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On the Approximate Controllability for some Explosive Parabolic Problems

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Abstract. We consider in this paper distributed systems governed by parabolic evolution equations which can blow up in finite time and which are controlled by initial conditions. We study here the following question : Can one choose the initial condition in such a way that the solution does not blow up before a given time T and which is, at time T, as close as we wish from a given state ? Some general results along these lines are presented here for semilinear second order parabolic equations as well as for a non local nonlinear problem. We also give some results proving that "the more the system will blow up" the "cheaper" it will be the control.

1. Introduction

We consider *distributed systems* of evolution, i.e. systems whose state (denoted by y) is given by the solution (or by a solution) of a Partial Differential Equation (PDE) of evolution. In this paper we consider distributed systems which, *if not controlled, can blow up in finite time.* We *conjecture* that these systems are *approximately controllable* (and even *exactly* controllable). In other words, by a *"suitable set of actions"* (the control), the system can be driven, in a finite time T, from an initial state y^0 to a neighborhood of the target y^T (or to reach exactly y^T). We also conjecture that "the more the system will blow-up", the *"cheaper"* it will be the control.

Of course, all this has to be made precise!

This is what we intend to do in the present paper, *when the control is the initial state.*

Before proceeding, let us notice that considering the initial state as a control is a standard point of view in the *assimilation of data* in Meteorology or in Climatology. Cf. Blayo, Blum and Verron [2], Le Dimet and Charpentier [10].

We consider first semilinear parabolic problems of the type

$$\begin{cases} y_t + Ay &= f(y) \quad \text{in } \Omega \times (0,T), \\ y &= 0 \quad \text{on } \Gamma_D \times (0,T), \\ \frac{\partial y}{\partial \nu} &= 0 \quad \text{on } \Gamma_N \times (0,T), \\ y(0,x) &= u(x) \quad \text{on } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded open regular set of R^N, $\partial\Omega = \Gamma_D \cup \Gamma_N$, A is a linear second order elliptic operator of the form

$$Ay = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial y}{\partial x_j}) + a_0 y,$$

where $a_{ij} \in C^1(\overline{\Omega}), a_0 \in C^0(\overline{\Omega})$, $a_0 \geq 0$ and there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \alpha \left|\xi\right|^2, \ \forall\xi = \{\xi_i\}_{i=1}^N \in R^N, \text{ a.e. } x \in \Omega.$$

In (1), vector ν denotes the associated *conormal vector*, $\nu = \{\nu_i\}_{i=1}^N$,

$$\nu_i = \sum_{j=1}^{N} a_{ij}(x) n_j,$$

with $\mathbf{n} = \{n_i\}_{i=1}^N$ the unit outward normal to Γ_N. Function f is assumed to be locally Lipschitz and u denotes the control function.

In (1) the function f is not necessarily decreasing (nor sublinear at infinity), so that the associate solutions can *blow up in a finite time.* Let us recall that by well-known results (see, e.g., Cazenave and Haraux [3]) given $u \in L^q(\Omega)$, for some $q \in [1, +\infty]$ there is a unique (local in time) solution, defined on a *maximal interval* $[0, T_m[$, $T_m = T_m(u)$ and $\|y(t)\|_{L^q(\Omega)} \nearrow +\infty$ if $t \nearrow T_m$ when $T_m(u) < +\infty$.

The problem of approximate controllability alluded to above can now be stated in the following fashion: let $T > 0$ be given and let $\mathcal{E}(T)$ be the set of elements $u \in L^q(\Omega)$ for which $T_m(u) > T$; is the set $\{y(T : u), u \in \mathcal{E}(T)\}$ dense in $L^q(\Omega)$?

Some positive answers will be presented in what follows for equations of type (1) and also for other *non local* models as we explain below.

Of course the problem of approximate (resp. exact) controllability can be raised for systems *not necessarily blowing up in finite time,* i.e. for systems such that $T_m(u) = +\infty$.

For the case of $A = -\Delta$ and Dirichlet boundary conditions (i.e., $\partial\Omega = \Gamma_D$), the linear case ($f(s) = as + b$) was already considered by Lattes and Lions [9] and Lions [11]. Concerning the nonlinear case, it seems that the first result in the literature was due to Bardos and Tartar [1]. They proved a negative answer: if $f(s) = -\left|s\right|^{p-1} s$, and $p > 1$ then any solution of (1) satisfies the "universal" estimate

$$|y(x,t)| \leq Ct^{-1/(p-1)}, \text{ for any } (x,t) \in \Omega \times (0,T). \tag{2}$$

for some positive constant C independent of u. So, in this case, the approximate controllability fails if $y^T \in L^q(\Omega)$ is such that

$$\left|y^T(x)\right| > CT^{-1/(p-1)}, \text{ on a positively measured subset of } \Omega.$$

A first positive result for nonlinear problems was due to Henry [8] who proved the L^2 approximate controllability when f is assumed to be globally Lipschitz. More recently his result was improved in Fabre, Puel and Zuazua [7] where the authors obtain the mentioned property in $L^q(\Omega)$,with q arbitrary $1 \leq q \leq \infty$, under the condition f locally Lipschitz and *sublinear at the infinity*, i.e. such that

$$|f(s)| \leq C(1+|s|), \text{ for } |s| \text{ large enough.} \tag{3}$$

The above results can be easily extended to more general elliptic operators A and when Γ_N is not empty.

We shall assume that

$$sf(s) \geq 0 \text{ for any } s \text{ with } |s| > s_0, \text{ for some } s_0 \geq 0 \tag{4}$$

so that solutions may blow up. A first result concerns the case of small time T

Theorem 1.1. *Assume (4) and*

$$\max\left\{\left|\int_{-\infty}^{-s_1} \frac{ds}{f(s)}\right|, \int_{s_1}^{+\infty} \frac{ds}{f(s)}\right\} < \infty \tag{5}$$

for some $s_1 > s_0$. *Let* $y^T \in L^q(\Omega)$, *for some* $q \in [1,+\infty]$, *and let* $\varepsilon \in (0,1)$. *Then, there exists* $u \in C^2(\overline{\Omega})$ *and* $\tau_0 > 0$ *such that* $T_m(u) > \tau_0$ *and*

$$\left\|y(\tau_0; u) - y^T\right\|_{L^q(\Omega)} \leq \varepsilon.$$

The proof of this result will be given in Section 2. Although several remarks and generalizations will be also given in that section, we point out that condition (5) requires a *superlinear* growth on f (when $|s|$ is large). In fact, it is easy to see that (5) is fulfilled in the cases of $f(s) = \lambda |s|^{p-1} s$, with $p > 1$, and, for instance, $|f(s)| \leq \lambda e^{|s|}$, for any $\lambda > 0$, for which *blow-up phenomena* may arise.

The proof of Theorem 1.1 shows that time τ_0 must be (in general) small enough. A natural question is to find conditions on the data in order to have $\tau_0 = T$ arbitrary. Two different results can be obtained in that direction. The first one concerns the case of the "pure" Neumann problem and y^T "near a constant":

Theorem 1.2. *Let* $\varepsilon > 0$ *be given. Assume (4), (5),* $\partial\Omega = \Gamma_N$ *and assume that* $y^T \in L^q(\Omega)$, *for some* $q \in [1,+\infty]$, *is such that*

$$\left\|y^T - M\right\|_{L^q(\Omega)} \leq \varepsilon/2, \text{ for some constant } M. \tag{6}$$

Then, for any $T > 0$ *and* $\varepsilon \in (0,1)$ *there exists* $u \in C^2\left(\overline{\Omega}\right)$ *with* $T_m(u) > T$ *and*

$$\left\|y(T; u) - y^T\right\|_{L^q(\Omega)} \leq \varepsilon.$$

If y^T is near a stationary state (even if it is an *unstable one*) we shall prove

Theorem 1.3. *Assume (4), (5) and let $y^T \in L^q(\Omega)$, for some $q \in [1,+\infty]$, be such that there exists $g^* \in L^\infty(\Omega)$ verifying that*

$$\begin{cases} Ag^* &= f(g^*) \quad \text{in } \Omega, \\ g^* &= 0 \quad \text{on } \Gamma_D, \\ \frac{\partial g^*}{\partial \nu} &= 0 \quad \text{on } \Gamma_N. \end{cases}$$

and

$$\| g^* - y^T \|_{L^q(\Omega)} \leq \varepsilon/2.$$

Then, for any $T > 0$ and $\varepsilon \in (0,1)$ there exists $u \in C^2(\overline{\Omega})$, different of g^*, with $T_m(u) > T$ and

$$\| y(T;u) - y^T \|_{L^q(\Omega)} \leq \varepsilon.$$

The proofs of these results are contained in Section 3 where, again, some generalizations and remarks will be given.

As a final question, it seems interesting to study the optimality of the control u. This is done in Section 4. A partial result in this direction is the following

Theorem 1.4. *Assume that the conditions of Theorem 1.2 or 1.3 are satisfied with $q = +\infty$. Then the set*

$$K = \Big\{ v : v \in L^\infty(\Omega) : \| y(T;v) - y^T \|_{L^\infty(\Omega)} \leq \varepsilon, \\ \| y(t;v) \|_{L^\infty(\Omega)} \leq \| y^T \|_{L^\infty(\Omega)} + 1, \forall t \in [0,T] \Big\}$$

is not empty. Moreover there exists $v_0 \in K$ such that

$$\| v_0 \|_{L^\infty(\Omega)} = \inf\{ \| v \|_{L^\infty(\Omega)}, v \in K \}.$$

We indicated above that we conjecture that the "more it blows up", the cheaper it will to (approximately) control the system. Such a result is provided by the following

Theorem 1.5. *Assume that the conditions of Theorem 1.2 holds true with $q = +\infty$ and $f(s) = \lambda F(s)$, $\lambda > 0$, $F(0) = 0$. Then*

$$\| v_0(\lambda) \|_{L^\infty(\Omega)} \searrow 0 \text{ when } \lambda \nearrow +\infty,$$

where $v_0(\lambda)$ denotes the control obtained in Theorem 1.4.

All the above results are improvements of the paper Díaz and Lions [6]. Our approach is based on a suitable use of the solution $Y(t)$ of the associated *backward* Cauchy problem

$$(CP : Y_d) \begin{cases} \frac{dY}{dt}(t) = f(Y(t)) - f(0), \ t < 0, \\ Y(0) = Y_d. \end{cases}$$

(Notice that in the above formulation we replaced the *final time* T by $t = 0$, which is possible since the ODE is autonomous, i.e., f is time independent).

We conclude this paper by some few remarks on another type of *non linear non local* system. Namely we consider the problem

$$\begin{cases} y_t + Ay = cy \int_\Omega y^2 dx & \text{in } \Omega \times (0,T), \\ y = 0 & \text{on } \partial\Omega \times (0,T), \\ y(0,x) = u(x) & \text{on } \Omega, \end{cases} \tag{7}$$

when A is as above but *symmetric*

$$a_{ij} = a_{ji}$$

and where $c > 0$.

We shall prove (by different methods) the following

Theorem 1.6. *System (7) is approximately controllable in* $L^2(\Omega)$.

Moreover we shall prove

Theorem 1.7. *The "cost of approximate controllability" decreases to* 0 *as* c *increases to* $+\infty$ *in (7).*

2. On the approximate controllability for small time

Proof of Theorem 1.1. Let $g^* \in C^2\left(\overline{\Omega}\right)$ be such that

$$g^* = 0 \text{ on } \Gamma_D, \ \frac{\partial g^*}{\partial \nu} = 0 \text{ on } \Gamma_N \text{ and } \left\| g^* - y^T \right\|_{L^q(\Omega)} \leq \varepsilon/2.$$

Assume, for the moment, that

$$f \in C^2(R).$$

First step. Let $x \in \overline{\Omega}$ arbitrary and let $Y_d = g^*(x)$. Since the ODE of the $(CP : Y_d)$ is an equation of "separable variables", it is easy to see that there exists $T(f, g^*) > 0$ such that the solution $Y(t : Y_d)$ of $(CP : Y_d)$ can be continued backwards to $[-T(f, g^*), 0]$. In particular, for any $\tau \in [0, T(f, g^*)]$ we can define the function

$$u_\tau(x) := Y(-\tau : g^*(x)), x \in \overline{\Omega}. \tag{8}$$

Clearly $Y(t : g^*(x))$ depends continuously on the initial data and therefore on $x \in \overline{\Omega}$. Thus, $u_\tau \in C(\overline{\Omega}.)$. Moreover, if $x \in \Gamma_D$, then $g^*(x) = 0$ and so $u_\tau(x) = 0$ since $Y_\infty = 0$ is an *equilibrium point* of the ODE

$$\frac{dY}{dt}(t) = f(Y(t)) - f(0).$$

On the other hand, we have

$$\nabla u_\tau(x) = \frac{\partial Y}{\partial \alpha}(-\tau : g^*(x)) \nabla g^*(x).$$

Thus $\frac{\partial g^*}{\partial \nu} = 0$ on Γ_N . Moreover

$$\frac{\partial^2 u_\tau}{\partial x_i \partial x_j}(x) = \frac{\partial^2 Y}{\partial \alpha^2}(-\tau : g^*(x))\frac{\partial g^*}{\partial x_i}(x)\frac{\partial g^*}{\partial x_j}(x) + \frac{\partial Y}{\partial \alpha}(-\tau : g^*(x))\frac{\partial^2 g^*}{\partial x_i \partial x_j}(x).$$

But from the Peano Theorem we know that $\frac{\partial Y}{\partial \alpha}(t : g^*(x))$ is given as the solution of the linear Cauchy problem

$$\begin{cases} \frac{dZ}{dt}(t) = f'(Y(t : g^*(x)))Z(t) \\ Z(0) = 1, \end{cases}$$

and so, $\frac{\partial Y}{\partial \alpha}(t : g^*(x))$ depends continuously on $x \in \overline{\Omega}$. Since $f \in C^2(R)$, applying again the Peano Theorem we get that $\frac{\partial^2 Y}{\partial \alpha^2}(t : g^*(x))$ also depends continuously on $x \in \overline{\Omega}$. In conclusion, $u_\tau \in C^2(\overline{\Omega})$. Let

$$M_1(T(f, g^*)) := \sup\{\|u_\tau\|_{C^2(\overline{\Omega})} : \tau \in [0, T(f, g^*)]\}.$$

Is is clear that $M_1(T(f, g^*)) < \infty$. Consider, now, the function $U \in C^2(\overline{\Omega} \times [0, \tau])$ given by

$$U(x, t) := Y(t - \tau : g^*(x)).$$

(notice that, in fact, U is defined on $\overline{\Omega} \times [-(T(f, g^*) - \tau), \tau]$). Then

$$U(x, 0) = Y(-\tau : g^*(x)) = u_\tau(x), \ x \in \overline{\Omega},$$

$$U(x, t) = 0, \text{if } x \in \Gamma_D \text{ and } \ \frac{\partial U}{\partial \nu}(x, t) = 0 \text{ if } x \in \Gamma_N. \ , \ t \in [0, T(f, g^*)],$$

$$U_t + AU - f(U) = h(x, t), \ x \in \Omega, \ t \in (0, T(f, g^*)),$$

where

$$h(x, t) := -f(0) \ + AU.$$

(notice that from the above arguments $h \in C(\overline{\Omega} \times [0, T(f, g^*)])$). Define

$$M_2(T(f, g^*)) := \sup\{\|h\|_{L^\infty(\Omega \times (0,\tau))} : \tau \in [0, T(f, g^*)]\}$$

We point out that $M_2(T(f, g^*)) < \infty$ and that

$$U(x, \tau) = Y(0 : g^*(x)) = g^*(x), \ x \in \overline{\Omega}.$$

Second step. Let us show that the solution $y(x, t : u_\tau)$, with u_τ given by (8), is a global solution on $\overline{\Omega} \times [0, \tau]$, i.e. that $T_m(u_\tau) > \tau$. More precisely, let us show that

$$\sup\{\|y(x, t : u_\tau)\|_{L^\infty(\Omega \times (0,\tau))} : \tau \in [0, T(f, g^*)]\} \leq M_3(T(f, g^*)), \tag{9}$$

for some $M_3(T(f, g^*)) < \infty$. Let us start by assuming that $f(0) \leq 0$. Define $m^+(t)$ by

$$m^+(t) = Y(t - \tau : \|[u_\tau]_+\|_{L^\infty(\Omega)}) \ ,$$

where, in general, $[u]_+(x) = \max(u(x), 0)$. From assumption (5) we know that $m_+(t)$ is defined at least on $[-(T(f,g^*)-\tau), \tau]$. Moreover we have

$$\begin{cases} y_t + Ay - f(y) &= 0 \le m_t^+ + Am^+ - f(m^+) & \text{in } \Omega \times (0,\tau), \\ y(t,x) &= 0 \le m^+(t) & \text{on } \Gamma_D \times (0,\tau), \\ \frac{\partial y}{\partial \nu} = \frac{\partial m^+}{\partial \nu} &= 0 & \text{on } \Gamma_N \times (0,\tau), \\ y(0,x) &= u_\tau(x) \le m^+(0) = \|[u_\tau]+\|_{L^\infty(\Omega)} & \text{on } \Omega. \end{cases}$$

Then, by the comparison principle (which holds for problem (1)), we conclude that $y(x,t) \le m^+(t)$, a.e. $x \in \Omega$, for any $t \in [0,\tau]$. If $f(0) > 0$ we replace the barrier function $m^+(t)$ by the solution of

$$\begin{cases} M_t^+(t) &= f(M^+(t)) \\ M^+(0) &= \|[u_\tau]+\|_{L^\infty(\Omega)} \end{cases}$$

and again we get that $y(x,t) \le M^+(t)$, a.e. $x \in \Omega$, for any $t \in [0,\tau]$ (notice that $m^+(t) \le M^+(t)$). In a similar way we can construct negative barrier functions $m^-(t)$ and $M^-(t)$ and the conclusion holds.

Third step. Given $\varepsilon \in (0,1)$ let $\tau_0 \in (0, T(f,g^*)]$ be such that

$$\tau_0 e^{K(T(f,g^*))\tau_0} M_2(T(f,g^*)) \le \varepsilon/2 \tag{10}$$

where

$$K(T(f,g^*)) := \max\{|f'(s)| : s \in [-M_3(T(f,g^*)), M_3(T(f,g^*))]\}. \tag{11}$$

Then, as f is globally Lipschitz on the interval $[-M_3(T(f,g^*)), M_3(T(f,g^*))]$, using the $L^\infty(\overline{\Omega} \times [0,\tau_0])$-estimates on functions $y(x,t:u_{\tau_0})$ and $U(x,t)$, and since the functional operator $\mathbf{A} : D(\mathbf{A}) \to L^s(\Omega)$ given by

$$\begin{aligned} &D(\mathbf{A}) = \{w \in W^{1,q}(\Omega) : w = 0 \text{ on } \Gamma_D,\ \tfrac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_N, Ay \in L^q(\Omega)\} \text{ and} \\ &\mathbf{A}y = Ay, \text{ if } y \in D(\mathbf{A}), \end{aligned} \tag{12}$$

generates a semigroup of contractions on $L^q(\Omega)$, using Gronwall's inequality, we conclude that, for any $t \in [0,\ T(f,g^*)]$, we have

$$\|y(.,t:u_{\tau_0}) - U(.,t)\|_{L^q(\Omega)} \le e^{Kt}\int_0^t \|h(.,s)\|_{L^q(\Omega)}\,ds \le te^{Kt}M_2(T(f,g^*)) \le \varepsilon/2 \tag{13}$$

In particular, making $t = \tau_0$ we get that

$$\left\|y(.,\tau_0:u_{\tau_0}) - y^T\right\|_{L^q(\Omega)} \le \|y(.,\tau_0:u_{\tau_0}) - g^*\|_{L^q(\Omega)} + \left\|\ g^* - y^T\right\|_{L^q(\Omega)} \le \varepsilon.$$

Fourth step. If $f \notin C^2(R)$ we approximate f by f_ε with $f_\varepsilon \in C^2(R)$ such that

$$|f(s) - f_\varepsilon(s)| \le \varepsilon/2, \text{ for any } s \in R.$$

We modify the definition of function $U(x,t)$ by replacing f by f_ε in the definition of $Y(t)$. The rest of the proof follows with obvious modifications.

Remark 2.1. *It is possible to obtain some expressions for the solution $Y(t)$ of $(CP : Y_d)$. We start by pointing out that, for the arguments of the proof of Theorem 1.1, we can assume without loss of generality that f is strictly increasing. Indeed, once that we have the estimate (9), we can replace the equation of (1) by*

$$y_t + \widetilde{A}y = \widetilde{f}(y) \quad \text{in } \Omega \times (0,T),$$

with

$$\widetilde{A}y = Ay + (K+1)y, \ \widetilde{f}(y) = f(y) + (K+1)y, \\ K = K(T(f, g^*))$$

and now $\widetilde{f}(y)$ is strictly increasing. Coming back to $(CP : Y_d)$ we have

$$\int_{Y(t)}^{Y_d} \frac{ds}{f(s) - f(0)} = -t \tag{14}$$

Then, if we define the (strictly decreasing) function

$$\Psi(r) := \int_r^\infty \frac{ds}{f(s) - f(0)}$$

and its inverse function

$$\eta = \Psi^{-1}, \quad \text{then (14) says that} \quad \Psi(Y(t)) - \Psi(Y_d) = -t.$$

In fact, it is clear that defining

$$Y(t) = \eta(\Psi(Y_d) - t)$$

we get the (unique) solution of $(CP : Y_d)$. In this way, the function $u_\tau(x)$ of the proof of Theorem 1.1 is given by

$$u_\tau(x) := \eta(\Psi(g^*(x)) + \tau), x \in \overline{\Omega},$$

and

$$U(x,t) := Y(t - \tau : g^*(x)) = \eta(\Psi(g^*(x)) + \tau - t).$$

If, for instance,

$$f(s) = s^3$$

then

$$\Psi(r) := \int_r^\infty \frac{ds}{f(s) - f(0)} = \frac{1}{2r^2},$$

$$\eta(s) = \Psi^{-1}(s) = \frac{1}{\sqrt{2s}},$$

$$u_\tau(x) := \eta(\Psi(g^*(x)) + \tau) = \frac{g^*(x)}{\sqrt{1 + 2\tau g^*(x)^2}}$$

$$U(x,t) := Y(t - \tau : g^*(x)) = \eta(\Psi(g^*(x)) + \tau - t) = \frac{g^*(x)}{\sqrt{1 + 2(\tau - t) g^*(x)^2}}.$$

In this special case it is also possible to check directly that if, for instance, $A = -\Delta$, *then*

$$\Delta U(x,t) = -\frac{6(\tau - t)g^*(x)\left|\nabla g^*(x)\right|^2}{2[1+2(\tau - t)g^*(x)^2]^{5/2}} + \frac{\Delta g^*(x)}{[1+2(\tau - t)g^*(x)^2]^{3/2}}.$$

Remark 2.2. *We point out that if (4) holds true with* $s_0 = 0$ *and* $q = +\infty$, *estimate (9) implies that*

$$\|y(t:u_\tau)\|_{L^\infty(\Omega)} \leq \left\|y^T\right\|_{L^\infty(\Omega)}, \text{ for any } t \in [0,\tau]$$

(it suffices to see that $Y(t)$ *and* $M(t)$ *are increasing functions and that we can assume that* $\|g^*\|_{L^\infty(\Omega)} \leq \left\|y^T\right\|_{L^\infty(\Omega)}$*).*

Remark 2.3. *The proof of Theorem 1.1 allows to see that assumptions (4), and (5) are merely used to be sure that the Cauchy problem* $(CP : Y_d)$ *has a solution backward continuable when we take* $Y_d = g^*(x)$ *with* $x \in \overline{\Omega}$ *arbitrary. In fact, assumption (5) implies that the continuation interval is "universal" (i.e. independent of the values of* $g^*(x)$*). More generally, the same proof applies to the case in which (4), and (5) are replaced by the following general condition on* f *and* y^T

$$H(f,y^T) \equiv \left\{ \begin{array}{l} \exists\ T(f,y^T) \in (0,T] \text{ such that the solution of } (CP:Y_d) \\ \text{is continuable to } [-T(f,y^T),0] \text{ for any } Y_d = y^T(x), x \in \overline{\Omega}. \end{array} \right.$$

as well as that $g^* = y^T$, *i.e.,*

$$y^T \in C^2\left(\overline{\Omega}\right) \quad \text{and } y^T = 0 \text{ on } \Gamma_D, \frac{\partial y^T}{\partial \nu} = 0 \text{ on } \Gamma_N.$$

It is easy to see that assumption $H(f,y^T)$ *is verified if* f *is sublinear (in that case* $T(f,y^T) = +\infty$*), as well as in the case in which* f *is a decreasing function, as , for instance,* $f(s) = -\left|s\right|^{p-1}s$, *with* $p > 1$. *In this last case* $T(f,y^T) < +\infty$ *and* $T(f,y^T)$ *strongly depends on the concrete values of* $Y_d = y^T(x)$. *So, if* $y^T \in L^\infty(\Omega)$ *we get the approximate controllability at least for some small* $\tau_0 \in (0, T(f,y^T)]$. *In some sense, this generalization of Theorem 1.1 covers the gap open by the negative results of Bardos and Tartar [1] and gives an answer to a conjecture posed in Fabre, Puel and Zuazua [7] concerning this special function* f.

Remark 2.4. *It is also possible to get other type of generalizations, this time concerning the elliptic operator* A. *In the proof of Theorem 1.1 we merely applied the* L^s*-continuous dependence and the comparison principle for the associated functional operator* $\mathbf{A} : D(\mathbf{A}) \to L^s(\Omega)$ *and the linearity of* $\mathbf{A}$ *was not used. So the results remain valid for other diffusion operators (some quasilinear operators as, for instance, the p-Laplacian, the minimal surface operator and some fully nonlinear operators).*

Remark 2.5. *The constructive nature of the proof of Theorem 1.1 supplies additional qualitative informations on the constructed control* u. *So, for instance,* u *vanishes (resp. is strictly positive, resp. strictly negative) on the same subset of* Ω

where g^* vanishes (resp. is strictly positive, resp. strictly negative). This type of qualitative informations were obtained in Díaz [4] *and Díaz, Henry and Ramos* [5] *for different nonexplosive semilinear problems.*

Remark 2.6. *It is easy to see that there is not uniqueness of the control u (in fact of the pair $\{u, \tau_0\}$).*

Remark 2.7. *A general comment concerns a different type of arguments in order to prove conclusions for short time such as the one presented in Theorem 1.1. In fact, we can consider the abstract semilinear Cauchy problem*

$$\begin{cases} \frac{dy}{dt}(t) + \mathbf{A}y(t) = f(y(t)) & t \in (0,T), \text{ in } X, \\ y(0) = u & \text{in } X, \end{cases}$$

where X denotes a Banach space. Under very general conditions (see, e.g., the exposition made in Vrabie [14] *it is well known that the solution depends continuously on the initial data and therefore the conclusion of Theorem 1.1 holds trivially by choosing $u = y^T$ (or $u = g^*$ with g^* a regularization of y^T). Although this kind of arguments would lead to very general results (even of a greater generality than Theorem 1.1) we point out that the proof of Theorem 1.1 has a constructive character which is very useful in order to study the approximate controllability for large time and other qualitative properties (see Sections 3 and 4 below).*

3. On the approximate controllability for large time

Proof of Theorem 1.2. Arguing as in the proof of Theorem 1.1 with $g^* = M$ we construct the function

$$v(x) := Y^*(-T : M), x \in \overline{\Omega},$$

where $Y^*(t : M)$ denotes the solution of the ordinary Cauchy problem

$$\begin{cases} \frac{dY^*}{dt}(t) = f(Y^*(t)), \ t < 0, \\ Y^*(0) = M. \end{cases}$$

Defining

$$U^*(x,t) := Y^*(t - T : M)$$

we have that

$$\frac{\partial U^*}{\partial \nu}(x,t) = 0 \text{ if } x \in \Gamma_N \ , \ t \in [0,T]$$

(notice that since Γ_D is empty we do not need to have U^* vanishing in any part of $\partial\Omega$: this is the reason why we replace $Y(t : M)$ by $Y^*(t : M)$). The rest of the arguments of the proof of Theorem 1.1 can be repeated but now with

$$h(x,t) := AU^*(x,t) = 0$$

and the conclusion holds for the initial control $u = v$.

Remark 3.1. *As in Remark 2.1, if we assume f strictly increasing (or more generally (4) with $s_0 = 0$) then we have that*

$$U^*(x,t) := Y^*(t-T:M) = \eta^*(\Psi^*(M) + T - t)$$

with

$$\Psi^*(r) := \int_r^\infty \frac{ds}{f(s)}, \text{ if } r > 0,$$

and

$$\eta^* = (\Psi^*)^{-1}.$$

Proof of Theorem 1.3. We introduce the positive constants

$$\omega = \max\{|f'(s)| : s \in [-\|g^*\|_{L^\infty(\Omega)} - 1, \|g^*\|_{L^\infty(\Omega)} + 1]\}$$

and

$$\widetilde{\varepsilon} = (\varepsilon/(2|\Omega|^{1/q}))\exp(-\omega T).$$

By Theorem 1.1 there exists $\{v_0, \tau_0\}$ such that

$$\|y(\tau_0; v_0) - g^*\|_{L^\infty(\Omega)} \leq \widetilde{\varepsilon}.$$

We can assume that $\tau_0 < T$ (otherwise the proof ends). For $t < T_m(v_0)$ and $x \in \overline{\Omega}$ we introduce

$$z(x,t) := y(x,t:v_0) - g^*(x).$$

We also define

$$T^* = \sup\{t \mid t \geq \tau_0, \|z(\widetilde{t})\|_{L^\infty(\Omega)} < 1 \text{ for any } \widetilde{t} \in (\tau_0, t)\}.$$

It is clear that $T^* < T_m(v_0)$. So, by construction, $\|z(t)\|_{L^\infty(\Omega)} < 1$ for any $t \in (\tau_0, T^*)$. Then, if we define

$$\phi(x,s) = \begin{cases} f(s+g^*(x)) - f(g^*(x)) & \text{if } |s| \leq 1, \\ f(1+g^*(x)) - f(g^*(x)) & \text{if } s > 1, \\ f(-1+g^*(x)) - f(g^*(x)) & \text{if } s < -1, \end{cases}$$

we get that $f(z(x,t)+g^*(x)) - f(g^*(x)) = \phi(x, z(x,t))$. We point out that $\phi(x,s)$ is a Lipschitz function of constant smaller or equal than ω. Since

$$\begin{cases} z_t + Az = \phi(x, z(x,t)) & \text{in } \Omega \times (\tau_0, T^*), \\ z(t,x) = 0 & \text{on } \Gamma_D \times (0,\tau), \\ \frac{\partial z}{\partial \nu} = 0 & \text{on } \Gamma_N \times (0,\tau), \\ \|z(\tau_0)\|_{L^\infty(\Omega)} \leq \widetilde{\varepsilon}, \end{cases}$$

we deduce that

$$\|z(t)\|_{L^\infty(\Omega)} \leq e^{\omega t}\|z(\tau_0)\|_{L^\infty(\Omega)} \text{ for any } t \in (\tau_0, T^*).$$

In particular,

$$\|z(T^*)\|_{L^\infty(\Omega)} \leq (\varepsilon/(2|\Omega|^{1/q}))e^{\omega(T^*-T)}$$

and so $T \leq T^*$ (otherwise we get a contradiction with the definition of T^*). Then

$$\|z(T)\|_{L^\infty(\Omega)} \leq \varepsilon/(2\,|\Omega|^{1/q})$$

and the proof is completed since, again,

$$\left\|y(.,T:v_0) - y^T\right\|_{L^q(\Omega)} \leq |\Omega|^{1/q}\,\|y(.,T:v_0) - g^*\|_{L^\infty(\Omega)} + \left\|\, g^* - y^T\right\|_{L^q(\Omega)} \leq \varepsilon.$$

Remark 3.2. *The same proof applies for more general functions $y^T \in L^q(\Omega)$, for some $q \in [1,+\infty]$, such that*

$$\begin{cases} \text{there exists } g^* \in L^\infty(\Omega) \text{ verifying that } \left\|\, g^* - y^T\right\|_{L^q(\Omega)} \leq \varepsilon/2 \text{ and} \\ \|Ag^* - f(g^*)\|_{L^\infty(\Omega)} \leq (\varepsilon/(2\,|\Omega|^{1/q}\,T))\exp(-\omega T) \text{ with} \\ \omega = \max\{|f'(s)| : s \in [-\|g^*\|_{L^\infty(\Omega)} - 1, \|g^*\|_{L^\infty(\Omega)} + 1]\}\ , \\ g^* = 0 \text{ on } \Gamma_D,\ \frac{\partial g^*}{\partial \nu} = 0 \text{ on } \Gamma_N. \end{cases}$$

4. On the "cost" of the controls

Proof of Theorem 1.4. By applying Theorems 1.2 or 1.3, with $q = +\infty$, and Remark 2.2 we have that the set

$$K = \begin{array}{l} \Big\{v : v \in L^\infty(\Omega) : \left\|y(T;v) - y^T\right\|_{L^\infty(\Omega)} \leq \varepsilon, \\ \|y(t;v)\|_{L^\infty(\Omega)} \leq \left\|y^T\right\|_{L^\infty(\Omega)} + 1, \forall t \in [0,T]\Big\} \end{array}$$

is not empty. Let us start by assuming that

$$f \text{ is strictly decreasing on } [-\left\|y^T\right\|_{L^\infty(\Omega)} - 1, \left\|y^T\right\|_{L^\infty(\Omega)} + 1]. \tag{15}$$

Then any solution of problem (1) with $u \in K$ coincides with the solution of a similar problem in which we replace the function f by the truncated function

$$\widetilde{f}(s) = \begin{cases} f(s) & \text{if } |s| \leq \left\|y^T\right\|_{L^\infty(\Omega)} + 1, \\ f(\left\|y^T\right\|_{L^\infty(\Omega)} + 1) & \text{if } s > \left\|y^T\right\|_{L^\infty(\Omega)} + 1, \\ f(-\left\|y^T\right\|_{L^\infty(\Omega)} - 1) & \text{if } s < -\left\|y^T\right\|_{L^\infty(\Omega)} - 1. \end{cases}$$

Now, let $\{v_n\}$ be a minimizing sequence. Obviously

$$\|v_n\|_{L^\infty(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)},$$

with v_0 the control given in the statement of Theorems 1.2 or 1.3. So, $v_n \rightharpoonup v$ weakly-star in $L^\infty(\Omega)$. By applying the theory of accretive operators generating compact semigroups (see, e.g. Vrabie [14]) it is easy to see that $y(t;v_n) \to y(t;v)$ strongly in $L^q(\Omega)$, for any $q \in (1,+\infty)$, and that $\|y(t;v)\|_{L^\infty(\Omega)} < \left\|y^T\right\|_{L^\infty(\Omega)} + 1$. Thus the conclusion follows. If (15) fails, we argue as in Remark 2.1, i.e. we replace the equation of (1) by

$$y_t + \widehat{A}y = \widehat{f}(y) \quad \text{in } \Omega \times (0,T),$$

with

$$\widetilde{A}y = Ay - (K+1)y, \ \widetilde{f}(y) = f(y) - (K+1)y,$$

for some $K > 0$ large enough and now $\widetilde{f}(y)$ becomes strictly increasing (notice that by the change of variables $y = e^{\lambda t}y^*$, we can always assume that $a_0 > K+1$).

Remark 4.1. *It would be interesting to know if the conclusion of Theorem 1.4 remains true after replacing the set K by the more general set*

$$\widetilde{K} = \{v \mid \ v \in L^\infty(\Omega), \left\|y(T;v) - y^T\right\|_{L^\infty(\Omega)} \leq \varepsilon\}$$

and if we also replace the exponent $q = +\infty$ by $q \in [1, +\infty)$ arbitrary (see some related results for $q = 2$ and $f(s) = s^3$ in Lions [12] *(Section 1.12)).*

Proof of Theorem 1.5. We can assume without loss of generality that F is a strictly increasing function. Using the notation of Remark 3.1 we have that

$$u(x) = U^*(x,0) = \eta_F^*(\Psi_F^*(M) + \lambda T)$$

where

$$\Psi_F^*(r) := \int_r^\infty \frac{ds}{F(s)}, \text{ if } r > 0,$$

and

$$\eta_F^* = (\Psi_F^*)^{-1}$$

(notice that $\Psi^*(r) = \Psi_F^*(r)/\lambda$). On the other hand, since F is a Lipschitz function near $r = 0$ and $F(0) = 0$ we have that

$$\Psi_F^*(r) \nearrow +\infty \text{ if } r \searrow 0$$

(and also $\Psi_F^*(r) \searrow 0$ if $r \nearrow +\infty$). Analogous properties hold if $r < 0$. Then $\|u\|_{L^\infty(\Omega)} \searrow 0$ if $\lambda \nearrow +\infty$ and, since by construction

$$\|v_0(\lambda)\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)},$$

we get the result.

Remark 4.2. *We conjecture that the conclusion of Theorem 1.5 remains true under more general conditions (more general functions y^T, more general boundary conditions, etc.).*

Remark 4.3. *For previous results along the lines of Theorem 1.5 with distributed (or boundary) control, cf. Lions and Zuazua (*[13]*) (cf. also Theorem 1.7 below).*

5. Approximate controllability of non local systems

We consider now system (7) of the Introduction and we are going to prove Theorems 1.6 and 1.7.

Let us first notice that given $u \in L^2(\Omega)$ the existence of a solution (local in time) defined on a *maximal interval* $[0, T_m[$ ($T_m = T_m(u)$ and $\|y(t)\|_{L^2(\Omega)} \nearrow +\infty$ if $t \nearrow T_m$ when $T_m(u) < +\infty$) can be proved by using compactness methods (see, e.g. Theorem 4.3.1. of Vrabie [14]). Notice also that problem (7) has a *superlinear* nature since if we denote

$$\mathbf{F}(y) = cy \int_\Omega y^2 dx$$

then $\mathbf{F} : L^2(\Omega) \to L^2(\Omega)$ and

$$\|\mathbf{F}(y)\|_{L^2(\Omega)} = c\,\|y\|^3_{L^2(\Omega)}\,.$$

On the other hand, if we set

$$\|y(t)\|^2 = \int_\Omega y(x,t)^2 dx,$$

we obtain, after multiplying (7) by y, that

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|^2 + a(y(t)) - c\,\|y(t)\|^4 = 0 \tag{16}$$

where

$$a(y(t)) = a(y(t), y(t)) = \int_\Omega \Big(\sum_{i,j=1}^N a_{ij}\frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_j} + a_0 y^2\Big)dx.$$

The estimate (16) shows the "unboundedness" of y as $t \to +\infty$ and the *"increase in instability"* as $c \nearrow +\infty$. This is made more precise below.

Before proving Theorems 1.6 and 1.7 let us show some examples in which $T_m(u) < +\infty$. For this and later purposes we are going to use the eigenfunctions

$$\begin{cases} Aw_i = \lambda_i w_i, & 0 < \lambda_1 \le \lambda_2 \le \ldots \\ w_i = 0 & \text{on } \partial\Omega, \\ \int_\Omega w_i w_j dx = \delta_{ij}. \end{cases} \tag{17}$$

We recall that $w_1(x) > 0$ a.e. $x \in \Omega$.

Proposition 5.1. *Let $u \in L^2(\Omega)$, $u \ge 0$ on Ω, be such that*

$$\Big(\int_\Omega u w_1 dx\Big)^2 \ge \frac{\lambda_1}{c}.$$

Then $T_m(u) < +\infty$.

Proof. By multiplying by $y_-(t) = \min(y(t), 0)$ and applying Gronwall's inequality we get that $y(t) \ge 0$ on Ω for any $t \in [0, T_m[$. We introduce the function

$$H(t) = \int_\Omega y(t) w_1 dx \text{ for } t \in [0, T_m[.$$

Then

$$H'(t) = \int_\Omega y_t(t) w_1 dx = -\lambda_1 H(t) + c(\int_\Omega y(t)^2 dx) H(t).$$

But

$$H(t)^2 \leq \int_\Omega y(t)^2 w_1 dx \leq \int_\Omega y(t)^2 dx.$$

Therefore

$$H'(t) \geq H(t)(-\lambda_1 + cH(t)^2), \text{ on }]0, T_m[$$

and the result follows in a standard way (see, e.g. Proposition 5.4.1 of Cazenave and Haraux [3]).

In order to prove Theorems 1.6 and 1.7 we introduce

$$E_m = \text{space generated by } w_i, \ 1 \leq i \leq m. \tag{18}$$

Let T be given. We shall have *approximate controllability* if we know that

$$\left.\begin{array}{l}\text{given } y^T \in E_m, \text{ there exists } u \in L^2(\Omega) \text{ such that (7) has} \\ \text{a solution defined in } \Omega \times (0,T) \text{ and which is such that } y(T) = y^T.\end{array}\right\} \tag{19}$$

Actually, we are going to verify (19) by choosing $u \in E_m$, i.e.

$$u = \sum_{j=1}^{m} u_j w_j. \tag{20}$$

The controls are now $\{u_j\} \in R^m$.

For the initial condition (20), the solutions of (7) can be computed as follows. The solution $y(t) \in E_m$, so that

$$y(t) = \sum_{j=1}^{m} y_j(t) w_j,$$

where

$$\left.\begin{array}{l} y_j'(t) + \lambda_j y_j(t) = c y_j(t) Y(t), \\ y_j(0) = u_j, \\ Y(t) = \sum_{j=1}^m y_j(t)^2. \end{array}\right\} \tag{21}$$

One has to find (if possible) u_j such that $T < T_m(u)$ and

$$y_j(T) = y_j^T \ (\text{where } y^T = \sum_{j=1}^{m} y_j^T w_j). \tag{22}$$

The solution of (21) is given by

$$\left.\begin{array}{l} y_j(t) = u_j \exp(-\lambda_j t + cZ(t)), \text{ where} \\ Z(t) = \int_0^t Y(s) ds. \end{array}\right\} \tag{23}$$

But (23) is still *implicit*. We make it explicit in the following way. Let us set

$$U(t) = \sum_{j=1}^{m} {u_j}^2 e^{-2\lambda_j t}.$$

Then (23) leads to

$$Y(t) = U(t) e^{2cZ(t)}$$

and since $Y(t) = Z'(t)$, it follows that

$$\begin{aligned} Z'(t)e^{-2cZ(t)} &= U(t), \text{ i.e.} \\ e^{-2cZ(t)} - 1 &= -2c \int_0^t U(s)ds = -2c \sum_{j=1}^{m} \left(\frac{1 - e^{-2\lambda_j t}}{2\lambda_j}\right) u_j^2. \end{aligned} \tag{24}$$

Formulae (23) (24) define "explicitly" y_j in function of the terms u_j. One has to find, if possible, u_j such that (22) holds true, i.e.

$$u_j e^{-\lambda_j T} = e^{-cZ(T)} y_j^T$$

i.e. finally

$$u_j = e^{\lambda_j T} y_j^T e^{-cZ(T)}.$$

Let us set

$$\mu_j(T) = \frac{1 - e^{-2\lambda_j T}}{\lambda_j}, \ z_j^T = e^{\lambda_j T} y_j^T.$$

Then one has to find, if possible, u_j such that

$$u_j = z_j^T (1 - c \sum_{j=1}^{m} \mu_j(T) u_j^2)^{1/2}. \tag{25}$$

It follows from (25) that

$$\sum_{j=1}^{m} \mu_j(T) u_j^2 = (\sum_{j=1}^{m} \mu_j(T)(z_j^T)^2)(1 - c \sum_{j=1}^{m} \mu_j(T) u_j^2).$$

If we set

$$\|z^T\| = \sum_{j=1}^{m} \mu_j(T)(z_j^T)^2,$$

it follows that

$$\sum_{j=1}^{m} \mu_j(T) u_j^2 = \frac{\|z^T\|^2}{1 + c\|z^T\|^2}, \tag{26}$$

hence

$$u_j = \frac{z_j^T}{(1 + c\|z^T\|^2)^{1/2}}.$$

This proves Theorem 1.6.

Moreover we can define as the *cost* the quantity $\sum_{j=1}^{m} \mu_j(T)u_j^2$. It is given by (26), which shows Theorem 1.7.

Remark 5.2. *By using (23) and (24) we get that the solution of (21) is given by*

$$y_j(t) = e^{-\lambda_j t} \frac{u_j}{(1 - c\sum_{j=1}^{m}(\frac{1-e^{-2\lambda_j t}}{\lambda_j})u_j^2)^{1/2}}.$$

It is easy to see that the associated function $y(t)$ blows up in a finite time (i.e. $T_m(u) < +\infty$) if the initial datum u is such that

$$\sum_{j=1}^{m} \frac{u_j^2}{\lambda_j} > \frac{1}{c}.$$

Remark 5.3. *In the above proof, the fact that $c > 0$ is essential. Things become different, for instance in (26), if $c < 0$.*

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