h,L}$ *is globally generated*
2) $H^1(M^{h,L} \otimes B_1^{\otimes b_1} \otimes B_2^{\otimes b_2}) = 0$ *for all* $b_1, b_2 \geq 0$
3) $H^1(M^{h,L} \otimes B_j^*) = 0$ *where* $j = 1, 2$
4) $H^1(M^{h,L} \otimes B_i \otimes B_j^*) = 0$ *where* $i = 1, 2$ *and* $j = 2, 1$
5) $H^1(M^{h,L} \otimes B_i^{\otimes 2} \otimes B_j^*) = 0$ *where* $i = 1, 2$ *and* $j = 2, 1$

In particular $H^1(M^{(h),L} \otimes L^{\otimes s}) = 0$ for all $h, s \geq 0$, and $R(L)$ is a Koszul k-algebra.

Proof. For the proof of the lemma we refer to [GP1]. Since now B_1 and B_2 are ample and $K_X \equiv 0$, we obtain all the vanishings of the groups $H^1(B_1^{\otimes a} \otimes B_2^{\otimes b})$ when $a, b \geq 0$ and $a + b \geq 1$ needed in the proof, by Kodaira vanishing Theorem, making therefore unnecessary to assume the vanishings of $H^1(B_1)$, $H^1(B_2)$ and $H^2(\mathcal{O}_X)$. □

4.9. *Proof of Theorem 4.7.* The result follows from Lemma 4.8. If X is bielliptic, the only problem we might have is if $B_1 = B_2 \otimes K_X$ or if $B_2 = B_1 \otimes K_X$. In the former case, choose a line bundle $E \in \mathrm{Pic}^0(X)$ such that $E^{-2} \neq 0$ and $K_X^{\otimes 2} \otimes E^{\otimes 2} \neq 0$. In the latter case choose E such that $E^{\otimes 2} \neq 0$ and $K_X^{\otimes 2} \otimes E^{-2} \neq 0$. Let then $B_1' = B_1 \otimes E$ and $B_2' = B_2 \otimes E^*$. The desired result follows if we apply Lemma 4.8 to B_1' and B_2' instead.

If X is an abelian surface, K_X is trivial, so the only problem applying Lemma 4.8 would appear when $B_1 = B_2$. This is solved analogously considering $B_1' = B_1 \otimes E$ and $B_2' = B_2 \otimes E^*$, where now E is taken to have nontrivial square. □

5. Surfaces of general type

In this chapter we study the pluricanonical morphism of a surface of general type S. It is known that under certain mild conditions $K_S^{\otimes n}$ is base-point-free if $n \geq 2$. The precise result, which is due to Bombieri, Francia, Reider and others, can be found in [Ca], Theorem 1.11 (i):

Theorem 5.1. *Let S be a surface of general type. Assume that either*

(1) $K_S^2 \geq 5$ *or*
(2) $K_S^2 \geq 2$ *and* $p_g \geq 1$*, but it does not happen that* $q = p_g = 1$ *and* $K_S^2 = 3$ *or* 4.

If $n \geq 2$*, then* $K_S^{\otimes n}$ *is base-point-free.*

Bombieri asked in [Bo] whether $|K_S^{\otimes 5}|$ maps S as a projectively normal variety. This question was answered affirmatively by Ciliberto in [Ci] (under basically the same assumptions of Theorem 5.1). We give here a proof (broken in two parts according to whether S is regular or not) which uses the techniques displayed throughout the article. From this starting point we go on to improve Ciliberto's result in the case of regular surfaces, obtaining Theorem 5.4 and to generalize it to higher syzygies.

Theorem 5.2. *Let S be an irregular surface of general type with $K_S^2 \geq 5$. If $n \geq 5$, then the image of S by the complete linear series $|K_S^{\otimes n}|$ is a projectively normal variety.*

Proof. We prove a fact which is more general than the projective normality of the image by $|K_S^{\otimes n}|$, namely, the surjectivity of the map

$$H^0(K_S^{\otimes l}) \otimes H^0(K_S^{\otimes m}) \xrightarrow{\alpha} H^0(K_S^{\otimes l+m}) \text{ for all } l \geq 5 \text{ and } m \geq 4\ .$$

Since the irregularity $q \neq 0$, we can find a divisor δ in the $\mathrm{Pic}(S)$ such that $\delta \equiv 0$ but $\delta^{\otimes 2} \neq 0$. Let $B_1 = K_S^{\otimes 2} \otimes \delta$. Then $K_S^{\otimes m} = B_1 \otimes \cdots \otimes B_s$, where $B_2 = \cdots = B_{s-1} = K_S^{\otimes 2}$ and $B_s = K_S^{\otimes 2} \otimes \delta^*$ or $K_S^{\otimes 3} \otimes \delta^*$, depending on whether m is even or odd. By Theorem 5.1, $B_2, \ldots, B_{s-1}$ are base-point-free. The line bundle B_1 is base-point-free by Reider's theorem. Indeed, otherwise there would exist E effective line bundle such that $(K_S \otimes \delta) \cdot E = K_S \cdot E = 0, 1$ and $E^2 = -1, 0$. Then by adjunction it would follow that $2p_a - 2$ is an odd number! It also follows from Reider's that B_s is base-point-free. By Observation 1.4.1 it is enough to show that the maps

$$H^0(K_S^{\otimes l} \otimes B_1 \otimes \cdots \otimes B_{i-1}) \otimes H^0(B_i) \xrightarrow{\beta} H^0(K_S^{\otimes l} \otimes B_1 \otimes \cdots \otimes B_i)$$

surject for all $1 \leq i \leq s$. If $l \geq 6$ or $i \geq 2$, then the map β surjects by (1.4.2) and Kawamata-Viehweg. If $l = 5$ and $i = 1$, the surjectivity of β will also follow from (1.4.2). We need to check the vanishing of $H^1(K_S^{\otimes 3} \otimes \delta^*)$ and $H^2(K_S \otimes \delta^{\otimes -2})$. The first group vanishes by Kawamata-Viehweg. The second vanishes because of our choice of δ. $\square$

Theorem 5.3. *Let S be a regular surface of general type with either $p_g \geq 1$ and $K_S^2 \geq 2$ or $K_S^2 \geq 5$. If $n \geq 5$, then the image of S by the complete linear series $|K_S^{\otimes n}|$ is a projectively normal variety.*

Proof. Again, we prove a stronger statement, namely, the surjectivity of the map

$$H^0(K_S^{\otimes l}) \otimes H^0(K_S^{\otimes m}) \xrightarrow{\alpha} H^0(K_S^{\otimes l+m}), \text{ for all } l \geq 5 \text{ and } m \geq 4\ .$$

Let $B = K_S^{\otimes 2}$ and $B' = K_S^{\otimes 3}$. By Theorem 5.1, both B and B' are base-point-free. The bundle $K_S^{\otimes m}$ can be written as $B_1 \otimes \cdots \otimes B_s$, where $B_1 = \cdots = B_{s-1} = B$

and $B_s = B$ or B', depending on whether m is even or odd. In view of Observation 1.4.1 it is enough to check the surjectivity of several maps, which are

$$H^0(K_S^{\otimes l}) \otimes H^0(B) \xrightarrow{\beta} H^0(K_S^{\otimes l+2})$$
$$H^0(K_S^{\otimes l+2}) \otimes H^0(B') \xrightarrow{\gamma} H^0(K_S^{\otimes l+5}), \text{ for all } l \geq 5 \ .$$

For the first family of maps, let $C \in |B|$ be a smooth curve, which exists by Bertini's Theorem. Since $q = 0$ and by Kawamata-Viehweg, $H^1(K_S^{\otimes r}) = 0$ for all $r \geq 0$. Hence we can apply Observation 2.3, so we only need to check the surjectivity of the following map on the curve C:

$$H^0(K_S^{\otimes l} \otimes \mathcal{O}_C) \otimes H^0(B \otimes \mathcal{O}_C) \xrightarrow{\zeta} H^0(K_S^{\otimes l+2} \otimes \mathcal{O}_C), \text{ for all } l \geq 5 \ .$$

Let $G = K_S^{\otimes l} \otimes \mathcal{O}_C$ and $G' = B \otimes \mathcal{O}_C$. By adjunction $6K_S^2 = 2g(C) - 2$, hence $\deg(G) + \deg(G') \geq 4g(C) - 4 + 2K_S^2$. Since $H^1(B) = H^2(B) = 0$ we have $p_g = h^1(B \otimes \mathcal{O}_C)$. Therefore, having in account the hypothesis on p_g and K_S^2, the surjectivity of ζ follows from Proposition 2.5. Note that the surjectivity of β if $l \geq 6$ follows also from (1.4.2) and Kawamata-Viehweg but it is crucial to use the argument of the "restriction to curves" for $l = 5$. To complete the proof we need to see the surjectivity of γ for all $l \geq 5$, which follows from the same arguments used for the surjectivity of β. □

As already said, we can improve these results when S is regular. The next theorem is the first step of the inductive argument which allows us to study the syzygies of pluricanonical models:

Theorem 5.4. *Let S be a regular surface of general type with $p_g \geq 3$. Then*

(1) $H^1(M_{K_S^{\otimes 4+k}} \otimes K_S^{\otimes 4+l}) = 0$ *for all* $k, l \geq 0$, *and*
(2) $H^1(M^{\otimes 2}_{K_S^{\otimes 4+k}} \otimes K_S^{\otimes 4+l}) = 0$ *for* $k, l = 0$, *for all* $k, l \geq 1$, *and for all* $k \geq 0, l \geq 2$.

In particular, if $n \geq 4$, then the image of S by the complete linear series $|K_S^{\otimes n}|$ is a projectively normal variety and its homogeneous ideal is generated by quadratic equations.

Proof. We first prove (1). It follows from Kawamata-Viehweg that this is equivalent to the surjectivity of these maps:

$$H^0(K_S^{\otimes 4+k}) \otimes H^0(K_S^{\otimes 4+l}) \longrightarrow H^0(L^{\otimes 8+k+l}), \text{ for all } k, l \geq 0 \ .$$

Let $B = K_S^{\otimes 2}$ and $B' = K_S^{\otimes 3}$. By Theorem 5.1(note that $K_S^2 \geq 2$ by Nöther's inequality) both B and B' are base-point-free. We will use again Observation 1.4.1. According to it, it suffices to see that the multiplication maps

$$H^0(K_S^{\otimes 4+k}) \otimes H^0(B) \xrightarrow{\alpha} H^0(K_S^{\otimes 6+k})$$
$$H^0(K_S^{\otimes 7+k}) \otimes H^0(B') \xrightarrow{\alpha'} H^0(K_S^{\otimes 10+k})$$

surjects for all $k \geq 0$. To check that α (respectively. α') surjects we consider C a smooth member in $|B|$ (respectively, $|B'|$). By Kawamata-Viehweg and because of the fact that S is regular, $H^1(B^{\otimes r}) = H^1({B'}^{\otimes r}) = 0$ for all $r \geq 0$, so we use Observation 2.3 and translate the problem to checking the surjectivity of a multiplication map on C as we did in Theorem 5.3 and argue analogously, i.e., using Proposition 2.5, having in account that $p_g \geq 3$. Note that the surjectivity of α and α' also follows from (1.4.2) if $k \geq 1$, but it is crucial to argue restricting to C in the case $k = 0$. Thus the proof of (1) is complete.

Now we prove (2). The cohomology groups in question fit in the sequences:

$$H^0(M_{K_S^{\otimes 4+k}} \otimes K_S^{\otimes 4+l}) \otimes H^0(K_S^{\otimes 4+k}) \xrightarrow{\gamma} H^0(M_{K_S^{\otimes 4+k}} \otimes K_S^{\otimes 8+k+l})$$
$$\longrightarrow H^1(M^{\otimes 2}_{K_S^{\otimes 4+k}} \otimes K_S^{\otimes 4+l}) \longrightarrow H^1(M_{K_S^{\otimes 4+k}} \otimes K_S^{\otimes 4+l}) \otimes H^0(K_S^{\otimes 4+k}) \; .$$

The last term of the sequence vanishes by (1), thus the required vanishings are equivalent to the surjectivity of γ for $k, l = 0$, for all $k, l \geq 1$, and for all $k \geq 0, l \geq 2$. To see the surjectivity we invoke Observation 1.4.1. It is enough to show that the maps

$$H^0(M_{K_S^{\otimes 4+k}} \otimes K_S^{\otimes 4+l}) \otimes H^0(B) \xrightarrow{\mu} H^0(M_{K_S^{\otimes 4+k}} \otimes K_S^{\otimes 6+l})$$
$$H^0(M_{K_S^{\otimes 4+k}} \otimes K_S^{\otimes 7+l}) \otimes H^0(B') \xrightarrow{\delta} H^0(M_{K_S^{\otimes 4+k}} \otimes K_S^{\otimes 10+l}) \text{ for all } l, k \geq 0$$

surject. Let $L_1 = K_S^{\otimes 4+k} \otimes \mathcal{O}_C$, $L_2 = K_S^{\otimes 4+l} \otimes \mathcal{O}_C$ and $L_3 = B \otimes \mathcal{O}_C$. To see the surjectivity of μ, by Observation 2.3 and Lemma 2.9 it is enough to check the surjectivity of the maps

$$H^0(M_{L_1} \otimes L_2) \otimes H^0(L_3) \xrightarrow{\mu'} H^0(M_{L_1} \otimes L_2 \otimes L_3)$$
$$H^0(L_2) \otimes H^0(L_3) \xrightarrow{\mu'} H^0(L_2 \otimes L_3) \; .$$

The argument to see the surjectivity of μ" is analogous to the one carried out in the proof of Theorem 5.3. For μ' note that since $\deg L_1 \geq 2g$, by [Bu], Theorem 1.12, the vector bundle $M_{L_1} \otimes L_2$ is semistable. To show the surjectivity of μ' we use Proposition 2.4 having in account the hypothesis on p_g and the fact that Lastly, to see the surjectivity of δ we argue similarly.

Finally the statements about the coordinate rings of the images of the pluricanonical maps follow from Theorem 1.2, (1), and (2). □

As already announced, the previous result starts an inductive process which allows us to get a result about higher syzygies for pluricanonical maps of regular general type surfaces. Before we state that result, we show another result about higher syzygies of pluricanonical maps of surfaces of general type, which is a consequence of Theorem 1.3:

Theorem 5.5. *Let S be a surface of general type satisfying the assumptions of Theorem 5.1. If $n \geq 2p + 4$, then the image of S by $|K_S^{\otimes n}|$ is projectively normal, its ideal is generated by quadrics and the resolution of its homogeneous coordinate ring is linear until the pth stage.*

Proof. The line bundle $K_S^{\otimes 2}$ is base-point-free by Theorem 5.1. On the other hand, $K_S^{\otimes 2}$ is 3-regular by the Kawamata-Viehweg Vanishing Theorem. Hence

2.15, writing $K_S^{\otimes n}$ as $B^{\otimes s-1} \otimes B'$, where $B = K_S^{\otimes 2}$ and $B' = K_S^{\otimes 3}$, which are base-point-free by Theorem 5.1.

With the extra hypotheses of $q = 0$ and on the number of global sections of K_S, we can do better:

Theorem 5.6. *Let S be a regular surface of general type with $p_g \geq 3$. Let $L = K_S^{\otimes 2p+2+l}$ and $L' = K_S^{\otimes 2p+2+k}$.*

(1) *If $p \geq 2$, $H^1(M_L^{\otimes p+1} \otimes L') = 0$ for all $k, l \geq 0$, and*
(2) *if $p = 1$, $H^1(M_L^{\otimes p+1} \otimes L') =$ for $k, l = 0$, for all $k, l \geq 1$, and for all $k \geq 0, l \geq 2$.*

In particular, if $n \geq 2p+2$ and $p \geq 1$, then the image of S by $|K_S^{\otimes n}|$ is projectively normal, its ideal is generated by quadrics and the resolution of its homogeneous coordinate ring is linear until the pth stage.

Proof. The proof is by induction on p. The statement for $p = 1$ is Theorem 5.4 (2). Let us assume the result to be true for $p - 1$ and prove the vanishing for p. Tensoring (1.1) with $M_L^{\otimes p} \otimes L'$ and taking global sections yields the following long exact sequence

$$H^0(L) \otimes H^0(M_L^{\otimes p} \otimes L') \xrightarrow{\eta} H^0(M_L \otimes (L \otimes L'))$$
$$\longrightarrow H^1(M_L^{\otimes p} \otimes L') \longrightarrow H^0(L) \otimes H^1(M_L^{\otimes p+1} \otimes L') \ .$$

The last term is zero by induction assumption, thus the vanishing is equivalent to showing the surjectivity of the multiplication map η. Again let $B = K_S^{\otimes 2}$ and $B' = K_S^{\otimes 3}$. By Observation 1.4.1 it suffices to show the surjectivity of several maps:

$$H^0(B) \otimes H^0(M_L^{\otimes p} \otimes L') \xrightarrow{\eta'} H^0(M_L \otimes (B \otimes L')), \text{ for all } l, k \geq 0 \text{ and,}$$
$$H^0(B') \otimes H^0(M_L^{\otimes p} \otimes L' \otimes B) \xrightarrow{\eta''} H^0(M_L \otimes (B \otimes B' \otimes L')), \text{ for all } l, k \geq 0 \ .$$

The surjectivity of η'' follows by (1.4.2), the vanishings required following by induction, from (1.1) and Kawamata-Viehweg. The surjectivity of η' also follows from (1.4.2) by the same reasons, but as in the proof of Theorem 2.14 we could alternatively argue restricting to a smooth curve C in $|B|$. We would have to eventually use Proposition 2.4 and the fact that the needed inequalities hold follows from adjunction and the assumption on p_g, having in account that $h^1(B \otimes \mathcal{O}_C) = h^2(\mathcal{O}_S)$.

Finally, the statement on the coordinate ring of the image of the pluricanonical maps follows from Theorem 5.4, the vanishings just proven and Theorem 1.2. □

As advanced in the introduction, the proofs of the results for Calabi-Yau threefolds rest on results for surfaces of general type. More precisely we need results for pluricanonical models of (regular) surfaces with (ample) and base-point-free canonical bundle. Moreover the theorem below improves another result by Ciliberto (cf. [Ci]).

Theorem 5.7. *Let S be a regular surface of general type with K_S base-point-free. Let $p_g \geq 4$. Let $L = K_S^{\otimes p+2+l}$ and $L' = K_S^{\otimes p+2+k}$. Then, if $p \geq 1$, $H^1(M_L \otimes L') = H^1(M_L^{\otimes p+1} \otimes L') = 0$ for all $k, l \geq 0$. Moreover, if $p \geq 1$ the image of S by $|L|$ is* *projectively normal, its ideal is generated by quadratic equations and the resolution of the homogeneous coordinate ring is linear until the pth stage.*

Proof. First we check the vanishing of $H^1(M_L \otimes L')$. By Kawamata-Viehweg it suffices to check the surjectivity of

$$H^0(L') \otimes H^0(L) \to H^0(L \otimes L') \ .$$

Recall that K_S is base-point-free. Let C be a smooth curve in $|K_S|$. By Observation 1.4.1 and Observation 2.3 it suffices to show the surjectivity of

$$H^0(\theta^{\otimes 3+l}) \otimes H^0(\theta) \xrightarrow{\alpha} H^0(\theta^{\otimes 4+l}) \text{ for all } l \geq 0 \ ,$$

where $\theta = K_S \otimes \mathcal{O}_C$. Thus θ is a theta-characteristic and $h^0(\theta) \geq 3$, so by Clifford's bound, $g(C) \geq 5$. Thus it follows either from Proposition 2.4 or Proposition 2.5 that α surjects. Now we prove the vanishing of $H^1(M_L^{\otimes p+1} \otimes L')$ by induction. For $p = 1$, by the vanishing just proved, it suffices to see the surjectivity of

$$H^0(M_L \otimes L') \otimes H^0(L) \to H^0(M_L \otimes L \otimes L') \ .$$

Using Observation 1.4.1, Observation 2.3, Lemma 2.9, and having in account the already proven surjectivity of α, we see that it suffices to check the surjectivity of

$$H^0(M_G \otimes G') \otimes H^0(\theta) \xrightarrow{\beta} H^0(M_G \otimes G \otimes G') \ ,$$

where $G = \theta^{\otimes 3+k}$ and $G' = \theta^{\otimes 3+l}$. Since $g(C) \geq 5$, then $\deg G > 2g$, so M_G is semistable. Then β surjects by Proposition 2.4. The proof for $p > 1$ follows the same pattern (see the proof of Theorem 5.6). We use Observation 1.4.1, Observation 2.3, Lemma 2.9, and Proposition 2.4 in similar fashion.

Lastly, the statement about the syzygies of the resolution of the pluricanonical models follow from Theorem 1.2 and from the vanishings just proven. □

Finally, we show that, when K_S is ample, the pluricanonical models considered in the previous theorem have Koszul coordinate rings:

Theorem 5.8. *Let S be a regular surface of general type with K_S ample and base-point-free. Let $p_g \geq 4$. Let $L = K_S^{\otimes n}$ Then, if $n \geq 3$, the coordinate ring of S embedded by $|L|$ is a Koszul ring.*

Proof. According to Lemma 3.2 we need to show that $M^{h,L}$ is globally generated and that

$$H^0(M^{h,L}) \otimes H^0(L^{\otimes s}) \xrightarrow{\alpha} H^0(M^{h,L} \otimes L^{\otimes s})$$

surjects for all $h \geq 0$ and $s \geq 1$. By Observation 1.4.1, it suffices to prove the surjectivity of

$$H^0(M^{h,L} \otimes K_S^{\otimes l}) \otimes H^0(K_S) \xrightarrow{\beta_l} H^0(M^{h,L} \otimes K_S^{\otimes l+1})$$

for all $l \geq 0$. We explain in some detail the border case

$$H^0(M^{h,L}) \otimes H^0(K_S) \xrightarrow{\beta} H^0(M^{h,L} \otimes K_S)$$

and leave the others to the reader. The proof of the surjectivity of β goes by induction and as in Theorem 3.5 , it is convenient to prove the vanishing of $H^1(M^{h,L}\otimes K_S^*)$ at the same time. If $h=0$, the surjectivity was shown in the course of proving projective normality in Theorem 5.7. Assume the statement to be true for $h-1$. Consider the sequence

$$\begin{aligned} &H^0(M^{h-1,L})\otimes H^0(K_S^{\otimes 2}) \xrightarrow{\gamma} H^0(M^{h-1,L}\otimes K_S^{\otimes 2})\\ &\to H^1(M^{h,L}\otimes K_S^*) \to H^1(M^{h-1,L})\otimes H^0(K_S^{\otimes 2})\ . \end{aligned}$$

The multiplication map γ is surjective by induction hypothesis. The group $H^1(M^{h-1,L})$ vanishes also by induction hypothesis, therefore $H^1(M^{h,L}\otimes K_S^*)=0$. Now since $H^1(\mathcal{O}_X)=0$, in order to see the surjectivity of β we proceed as in Theorem 3.5 , K_S playing the role B_1 plays there and restricting to a smooth curve $C\in|K_S|$, which plays the same role as $\mathfrak{b}_1$. In order to obtain the inequalities needed to apply Lemma 2.10 and Proposition 2.4, note that $K_S^{\otimes 2}\otimes\mathcal{O}_C=K_C$ and that $\deg K_S\otimes\mathcal{O}_C=K_S^2\geq 4$ by Nöther's formula. □

It is worth remarking that arguments similar to those used to prove Theorems Theorem 5.2, Theorem 5.3 and Theorem 5.4 give somewhat more general results on Koszul cohomology and syzygies of adjoint linear series. For instance, for regular surface of general type we have the following

Theorem 5.9. *Let S be a regular surface of general type with $p_g\geq 4$. Let B be an ample and base-point-free line bundle such that $H^1(B)=0$ and $B^2\geq B\cdot K_S$. Then $K_S\otimes B^{\otimes k+2}$ embeds S as a projectively normal variety for all $k\geq 0$.*

Sketch of proof. We sketch now the proof for $K_S\otimes B^{\otimes k+2}$. By Observation 1.4.1 it suffices to show that

$$\begin{gathered} H^0(K_S\otimes B^{\otimes n})\otimes H^0(B)\longrightarrow H^0(K_S\otimes B^{\otimes n+1})\\ H^0(K_S\otimes B^{\otimes m})\otimes H^0(K_S\otimes B)\longrightarrow H^0(K_S\otimes B^{\otimes m+1}),\ \text{for all } n\geq 2, m\geq 3 \end{gathered}$$

surject. Now one has the following two facts:

(5.9.1) Let S be a surface of general type and B an ample and base-point-free line bundle with $H^1(B)=0$ and $B^2\geq B\cdot K_S$. Then $H^1(B^{\otimes m})=0$ for all $m\geq 1$.
Proof of (5.9.1). Let C smooth curve in $|B|$. Since $\deg B^{\otimes m}\otimes\mathcal{O}_C>2g(C)-2$ when $m\geq 3$, we only have to prove $H^1(B^{\otimes 2})=0$. If $B^2>B\cdot K_S$ or $B^{\otimes 2}\otimes\mathcal{O}_C\neq K_C\otimes B$, then $H^1(B^{\otimes 2}\otimes\mathcal{O}_C)=0$, hence $H^1(B^{\otimes 2})=0$ because $H^1(B)=0$. If $B^{\otimes 2}\otimes\mathcal{O}_C=K_C$, then $B\otimes\mathcal{O}_C=K_S\otimes\mathcal{O}_C$. Consider the sequence

$$0\to H^0(K_S^*)\to H^0(B\otimes K_S^*)\to H^0(B\otimes K_S^*\otimes\mathcal{O}_C)\to H^1(K_S^*)\ .$$

Since $H^0(K_S^*)=H^1(K_S^*)=0$, it follows that $B\otimes K_S^*$ is effective and since B is ample, it must be $B\otimes K_S^*=\mathcal{O}_S$. Hence $H^1(B^{\otimes 2})=H^1(K_S^{\otimes 2})=0$.

(5.9.2) Let S be a regular surface of general type with $p_g\geq 4$ and B an ample and base-point-free line bundle such that $B^2\geq B\cdot K_S$. Then $B^2\geq 4$ and if $B\neq K_S$, $B^2\geq 5$, and in both cases, $K_S\otimes B$ is base-point-free.

Hence there exist smooth curves $C\in|B|$ and $C'\in|K_S\otimes B|$. Since $H^1(\mathcal{O}_S)=0$ and by (5.9.1) and Kodaira vanishing theorem, we may apply Observation 2.3 and conclude that it is enough to check that

$$H^0(K_S \otimes B^{\otimes n} \otimes \mathcal{O}_C) \otimes H^0(B \otimes \mathcal{O}_C) \longrightarrow H^0(K_S \otimes B^{\otimes n+1} \otimes \mathcal{O}_C)$$
$$H^0(K_S \otimes B^{\otimes m} \otimes \mathcal{O}_{C'}) \otimes H^0(K_S \otimes B \otimes \mathcal{O}_{C'}) \longrightarrow H^0(K_S \otimes B^{\otimes m+1} \otimes \mathcal{O}_{C'})$$

surject for all $n \geq 2, m \geq 3$. This follows from Proposition 2.4 or Proposition 2.5. $\square$.

One also has the following

Theorem 5.10. *Let S be a regular surface of general type with $p_g \geq 4$. Let B be an ample and base-point-free line bundle such that $H^1(B) = 0$ and $B^2 \geq 2B \cdot K_S$. Then $K_S \otimes B^{\otimes k+2}$ and $B^{\otimes l+2}$ satisfy property N_1 for all $k, l \geq 0$.*

Sketch of proof. We have already proven (cf. Theorem 5.9) that $K_S \otimes B^{\otimes k+2}$ satisfies property N_0. One can similarly prove that $B^{\otimes l+2}$ satisfies property N_0 (in proving this, when applying Proposition 2.5, it can be seen that the hypothesis $B^2 \geq K_S \cdot B$ of Theorem 5.9 is not sufficient). Let L_1 be $K_S \otimes B^{\otimes k+2}$ and let L_2 be $B^{\otimes l+2}$. It rests to show that $H^1(M_{L_i}^{\otimes 2} \otimes L_i^{\otimes s}) = 0$ for all $s \geq 1$. By Observation 1.4.1 it is enough to show that the following maps

$$H^0(M_{L_1} \otimes L_1) \otimes H^0(B) \xrightarrow{\alpha_1} H^0(M_{L_1} \otimes L_1 \otimes B)$$
$$H^0(M_{L_1} \otimes L_1 \otimes B) \otimes H^0(K_S \otimes B) \xrightarrow{\alpha_2} H^0(M_{L_1} \otimes L_1 \otimes K_S \otimes B^{\otimes 2})$$
$$H^0(M_{L_2} \otimes L_2) \otimes H^0(B) \xrightarrow{\alpha_3} H^0(M_{L_2} \otimes L_2 \otimes B)$$

surject. We only sketch in some detail the proof of the surjectivity of α_1, as the proofs for the other two maps are analogous. We will use Observation 2.3 and Lemma 2.9. For that we need to check that $H^1(M_{L_1} \otimes L_1 \otimes B^*) = 0$. This follows from the surjectivity of

$$H^0(L_1) \otimes H^0(B)) \xrightarrow{\beta_1} H^0(L_1 \otimes B)$$
$$H^0(L_1 \otimes B) \otimes H^0(K_S \otimes B) \xrightarrow{\beta_2} H^0(L_1 \otimes B)\ .$$

The surjectivity of β_1 was already shown in the proof of Theorem 5.9. The surjectivity of β_2. It follows from (5.9.2) that $K_S \otimes B$ is ample and base-point-free, hence there exists a smooth curve $C \in |K_S \otimes B|$. Now by Observation 2.3 it suffices to see that

$$H^0(L_1 \otimes \mathcal{O}_C) \otimes H^0(K_S \otimes B \otimes \mathcal{O}_C) \longrightarrow H^0(L_1 \otimes K_S \otimes B \otimes \mathcal{O}_C)$$

surjects. This follows by Proposition 2.4 or Proposition 2.5, using the hypothesis and the following fact:

(5.10.1) Let S be an algebraic surface with Kodaira dimension bigger than 0 and let B be an ample line bundle. Let $m \geq 1$. If $B^2 \geq mK_S \cdot B$, then $K_S \cdot B \geq mK_S^2$.
Proof of (5.10.1). We assume the contrary, i.e., that $K_S \cdot B < mK_S^2$, and get a contradiction. Let $L = B \otimes K_S^{\otimes -m}$. We have that $L^2 > 0$. By Riemann-Roch

$$h^0(L^{\otimes n}) \geq \frac{n^2L^2 - nK_S \cdot L}{} + \chi(\mathcal{O}_S) - h^0(K_S \otimes L^{\otimes -n})\ .$$

If $B^2 > mK_S \cdot B$, $(K_S \otimes L^{\otimes -n}) \cdot B < 0$, for n large enough, and since B is ample, $K_S \otimes L^{\otimes -n}$ is not effective, so finally $L^{\otimes n}$ is effective for n large enough. But in that case $nK_S \cdot L \geq 0$, because K_S is nef, contradicting our assumption.

Now if $B^2 = mK_S \cdot B$, we have that $L^2 > 0$, $B^2 > 0$ (because B is ample), and $L \cdot B = 0$, but this is impossible by the Hodge index theorem.

Returning to the proof of the surjectivity of α_1, we may apply Observation 2.3 and Lemma 2.9 and therefore it suffices to check the surjectivity of

$$H^0(M_{L_1 \otimes \mathcal{O}_C} \otimes L_1 \otimes \mathcal{O}_C) \otimes H^0(B \otimes \mathcal{O}_C) \longrightarrow H^0(M_{L_1 \otimes \mathcal{O}_C} \otimes L_1 \otimes B \otimes \mathcal{O}_C)\ ,$$

which essentially follows from Proposition 2.4. □

As a corollary of Theorem 5.10 we recover Theorem 5.4 when $p_g(S) \geq 4$. Note that Theorem 5.4 is stronger since we only require there $p_g \geq 3$; this is sufficient for pluricanonical models because, among other things, $K_S^{\otimes 2}$ and $K_S^{\otimes 3}$ are known to be base-point-free for different reasons than those used in (5.9.2) (cf. Theorem 5.1).

We deal know with irregular general type surfaces. As a matter of fact, one can state a result for surfaces of positive Kodaira dimension:

Theorem 5.11. *Let S be an algebraic surface such that $\kappa(S) \geq 1$ and $q(S) > 0$. Let B be an ample and base-point-free line bundle satisfying*

1. *$B^2 > K_S \cdot B$ if $\kappa(S) = 1$ and $B^2 \geq 2K_S \cdot B$ if $\kappa(S) = 2$;*
2. *$H^1(B') = 0$ for all B' homologous to B;*
3. *$B^2 \geq 5$;*
4. *B' is base-point-free for all B' homologous to B.*

Then $H^1(M_{K_S \otimes B^{\otimes l+2}} \otimes K_S \otimes B^{\otimes k+2}) = 0$ and $H^1(M^{\otimes 2}_{K_S \otimes B^{\otimes l+3}} \otimes K_S \otimes B^{\otimes k+3}) = 0$ for all $k, l \geq 0$. In particular $K_S \otimes B^{\otimes k+2}$ satisfies property N_0 and $K_S \otimes B^{\otimes k+3}$ satisfies property N_1 for all $k \geq 0$ (if $\kappa(S) = 2$, it is enough that $B^2 \geq K_S \cdot B$ for the latter to happen).

If $\kappa(S) = 1$ and $q(S) = 0$, $p_g \geq 4$ and B is an ample and base-point-free line bundle with $B^2 > K_S \cdot B$, then $K_S \otimes B^{\otimes n}$ satisfies property N_1 for all $n \geq 2$.

Sketch of proof. The proofs uses (5.9.1), (1) and similar arguments as the proof of Theorem 5.2 or those in Section 4.

The proof of the statement for regular elliptic surfaces goes along the same lines as the proof of Theorem 5.10. □.

As a corollary we recover Theorem 5.2. Another quite interesting consequence of Theorems 5.9, 5.10 and 5.11 and of a result by Fernández del Busto is the following effective bound along the lines of Mukai's conjecture:

Theorem 5.12. *Let S be an algebraic surface with positive Kodaira dimension, let A be an ample line bundle and let $m = \left[\frac{(A \cdot (K_S + 4A) + 1)^2}{2A^2}\right]$. Let $L = K_S \otimes A^{\otimes n}$. If $n \geq 2m$, then L satisfies property N_0. If $n \geq 3m$, then L satisfies property N_1. If S is a regular surface and $n \geq 2m$, then L satisfies property N_1.*

Sketch of proof. The key observation is the fact that if $k \geq m$ then it follows from [FdB], Section 2 that $A^{\otimes m}$ is base-point-free and $H^1(A^{\otimes m}) = 0$. Then we take $A^{\otimes m}$ as the base-point-free line bundle B in Theorems 5.9, 5.10, 5.11. One can easily verify that the numerical conditions in the statements are satisfied. For (2) and (4) in Theorem 5.11 note that $\mathrm{Pic}^0(S)$ is divisible, then Fernández del Busto's result applies also to B'. Therefore the theorem is either a consequence of Theorems 5.9, 5.10 and 5.11 or follows from slight modifications of the arguments involved to prove those theorems, for we are in a situation similar to the proof of Corollary 2.15. $\square$.

There is also a generalization of Theorem 5.7 which deals with adjoint bundles:

Theorem 5.13. *Let S be a regular surface of general type with $p_g \geq 4$ and base-point-free canonical bundle. Let B be an ample line bundle on S with $H^1(B) = 0$ and let $B^2 \geq B \cdot K_S$. If $n \geq p+1$, then $K_S \otimes B^{\otimes n}$ satisfies the property N_p.*

Sketch of proof. By (5.9.1) we have $H^1(B^{\otimes m}) = 0$ for all $m \geq 2$. Given $C \in |B|$ smooth, observe also that $h^0(B \otimes \mathcal{O}_C) \geq p_g - 1 \geq 3$. Then the wanted result will follow from these observations and Kodaira vanishing, (1.4.2), Observation 1.4.1, Observation 2.3, Proposition 2.5, Proposition 2.4and Lemma 2.9 , used in similar fashion as in the proofs of the previous theorems. $\square$.

6. Calabi-Yau threefolds

To set the stage we start with a general result for varieties of arbitrary dimension which is a direct consequence of Theorem 1.3:

Theorem 6.1. *Let X be a variety of dimension m with $K_X \equiv 0$ and let B be ample and base-point-free line bundle. Let $L = B^{\otimes n+1}$. If $n \geq p+m-1$, then L satisfies property N_p.*

Proof. The result is a straight forward consequence of Theorem 1.3, since by Kodaira vanishing Theorem, B is $(n+1)$-regular. $\square$

Corollary 6.2. *Let X be a Calabi-Yau threefold, and B an ample and base-point-free line bundle on X. If $n \geq p+3$ and $p \geq 1$ then $B^{\otimes n}$ satisfies property N_p.*

The previous corollary tells us in particular that $B^{\otimes 4}$ satisfies property N_0. Now we want to find conditions so that smaller powers of B satisfy N_0. We begin with the following

Theorem 6.3. *Let X be a Calabi-Yau threefold. Let B be an ample and base-point-free divisor with $h^0(B) \geq 5$. Then $B^{\otimes 3}$ is very ample and $|B^{\otimes 3}|$ embeds X as a projectively normal variety.*

Proof. The proof is by induction on the dimension. A smooth divisor of $|B|$ is by adjunction a regular surface of general type S and a smooth divisor of $|B \otimes \mathcal{O}_S|$ is a curve C such that $(B \otimes \mathcal{O}_C)^{\otimes 2} = K_C$. Thus the result eventually rests on checking the surjectivity of multiplication maps on C. We did already carry out part of this inductive process in Theorem 5.7. Indeed the proof goes as follows: by Observation 1.4.1 it is enough to see that the map

$$H^0(B^{\otimes 3+l}) \otimes H^0(B) \to H^0(B^{\otimes 4+l}) \text{ for all } l \geq 0$$

surjects and by Observation 2.3 it is then enough to show that the map

$$H^0(K^{\otimes 3+l}) \otimes H^0(K) \to H^0(K^{\otimes 4+l}) \text{ for all } l \geq 0$$

surjects. We recall that the latter was shown in the proof of Theorem 5.7. □

Now we deal with the same problem relaxing the hypothesis of Theorem 6.3. Concretely we study what happens if $h^0(B) = 4$. We will also look for conditions so that $B^{\otimes 2}$ be normally generated.

Theorem 6.4. *Let X be a Calabi-Yau threefold and let B be an ample and base-point-free line bundle with $h^0(B) = 4$. If there exists a smooth, nonhyperelliptic curve C and a smooth surface S such that $S \in |B|$ and $C \in |B \otimes \mathcal{O}_S|$, then $B^{\otimes 3}$ is very ample and $|B^{\otimes 3}|$ embeds X as a projectively normal variety.*

Proof. It suffices to see the surjectivity of

$$H^0(B^{\otimes 3+l}) \otimes H^0(B^{\otimes 3+k}) \to H^0(B^{\otimes 6+k+l}) \text{ for all } l, k \geq 0 \ .$$

If $l \geq 1$ or $k \geq 1$, the surjectivity follows from Observation 1.4.1, (1.4.2) and Kodaira vanishing Theorem. If $l = k = 0$, it follows from Observation 1.4.1 that it is enough to check the surjectivity of

$$H^0(B^{\otimes 3}) \otimes H^0(B^{\otimes 2}) \xrightarrow{\alpha} H^0(B^{\otimes 5})$$
$$H^0(B^{\otimes 5}) \otimes H^0(B) \xrightarrow{\beta} H^0(B^{\otimes 6}) \ .$$

Note that we cannot follow the path we took other times because the map $H^0(B^{\otimes 3}) \otimes H^0(B) \to H^0(B^{\otimes 4})$ is actually non surjective, for otherwise the map

$$H^0(K_C \otimes \theta) \otimes H^0(\theta) \to H^0(K_C^{\otimes 2}),$$

where $\theta = B \otimes \mathcal{O}_C$ is a theta-characteristic, would also surject, and the latter is false by base-point-free pencil trick. Now the map β surjects by (1.4.2) and Kodaira vanishing Theorem. The surjectivity of α will follow from the surjectivity of γ and δ in the following diagram :

$$\begin{array}{ccccc} H^0(B^{\otimes 2}) \otimes H^0(B^{\otimes 2}) & \hookrightarrow & H^0(B^{\otimes 2}) \otimes H^0(B^{\otimes 3}) & \twoheadrightarrow & H^0(B^{\otimes 2}) \otimes H^0(K_S^{\otimes 3}) \\ \big\downarrow \gamma & & \big\downarrow \alpha & & \big\downarrow \delta \\ H^0(B^{\otimes 4}) & \hookrightarrow & H^0(B^{\otimes 5}) & \twoheadrightarrow & H^0(K_S^{\otimes 5}) \ , \end{array}$$

obtained from the sequence

$$0 \to B^* \to \mathcal{O}_X \to \mathcal{O}_S \ .(6.4.1)$$

To see the surjectivity of γ we construct yet another two similar diagrams arising from (6.4.1). After doing that and since $H^1(B^{\otimes r}) = 0$ for all $r \geq 0$, the surjectivity of γ is reduced to see the surjectivity of

$$H^0(K_S^{\otimes 2}) \otimes H^0(K_S^{\otimes 2}) \xrightarrow{\epsilon} H^0(K_S^{\otimes 4})$$
$$H^0(K_S^{\otimes 2}) \otimes H^0(K_S) \xrightarrow{\eta} H^0(K_S^{\otimes 3}) \ .$$

On the other hand in order to see the surjectivity of δ, again by the vanishing of $H^1(B^{\otimes r})$ it is enough to check the surjectivity of

$$H^0(K_S^{\otimes 3}) \otimes H^0(K_S^{\otimes 2}) \xrightarrow{\varphi} H^0(K_S^{\otimes 5})$$

For the surjectivity of ϵ, η and φ we build commutative diagrams like the one above, now upon the sequence

$$0 \to K_S^* \to \mathcal{O}_S \to \mathcal{O}_C \to 0 \; .$$

Since $H^1(K_S^{\otimes r}) = 0$ for all $r \geq 0$, the surjectivity of ϵ, η and φ will follow from the surjectivity of the maps

$$\begin{gathered} H^0(K_C) \otimes H^0(K_C) \to H^0(K_C^{\otimes 2}) \\ H^0(K_C) \otimes H^0(\theta) \to H^0(K_C \otimes \theta) \text{ and} \\ H^0(K_C) \otimes H^0(K_C \otimes \theta) \to H^0(K_C^{\otimes 2} \otimes \theta) \; . \end{gathered}$$

The first is surjective by Nöther's theorem, since C is nonhyperelliptic. For the second, recall that θ is a theta-characteristic, that it is base-point-free and $h^0(\theta) = 2$. Thus the surjectivity follows from the base-point-free pencil trick. Finally the third one follows from Proposition 2.4. □

The following example shows that one cannot drop the hypothesis of C being non-hyperelliptic:

Example 6.5 Let X be the double cover of $\mathbf{P}^3$ ramified along a smooth degree 8 surface. The threefold X is Calabi-Yau. Let L be the pullback of $\mathcal{O}_{\mathbf{P}^3}(1)$. Then neither $L^{\otimes 2}$ nor $L^{\otimes 3}$ is very ample, and hence neither do they satisfy property N_0. Indeed, consider a smooth divisor C of $|L|$. The curve C is hyperelliptic and has genus 3 by construction and the restriction of $L^{\otimes 2}$ to C is the canonical bundle of C, which is not very ample. On the other hand the restriction of $L^{\otimes 3}$ to C is the tensor product of the canonical with an effective line bundle of degree 2 (namely, the restriction of L to C).

We inquire now under what conditions $B^{\otimes 2}$ satisfies property N_0. We need a definition first. We say that a polarized Calabi-Yau threefold (X, B), where B is base-point-free, is of *type* n if the morphism defined by $|B|$ is generically $n : 1$ onto the image. We prove two theorems, the first characterizes in particular the very ampleness and property N_0 of $B^{\otimes 2}$ for Calabi-Yau threefolds of *type* 2.

Theorem 6.6. *Let X be a Calabi-Yau threefold and let B be an ample and base-point-free line bundle such that $h^0(B) \geq 4$. Assume that either*

(1) *(X, B) is of type 2 or*
(2) *$h^0(B) = 4$.*

Then $B^{\otimes 2}$ satisfies property N_0 if and only if it is very ample, if and only if for every smooth S in $|B|$ the smooth curves in $|B \otimes \mathcal{O}_S|$ are non hyperelliptic, if and only if there exists a smooth, non-hyperelliptic curve C in $|B \otimes \mathcal{O}_S|$ for some S smooth surface in $|B|$.

Proof. If there existed some smooth hyperelliptic curve C in $|B \otimes \mathcal{O}_S|$ for some smooth surface S in $|B|$, then $B^{\otimes 2}$ cannot be very ample, because by adjunction that would imply that K_C would be very ample. Hence there is only left to prove the fact that if there exists a non-hyperelliptic curve C in $|B \otimes \mathcal{O}_C|$ for a smooth surface S in X, then $B^{\otimes 2}$ satisfies property N_0. It is enough to show that the multiplication map

$$H^0(B^{\otimes 2l}) \otimes H^0(B^{\otimes 2}) \xrightarrow{\alpha} H^0(B^{\otimes 2l+2})$$

surjects for all $l \geq 1$. If $l \geq 2$, the surjectivity of α follows from Observation 1.4.1, Kodaira vanishing and (1.4.2). If $l = 1$ and $h^0(B) = 4$, the surjectivity of α was shown in the proof of Theorem 6.4. If $h^0(B) \geq 5$, by Observation 1.4.1 it suffices to prove the surjectivity of

$$H^0(B^{\otimes 2}) \otimes H^0(B) \xrightarrow{\beta} H^0(B^{\otimes 3}) \text{ and}$$
$$H^0(B^{\otimes 3}) \otimes H^0(B) \xrightarrow{\gamma} H^0(B^{\otimes 4}) \ .$$

The surjectivity of γ was shown in the proof of Theorem 6.3. For the surjectivity of β we use Observation 2.3. Then, since $H^1(B) = 0$ and by adjunction $B \otimes \mathcal{O}_S = K_S$, it is enough to prove the surjectivity of

$$H^0(K_S) \otimes H^0(K_S^{\otimes 2}) \xrightarrow{\delta} H^0(K_S^{\otimes 3}) \ .$$

Since (X, B) is of type 2, the image of S under the morphism defined by $|K_S|$ cannot be a surface of minimal degree, for otherwise C would be a double cover of $\mathbf{P}^1$, hence hyperelliptic. Thus, since $h^0(K_S) \geq 4$ and $H^1(\mathcal{O}_S) = 0$, by [G], Theorem 3.9.3, the map δ surjects. □

Note that the same arguments used for Theorem 6.6 prove the following result, which has as a corollary the fact that the square of a very ample line bundle gives a projectively normal embedding:

Theorem 6.7. *Let X be a Calabi-Yau threefold. Let B be an ample and base-point-free line bundle on X with $h^0(B) \geq 5$. Assume that the image of X by the morphism induced by $|B|$ is not a variety of minimal degree. Then $B^{\otimes 2}$ is very ample and $|B^{\otimes 2}|$ embeds X as a projectively normal variety.*

We go back now to the study of higher syzygies. Making the same assumption on $h^0(B)$ as in Theorem 6.3 we can improve Corollary 6.2. The proof is, as for Theorem 6.3, by induction on the dimension and rests on Theorem 5.7:

Theorem 6.8. *Let X be a Calabi-Yau threefold. Let B be an ample and base-point-free divisor with $h^0(B) \geq 5$. Let $L = B^{\otimes p+2+k}$ and $L' = B^{\otimes p+2+l}$. If $k, l \geq 0$ and $p \geq 1$, then $H^1(M_L^{\otimes p+1} \otimes L') = 0$ and L satisfies property N_p.*

Sketch of proof. The proof is by induction on p. For $p = 1$ consider the sequence

$$H^0(M_L \otimes L') \otimes H^0(L) \xrightarrow{\alpha} H^0(M_L \otimes L' \otimes L)$$
$$\to H^0(M_L^{\otimes 2} \otimes L') \to H^1(M_L \otimes L') \otimes H^0(L)$$

The last term of the sequence vanishes by Theorem 6.3, so we need to prove the surjectivity of α, which follows combining Observation 1.4.1, Observation 2.3, Lemma 2.9, and Theorem 5.7. For $p > 1$, we write a similar sequence. The place of $H^1(M_L \otimes L')$ is taken now by $H^1(M_L^{\otimes p} \otimes L')$, which vanishes by induction hypothesis. The surjectivity of the map in the place of α follows from Observation 1.4.1, Observation 2.3, Lemma 2.9 and Theorem 5.7.

Finally, the fact that L satisfies property N_p follows from the vanishings just proven, Theorem 6.3 and Theorem 1.2. □

To end this section we show that the line bundles considered in the previous theorem embed X with homogeneous Koszul coordinate ring:

Theorem 6.9. *Let X be a Calabi-Yau threefold. Let B be an ample and base-point-free divisor with $h^0(B) \geq 5$. Let $L = B^{\otimes p+2+k}$ and $L' = B^{\otimes p+2+l}$. If $k, l \geq 0$ and $p \geq 1$, then the coordinate ring of X embedded by $|L|$ is Koszul.*

Sketch of proof. The result follows, as in the case of Theorem 6.8, from results proven for surfaces of general type. Precisely it follows as a corollary of Theorem 5.8 (case $q = 0$), using Lemma 3.4 and Observation 2.3 with the same strategy employed to prove Theorem 3.5. □

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