PROPER POLYNOMIAL MAPS: THE REAL CASE

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Introduction. Let R be a real closed field. If $x = (x_1,\dots,x_n) \in R^n$, $r \in R$, $r>0$, we denote

$\|x\| = \sqrt{x_1^2+\dots+x_n^2}$; $B_n(x,r) = \{y \in R^n: \|y-x\| < r\}$ (open ball);

$\overline{B}_n(x,r) = \{y \in R^n: \|y-x\| \le r\}$ (closed ball) ; $R^+ = \{r \in R: r > 0\}$;

$S^{n-1}(x,r) = \{y \in R^n: \|y-x\| = r\}$ (n-1-sphere); $p_N^n = (0,\dots,0,1)$

The <u>euclidean topology</u> on R^n is the topology in which the open balls are a basis of open sets. In what follows R^n will be always considered endowed with its euclidean topology. The <u>semialgebraic</u> subsets of R^n form the smallest collection of subsets of R^n containing those of type

$$U(f) = \{x \in R^n: f(x) > 0\}$$

where $f \in R[X_1,\dots,X_n]$ is a polynomial, and stable under finite union and intersection, and complementation. A continuous map $f:X \longrightarrow Y$ between semialgebraic subsets $X \subset R^n$ and $Y \subset R^m$ is <u>semialgebraic</u> if its graph $Gr(f)$ is a semialgebraic subset of R^{n+m}. For a semialgebraic function $f:U \longrightarrow R$, where U is an open semialgebraic subset of R^n, we copy from the case $R = \mathbb{R}$ the notion of derivability. The ring $S^\infty(U)$ is

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the set of semialgebraic functions $f:U \longrightarrow R$ which have partial derivatives of each order and all of them are (continuous) semialgebraic functions. An element f in $S^{\infty}(U)$ is called a Nash function. A map $f = (f_1,\dots,f_m):U \longrightarrow R^m$ is a Nash map if each coordinate function f_j is a Nash function. If each f_j is a polynomial we say that f is a polynomial map. The notion of proper semialgebraic map was introduced in [DK_1,pg. 192]: a semialgebraic map $f:X \longrightarrow Y$ is called semialgebraically proper if for every semialgebraic map $g:Z \longrightarrow Y$, the canonical projection

$$p: X x_Y Z = \{(x,z) \in XxZ: f(x) = g(y)\} \longrightarrow Z$$

is semialgebraically closed, i.e. p maps every closed semialgebraic subset C of Xx_YZ onto a closed semialgebraic subset p(C) of Z. Taking Z = Y and g the identity map on Y, it follows immediately that semialgebraically proper maps are semialgebraically closed. Also, from 9.1 and 9.4 in [D-K_1] -see also [Br., 8.13.5]- it follows that the constant map on a closed semialgebraic subset X of R^n is proper if and only if X is bounded. This together with [D-K_1, Thm 12.5] provide us the following characterization of semialgebraically proper maps, which is the starting point of our considerations in this note:

Theorem 1.- Let $f:X \longrightarrow Y$ be a semialgebraic map between the semialgebraic subsets $X \subset R^n$, $Y \subset R^m$. The following statements are equivalent:

(1) f is semialgebraically proper.
(2) f is semialgebraically closed and its fibers are closed in R^n and bounded.

Our main results are refinements of this theorem for Nash or polynomial mappings: