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OPTIMAL CONTROL AND PERFORMANCE ANALYSIS OF AN $M^X/M/1$ QUEUE WITH BATCHES OF NEGATIVE CUSTOMERS

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Abstract. We consider a Markov decision process for an $M^X/M/1$ queue that is controlled by batches of negative customers. More specifically, we derive conditions that imply threshold-type optimal policies, under either the total discounted cost criterion or the average cost criterion. The performance analysis of the model when it operates under a given threshold-type policy is also studied. We prove a stability condition and a complete stochastic comparison characterization for models operating under different thresholds. Exact and asymptotic results concerning the computation of the stationary distribution of the model are also derived.

Keywords. Queueing, Markov decision processes, negative customers, stationary distribution, stochastic comparison.

1. INTRODUCTION

Queueing systems with negative customers have attracted the interest of many investigators during the last fifteen years, since they can be successfully used for modeling and studying queueing systems with signalling mechanisms. The idea of negative customers was originated by Gelenbe [8] in the context of neural networks and subsequently was developed within the area of queueing networks (Gelenbe [9]). There now exists a significant amount of results about product-form

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queueing networks with negative customers and other relevant entities as triggers, signals etc. A broad review of such results can be found in Gelenbe and Pujolle [13] and Chao *et al.* [5].

The notion of negative customers was also used in more intricate models with non-Markovian assumptions. In this sense, we mention Harrison and Pitel [14] who studied the M/G/1 queue with negative customers while Yang *et al.* [30] studied an M/G/1 stochastic clearing system. Artalejo [1] summarizes several diverse applications of negative customers as a mechanism for work removal in queueing systems.

However, although the performance evaluation of such systems has been extensively studied, it seems that there do not exist results in the literature on the control of queueing systems using negative customers. The purpose of the present paper is to study in detail a Markov decision problem for the basic model of an $M^X/M/1$ queue that is controlled by batches of negative customers. The necessity of dealing with a Markovian model should be understandable since the analysis of the M/G/1 model is very complicated (see Harrison and Pitel [14]) and the control problem for such a model seems too difficult to solve.

Queueing systems with batch transitions (arrivals or departures) that incorporate some kind of dynamic control mechanism appear in many Operations Research problems and more specifically in assembly and transportation problems occurring in manufacturing, inventory control etc. These models are inherently much more complex than the single transition models and in most cases we cannot characterize their optimal policy nor can we extract an exact formula for their stationary distribution under a given policy.

Most of the reported results concern either systems with batch arrivals, single departures and a control mechanism that affects the speed (rate) of the service (see for example Federgruen and Tijms [7], Nishigaya *et al.* [18], Nishimura and Jiang [19] and Nobel and Tijms [20]) or systems with a control mechanism that introduces total catastrophes which remove all the units of the system (see for example Kyriakidis [16, 17] and Economou [6]). Moreover, Deb and Serfozo [4] and Deb [3] studied optimal control problems for bulk queues. Teghem [28] summarizes the early results about the control of service in queueing systems. The dynamic programming approach prevails in this kind of problems. Standard references are Ross [22, 23] and Bertsekas [2]. For a recent account and queueing applications see Hernandez-Lerma and Lasserre [15], Puterman [21] and Sennott [24].

A flow of negative arrivals can be viewed as a second flow of departures. From a performance analysis point of view for a Markovian queueing system with both regular services and negative arrivals, the model is equivalent to a model with only one service flow consisting in the superposition of regular (service) and negative departures. However, from the control point of view there exists essential difference in the cost structure. In a system with negative arrivals, it seems natural to assign costs associated with the erased customers. These costs are accumulated every time that one or several customers are removed by negative arrivals and correspond to loss-of-profit, reimbursement or removal costs that should be paid

by the administrator of the system. It is exactly this enriched cost structure that differentiates our study radically from the above mentioned works.

The following service system is considered: groups of customers arrive at a service station according to a Poisson process at rate λ . The sizes of successive arriving groups are independent identically distributed random variables and independent of the arrival times. Let $(g_j : j = 1, 2, \dots)$ be the group size discrete probability mass function. There is a single server who serves one customer at a time. The service times are independent exponentially distributed random variables with parameter μ , independent of everything else. The system has an infinite waiting room and is equipped with a negative batch mechanism (batch removing mechanism) which can be in one of two modes: on or off. The mechanism is characterized by its capability N and its rate ν . Whenever the mechanism is off it does not have any influence to the system. If it is on then it produces batches of N negative customers at rate ν . A batch of N negative customers remove immediately at most N of the present customers, *i.e.* if there exist n present customers in the system, it removes $\min(n, N)$ of them. The controller may turn the negative batch mechanism on or off at any transition epoch. This system will be referred as the $M^X/M/1$ queue with a negative batch control mechanism. The negative batch mechanism can be thought of as a “vehicle” with capacity N that is activated to move some customers elsewhere when the system is congested. Alternatively, it can be thought of as a signaling mechanism that produces batches of N negative arrivals that remove (or cancel) regular customers of the system.

In the present study, we are interested in characterizing the optimal policy under a natural cost structure, in computing the stationary distribution and in proving qualitative properties concerning stochastic comparison and asymptotic questions. The complete understanding of such a simple system may facilitate the study of more complex systems, in particular the study of tree-like networks that occur in several fields of applications.

It should be noted that the batch mechanism consisting in removing $\min(n, N)$ of the n present customers can be viewed as a particular case of the batch removal mechanism investigated by Gelenbe [10], in the more general context of an open G-network with a finite number of nodes. In Gelenbe [10] a negative arrival is allowed to remove a batch of random size. Since we assume a batch arrival input, it is mathematically convenient to reduce the batch removal size to the constant N ; otherwise, the underlying matricial structure could yield to a matrix having all its elements strictly positive. A reduction to the single arrival case would allow us to consider more general batch removal distributions. The investigation of optimal control problems for models of type $M/M^X/1$ could be the subject matter of any subsequent paper.

The paper is organized as follows. In Section 2 we describe a natural cost structure and we deduce conditions that imply the existence of an optimal stationary policy of threshold type under the expected total discounted cost criterion. In Section 3 we study the stability of the system and its stationary distribution under an arbitrary threshold policy. More specifically, we present an exact and an

asymptotic result for the computation of the stationary distribution in the general case. In Section 4 we also illustrate a more efficient way of computation for the single arrivals case. In Section 5 we prove a complete stochastic comparison characterization for models with identical parameters that operate under different thresholds. In Section 6 we examine the original control problem under the average cost criterion.

2. THE TOTAL DISCOUNTED COST MARKOV DECISION PROBLEM

We consider the system that we described in the introduction with the following cost structure: there is a running cost s per time unit (*i.e.* a cost that incurred at rate s whenever the server is busy) and a holding cost c per customer and time unit. Whenever the negative batch mechanism is in the on mode there is a cost d per time unit (this encompasses the cost of power for running the mechanism, labor costs for maintaining the mechanism etc.). In addition there exists a cost e per erased customer. Note that costs s , c and d are accumulated in a continuous manner, during a sojourn time (*i.e.* a transition interval). However, cost e per erased customer is charged at the end of a sojourn time in state i if the negative batch mechanism is on and a negative arrival has occurred.

Let $\{X(t)\}$ be the stochastic process that describes the evolution of the number of the customers in the system (state of the system). Its state space is the set of non-negative integers \mathbb{Z}_0^+ . Now whenever a transition occurs and state i is entered, we can take one of two possible control actions: set the batch removing mechanism on or off. Define

$$f = \begin{cases} 1 & \text{when the on mode is chosen,} \\ 0 & \text{when the off mode is chosen.} \end{cases} \quad (1)$$

We consider the problem of finding an optimal policy under the expected total discounted cost criterion. To this end we define $V_\beta^\pi(i)$ to be the total expected β -discounted cost under a given arbitrary policy π , starting from an initial state i . By well-known theorems on the expected total discounted cost criterion (see *e.g.* Ross [23]), we conclude that there exists an optimal stationary policy. We use the standard uniformization technique; for details see Serfozo [26]. We choose the uniformization rate $\gamma = \lambda + \mu + \nu$ and set $\alpha = \gamma/(\beta + \gamma)$, $p_k = \lambda g_k/\gamma$, $k \geq 1$, $p = \lambda/\gamma$, $q = \mu/\gamma$ and $r = \nu/\gamma$. Consider an arbitrary stationary policy $f = (f(i) : i = 0, 1, \dots)$. Then

$$V_\beta^f(i) = \frac{1}{\beta + \gamma} E \left[\sum_{k=0}^{\infty} \alpha^k c(X_k, f(X_k)) | X_0 = i \right], \quad (2)$$

where X_k is the corresponding uniformized discrete-time Markov decision process and $c(i, f)/(\beta + \gamma)$ is the expected discounted one-step cost when taking action f in state i in the uniformized model. Applying Serfozo [26] formulas about the cost

structure of the uniformized model we obtain

$$c(i, f) = \min(i, 1)s + ci + df + \min(i, N)evf. \tag{3}$$

The transition probabilities of the uniformized discrete-time Markov decision process $\{X_k\}$ are given by

$$\begin{aligned}
 p_{0j}(f) &= \begin{cases} p_k & \text{if } j = k, k \geq 1 \\ r + q & \text{if } j = 0, \end{cases} \\
 p_{ij}(f) &= \begin{cases} p_k & \text{if } j = i + k, k \geq 1 \\ r(1 - f) & \text{if } j = i \\ q & \text{if } j = i - 1 \\ rf & \text{if } j = 0, \end{cases} & 1 \leq i \leq N - 1, \\
 p_{ij}(f) &= \begin{cases} p_k & \text{if } j = i + k, k \geq 1 \\ r(1 - f) & \text{if } j = i \\ q & \text{if } j = i - 1 \\ rf & \text{if } j = i - N, \end{cases} & i \geq N. \tag{4}
 \end{aligned}$$

Therefore the problem reduces to find an optimal policy for the corresponding α -discounted discrete time model, that is to minimize

$$E \left[\sum_{k=0}^{\infty} \alpha^k c(X_k, f(X_k)) | X_0 = i \right].$$

The cost function $c(i, f)$ is unbounded due to its linear dependence on the queue length i . However since all costs are non-negative, the standard optimality equation for the total expected discounted cost criterion is applicable (see Ross [22], Sect. 6.4). Let $V_\alpha(i)$ be the minimum expected α -discounted cost when the initial state of the process is i . Then

$$V_\alpha(0) = \alpha \sum_{k=1}^{\infty} p_k V_\alpha(k) + \alpha(r + q)V_\alpha(0) + \min_{f \in \{0,1\}} [fd], \tag{5}$$

$$\begin{aligned}
 V_\alpha(i) &= ci + s + \alpha \sum_{k=1}^{\infty} p_k V_\alpha(i + k) + \alpha r V_\alpha(i) + \alpha q V_\alpha(i - 1) \\
 &\quad + \min_{f \in \{0,1\}} [f(d + iev - \alpha r(V_\alpha(i) - V_\alpha(0)))], \quad 1 \leq i \leq N - 1, \tag{6}
 \end{aligned}$$

$$\begin{aligned}
 V_\alpha(i) &= ci + s + \alpha \sum_{k=1}^{\infty} p_k V_\alpha(i + k) + \alpha r V_\alpha(i) + \alpha q V_\alpha(i - 1) \\
 &\quad + \min_{f \in \{0,1\}} [f(d + Nev - \alpha r(V_\alpha(i) - V_\alpha(i - N)))], \quad i \geq N. \tag{7}
 \end{aligned}$$

Let $f_\alpha^* = (f_\alpha^*(i))$ be an optimal stationary policy which attains the minimum in the right side of $V_\alpha(i)$. Set

$$\Delta V_\alpha(i) = V_\alpha(i) - V_\alpha((i - N)_+) = \begin{cases} V_\alpha(i) - V_\alpha(0) & \text{if } i \leq N - 1 \\ V_\alpha(i) - V_\alpha(i - N) & \text{if } i \geq N. \end{cases} \quad (8)$$

We can immediately conclude that the optimal control action at state i is defined by:

$$\begin{aligned} f_\alpha^*(0) &= 0 \\ f_\alpha^*(i) &= 1 \iff F_\alpha(i) < 0, \quad i \geq 1, \end{aligned} \quad (9)$$

where

$$F_\alpha(i) = d + \min(i, N)e\nu - \alpha r \Delta V_\alpha(i), \quad i \geq 0. \quad (10)$$

We are interested in finding conditions that assure that the optimal policy is of threshold type (also known as control-limit type). A stationary policy $f_\alpha^* = (f_\alpha^*(i))$ is said to be of threshold type if there exists a number i_α^* (the threshold) such that

$$f_\alpha^*(i) = 1 \iff i \geq i_\alpha^*.$$

We will use the so-called method of successive approximations of the value function $V_\alpha(i)$, *i.e.* we consider the sequence of the corresponding finite-horizon versions of the problem. We denote by $V_{n,\alpha}(i)$ the minimum n -step expected α -discounted cost, starting from state i . Similarly we define $\Delta V_{n,\alpha}(i) = V_{n,\alpha}(i) - V_{n,\alpha}((i - N)_+)$ and $F_{n,\alpha}(i) = d + \min(i, N)e\nu - \alpha r \Delta V_{n,\alpha}(i)$, for $i \geq 0$.

The proofs of this section use inductive arguments and so we need first to establish recursive relations for the quantities $V_{n,\alpha}(i)$, $\Delta V_{n,\alpha}(i)$ and $F_{n,\alpha}(i)$. Denote by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

the Kronecker's symbol. We have:

$$\begin{aligned} V_{0,\alpha}(i) &= 0, \quad i \geq 0, \\ V_{n+1,\alpha}(i) &= ci + s(1 - \delta_{i0}) + \alpha \sum_{k=1}^{\infty} p_k V_{n,\alpha}(i + k) + \alpha r V_{n,\alpha}(i) \\ &\quad + \alpha q V_{n,\alpha}((i - 1)^+) + \min[d + \min(i, N)e\nu - \alpha r \Delta V_{n,\alpha}(i), 0], \\ &\quad i \geq 0, \quad n \geq 0. \end{aligned} \quad (11)$$

For the difference $V_{n,\alpha}(i+1) - V_{n,\alpha}(i)$ of two successive terms of the value function we obtain the recursion:

$$V_{0,\alpha}(i+1) - V_{0,\alpha}(i) = 0, \quad i \geq 0,$$

$$\begin{aligned} V_{n+1,\alpha}(i+1) - V_{n+1,\alpha}(i) &= c + s\delta_{i0} + \alpha \sum_{k=1}^{\infty} p_k (V_{n,\alpha}(i+k+1) - V_{n,\alpha}(i+k)) \\ &\quad + \alpha r (V_{n,\alpha}(i+1) - V_{n,\alpha}(i)) + \alpha q (V_{n,\alpha}(i) - V_{n,\alpha}((i-1)^+)) \\ &\quad + \min[d + \min(i+1, N)e\nu - \alpha r \Delta V_{n,\alpha}(i+1), 0] \\ &\quad - \min[d + \min(i, N)e\nu - \alpha r \Delta V_{n,\alpha}(i), 0], \end{aligned}$$

$$i \geq 0, \quad n \geq 0. \quad (12)$$

For obtaining recursive relations for the function $\Delta V_{n,\alpha}$ we should consider two different cases for $i \leq N-1$ and $i \geq N$. We have:

$$\Delta V_{0,\alpha}(i) = 0, \quad i \geq 0,$$

$$\begin{aligned} \Delta V_{n+1,\alpha}(i) &= ci + s(1 - \delta_{i0}) + \alpha \sum_{k=1}^{\infty} p_k (V_{n,\alpha}(i+k) - V_{n,\alpha}(k)) \\ &\quad + \alpha r \Delta V_{n,\alpha}(i) + \alpha q \Delta V_{n,\alpha}((i-1)^+) \\ &\quad + \min[d + \min(i, N)e\nu - \alpha r \Delta V_{n,\alpha}(i), 0], \end{aligned}$$

$$i \leq N-1, \quad n \geq 0, \quad (13)$$

$$\begin{aligned} \Delta V_{n+1,\alpha}(i) &= cN + s\delta_{iN} + \alpha \sum_{k=1}^{\infty} p_k \Delta V_{n,\alpha}(i+k) \\ &\quad + \alpha r \Delta V_{n,\alpha}(i) + \alpha q \Delta V_{n,\alpha}(i-1) + \min[d + Ne\nu - \alpha r \Delta V_{n,\alpha}(i), 0] \\ &\quad - \min[d + \min(i-N, N)e\nu - \alpha r \Delta V_{n,\alpha}(i-N), 0], \end{aligned}$$

$$i \geq N, \quad n \geq 0. \quad (14)$$

Then for the differences $\Delta V_{n,\alpha}(i+1) - \Delta V_{n,\alpha}(i)$ of successive terms of the function $\Delta V_{n,\alpha}(i)$ we obtain the recursion:

$$\begin{aligned} \Delta V_{0,\alpha}(i+1) - \Delta V_{0,\alpha}(i) &= 0, \quad i \geq 0, \\ \Delta V_{n+1,\alpha}(i+1) - \Delta V_{n+1,\alpha}(i) &= c + s\delta_{i0} + \alpha \sum_{k=1}^{\infty} p_k (V_{n,\alpha}(i+k+1) - V_{n,\alpha}(i+k)) \\ &\quad + \alpha r (\Delta V_{n,\alpha}(i+1) - \Delta V_{n,\alpha}(i)) \\ &\quad + \alpha q (\Delta V_{n,\alpha}(i) - \Delta V_{n,\alpha}((i-1)^+)) \\ &\quad + \min[d + (i+1)e\nu - \alpha r \Delta V_{n,\alpha}(i+1), 0] \\ &\quad - \min[d + ie\nu - \alpha r \Delta V_{n,\alpha}(i), 0], \\ &\quad i \leq N-1, \quad n \geq 0, \end{aligned} \quad (15)$$

$$\begin{aligned} \Delta V_{n+1,\alpha}(i+1) - \Delta V_{n+1,\alpha}(i) &= -s\delta_{iN} + \alpha \sum_{k=1}^{\infty} p_k (\Delta V_{n,\alpha}(i+k+1) - \Delta V_{n,\alpha}(i+k)) \\ &\quad + \alpha r (\Delta V_{n,\alpha}(i+1) - \Delta V_{n,\alpha}(i)) \\ &\quad + \alpha q (\Delta V_{n,\alpha}(i) - \Delta V_{n,\alpha}(i-1)) \\ &\quad + [\min[d + Ne\nu - \alpha r \Delta V_{n,\alpha}(i+1), 0] \\ &\quad - \min[d + Ne\nu - \alpha r \Delta V_{n,\alpha}(i), 0]] \\ &\quad - [\min[d + \min(i+1-N, N)e\nu \\ &\quad - \alpha r \Delta V_{n,\alpha}(i+1-N), 0] \\ &\quad - \min[d + \min(i-N, N)e\nu - \alpha r \Delta V_{n,\alpha}(i-N), 0]], \\ &\quad i \geq N, \quad n \geq 0. \end{aligned} \quad (16)$$

Since we are interested in monotonicity and non-negativity properties of $F_{n,\alpha}(i)$, we will also use the relation

$$F_{n,\alpha}(i+1) - F_{n,\alpha}(i) = \begin{cases} e\nu - \alpha r (\Delta V_{n,\alpha}(i+1) - \Delta V_{n,\alpha}(i)) & \text{if } i \leq N-1 \\ -\alpha r (\Delta V_{n,\alpha}(i+1) - \Delta V_{n,\alpha}(i)) & \text{if } i \geq N \end{cases} \quad (17)$$

and the equivalences

$$F_{n,\alpha}(i+1) \leq F_{n,\alpha}(i) \iff \begin{cases} \Delta V_{n,\alpha}(i+1) - \Delta V_{n,\alpha}(i) \geq \frac{e\nu}{\alpha r} & \text{if } i \leq N-1 \\ \Delta V_{n,\alpha}(i+1) - \Delta V_{n,\alpha}(i) \geq 0 & \text{if } i \geq N \end{cases} \quad (18)$$

and

$$F_{n,\alpha}(i) \leq 0 \iff \Delta V_{n,\alpha}(i) \geq \frac{d + \min(i, N)e\nu}{\alpha r}. \quad (19)$$

We will also frequently use the elementary fact that for any $x, y \in \mathbb{R}$:

$$\min(x, 0) - \min(y, 0) \geq \min(x - y, 0). \quad (20)$$

We are now in position to prove the first proposition that states a monotonicity result about $V_{n,\alpha}(i)$.

Proposition 1. $V_{n,\alpha}(i)$ is non-decreasing in i , for all $n = 0, 1, \dots$

Proof. We use induction on n . For $n = 0$ the proposition is true trivially since $V_{0,\alpha}(i) = 0$ for all $i \geq 0$. Suppose that the proposition is true for some fixed n , i.e. $V_{n,\alpha}(i)$ is non-decreasing in i . Next we make use of (12) and (20) to get that

$$\begin{aligned} V_{n+1,\alpha}(i+1) - V_{n+1,\alpha}(i) &\geq c + s\delta_{i0} + \alpha \sum_{k=1}^{\infty} p_k (V_{n,\alpha}(i+k+1) - V_{n,\alpha}(i+k)) \\ &\quad + \alpha r (V_{n,\alpha}(i+1) - V_{n,\alpha}(i)) + \alpha q (V_{n,\alpha}(i) - V_{n,\alpha}((i-1)^+)) \\ &\quad + \min[\min(i+1, N)e\nu - \min(i, N)e\nu \\ &\quad - \alpha r (\Delta V_{n,\alpha}(i+1) - \Delta V_{n,\alpha}(i)), 0], \quad i \geq 0. \end{aligned} \quad (21)$$

We consider two cases. If the minimum on the right hand side of expression (21) is zero then $V_{n+1,\alpha}(i+1) - V_{n+1,\alpha}(i) \geq 0$ by the inductive hypothesis. In the other case, by replacing $\Delta V_{n,\alpha}(i)$ by $V_{n,\alpha}(i) - V_{n,\alpha}((i-N)^+)$, expression (21) is written as

$$\begin{aligned} V_{n+1,\alpha}(i+1) - V_{n+1,\alpha}(i) &\geq c + s\delta_{i0} + \alpha \sum_{k=1}^{\infty} p_k (V_{n,\alpha}(i+k+1) - V_{n,\alpha}(i+k)) \\ &\quad + \alpha r (V_{n,\alpha}((i+1-N)^+) - V_{n,\alpha}((i-N)^+)) \\ &\quad + \alpha q (V_{n,\alpha}(i) - V_{n,\alpha}((i-1)^+)) \\ &\quad + \min(i+1, N)e\nu - \min(i, N)e\nu, \end{aligned}$$

which is non-negative due to the inductive hypothesis and the inequality $\min(i+1, N)e\nu - \min(i, N)e\nu \geq 0$ which is obviously valid. \square

Since the method of successive approximations relies on inductive arguments, we need conditions that assure that the properties of the optimal policy (equivalently properties of $F_{n,\alpha}(i)$) are preserved for the various values of n , as $n \rightarrow \infty$. We have the following two lemmas:

Lemma 2. *If*

$$e\nu \leq \min \left\{ \frac{\alpha r(c+s)}{1-\alpha r}, \frac{\alpha r c}{1-\alpha(1-p)} \right\} \quad (22)$$

and for some $n_0 \geq 1$, $F_{n_0,\alpha}(i)$ is non-increasing in the domain $i \in \{0, \dots, N\}$ then $F_{n,\alpha}(i)$ is non-increasing in the domain $i \in \{0, \dots, N\}$ for all $n \geq n_0$.

Proof. We use induction on n . By the statement of the lemma we have that for $n = n_0$, $F_{n,\alpha}(i)$ is non-increasing in the domain $i \in \{0, \dots, N\}$. Suppose that it is valid for some fixed $n \geq n_0$.

Then by (15) we have that for $i \leq N - 1$

$$\begin{aligned} \Delta V_{n+1,\alpha}(i+1) - \Delta V_{n+1,\alpha}(i) &= c + s\delta_{i0} + \alpha \sum_{k=1}^{\infty} p_k (V_{n,\alpha}(i+k+1) - V_{n,\alpha}(i+k)) \\ &\quad + \alpha r (\Delta V_{n,\alpha}(i+1) - \Delta V_{n,\alpha}(i)) \\ &\quad + \alpha q (\Delta V_{n,\alpha}(i) - \Delta V_{n,\alpha}((i-1)^+)) \\ &\quad + \min(F_{n,\alpha}(i+1), 0) - \min(F_{n,\alpha}(i), 0). \end{aligned} \quad (23)$$

Note that $\sum_{k=1}^{\infty} p_k (V_{n,\alpha}(i+k+1) - V_{n,\alpha}(i+k)) \geq 0$ because of Proposition 1. Moreover, using (20) we have

$$\begin{aligned} \min(F_{n,\alpha}(i+1), 0) - \min(F_{n,\alpha}(i), 0) &\geq \min(F_{n,\alpha}(i+1) - F_{n,\alpha}(i), 0) \\ &= F_{n,\alpha}(i+1) - F_{n,\alpha}(i) \\ &= e\nu - \alpha r (\Delta V_{n,\alpha}(i+1) - \Delta V_{n,\alpha}(i)), \end{aligned}$$

because of the inductive hypothesis and (17). We consider two cases:

Case I: $1 \leq i \leq N - 1$. Then (23) implies that

$$\begin{aligned} \Delta V_{n+1,\alpha}(i+1) - \Delta V_{n+1,\alpha}(i) &\geq c + \alpha r (\Delta V_{n,\alpha}(i+1) - \Delta V_{n,\alpha}(i)) \\ &\quad + \alpha q (\Delta V_{n,\alpha}(i) - \Delta V_{n,\alpha}((i-1)^+)) \\ &\quad + e\nu - \alpha r (\Delta V_{n,\alpha}(i+1) - \Delta V_{n,\alpha}(i)) \\ &\geq c + \alpha q \frac{e\nu}{\alpha r} + e\nu, \end{aligned}$$

where the last inequality results from the inductive hypothesis and (18). By the Condition (22) $\left(e\nu \leq \frac{\alpha r c}{1 - \alpha(1-p)}\right)$ we obtain $\Delta V_{n+1,\alpha}(i+1) - \Delta V_{n+1,\alpha}(i) \geq \frac{e\nu}{\alpha r}$. But now (18) implies that $F_{n+1,\alpha}(i+1) \leq F_{n+1,\alpha}(i)$.

Case II: $i = 0$. In that case $\Delta V_{n+1,\alpha}(i) = 0$ and therefore

$$\begin{aligned} \Delta V_{n+1,\alpha}(1) - \Delta V_{n+1,\alpha}(0) &\geq c + s + \alpha r \Delta V_{n,\alpha}(1) + (e\nu - \alpha r \Delta V_{n,\alpha}(1)) \\ &= c + s + e\nu. \end{aligned}$$

By the Condition (22) $\left(e\nu \leq \frac{\alpha r(c+s)}{1 - \alpha r}\right)$ and (17) we easily find that $F_{n+1,\alpha}(1) \leq F_{n+1,\alpha}(0)$. \square

Lemma 3. *If*

$$d + Ne\nu \leq \min \left\{ \frac{\alpha r c N}{1 - \alpha}, \frac{\alpha r (cN + s)}{1 - \alpha(1 - q)} \right\} \quad (24)$$

and for some $n_0 \geq 1$, $F_{n_0,\alpha}(i) \leq 0$ for $i \geq N$ then for all $n \geq n_0$ we have $F_{n,\alpha}(i) \leq 0$ for $i \geq N$.

Proof. We use induction on n . By the statement of the lemma we have the result for $n = n_0$. Assume that it is valid for some $n \geq n_0$. Then by (14) we have

$$\begin{aligned} \Delta V_{n+1,\alpha}(N) &= cN + s + \alpha \sum_{k=1}^{\infty} p_k \Delta V_{n,\alpha}(N+k) + \alpha r \Delta V_{n,\alpha}(N) \\ &\quad + \alpha q \Delta V_{n,\alpha}(N-1) + \min(F_{n,\alpha}(N), 0) - \min(F_{n,\alpha}(0), 0). \end{aligned} \quad (25)$$

But $\Delta V_{n,\alpha}(N+k) \geq (d + Ne\nu)/\alpha r$, $k = 1, 2, \dots$ by the inductive hypothesis and (19). Moreover the inductive hypothesis implies that $\min(F_{n,\alpha}(N), 0) = F_{n,\alpha}(N) = d + Ne\nu - \alpha r \Delta V_{n,\alpha}(N)$. We have also that $\min(F_{n,\alpha}(0), 0) = 0$ and by Proposition 1 that $\alpha q \Delta V_{n,\alpha}(N-1) \geq 0$. Therefore (25) implies that

$$\Delta V_{n+1,\alpha}(N) \geq cN + s + \alpha \sum_{k=1}^{\infty} p_k \frac{d + Ne\nu}{\alpha r} + d + Ne\nu. \quad (26)$$

By the Condition (24) $\left(d + Ne\nu \leq \frac{\alpha r(cN+s)}{1-\alpha(1-q)}\right)$ we obtain

$$\Delta V_{n+1,\alpha}(N) \geq \frac{d + Ne\nu}{\alpha r}. \quad (27)$$

Now (19) and (27) imply that $F_{n+1,\alpha}(N) \leq 0$.

We now analyse $F_{n+1,\alpha}(i)$ for $i \geq N+1$. Using (14) we have

$$\begin{aligned} \Delta V_{n+1,\alpha}(i) &= cN + \alpha \sum_{k=1}^{\infty} p_k \Delta V_{n,\alpha}(i+k) + \alpha r \Delta V_{n,\alpha}(i) + \alpha q \Delta V_{n,\alpha}(i-1) \\ &\quad + \min(F_{n,\alpha}(i), 0) - \min(F_{n,\alpha}(i-N), 0). \end{aligned} \quad (28)$$

But $\Delta V_{n,\alpha}(i-1) \geq (d + Ne\nu)/\alpha r$ and $\Delta V_{n,\alpha}(i+k) \geq (d + Ne\nu)/\alpha r$, $k = 1, 2, \dots$, by the inductive hypothesis and (19). Moreover $\min(F_{n,\alpha}(i), 0) = F_{n,\alpha}(i) = d + Ne\nu - \alpha r \Delta V_{n,\alpha}(i)$ by the inductive hypothesis and $-\min(F_{n,\alpha}(i-N), 0) \geq 0$. Then, using Condition (24) $\left(d + Ne\nu \leq \frac{\alpha r cN}{1-\alpha}\right)$, (28) implies that

$$\begin{aligned} \Delta V_{n+1,\alpha}(i) &\geq cN + \alpha \sum_{k=1}^{\infty} p_k \frac{d + Ne\nu}{\alpha r} + \alpha q \frac{d + Ne\nu}{\alpha r} + d + Ne\nu \\ &= cN + \frac{d + Ne\nu}{r} \geq \frac{d + Ne\nu}{\alpha r}. \end{aligned}$$

Then by (19) we obtain that $F_{n+1,\alpha}(i) \leq 0$, $i \geq N+1$. \square

The conditions of Lemma 2 establish the monotonicity of $F_{\alpha}(i)$ in the domain $i \in \{0, \dots, N\}$. In contrast the conditions of Lemma 3 imply the non-positivity of $F_{\alpha}(i)$ in the domain $i \in \{N, N+1, \dots\}$. By combining Lemmas 2 and 3 we obtain the following.

Corollary 4. *If the Condition (22) holds and the strict inequality is given in Condition (24), and for some $n_0 \geq 1$ we have that $F_{n_0, \alpha}(i)$ is non-increasing in the domain $i \in \{0, \dots, N\}$ and $F_{n_0, \alpha}(i) \leq 0$ for $i \geq N$ then*

$$F_\alpha(N) \leq F_\alpha(N-1) \leq \dots \leq F_\alpha(1) \leq F_\alpha(0) \quad (29)$$

and

$$F_\alpha(i) < 0, \quad i \geq N. \quad (30)$$

Proof. The conditions of Lemmas 2 and 3 are valid, so we conclude that for all $n \geq n_0$ $F_{n, \alpha}(i)$ is non-increasing in the domain $i \in \{0, \dots, N\}$ and $F_{n, \alpha}(i) \leq 0$, for $i \geq N$. It is also known that $V_{n, \alpha}(i) \rightarrow V_\alpha(i)$ as $n \rightarrow \infty$; hence we also obtain that $\Delta V_{n, \alpha}(i) \rightarrow \Delta V_\alpha(i)$ and $F_{n, \alpha}(i) \rightarrow F_\alpha(i)$ as $n \rightarrow \infty$ and we obtain (29) and $F_\alpha(i) \leq 0$ for $i \geq N$. To obtain (30) we have to prove that $F_\alpha(i) \neq 0$ for $i \geq N$.

To a contradiction, suppose there exists an $\tilde{i} \geq N$ such that $F_\alpha(\tilde{i}) = 0$, or equivalently $\Delta V_\alpha(\tilde{i}) = (d + Ne\nu)/\alpha r$. Then we observe that

$$\begin{aligned} \frac{d + Ne\nu}{\alpha r} = \Delta V_\alpha(\tilde{i}) &= cN + s\delta_{\tilde{i}N} + \alpha \sum_{k=1}^{\infty} p_k \Delta V_\alpha(\tilde{i} + k) + \alpha r \Delta V_\alpha(\tilde{i}) \\ &\quad + \alpha q \Delta V_\alpha(\tilde{i} - 1) + \min(d + Ne\nu - \alpha r \Delta V_\alpha(\tilde{i}), 0) \\ &\quad - \min(d + \min(\tilde{i} - N, N)e\nu - \alpha r \Delta V_\alpha(\tilde{i} - N), 0). \end{aligned}$$

But $d + Ne\nu - \alpha r \Delta V_\alpha(\tilde{i}) = 0$ and $-\min(d + \min(\tilde{i} - N, N)e\nu - \alpha r \Delta V_\alpha(\tilde{i} - N), 0) \geq 0$, so we obtain

$$\frac{d + Ne\nu}{\alpha r} \geq cN + s\delta_{\tilde{i}N} + \alpha \sum_{k=1}^{\infty} p_k \Delta V_\alpha(\tilde{i} + k) + d + Ne\nu + \alpha q \Delta V_\alpha(\tilde{i} - 1). \quad (31)$$

We distinguish the following two cases:

Case I: $\tilde{i} = N$. Then (31) and Condition (24) $\left(d + Ne\nu < \frac{\alpha r(cN + s)}{1 - \alpha(1 - q)}\right)$ imply that

$$\frac{d + Ne\nu}{\alpha r} \geq cN + s + \alpha \sum_{k=1}^{\infty} p_k \frac{d + Ne\nu}{\alpha r} + d + Ne\nu > \frac{d + Ne\nu}{\alpha r},$$

a contradiction.

Case II: $\tilde{i} \geq N + 1$. Then, similarly to case I, (31) and Condition (24) $\left(d + Ne\nu < \frac{\alpha r cN}{1 - \alpha}\right)$ imply that

$$\frac{d + Ne\nu}{\alpha r} \geq cN + \alpha \sum_{k=1}^{\infty} p_k \frac{d + Ne\nu}{\alpha r} + d + Ne\nu + \alpha q \frac{d + Ne\nu}{\alpha r} > \frac{d + Ne\nu}{\alpha r},$$

a contradiction. □

We are now in position to prove the following.

Theorem 5. *Under the infinite horizon total discounted cost criterion and the condition*

$$d \leq N(\alpha rc - e\nu) \quad (32)$$

there exists a threshold type optimal stationary policy of the form

$$f_\alpha^*(i) = 1 \iff i \geq i_\alpha^*, \quad (33)$$

where $i_\alpha^* \in \{1, \dots, N\}$.

Proof. We first consider the one-step α -discounted problem. We have that

$$\begin{aligned} V_{1,\alpha}(i) &= ci + s(1 - \delta_{i0}), \\ \Delta V_{1,\alpha}(i) &= \begin{cases} ci + s(1 - \delta_{i0}) & \text{if } i \leq N - 1 \\ cN + s\delta_{iN} & \text{if } i \geq N \end{cases} \end{aligned}$$

and we obtain that

$$\begin{aligned} F_{1,\alpha}(0) &= d, \\ F_{1,\alpha}(i) &= d + ie\nu - \alpha r(ci + s), \quad 1 \leq i \leq N, \\ F_{1,\alpha}(i) &= d + Ne\nu - \alpha rcN, \quad i \geq N + 1. \end{aligned} \quad (34)$$

The Condition (32) implies that $\alpha rc - e\nu \geq 0$ so $F_{1,\alpha}(i)$ is non-increasing in the domain $i \in \{0, \dots, N\}$ and that $F_{1,\alpha}(i) \leq 0$ for $i \geq N$. Therefore we have the desired properties for the function $F_{n_0,\alpha}(i)$ of Corollary 4, for $n_0 = 1$. Note also that

$$\begin{aligned} (1 - \alpha r)e\nu < e\nu \leq \alpha rc \leq \alpha r(c + s) &\implies e\nu < \frac{\alpha r(c + s)}{1 - \alpha r}, \\ (1 - \alpha(1 - p))e\nu < e\nu \leq \alpha rc &\implies e\nu < \frac{\alpha rc}{1 - \alpha(1 - p)} \end{aligned}$$

i.e. (22) holds. Similarly we have

$$\begin{aligned} (d + Ne\nu)(1 - \alpha) < d + Ne\nu \leq \alpha rcN &\implies d + Ne\nu < \frac{\alpha rcN}{1 - \alpha}, \\ (d + Ne\nu)(1 - \alpha(1 - q)) < d + Ne\nu \leq \alpha r(cN + s) &\implies d + Ne\nu < \frac{\alpha r(cN + s)}{1 - \alpha(1 - q)}, \end{aligned}$$

and we conclude that (24) holds and the inequality is strict. Therefore Corollary 4 is applicable and we conclude that (29) and (30) hold. Moreover $F_\alpha(0) = d \geq 0$ and $F_\alpha(N) < 0$ so we conclude that there exists $i_\alpha^* \in \{1, \dots, N\}$ such that

$$F_\alpha(i) = \begin{cases} \geq 0 & \text{for } i = 0, \dots, i_\alpha^* - 1 \\ < 0 & \text{for } i = i_\alpha^*, i_\alpha^* + 1, \dots \end{cases}$$

and the optimal policy is of the form (33). \square

Remark 6. A careful examination of Lemmas 2 and 3 and the proof of Theorem 5 reveals that the Condition (32) is sufficient for having

$$F_{n,\alpha}(N) \leq F_{n,\alpha}(N-1) \leq \dots \leq F_{n,\alpha}(1) \leq F_{n,\alpha}(0) \quad (35)$$

$$F_{n,\alpha}(i) \leq 0, \quad i \geq N, \quad (36)$$

for all $n \geq 1$. On the other hand, if (35) and (36) hold for all $n \geq 1$, we have in particular $F_{1,\alpha}(N+1) \leq 0$ which implies (32). Thus, Condition (32) is necessary and sufficient for the validity of (35) and (36) for all $n \geq 1$. This shows that Condition (32) is in some sense minimal and cannot be weakened if one uses the method of successive approximations for deriving the above structural properties of the optimal function.

Remark 7. Another approach that seems more natural at first glance than ours is to try to find conditions that imply that $F_{n,\alpha}(i)$ is non-increasing in the domain $i \in \{0, 1, \dots\}$ for all $n \geq 1$. In the classical control problems in queueing (e.g. M/M/1 queue with controlled service rate, M/M/1 queue with controlled arrival rate, routing in a two station network etc.) that are summarized in Bertsekas [2] a similar argument is used. However, in the framework of the present model this approach does not work. The reason is that the function $F_{1,\alpha}(i)$ is not monotone as it readily seen by (34):

$$F_{1,\alpha}(0) \geq F_{1,\alpha}(1) \geq \dots \geq F_{1,\alpha}(N) \leq F_{1,\alpha}(N+1) = F_{1,\alpha}(N+2) = \dots$$

This singularity at $i = N$ for $n = 1$ is transferred to greater n 's and destroys completely the monotonicity behavior of $F_{n,\alpha}(i)$ for $i \geq N$. This is why it is needed to handle differently the cases $i \leq N$ and $i \geq N$ in the lines of Lemmas 2 and 3.

In the case where $s = 0$ this singularity disappears and we can prove that $F_{n,\alpha}(i)$ is non-increasing in i for all $n \geq 1$, under the condition $e\nu - \alpha r c \leq 0$.

Theorem 5 assures that for sufficiently small d the optimal policy is of threshold type. On the other extreme, that is when d becomes large, we have that the trivial off-policy is optimal.

Theorem 8. *If*

$$d \geq \frac{\alpha r(cN + s)}{1 - \alpha} \quad (37)$$

then $F_{n,\alpha}(i) \geq 0$ and $\Delta V_{n,\alpha}(i) \leq \frac{d}{\alpha r}$, for $i \geq 0$, $n \geq 0$. Hence the trivial off-policy $f_\alpha^(i) = 0$, $i \geq 0$, is optimal.*

Proof. By (34) we have that $F_{1,\alpha}(i) \geq d - \alpha r(cN + s)$ and $\Delta V_{1,\alpha}(i) \leq cN + s$, for $i \geq 0$. Using Condition (37) we can easily see that $F_{1,\alpha}(i) \geq 0$, and $\Delta V_{1,\alpha}(i) \leq d/\alpha r$, for $i \geq 0$, i.e. the result is valid for $n = 1$. Suppose that it is valid for some $n \geq 1$, i.e. $F_{n,\alpha}(i) \geq 0$, and $\Delta V_{n,\alpha}(i) \leq d/\alpha r$, for $i \geq 0$. We analyse the case for $n + 1$.

Case I: $i \leq N - 1$. By (13) we have that

$$\begin{aligned} \Delta V_{n+1,\alpha}(i) &= ci + s(1 - \delta_{i0}) + \alpha \sum_{k=1}^{\infty} p_k (V_{n,\alpha}(i+k) - V_{n,\alpha}(k)) \\ &\quad + \alpha r \Delta V_{n,\alpha}(i) + \alpha q \Delta V_{n,\alpha}((i-1)^+) \\ &\quad + \min(F_{n,\alpha}(i), 0), \quad i \leq N - 1. \end{aligned} \quad (38)$$

Note that because of Proposition 1 we have that

$$V_{n,\alpha}(i+k) - V_{n,\alpha}(k) \leq V_{n,\alpha}(i+k) - V_{n,\alpha}((i+k-N)^+) = \Delta V_{n,\alpha}(i+k)$$

and by the inductive hypothesis $\Delta V_{n,\alpha}(i) \leq \frac{d}{\alpha r} \leq \frac{d+N\epsilon\nu}{\alpha r}$ and $\min(F_{n,\alpha}(i), 0) = 0$. Hence (37) and (38) imply that

$$\begin{aligned} \Delta V_{n+1,\alpha}(i) &\leq cN + s + \alpha \left(\sum_{k=1}^{\infty} p_k + r + q \right) \frac{d}{\alpha r} \\ &= cN + s + \frac{d}{r} \leq \frac{d}{\alpha r}, \end{aligned}$$

and we obtain $\Delta V_{n+1,\alpha}(i) \leq \frac{d+\min(i,N)\epsilon\nu}{\alpha r}$, *i.e.* $F_{n+1,\alpha}(i) \geq 0$.

Case II: $i \geq N$. By (14) we have that

$$\begin{aligned} \Delta V_{n+1,\alpha}(i) &= cN + s\delta_{iN} + \alpha \sum_{k=1}^{\infty} p_k \Delta V_{n,\alpha}(i+k) + \alpha r \Delta V_{n,\alpha}(i) \\ &\quad + \alpha q \Delta V_{n,\alpha}(i-1) + \min(F_{n,\alpha}(i), 0) \\ &\quad - \min(F_{n,\alpha}(i-N), 0), \quad i \geq N. \end{aligned}$$

But by the inductive hypothesis and the Condition (37) we obtain as in Case I that $\Delta V_{n+1,\alpha}(i) \leq \frac{d}{\alpha r}$ so $F_{n+1,\alpha}(i) \geq 0$. \square

3. MODEL STABILITY AND STATIONARY DISTRIBUTION

In this section we study the stability (*i.e.* the positive recurrence) of a model that operates under a given threshold policy and its stationary distribution. Let i^* be the given threshold. Then the infinitesimal rates q_{ij} of the process $\{X(t)\}$ that describes the evolution of the number of customers in the system are given by

$$q(i, j) = \begin{cases} \lambda g_{j-i} & \text{if } i \geq 0, j \geq i+1 \\ \mu & \text{if } i \geq 1, j = i-1 \\ \nu & \text{if } i \geq i^*, j = (i-N)^+ \\ 0 & \text{otherwise.} \end{cases} \quad (39)$$

Proposition 9. *The queueing model with transition rates (39) is positive recurrent if and only if*

$$\lambda\bar{g} < \mu + \nu N. \quad (40)$$

Proof. Let $\{\hat{X}_n : n = 0, 1, \dots\}$ be the embedded discrete-time Markov chain of $\{X(t)\}$ at its transition epochs. Let also q_i be the rate of the exponential distribution that governs the length of a sojourn time in state i for the process $\{X(t)\}$. We observe that $\sup_{i \geq 0} \{q_i\} = \lambda + \mu + \nu < \infty$ and $\inf_{i \geq 0} \{q_i\} = \lambda > 0$. Thus, $\{X(t)\}$ is positive recurrent if and only if $\{\hat{X}_n\}$ is positive recurrent. To investigate the positive recurrence of $\{\hat{X}_n\}$ we use the classical criteria based on the mean drifts of $\{\hat{X}_n\}$.

Sufficiency of Condition (40) for positive recurrence:

a sufficient condition for the ergodicity of $\{\hat{X}_n\}$ is the existence of a non-negative function $f(i)$, $i \in S = \{0, 1, \dots\}$ (known as Lyapunov function), a positive number $\epsilon > 0$ and a finite subset $H \subseteq S$ such that the mean drift $\gamma_i = E[f(\hat{X}_{n+1})|\hat{X}_n = i] - f(i)$ is finite for all $i \in S$ and $\gamma_i \leq -\epsilon$ for all $i \notin H$ (Foster's criterion). Let us consider $f(i) = i$, $i \in S$. Then

$$\begin{aligned} \gamma_0 &= \sum_{k=1}^{\infty} \frac{\lambda g_k}{\lambda} k = \bar{g} > 0 \\ \gamma_i &= \sum_{k=1}^{\infty} \frac{\lambda g_k}{\lambda + \mu} (i + k) + \frac{\mu}{\lambda + \mu} (i - 1) - i = \frac{\lambda\bar{g} - \mu}{\lambda + \mu}, \quad 1 \leq i \leq i^* - 1 \\ \gamma_i &= \sum_{k=1}^{\infty} \frac{\lambda g_k}{\lambda + \mu + \nu} (i + k) + \frac{\mu}{\lambda + \mu + \nu} (i - 1) + \frac{\nu}{\lambda + \mu + \nu} (i - N)^+ - i \\ &= \frac{\lambda\bar{g} - \mu - \min(i, N)\nu}{\lambda + \mu + \nu}, \quad i \geq i^*. \end{aligned}$$

Now we can take $H = \{0, 1, \dots, N-1\}$ and $\epsilon = \frac{1}{2} \frac{\nu N + \mu - \lambda\bar{g}}{\lambda + \mu + \nu}$. Since $\gamma_i < -\epsilon$, $i \notin H$, by the Foster's criterion it follows that Condition (40) is sufficient for the ergodicity of $\{\hat{X}_n\}$.

Necessity of Condition (40) for positive recurrence:

We use Sennott *et al.* [25] Theorem 1 that states the following: let (\hat{p}_{ij}) be the transition probability matrix associated to an irreducible, aperiodic discrete-time Markov chain $\{\hat{X}_n\}$ with state-space S . If $E[\hat{X}_{n+1}|\hat{X}_n = i] - i$ is finite for all $i \in S$, $\delta_i = \sum_{j \leq i} (j - i)\hat{p}_{ij} \geq -c$ (a constant) for all $i \in S$ and there exists N such that $E[\hat{X}_{n+1}|\hat{X}_n = i] - i \geq 0$ for all $i \geq N$ then \hat{X}_n is not ergodic.

If $\lambda\bar{g} \geq \mu + \nu N$ then $\gamma_i \geq 0$ for all i . We also note that $\hat{p}_{ij} = 0$ for $j < i - N$. This guarantees that the condition on δ_i is satisfied so Sennott *et al.* theorem applies and we conclude that \hat{X}_n is not ergodic. Therefore Condition (40) is necessary for positive recurrence. \square

If we reduce to the single arrival case (*i.e.*, $g_1 = 1$, $g_k = 0$ for $k \geq 2$), then it is possible to check that the stability Condition (40) agrees with appropriate

particularizations of the stability results given by Gelenbe and Schassberger [12] in the context of G-networks and by Gelenbe *et al.* [11] for single node queues with more general descriptions of arrival and service processes.

Suppose now that the stability Condition (40) holds. We are interested in determining the stationary distribution $(\pi_i : i = 0, 1, \dots)$ of the model. The balance equations for the model are

$$\lambda\pi_0 = \mu\pi_1 + \nu \sum_{i=i^*}^N \pi_i$$

$$(\lambda + \mu + \nu 1[i \geq i^*])\pi_i = \lambda \sum_{k=0}^{i-1} g_{i-k}\pi_k + \mu\pi_{i+1} + \nu\pi_{i+N} 1[i + N \geq i^*], \quad i \geq 1. \quad (41)$$

We introduce the generating functions $\Pi(z) = \sum_{i=0}^{\infty} \pi_i z^i$ and $G(z) = \sum_{i=1}^{\infty} g_i z^i$, $|z| \leq 1$. As it will be apparent below, the computation of $\Pi(z)$ requires the study of the function

$$D(z) = \lambda(1 - G(z))z^N + \mu(z - 1)z^{N-1} + \nu(z^N - 1). \quad (42)$$

Lemma 10. *If the stability Condition (40) holds, then the equation $D(z)$ has N zeros z_1, z_2, \dots, z_N in the unit disk $\{z : |z| \leq 1\}$. Order them so as $|z_1| \leq |z_2| \leq \dots \leq |z_N|$. Then $z_N = 1$ and $|z_r| < 1$, for $1 \leq r \leq N - 1$.*

Proof. We have that

$$D(z) = 0 \Leftrightarrow z^N = \frac{\mu z^{N-1} + \nu}{\lambda(1 - G(z)) + \mu + \nu}. \quad (43)$$

To investigate the roots of this equation we will use Rouché’s theorem: if $f(z)$ and $g(z)$ are analytic functions of z inside and on a closed contour \mathcal{C} on the complex z -plane and if $|g(z)| < |f(z)|$ on \mathcal{C} , then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside \mathcal{C} .

Take $f(z) = z^N$ and $g(z) = -(\mu z^{N-1} + \nu)/(\lambda(1 - G(z)) + \mu + \nu)$. Note that

$$-g(z) = \frac{\mu z^{N-1} + \nu}{\mu + \nu} \frac{\mu + \nu}{\lambda(1 - G(z)) + \mu + \nu}. \quad (44)$$

The first factor in (44) is the probability generating function of a random variable with probability masses $\frac{\mu}{\mu + \nu}$ at $N - 1$ and $\frac{\nu}{\mu + \nu}$ at 0 respectively, while the second factor is the probability generating function of a random variable representing the number of events of a compound Poisson process with rate λ and group-size distribution (g_k) during an exponentially distributed interval with rate $\mu + \nu$. Hence $-g(z)$ is a probability generating function and we have that

$$-g(z) = \sum_{k=0}^{\infty} a_k z^k, \quad (45)$$

with $a_k \geq 0, k = 0, 1, \dots$. The mean $\sum_{k=0}^{\infty} k a_k$ of the underlying distribution is obviously $\frac{\mu}{\mu+\nu}(N-1) + \lambda \bar{g} \frac{1}{\mu+\nu}$, because of the above interpretation of the factors in (44). Consider the function $h(z) = z^N - \sum_{k=0}^{\infty} a_k z^k$. Then $h(1) = 0$ and $h'(1) = N - \frac{\mu(N-1) + \lambda \bar{g}}{\mu+\nu}$. The stability Condition(40) implies that $h'(1) > 0$ and we conclude that for sufficient small $\epsilon > 0$ we have $h(1 + \epsilon) > 0$, that is

$$\sum_{k=0}^{\infty} a_k (1 + \epsilon)^k < (1 + \epsilon)^N. \tag{46}$$

Consider the contour $\mathcal{C} = \{z : |z| = 1 + \epsilon\}$. Then for z on this contour we have

$$|g(z)| \leq \left| \sum_{k=0}^{\infty} a_k z^k \right| \leq \sum_{k=0}^{\infty} a_k |z|^k = \sum_{k=0}^{\infty} a_k (1 + \epsilon)^k < (1 + \epsilon)^N = |f(z)|,$$

using (45) and (46). Rouché’s theorem is applicable and we have that $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside \mathcal{C} . Clearly $f(z)$ has N zeros inside \mathcal{C} , so by (43) $D(z)$ has also N zeros inside \mathcal{C} . By letting $\epsilon \rightarrow 0$ we obtain that $D(z)$ has N zeros in $\{z : |z| \leq 1\}$. One of them is $z_N = 1$. There do not exist other zeros on the unit circle $\{z : |z| = 1\}$. Indeed consider a zero z_r with $|z_r| \leq 1$ and $z_r \neq 1$. Then we have $\text{Re}(G(z_r)) < 1$ which gives $\text{Re}(\lambda(1 - G(z_r))) > 0$ and we conclude that

$$|z_r| = \left| \frac{\mu z_r^{N-1} + \nu}{\lambda(1 - G(z_r)) + \mu + \nu} \right|^{1/N} < 1.$$

We have proved that $N - 1$ zeros are in the unit open disk $\{z : |z| < 1\}$ while another one is 1. □

We are now in position to obtain an exact expression for $\Pi(z)$. We consider two cases according to if $i^* < N$ or $i^* \geq N$.

Theorem 11. *If the stability Condition (40) holds and $i^* < N$ then*

$$\Pi(z) = \frac{\mu(z-1)z^{N-1}\pi_0 + \nu(z^N-1)\sum_{k=0}^{i^*-1}\pi_k z^k + \nu\sum_{k=i^*}^{N-1}\pi_k(z^N-z^k)}{\lambda(1-G(z))z^N + \mu(z-1)z^{N-1} + \nu(z^N-1)}. \tag{47}$$

The stationary probabilities $\pi_i, i = 0, \dots, N - 1$, are determined by solving the following $N \times N$ linear system of equations

$$\begin{aligned} \mu(z_r-1)z_r^{N-1}\pi_0 + \nu(z_r^N-1)\sum_{k=0}^{i^*-1}\pi_k z_r^k + \nu\sum_{k=i^*}^{N-1}\pi_k(z_r^N-z_r^k) &= 0, \\ r &= 1, \dots, N-1 \end{aligned} \tag{48}$$

and

$$\mu\pi_0 + \nu N \sum_{k=0}^{i^*-1}\pi_k + \nu \sum_{k=i^*}^{N-1}\pi_k(N-k) = \mu + \nu N - \lambda \bar{g}. \tag{49}$$

Proof. By multiplying the i th equation (41) by z^i and adding all of them we obtain after some algebra

$$\begin{aligned}
 (\lambda + \mu + \nu)\Pi(z) &= \lambda G(z)\Pi(z) + \mu \left(1 - \frac{1}{z}\right) \pi_0 + \frac{\mu}{z}\Pi(z) \\
 &+ \nu \sum_{k=0}^{i^*-1} \pi_k z^k + \nu \sum_{k=i^*}^N \pi_k - \frac{\nu}{z^N} \sum_{k=0}^N \pi_k z^k + \frac{\nu}{z^N}\Pi(z). \quad (50)
 \end{aligned}$$

Multiplying (50) by z^N and solving for $\Pi(z)$ results in (47).

We are now able to obtain the unknowns π_0, \dots, π_{N-1} . From Lemma 10 we have that the denominator $D(z)$ of $\Pi(z)$ in (47) has $N-1$ roots z_1, \dots, z_{N-1} in the open unit disk $\{z : |z| < 1\}$. These roots must also be roots of the numerator $N(z)$ because $\Pi(z)$ is analytic on the open unit disk. Hence we obtain the equations (48). Another equation (49) is provided by the normalization equation $\Pi(1) = 1$, using the L' Hopital rule. \square

Theorem 12. *If the stability Condition (40) holds and $i^* \geq N$ then*

$$\Pi(z) = \frac{\mu(z-1)z^{N-1}\pi_0 + \nu(z^N-1)\sum_{k=0}^{i^*-1} \pi_k z^k}{\lambda(1-G(z))z^N + \mu(z-1)z^{N-1} + \nu(z^N-1)}. \quad (51)$$

The stationary probabilities $\pi_i, i = 0, \dots, i^ - 1$ are determined by solving the $i^* \times i^*$ linear system of the equations*

$$\begin{aligned}
 \mu(z_r-1)z_r^{N-1}\pi_0 + \nu(z_r^N-1)\sum_{k=0}^{i^*-1} \pi_k z_r^k &= 0, \\
 r &= 1, \dots, N-1 \quad (52)
 \end{aligned}$$

$$\mu\pi_0 + \nu N \sum_{k=0}^{i^*-1} \pi_k = \mu + \nu N - \lambda\bar{g}, \quad (53)$$

and the balance equations (41) corresponding to indexes $i = 0, \dots, i^ - N - 1$.*

Proof. The proof is similar to the case $i^* < N$ so it is omitted. \square

Remark 13. For the exact computation of the stationary distribution when $i^* < N$, we first compute the initial $\pi_i, i = 0, \dots, N-1$ by solving the system (48), (49). Then the probabilities π_i , for $i \geq N$, are obtained recursively using the balance equations (41) for $i = 0, 1, \dots$. Similarly for the case $i^* \geq N$ we compute $\pi_i, i = 0, \dots, i^* - 1$ by solving the system integrated by (52)–(53) and the balanced equations associated to indexes $i = 0, \dots, i^* - N - 1$, and then the probabilities π_i for $i \geq i^*$ are obtained recursively using the balance equations (41) for $i = i^* - N, \dots$

Remark 14. The expression (51) agrees with formula (47), by noting that sums of the type $\sum_{k=i^*}^{N-1}$ are 0 for $i^* \geq N$. This enables us to treat the two cases $i^* \geq N$ and $i^* < N$ simultaneously, by using the formula (47).

We will now derive an asymptotic expression for the stationary probabilities of the model using the representation (47) of the generating function $\Pi(z)$.

Theorem 15. *Under the stability Condition (40), the equation $D(z) = 0$ has a unique real solution η in $(1, \infty)$. The stationary distribution (π_j) is asymptotically geometric with parameter η^{-1} . More specifically*

$$\pi_j \sim -\frac{N(\eta)}{\eta D'(\eta)} \eta^{-j}, \quad j \rightarrow \infty, \quad (54)$$

where $N(z)$ and $D(z)$ are the numerator and the denominator of $\Pi(z)$ in the representation (47).

Proof. We will use the following proposition of Tijms [29] (p. 453, Th. C.1). Let $P(z)$ be a probability generating function represented as

$$P(z) = \frac{N(z)}{D(z)}, \quad |z| \leq 1,$$

where $N(z)$ and $D(z)$ are analytic functions on a domain $\{z : |z| < R\}$, for some $R > 1$. Under the conditions: (i) $D(z) = 0$ has a real root z_0 in $(1, R)$; (ii) $D(z)$ has no zeros in the domain $\{z : 1 < |z| < z_0\}$; (iii) the zero $z = z_0$ of $D(z)$ is of multiplicity 1 and is the only zero of $D(z)$ on the circle $\{z : |z| = z_0\}$; we have that p_j is asymptotically of the form given in (54), as $j \rightarrow \infty$.

We will apply this proposition for $P(z) = \Pi(z)$. We have to check that the conditions (i)–(iii) are valid for the function $D(z)$ given by (42). Note that $D(z) = (\lambda + \mu + \nu)z^N D_1(z)$, where

$$D_1(z) = 1 - \frac{\lambda}{\lambda + \mu + \nu} G(z) - \frac{\mu}{\lambda + \mu + \nu} z^{-1} - \frac{\nu}{\lambda + \mu + \nu} z^{-N}.$$

Equivalently, we can check conditions (i)–(iii) for $D_1(z)$. Let $R = \sup\{|z| : |G(z)| < \infty\}$. Then for z with $1 < |z| < R$ we have that $|D_1(z)| < \infty$.

Set

$$f(x) = 1 - \frac{\lambda}{\lambda + \mu + \nu} G(x) - \frac{\mu}{\lambda + \mu + \nu} x^{-1} - \frac{\nu}{\lambda + \mu + \nu} x^{-N}, \quad x \in [1, R).$$

Then

$$\begin{aligned} f'(x) &= -\frac{\lambda}{\lambda + \mu + \nu} G'(x) + \frac{\mu}{\lambda + \mu + \nu} x^{-2} + \frac{N\nu}{\lambda + \mu + \nu} x^{-N-1}, \quad x \in [1, R), \\ f''(x) &= -\frac{\lambda}{\lambda + \mu + \nu} G''(x) - \frac{2\mu}{\lambda + \mu + \nu} x^{-3} - \frac{N(N+1)\nu}{\lambda + \mu + \nu} x^{-N-2}, \quad x \in [1, R). \end{aligned}$$

Hence $f''(x) < 0$ for $x \in [1, R)$ ($f'(x)$ decreasing, $f(x)$ concave). Moreover $f'(1) > 0$ (because of the stability Condition (40)) while $\lim_{x \rightarrow R^-} f'(x) = -\infty$ (because of the definition of R). Therefore there exists a unique $s \in (1, R)$ such that $f'(s) = 0$ in which the maximum of f occurs. Note that $f(s) > f(1) = 0$.

We have that f is strictly increasing in $(1, s)$ and $f(1) = 0$ so we conclude that there does not exist a zero of $f(x)$ in $(1, s)$. On the other hand, f is strictly decreasing in (s, R) and $f(s) \lim_{x \rightarrow R^-} f(x) < 0$, so we have that there exists a unique root $\eta \in (s, R)$ of $f(x)$. Therefore, η is the unique real root in $(1, R)$ of $D_1(z)$ and condition (i) is satisfied.

Since the coefficients of $G(z)$ are non-negative we have

$$|1 - D_1(z)| \leq \frac{\lambda}{\lambda + \mu + \nu} G(|z|) + \frac{\mu}{\lambda + \mu + \nu} |z|^{-1} + \frac{\nu}{\lambda + \mu + \nu} |z|^{-N} = 1 - f(|z|).$$

But f is strictly positive in $(1, \eta)$, hence for z with $1 < |z| < \eta$ we have $|1 - D_1(z)| < 1$ and we conclude that $D_1(z)$ does not have roots in the domain $\{z : 1 < |z| < \eta\}$, *i.e.* condition (ii) is satisfied.

For condition (iii), note that η has multiplicity 1, since the only root of $f'(x)$ is s (hence $D_1'(\eta) = f'(\eta) \neq 0$). Moreover η is the only zero of $D_1(z)$ on the circle $\{z : |z| = \eta\}$. Indeed, consider z with $|z| = \eta$ such that $D_1(z) = 0$, *i.e.*

$$\frac{\lambda}{\lambda + \mu + \nu} G(z) + \frac{\mu}{\lambda + \mu + \nu} z^{-1} + \frac{\nu}{\lambda + \mu + \nu} z^{-N} = 1.$$

Since the coefficients of $A(z)$ are all non-negative we have

$$\begin{aligned} 1 &= \frac{\lambda}{\lambda + \mu + \nu} \sum_{k=1}^{\infty} g_k \operatorname{Re}(z^k) + \frac{\mu}{\lambda + \mu + \nu} \operatorname{Re}(z^{-1}) + \frac{\nu}{\lambda + \mu + \nu} \operatorname{Re}(z^{-N}) \\ &\leq \frac{\lambda}{\lambda + \mu + \nu} G(\eta) + \frac{\mu}{\lambda + \mu + \nu} \eta^{-1} + \frac{\nu}{\lambda + \mu + \nu} \eta^{-N} = 1. \end{aligned}$$

Therefore we have $\operatorname{Re}(z^{-1}) = |z^{-1}| = \eta^{-1}$, *i.e.* z^{-1} is real and $z = \eta$, so the only zero of $D_1(z)$ with $|z| = \eta$ is $z = \eta$ and condition (iii) is satisfied. \square

The usefulness of the asymptotic expression is that enables us to use the easily computable approximation (54) for large j instead of computing π_j from the recursive scheme described in Remark 13.

4. THE SINGLE ARRIVALS CASE

The stationary distribution of the special case with single arrivals is not only asymptotically geometric as it was proved above in Theorem 15 but it is exactly geometric from a point and thereafter. In that case there is no need to compute the roots of the denominator of the probability generating function $\Pi(z)$ in the unit disk to obtain the stationary probabilities. More specifically we have the following.

Theorem 16. *In the case of single arrivals ($g_1 = 1$, $g_k = 0$ for $k \geq 2$) the stationary probabilities π_j are given by*

$$\pi_j = \begin{cases} cr_j, & 0 \leq j \leq i^* - 2 \\ c\eta^{-j} & j \geq i^* - 1, \end{cases} \quad (55)$$

where η is the asymptotic parameter of Theorem 15, i.e. η is the unique solution in $(1, \infty)$ of $D(z) = 0$, and r_j are computed recursively starting from

$$r_{i^*-2} = \frac{\eta^{-i^*+1}}{\lambda} \left(\mu + \nu \frac{1 - \eta^{1-N}}{\eta - 1} \right) \quad (56)$$

by the backward recursion for $j = i^* - 3, \dots, 0$:

$$r_j = \begin{cases} \frac{\mu}{\lambda} r_{j+1} & \text{if } N + j < i^* \\ \frac{\mu}{\lambda} r_{j+1} + \frac{\nu}{\lambda} \eta^{-i^*+1} \frac{1 - \eta^{i^*-N-j-1}}{\eta - 1} & \text{if } N + j \geq i^* \end{cases} \quad (57)$$

and

$$c = \left(\sum_{j=0}^{i^*-2} r_j + \frac{\eta^{-i^*+1}}{1 - \eta^{-1}} \right)^{-1}. \quad (58)$$

Proof. The balance equations (41) are equivalent to the cross balance equations which result by equating the rates between the sets of states $\{0, \dots, j\}$ and $\{j+1, \dots\}$ for all j :

$$\lambda \pi_j = \mu \pi_{j+1} + \nu \sum_{k=\max(j+1, i^*)}^{N+j} \pi_k, \quad j = 0, 1, \dots \quad (59)$$

For $j \geq i^* - 1$ by substituting π_j given by (55) in (59) we obtain

$$\lambda c \eta^{-j} = \mu c \eta^{-j-1} + \nu \sum_{k=j+1}^{N+j} c \eta^{-k}$$

which after some simplifications is reduced to $\lambda(1 - \eta)\eta^N + \mu(\eta - 1)\eta^{N-1} + \nu(\eta^N - 1) = 0$ which is valid from the definition of η .

For $j = i^* - 2$ by substituting π_j given by (55) in (59) we obtain

$$\lambda c r_{i^*-2} = \mu c \eta^{-i^*+1} + \nu \sum_{k=i^*}^{N+i^*-2} c \eta^{-k}$$

which reduced easily to (56).

For $j \leq i^* - 3$ by substituting π_j in (59) we obtain

$$\lambda c r_j = \mu c r_{j+1} + \nu \sum_{k=i^*}^{N+j} c \eta^{-k}.$$

But

$$\sum_{k=i^*}^{N+j} \eta^{-k} = \begin{cases} 0 & \text{if } N + j < i^* \\ \eta^{-i^*} + 1 \frac{1 - \eta^{i^* - N - j - 1}}{\eta - 1} & \text{if } N + j \geq i^* \end{cases}$$

and we arrive at (57). Finally the normalization equation yields (58). □

5. STOCHASTIC COMPARISON OF MODELS WITH DIFFERENT THRESHOLDS

In this section we study the stochastic domination of $M^X/M/1$ systems with a negative batch mechanism and identical parameters, operating under different threshold policies. Recall that if $P(t) = (p_{xy}(t) : x, y = 0, 1, \dots)$ and $P'(t) = (p'_{xy}(t) : x, y = 0, 1, \dots)$ are the families of the transition probability matrices of two continuous-time Markov chains, then P is said to be stochastically dominated by P' (denoted as $P \leq_{st} P'$) if $x \leq y$ implies that $p_{x.}(t) \leq_{st} p_{y.}(t)$ for all $t \geq 0$. This is equivalent to $(X(t)|X(0) = x) \leq_{st} (X'(t)|X'(0) = y)$ for every x and y with $x \leq y$. Note also that $P \leq_{st} P'$ implies that $\bar{\pi} \leq_{st} \bar{\pi}'$, where $\bar{\pi}, \bar{\pi}'$ are the stationary distributions of P and P' respectively (when both exist). We have the following.

Theorem 17. *Consider two $M^X/M/1$ systems with a negative batch mechanism and identical parameters $\lambda, \{g_k\}, \mu, \nu$ and N , operating under different threshold policies. Let $P = (P(t))$ and $P' = (P'(t))$ be their transition probability matrices corresponding to the thresholds I and I' respectively, $I < I'$. Then we have:*

- *Case I: $I = 1$ or $I' = \infty$. Then $P \leq_{st} P'$;*
- *Case II: $2 \leq I < I' < \infty$. Then $P \leq_{st} P' \iff (I' - I \geq N - 1)$ or $(I' - I = N - 2 \text{ and } \mu \geq \nu)$ or $(I' - I \leq N - 3 \text{ and } I = 2 \text{ and } \mu \geq \nu)$.*

Proof. We will apply the following characterization of the stochastic domination (see e.g. Stoyan [27]): let P and P' be transition probability matrices corresponding to the infinitesimal generators $Q = (q(x, y))$ and $Q' = (q'(x, y))$ respectively. Then $P \leq_{st} P'$ if and only if both of the following conditions hold:

- (i) $\sum_{z \geq w} q(x, z) \leq \sum_{z \geq w} q'(y, z)$, for every x, y, w with $x \leq y < w$;
- (ii) $\sum_{z \leq w} q(x, z) \geq \sum_{z \leq w} q'(y, z)$, for every x, y, w with $w < x \leq y$.

In the context of our model, we have that for $x \leq y < w$:

$$\sum_{z \geq w} q(x, z) = \sum_{k=w-x}^{\infty} \lambda g_k \text{ and } \sum_{z \geq w} q'(y, z) = \sum_{k=w-y}^{\infty} \lambda g_k.$$

Since $w - y \leq w - x$ we have $\sum_{z \geq w} q(x, z) \leq \sum_{z \geq w} q'(y, z)$, i.e. conditions (i) always hold.

For $w < x \leq y$ we have

$$\begin{aligned}\sum_{z \leq w} q(x, z) &= \mu 1[w = x - 1] + \nu 1[x \geq I] 1[w \geq x - N] \\ \sum_{z \leq w} q'(y, z) &= \mu 1[w = y - 1] + \nu 1[y \geq I'] 1[w \geq y - N].\end{aligned}$$

Case Ia: $I = 1$. We have that

$$1[x \geq I] = 1[x \geq 1] = 1 \geq 1[y \geq I'].$$

Moreover, $1[w = y - 1] = 1$ means that $w = y - 1$ which implies $y - 1 < x \leq y$. Then we obtain $x = y$ and we conclude that $1[w = x - 1] = 1$. Hence

$$1[w = x - 1] \geq 1[w = y - 1].$$

We also have that $1[w \geq y - N] = 1$ implies that $1[w \geq x - N] = 1$, because of $y \geq x$, and we obtain

$$1[w \geq x - N] \geq 1[w \geq y - N].$$

Therefore $\sum_{z \leq w} q(x, z) \geq \sum_{z \leq w} q'(y, z)$, *i.e.* conditions (ii) hold for all $w < x \leq y$ and we obtain that $P \leq_{st} P'$.

Case Ib: $I' = \infty$. We have that

$$1[x \geq I] \geq 0 = 1[y \geq I'],$$

and identically to Case Ia we have that $1[w = x - 1] \geq 1[w = y - 1]$ and $1[w \geq x - N] \geq 1[w \geq y - N]$. We obtain again that (ii) holds for all $w < x \leq y$ and $P \leq_{st} P'$.

For the Case II we will prove that

- a) $I' - I \geq N - 1 \implies P \leq_{st} P'$;
- b) $I' - I = N - 2, \mu \geq \nu \implies P \leq_{st} P'$;
- c) $I' - I = N - 2, \mu < \nu \implies P \not\leq_{st} P'$;
- d) $I' - I \leq N - 3, I \geq 3 \implies P \not\leq_{st} P'$;
- e) $I' - I \leq N - 3, I = 2, \mu \geq \nu \implies P \leq_{st} P'$;
- f) $I' - I \leq N - 3, I = 2, \mu < \nu \implies P \not\leq_{st} P'$.

These 6 subcases cover all possible relative values of I, I', μ and ν for Case II and show the claimed equivalence.

Case IIa: $I' - I \geq N - 1$. Consider $w < x \leq y$. Identically to Case Ia we have that

$$1[w = x - 1] \geq 1[w = y - 1].$$

Moreover $1[y \geq I']1[w \geq y - N] = 1$ means that $y \geq I'$ and $w \geq y - N$. But then we have that

$$x > w \geq y - N \geq I' - N \geq I - 1,$$

where the last inequality holds because of the condition $I' - I \geq N - 1$. Hence $x \geq I$. On the other hand we have that

$$w \geq y - N \geq x - N,$$

and we conclude that $1[x \geq I]1[w \geq x - N] = 1$. Therefore

$$1[x \geq I]1[w \geq x - N] \geq 1[y \geq I']1[w \geq y - N],$$

and conditions (ii) hold for all $w < x \leq y$. We obtain $P \leq_{st} P'$.

Case IIb: $I' - I = N - 2$ and $\mu \geq \nu$. Consider $w < x \leq y$. We consider two subcases: $w \leq x - 2$ (Case IIb-i) and $w = x - 1$ (Case IIb-ii).

IIb-i) Suppose that $w \leq x - 2$. Then

$$\mu 1[w = x - 1] = \mu 1[w = y - 1] = 0,$$

since $y - 1 \geq x - 1 > x - 2 \geq w$. Moreover $1[y \geq I']1[w \geq y - N] = 1$ means that $y \geq I'$ and $w \geq y - N$. But then we have that

$$x - 2 \geq w \geq y - N \geq I' - N = I - 2.$$

Hence $x \geq I$. We also have that $w \geq y - N \geq x - N$, and we conclude that $1[x \geq I]1[w \geq x - N] = 1$. Therefore

$$1[x \geq I]1[w \geq x - N] \geq 1[y \geq I']1[w \geq y - N],$$

and conditions (ii) hold for all $w < x \leq y$.

IIb-ii) Suppose that $w = x - 1$. If $x = y$ then we have

$$\mu 1[w = x - 1] = \mu 1[w = y - 1] = \mu,$$

while

$$1[x \geq I]1[w \geq x - N] \geq 1[x \geq I']1[w \geq x - N] = 1[y \geq I']1[w \geq y - N]$$

and conditions (ii) hold. If $x < y$ then $1[w = y - 1] = 0$ and we obtain

$$\begin{aligned} \sum_{z \leq w} q(x, z) &\geq \mu 1[w = x - 1] = \mu \geq \nu \\ &\geq \mu 1[w = y - 1] + \nu 1[y \geq I']1[w \geq y - N] = \sum_{z \leq w} q'(y, z), \end{aligned}$$

where the second inequality holds because of the condition $\mu \geq \nu$. Again conditions (ii) hold for all $w < x \leq y$ and we obtain $P \leq_{st} P'$ in either subcase.

Case IIc: $I' - I = N - 2$, $\mu < \nu$. It suffices to find a particular choice for x , y and w for which the conditions (ii) fail. Indeed, take $y = I'$, $x = I - 1$ and $w = I - 2$. Then

$$\sum_{z \leq w} q(x, z) = \mu \text{ and } \sum_{z \leq w} q'(y, z) \geq \nu.$$

Indeed the last equality holds since $y - w = I' - (I - 2) = N$ and therefore there exists $z \leq w$ with $q'(y, z) = \nu$. Hence for the choice $(x, y, w) = (I - 1, I', I - 2)$ condition (ii) fails and we obtain $P \not\leq_{st} P'$.

Case IIId: $I' - I \leq N - 3$, $I \geq 3$. Again it suffices to find a particular choice for x , y and w for which the conditions (ii) fail. In this case we take $y = I'$, $x = I - 1$ and $w = I - 3$. Then

$$\sum_{z \leq w} q(x, z) = 0 \text{ and } \sum_{z \leq w} q'(y, z) = \nu.$$

Here, the last inequality holds because $y - w = I' - (I - 3) \leq N$ and therefore there exists $z \leq w$ with $q'(y, z) = \nu$. For $(x, y, w) = (I - 1, I', I - 3)$ condition (ii) fails and we obtain $P \not\leq_{st} P'$.

Case IIe: $I' - I \leq N - 3$, $I = 2$, $\mu \geq \nu$. We consider 3 subcases: $x \geq I = 2$, $w \leq x - 2$ (Case IIe-i), $x \geq I = 2$, $w = x - 1$ (Case IIe-ii) and $x < I = 2$ (Case IIe-iii).

IIe-i) Suppose that $x \geq I = 2$, $w \leq x - 2$. Then

$$\mu 1[w = x - 1] = \mu 1[w = y - 1] = 0,$$

since $y - 1 \geq x - 1 > x - 2 \geq w$. Moreover $1[y \geq I'] 1[w \geq y - N] = 1$ easily implies $1[x \geq I] 1[w \geq x - N] = 1$. Indeed, we have that $w \geq y - N \geq x - N$ and $x \geq I$ by assumption and we conclude that

$$1[x \geq I] 1[w \geq x - N] \geq 1[y \geq I'] 1[w \geq y - N],$$

i.e. conditions (ii) hold.

IIe-ii) Suppose that $x \geq I = 2$, $w = x - 1$. Then

$$\mu 1[w = x - 1] = \mu \geq \mu 1[w = y - 1]$$

and $1[x \geq I] 1[w \geq x - N] \geq 1[y \geq I'] 1[w \geq y - N]$, as in IIe-i. Conditions (ii) hold.

IIe-iii) Suppose that $x < I = 2$. The only possible values for x and w are $x = 1$, $w = 0$ and we have that $\sum_{z \leq w} q(x, z) = q(1, 0) = \mu$.

If $y = x$ then we have that $\sum_{z \leq w} q'(y, z) = q'(1, 0) = \mu$ and condition (ii) holds. If $y > x$ then $1[w = y - 1] = 0$, hence

$$\sum_{z \leq w} q(x, z) = \mu \geq \nu \geq \sum_{z \leq w} q'(y, z),$$

i.e. condition (ii) holds.

Case III: $I' - I \leq N - 3$, $I = 2$, $\mu < \nu$. Take $(x, y, w) = (1, I', 0)$. Then $\sum_{z \leq w} q(x, z) = q(1, 0) = \mu$, while $\sum_{z \leq w} q'(y, z) = q'(I', 0) = \nu$. Condition (ii) fails since $\mu < \nu$ and we conclude that $\bar{P} \not\leq_{st} P'$. \square

The above theorem gives a complete characterization of the stochastic domination within the class of models with identical parameters that operate under different threshold policies.

6. THE AVERAGE COST MARKOV DECISION PROBLEM

In this section we consider the model with the same cost structure of Section 2, but under the average cost criterion. We use again the uniformization technique for reducing the original continuous time problem to a discrete control problem and stationary policies; for details see Serfozo [26]. Our approach relies on certain theorems that allow one to obtain an average cost optimal policy as the limit point of a sequence of total discounted cost optimal policies. This enables us to use the results that we have established in Section 2. Our standard reference in this section is Sennott [24] Chapter 7. For convenience we summarize the basic notions and the necessary results below.

A stationary policy f is said to be a limit point of a sequence of α_n -discounted optimal policies f_{α_n} with $\alpha_n \rightarrow 1$ if there exists a subsequence $\{\beta_n\}$ of $\{\alpha_n\}$ such that $\lim_{n \rightarrow \infty} f_{\beta_n} = f$. This means that for a given i and sufficiently large n (dependent on i), we have that $f_{\beta_n}(i) = f(i)$ (Sennott [24], Def. 7.2.2).

In a general context, several authors have reported conditions that assure the existence of an average cost optimal policy that can be obtained as a limit of discounted optimal policies. We will use Sennott's [24] 7.2 conditions:

- (SEN1) for $\alpha \in (0, 1)$, $(1 - \alpha)V_\alpha(z) < \infty$, for any distinguished state z ;
- (SEN2) there exists a non-negative function M such that $V_\alpha(i) - V_\alpha(z) \leq M(i)$ for all $i \geq 0$ and $\alpha \in (0, 1)$;
- (SEN3) there exists a non-negative constant L such that $-L \leq V_\alpha(i) - V_\alpha(z)$ for all $i \geq 0$ and $\alpha \in (0, 1)$.

Then we have the following basic result (Sennott [24], Th. 7.2.3(i) and (iii)): if $\{X_n\}$ is a Markov decision chain for which the (SEN) assumptions hold then

- there exists a finite constant $g = \lim_{\alpha \rightarrow 1^-} (1 - \alpha)V_\alpha(i)$ for $i \geq 0$;
- any limit point f of a sequence of discounted optimal policies is average cost optimal with average cost g .

The verification of (SEN1)-(SEN3) assumptions is in general difficult. In the case of our model we can exploit the monotonicity of the value function that has been proved in Proposition 1. Recall that a policy for a discrete-time Markov decision chain is said to be 0-standard if $m_{i0} < \infty$ and $c_{i0} < \infty$ for all states i , where m_{i0} is the expected first passage time from i to state 0 and c_{i0} is the expected cost incurred during such a passage time. Then the following result holds (Sennott [24], Corollary 7.5.4): if $\{X_n\}$ is a Markov decision chain with state space $\{0, 1, \dots\}$ and V_α is increasing in i for $\alpha \in (0, 1)$ and exists a 0-standard policy, then the (SEN) assumptions hold.

We will apply this result for our model. We have already proved that V_α is increasing in i in Proposition 1 so it remains to prove that there exists a 0-standard policy.

We first show that the value function under the total discounted cost criterion is always finite.

Theorem 18. *For every $\alpha \in (0, 1)$*

$$V_\alpha(i) < \infty, \quad i = 0, 1, \dots \quad (60)$$

Proof. Let

$$V_{f,\alpha}(i) = E \left[\sum_{k=0}^{\infty} \alpha^k c(X_k, f(X_k)) | X_0 = i \right]$$

be the minimum expected α -discounted cost under a stationary policy f for the uniformized model of Section 2, where $c(i, f)$ is given by (3). We have obviously that

$$V_{f,\alpha}(i) \leq c \sum_{k=0}^{\infty} \alpha^k E_f[X_k | X_0 = i] + \frac{M}{1-\alpha}, \quad (61)$$

where $M = d + s + Ne\nu$. The uniformized chain $\{X_k\}$ exhibits transitions to the right only when batch arrivals occur (see $p_{ij}(f)$ given by (4)). Thus

$$X_k \leq_{st} X_0 + \sum_{l=1}^k \xi_l \quad (62)$$

where ξ_1, ξ_2, \dots are independent, identically distributed random variables with $\Pr[\xi_l = 0] = r + q$ and $\Pr[\xi_l = j] = p_j = \lambda g_j / \gamma$ for $j = 1, 2, \dots$. Hence $E[\xi_l] = \lambda \bar{g} / \gamma$ and (62) yields

$$E_f[X_k | X_0 = i] \leq i + k \lambda \bar{g} / \gamma. \quad (63)$$

By (61) and (63) we obtain

$$V_{f,\alpha}(i) \leq c \sum_{k=0}^{\infty} \alpha^k (i + k \lambda \bar{g} / \gamma) + \frac{M}{1-\alpha} = \frac{ci + M}{1-\alpha} + \frac{c \lambda \bar{g} \alpha}{\gamma(1-\alpha)^2} < \infty,$$

and we conclude that $V_\alpha(i) = \inf_f V_{f,\alpha}(i) < \infty$ for $i \geq 0$. \square

We will now show that the policy $d(i) = 1 - \delta_{i0}$, $i \geq 0$, *i.e.* switch on the removing mechanism as soon as the queue is non empty, is 0-standard. Denote by $(\pi_d(k) : k = 0, 1, \dots)$ the stationary distribution that corresponds to the Markov chain induced by d and by $c(k) = c(k, d(k))$ the cost structure associated with d .

Lemma 19. *The discrete-time Markov chain induced by d is positive recurrent if and only if the stability Condition (40) holds. Let assume that the size of a batch arrival has second moment $\bar{g}_2 < \infty$. Then*

$$\sum_{k=0}^{\infty} c(k)\pi_d(k) < \infty. \tag{64}$$

Proof. It is well known that any uniformizable continuous-time Markov chain is positive recurrent if and only if the associated uniformized discrete-time Markov chain is positive recurrent. Moreover, both Markov chains have the same stationary distribution. Since (40) is the positive recurrence condition of the continuous-time Markov chain with threshold $i^* = 1$, we conclude that it is also the positive recurrence condition for the induced discrete-time Markov chain. We now observe that

$$\sum_{k=0}^{\infty} c(k)\pi_d(k) < c \sum_{k=0}^{\infty} k\pi_d(k) + s + d + Nev.$$

The quantity $\sum_{k=0}^{\infty} k\pi_d(k)$ is the mean queue length in the $M^X/M/1$ queue with a removing mechanism and threshold $i^* = 1$. However, the probability generating function of the stationary distribution of this model is of the form (47). The expected value $\sum_{k=0}^{\infty} k\pi_d(k) = \Pi'(1)$ is calculated using L'Hopital rule. In that process the second derivative of the denominator $D(z)$ is involved and consequently $G''(1) = \bar{g}_2 - \bar{g}$ arises. It is now clear that the finiteness condition $\bar{g}_2 < \infty$ guarantees that $\sum_{k=0}^{\infty} k\pi_d(k) < \infty$, hence $\sum_{k=0}^{\infty} c(k)\pi_d(k) < \infty$. \square

Lemma 20. *If the stability Condition (40) holds and $\bar{g}_2 < \infty$ then the policy d is 0-standard.*

Proof. Under the Condition (40) the discrete-time Markov chain induced by d is positive recurrent so all the expected first passage times m_{ij} are finite (see *e.g.* Sennott [24], Prop. C.1.4(v)). On the other hand, positive recurrence and the relation (64) give that the expected costs c_{ij} during first passage times are finite for all i, j (see *e.g.* Sennott [24], Prop. C.2.2(iv)). In particular $c_{i0} < \infty$ and $m_{i0} < \infty$ for all $i \geq 0$ and we obtain that the policy d is 0-standard. \square

We can now apply the aforementioned propositions of Sennott [24] and conclude the discussion about the average cost criterion for the $M^X/M/1$ model with the removing mechanism. We have the following

Theorem 21. *Suppose that the stability Condition (40) holds and $\bar{g}_2 < \infty$ and Condition (32) for existence of α -policy holds. Then there exists a threshold type stationary average cost optimal policy f^* of the form*

$$f^*(i) = 1 \iff i \geq i^*,$$

where $i^* \in \{1, \dots, N\}$.

Proof. Proposition 1 establishes that $V_{n,\alpha}(i)$ is non-decreasing in i for all n . Hence $V_\alpha(i) = \lim_{n \rightarrow \infty} V_{n,\alpha}(i)$ is non-decreasing for $\alpha \in (0, 1)$. In addition there exists an 0-standard policy (Lem. 20). Hence the conditions of Sennott [24] (Cor. 7.5.4) are fulfilled and we obtain that the SEN assumptions hold. Then by Sennott [24] (Th. 7.2.3) we have that there exists a finite constant $g = \lim_{\alpha \rightarrow 1^-} (1 - \alpha)V_\alpha(i)$ for $i \geq 0$, which is the minimum average cost.

Condition (32) is meaningful for every discount factor $\alpha \in [\alpha_0, 1)$, where $\alpha_0 = \frac{d+Ne\nu}{Nrc} \in (0, 1)$. Now, for every $\alpha \in [\alpha_0, 1)$ let f_α^* be an optimal threshold policy for the α -discounted problem with threshold $i_\alpha^* \in \{1, \dots, N\}$ (which exists from Th. 5 since Condition (32) holds). Take a sequence $\{\alpha_n\}$ with $\alpha_n \rightarrow 1$. The set of threshold policies with threshold in $\{1, \dots, N\}$ is finite. This means that there exists a subsequence $\{\beta_n\}$ of $\{\alpha_n\}$ such that $f_{\beta_n}^*(i) = f^*(i)$ for n sufficiently large, that is f^* is one of the f_α^* policies. Hence we have find a stationary policy f^* which is a limit point of a sequence of discounted optimal policies. Again by Sennott [24] (Th. 7.2.3) we have that f^* is average cost optimal with average cost g . \square

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