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Approximate roots, toric resolutions and deformations of a plane branch

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Abstract. We analyze the expansions in terms of the approximate roots of a Weierstrass polynomial $f \in \boldsymbol{C}\{x\}[y]$, defining a plane branch $(C, 0)$, in the light of the toric embedded resolution of the branch. This leads to the definition of a class of (non-equisingular) deformations of a plane branch $(C, 0)$ supported on certain monomials in the approximate roots of f, which are essential in the study of Harnack smoothings of real plane branches by Risler and the author. Our results provide also a geometrical approach to Abhyankar's irreducibility criterion for power series in two variables and also a criterion to determine if a family of plane curves is equisingular to a plane branch.

Introduction.

The use of *approximate roots* in the study of plane algebraic curves, initiated by Abhyankar and Moh in [**A-M**], was essential in the proof of the famous *embedding line theorem* in [**A-M2**]. Let $(C, 0) \subset (\boldsymbol{C}^2, 0)$ be a germ of analytically irreducible plane curve, a *plane branch* in what follows. Certain *approximate roots* of the Weierstrass polynomial defining $(C, 0)$ are *semi-roots*, i.e., they define *curvettes* at certain exceptional divisors of the minimal embedded resolution. A'Campo and Oka describe the embedded resolution of a plane branch by a sequence of toric modifications using approximate roots in [**A'C-Ok**] and give topological proofs of some of the results of Abhyankar and Moh. See [**Abh3**], [**PP**], [**G-P**], [**As-B**], [**Pi**] for an introduction to the notion of approximate root and its applications.

We consider canonical local coordinates at an infinitely near point of the toric embedded resolution, which are defined by the strict transform of a suitable approximate root (or more generally a *semi-root*) and the exceptional divisor. In Section 2 we introduce an injective correspondence between monomials in these coordinates and monomials in the approximate roots (see Proposition 2.4). From

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this natural correspondence we derive two applications.

The first application, given in Section 3, is based on the relations of the expansions in terms of semi-roots and Abhyankar's *straight line condition* for the *generalized Newton polygons* associated to a plane branch. These relations are better understood by passing through the toric embedded resolution of the branch (see Theorem 3.1 and Corollary 3.6). In particular, we prove that the generalized Newton polygons arise precisely from the Newton polygons of the strict transform of $(C, 0)$ at the infinitely near points of the toric embedded resolution of $(C, 0)$ (see Remark 3.9). We have revisited Abhyankar's irreducibility criterion for power series in two variables (see [**Abh4**]). We give a proof of Abhyankar's criterion by using the toric geometry tools we have previously introduced. As an application we obtain an algorithmic procedure to decide if family of plane curves is equisingular to a plane branch (see Algorithm 3.10). This procedure generalizes the criterion given by A'Campo and Oka in [**A'C-Ok**].

The second one is the definition of a class of (non equisingular) multi-parametric deformations $C_{\underline{t}}$ of the plane branch, which we call multi-semi-quasi-homogeneous (msqh). We explain its basic properties in Section 4. The terms appearing in this deformation are monomials in the semi-roots of f. The deformation may be seen naturally as a deformation of Teissier's embedding of the plane branch C in a higher dimensional affine space (see [**T2**]). If the deformation $C_{\underline{t}}$ is *generic* the Milnor number of $(C, 0)$ is related to the sum of the Milnor numbers of some curves defined from $C_{\underline{t}}$ at the infinitely near points of the toric resolution of $(C, 0)$ (see Proposition 4.6). As a consequence we obtain a formula for the Milnor number, which can be seen as a geometrical realization of the delta invariant of the singularity in terms of this class of deformations. In a recent joint work with Risler we apply this class of deformations in the study of the topological types of smoothings of real plane branches with the maximal number of connected components (see [**GP-R**]).

The paper is organized as follows: Section 1 introduce basic results and definitions. Section 4 only depends on Sections 1 and 2.

1. Plane branches, semi-roots and toric resolution.

See [**Z2**], [**W**], [**PP**], [**T2**], [**Abh3**], [**C**], [**Ca**], [**T3**], for references on singularities of algebraic or analytic curves.

NOTATION 1.1. The ring of formal (resp. convergent) power series in x, y is denoted by $\boldsymbol{C}[[x, y]]$ (resp. by $\boldsymbol{C}\{x, y\}$). The *Newton polygon* $\mathcal{N}(h)$ of a non zero series $h = \sum_{i,j} \alpha_{i,j} x^i y^j \in \boldsymbol{C}[[x, y]]$ is the convex hull of the set $\bigcup_{\alpha_{i,j} \neq 0} \{(i, j) + \boldsymbol{R}^2_{\geq 0}\}$. If $\Lambda \subset \boldsymbol{R}^2$ the *symbolic restriction* of h to Λ is the polynomial $\sum_{(i,j) \in \Lambda \cap \boldsymbol{Z}^2} \alpha_{i,j} x^i y^j$.

If $(C_i, 0) \subset (\boldsymbol{C}^2, 0)$, $i = 1, 2$ are plane curve germs defined by $h_i(x, y) = 0$, for $h_i \in \boldsymbol{C}\{x, y\}$, we denote by $(C_1, C_2)_0$ or by $(h_1, h_2)_0$ the intersection multiplicity $\dim_{\boldsymbol{C}} \boldsymbol{C}\{x, y\}/(h_1, h_2)$.

1.1. Expansions and approximate roots.

Abhyankar and Moh have applied and developed the expansions using approximate roots in the study of algebraic curves (see for instance [**A-M**], [**Abh4**], [**Abh2**], [**A-M2**]). See the surveys [**PP**], [**Pi**], [**A'C-Ok**], [**G-P**] on the applications of the approximate roots in the study of plane curves.

Let A be a integral domain. Let $H \in A[y]$ be a monic polynomial in y of degree $\deg H > 0$. Any polynomial $F \in A[y]$ has a unique *H-adic expansion* of the form:

$$F = a_s + a_{s-1} H + \cdots + a_1 H^{s-1} + a_0 H^s, \tag{1}$$

where $a_i \in A[y]$, $\deg a_i < \deg H$ and $s = [\deg F / \deg H]$. The symbol $[a]$ denotes the integral part of $a \in \boldsymbol{R}$. This expansion is obtained by iterated Euclidean division by H (see [**Z2**]).

Proposition 1.2 (see [**Abh2**] and [**PP**]). *Let $n_1, \ldots, n_g$ be integers > 1. If $F_1, \ldots, F_{g+1} \in A[y]$ are polynomials of degrees $1, n_1, n_1 n_2, \ldots, n_1 \cdots n_g$ respectively, then any polynomial $F \in A[y]$ has a unique expansion of the form*:

$$F = \sum_I \alpha_I F_1^{i_1} \cdots F_g^{i_g} F_{g+1}^{i_{g+1}}, \ \textit{with } \alpha_I \in A, \tag{2}$$

where the components of the index $I = (i_1, \ldots, i_{g+1})$ verify that $0 \leq i_1 < n_1, \ldots, 0 \leq i_g < n_g$, $0 \leq i_{g+1} \leq [\deg_y F / \deg_y F_{g+1}]$. Moreover, the degrees in y of the terms $F_1^{i_1} \cdots F_{g+1}^{i_{g+1}}$ are all distinct.

Proof. Consider the F_{g+1}-adic expansion, of the form (1), of the polynomial F. Iterate the procedure by taking recursively F_j-adic expansions of the coefficients obtained for $j = 1, \ldots, g$ in decreasing order. The assertion of the degrees in y is consequence of the following elementary property of the sequence of integers $(n_1, \ldots, n_g)$ (see [**PP**, proof of Corollary 1.5.4]). □

Remark 1.3. Let $n_1, \ldots, n_g$ be integers greater than 1. We set

$$\mathscr{A}_{g+1} := \{I = (i_1, \ldots, i_{g+1}) \mid 0 \leq i_1 < n_1, \ldots, 0 \leq i_g < n_g, 0 \leq i_{g+1}\}.$$

The map $\mathscr{A}_{g+1} \to \boldsymbol{Z}$, given by $I \mapsto q_I := i_1 + n_1 i_2 + \cdots + n_1 \cdots n_g i_{g+1}$, is injective.

Suppose that the integral domain A contains $\boldsymbol{Q}$. Denote by $\mathscr{B}_m \subset A[y]$ the set of monic polynomials of degree $m > 0$ in y. Let $F \in A[y]$ be a monic polynomial of degree N divisible by m. Suppose that $N = mk$ for some integer $k \geq 1$. The *Tschirnhausen operator* $\tau_F : \mathscr{B}_m \to \mathscr{B}_m$ is defined by $\tau_F(H) = H + a_1/k$ where a_1 is the coefficient of H^{k-1} in the H-adic expansion (1) of F (in this case notice that $s = k$ in (1) since $\deg H = m$). For instance, if $m = 1$, $H = y$ and $y' := y + a_1/N$, then the coefficient of $(y')^{N-1}$ in the y'-expansion of F is zero. Setting $y' = \tau_F(y)$ defines a change of coordinates, which is classically called the *Tschirnhausen transformation.*

DEFINITION 1.4. Let A a domain containing $\boldsymbol{Q}$. Let $F \in A[y]$ a monic polynomial of degree N and suppose $N = mk$. An approximate root G of degree m of the polynomial F is a monic polynomial in $A[y]$ such that $\deg(F - G^k) < N - m$.

The approximate root G of degree m of F exists and is unique. It is determined algorithmically in terms of Euclidean division of polynomials by: $G = \tau_F \circ \overset{(m)}{\cdots} \circ \tau_F(H)$, $\forall\, H \in \mathscr{B}_m$.

1.2. Local toric embedded resolution of a plane branch.

In this paper $(C, 0)$ denotes a germ of analytically irreducible plane curve, a *plane branch* for short, defined by an irreducible element in the ring $\boldsymbol{C}\{x, y\}$ of germs of holomorphic functions at the origin of $\boldsymbol{C}^2$. We recall the construction of a *local toric embedded resolution of singularities* of the plane branch $(C, 0)$ by a sequence of monomial maps. For a complete description see [**A'C-Ok**]. See [**Ok1**], [**Ok2**], [**L-Ok**], [**G-T**] for more on toric geometry and plane curve singularities.

We define a sequence of birational monomial maps $\pi_j : Z_{j+1} \to Z_j$, where Z_{j+1} is an affine plane $\boldsymbol{C}^2$ for $j = 1, \dots, g$, such that the composition $\Pi := \pi_1 \circ \cdots \circ \pi_g$ is a local *embedded resolution* of the plane branch $(C, 0)$, that is, Π is an isomorphism over $\boldsymbol{C}^2 \setminus \{(0, 0)\}$ and the *strict transform* C' of the plane branch C (defined as the closure of the pre-image by Π^{-1} of the punctured curve $C \setminus \{0\}$) is a smooth curve on Z_{g+1} which intersects the exceptional fiber $\Pi_1^{-1}(0)$ transversally. Notice that the map Π is not proper. The map Π can be seen as an affine chart of certain sequence of blow-ups of points.

We consider local coordinates (x, y) for $(\boldsymbol{C}^2, 0)$. We say that $y' \in \boldsymbol{C}\{x, y\}$ is *good* with respect to $(C, 0)$ and $\{x = 0\}$ if setting $(x_1, y_1) := (x, y')$ defines a pair of local coordinates at the origin and the germ $(C, 0)$ is defined by an equation $f = 0$ where,

$$f = \left(y_1^{n_1} - \theta_1 x_1^{m_1}\right)^{e_1} + \cdots, \tag{3}$$

in such a way that $\theta_1 \in \boldsymbol{C}^*$, $\gcd(n_1, m_1) = 1$ and the terms which are not written have exponents (i, j) such that $in_1 + jm_1 > n_1m_1e_1$, i.e., they lie above the compact edge $\Gamma_1 := [(0, n_1e_1), (m_1e_1, 0)]$ of the Newton polygon of f. Notice that $e_0 := e_1n_1$ is the intersection multiplicity of $(C, 0)$ with the line $\{x_1 = 0\}$.

Such a choice of y_1 is not unique. The choice $y_1 := y + \tau_f(y)$, defined by the *Tschirnhausen transformation*, is good with respect to $\{x_1 = 0\}$ and $(C, 0)$. We assume without loss of generality that f is a Weierstrass polynomial in y_1.

The vector $\vec{p}_1 = (n_1, m_1)$ is orthogonal to Γ_1 and defines a subdivision of the positive quadrant $\boldsymbol{R}^2_{\geq 0}$, which is obtained by adding the ray $\vec{p}_1\boldsymbol{R}_{\geq 0}$. The quadrant $\boldsymbol{R}^2_{\geq 0}$ is subdivided in two cones, $\tau_i := \vec{e}_i\boldsymbol{R}_{\geq 0} + \vec{p}_1\boldsymbol{R}_{\geq 0}$ for $i = 1, 2$ where $\{\vec{e}_1, \vec{e}_2\}$ is the canonical basis of $\boldsymbol{Z}^2$. We define the *minimal regular subdivision* Σ_1 of $\boldsymbol{R}^2_{\geq 0}$ which contains the ray $\vec{p}_1\boldsymbol{R}_{\geq 0}$ by adding the rays defined by those integral vectors in $\boldsymbol{R}^2_{>0}$, which belong to the boundary of the convex hull of the sets $(\tau_i \cap \boldsymbol{Z}^2)\backslash\{0\}$, for $i = 1, 2$. There is a unique cone $\sigma_1 = \vec{p}_1\boldsymbol{R}_{\geq 0} + \vec{q}_1\boldsymbol{R}_{\geq 0}$ in the subdivision Σ_1 such that $\vec{q}_1 = (c_1, d_1)$ satisfies that:

$$c_1m_1 - d_1n_1 = 1. \tag{4}$$

By convenience we denote $\boldsymbol{C}^2$ by Z_1, the coordinates (x, y) by (x_1, y_1) and the origin $0 \in \boldsymbol{C}^2 = Z_1$ by o_1. We also denote f by $f^{(1)}$ and C by $C^{(1)}$. The map $\pi_1 : Z_2 \to Z_1$ is defined by

$$\begin{aligned} x_1 &= u_2^{c_1} x_2^{n_1}, \\ y_1 &= u_2^{d_1} x_2^{m_1}, \end{aligned} \tag{5}$$

where u_2, x_2 are coordinates in the affine plane $Z_2 := \boldsymbol{C}^2$. The components of the exceptional fiber $\pi_1^{-1}(0)$ are $\{x_2 = 0\}$ and $\{u_2 = 0\}$. The pull-back of $C^{(1)}$ by π_1 is defined by $f^{(1)} \circ \pi_1 = 0$. The term $f^{(1)} \circ \pi_1$ decomposes as:

$$f^{(1)} \circ \pi_1 = \operatorname{Exc}(f^{(1)}, \pi_1)\, \bar{f}^{(2)}(x_2, u_2), \text{ where } \bar{f}^{(2)}(0, 0) \neq 0, \tag{6}$$

and $\operatorname{Exc}(f^{(1)}, \pi_1) := y_1^{e_0} \circ \pi_1 = u_2^{d_1e_0}x_2^{m_1e_0}$. The polynomial $\bar{f}^{(2)}(x_2, u_2)$ (resp. $\operatorname{Exc}(f^{(1)}, \pi_1)$) defines the *strict transform* $C^{(2)}$ of $C^{(1)}$ (resp. the *exceptional divisor*). By formula (3) we find that $\bar{f}^{(2)}(x_2, 0) = 1$, hence the exceptional line $\{u_2 = 0\}$ does not meet the strict transform. Since

$$\bar{f}^{(2)}(0, u_2) = \left(1 - \theta_1 u_2^{c_1m_1 - d_1n_1}\right)^{e_1} \overset{(4)}{=} (1 - \theta_1 u_2)^{e_1},$$

it follows that $\{x_2 = 0\}$ is the only component of the exceptional fiber of π_1

which intersects the strict transform $C^{(2)}$ of $C^{(1)}$, precisely at the point o_2 with coordinates $x_2 = 0$ and $u_2 = \theta_1^{-1}$ and with intersection multiplicity equal to e_1. If $e_1 = 1$ then the map π_1 is a local embedded resolution of the germ $(C, 0)$. If $e_1 > 1$ we consider a pair of coordinates (x_2, y_2) at the point o_2, with y_2 good for $\{x_2 = 0\}$ and $(C^{(2)}, o_2)$. It follows that $C^{(2)}$ is defined by a term, which we call *the strict transform function*, of the form:

$$f^{(2)}(x_2, y_2) = \left(y_2^{n_2} - \theta_2 x_2^{m_2}\right)^{e_2} + \cdots , \tag{7}$$

where $\theta_2 \in \boldsymbol{C}^*$, $\gcd(n_2, m_2) = 1$ and the terms which are not written have exponents (i, j) such that $in_2 + jm_2 > n_2 m_2 e_2$. Notice that $e_1 = e_2 n_2$.

We iterate this procedure defining for $j > 2$ a sequence of monomial birational maps $\pi_{j-1} : Z_j \to Z_{j-1}$, which are described by replacing the index 1 by $j-1$ and the index 2 by j above. In particular when we refer to a formula, like (4) at level j, we mean after making this replacement. We denote by $\mathrm{Exc}(f^{(1)}, \pi_1 \circ \cdots \circ \pi_j)$ the *exceptional function* defining the exceptional divisor of the pull-back of C by $\pi_1 \circ \cdots \circ \pi_j$. Notice that

$$\mathrm{Exc}\big(f^{(1)}, \pi_1 \circ \cdots \circ \pi_j\big) = \big(y_1^{e_0} \circ \pi_1 \circ \cdots \circ \pi_j\big) \cdots \big(y_j^{e_{j-1}} \circ \pi_j\big). \tag{8}$$

Since by construction we have that $e_j | e_{j-1} | \cdots | e_1 | e_0$ (for $|$ denoting divides), at some step we reach a first integer g such that $e_g = 1$ and then the process stops. The composition $\pi_1 \circ \cdots \circ \pi_g$ is a local toric embedded resolution of the germ $(C, 0)$.

REMARK 1.5. Given $e_0 = (x_1, f)_0$, the sequence of pairs $\{(m_j, n_j)\}_{j=1}^{g}$ determines and it is determined by the *characteristic pairs* or the *Puiseux exponents* of the plane branch $(C, 0)$, which are obtained when the line $\{x_1 = 0\}$ is not tangent to C at the origin (see [**A'C-Ok**] and [**Ok1**]). These pairs classify the *embedded topological type* of the germ $(C, 0) \subset (\boldsymbol{C}^2, 0)$, or equivalently its complex *equisingularity type.*

NOTATION 1.6. We set $n_0 := 1$. We denote by f_j' the approximate root of the polynomial $f \in \boldsymbol{C}\{x_1\}[y_1]$, of degree $n_0 \cdots n_{j-1}$ in y_1, for $j = 1, \ldots, g$. The integers n_i are those of Remark 1.5. We consider the sequence of intersection multiplicities given by:

$$\bar{b}_0 := e_0 = (x, f)_0, \ \ \bar{b}_j := (f_j', f)_0, \text{ for } j = 1, \ldots, g. \tag{9}$$

DEFINITION 1.7. A j^{th}-semi-root $(C_j, 0)$ of $(C, 0)$ with respect to the line $\{x_1 = 0\}$, is a germ $(C_j, 0)$ of curve such that $(C_j, C)_0 = \bar{b}_j$ and $(C_j, x_1)_0 =$

$n_0 \cdots n_{j-1}$, for $0 \leq j \leq g$. We convey that $C_{g+1} := C$. The sequence $\{(C_j, 0)\}_{j=1}^{g+1}$ is called the characteristic sequence of semi-roots of $(C, 0)$ with respect to $\{x_1 = 0\}$.

Remark 1.8. For simplicity we have defined semi-roots in terms of approximate roots, i.e., without passing by Abhyankar and Moh Theorem ([**A-M**]). For a definition of semi-roots in terms of Puiseux exponents and related results see [**PP**], for instance.

Notation 1.9. Let us fix a sequence of semi-roots $(C_j, 0)$ of the plane branch $(C, 0)$ with respect to $\{x_1 = 0\}$, for $j = 1, \ldots, g+1$. Each curve C_j is defined by a Weierstrass polynomial $f_j \in \boldsymbol{C}\{x_1\}[y_1]$ of degree $n_0 \cdots n_{j-1}$, which we call also *semi-root* by a slight abuse of terminology. We will assume that $f_1 = y_1$ and $f_{g+1} = f$.

Definition 1.10. Let us fix $2 \leq j \leq g$. A germ $(D, 0) \subset (\boldsymbol{C}^2, 0)$ is called a j^{th}*-curvette* for $(C, 0)$ and $\{x_1 = 0\}$ if it is analytically irreducible and the strict transform of D by $\pi_1 \circ \cdots \circ \pi_{j-1}$ is smooth and intersects transversally the exceptional divisor $\{x_j = 0\}$ at the point $o_j \in \{x_j = 0\}$. The branch $(D, 0)$ is a j^{th}*-curvette with maximal contact* if in addition the strict transform of $(D, 0)$ by $\pi_1 \circ \cdots \circ \pi_{j-1}$ is defined by $y'_j = 0$ where y'_j is good with respect to $\{x_j = 0\}$ and $(C^{(j)}, 0)$.

Proposition 1.11 (see [**Z1**], [**A'C-Ok**], [**PP**] and [**GP**, Section 3.4]).

(i) *If C_j is a j^{th}-semi-root of $(C, 0)$ with respect to $\{x_1 = 0\}$ then $(C_j, 0)$ is a j^{th}-curvette with maximal contact, for $j = 2, \ldots, g$.*

(ii) *We denote by $C_2^{(2)}, \ldots, C_g^{(2)}, C_{g+1}^{(2)} = C^{(2)}$ the strict transforms by the monomial map π_1 of the semi-roots $C_2, \ldots, C_g, C_{g+1} = C$ of the plane branch $(C, 0)$. The sequence $C_2^{(2)}, \ldots, C_{g+1}^{(2)}$ is a characteristic sequence of semi-roots of the branch $(C^{(2)}, o_2)$ with respect to the line $\{x_2 = 0\}$.*

Remark 1.12. We will assume in the rest of the paper that the local coordinate y_j, in the local embedded resolution of $(C, 0)$ introduced above, is the strict transform function of the semi-root f_j, for $j = 2, \ldots, g$ (we can do this by Proposition 1.11). This implies that y_j is of the form:

$$y_j = 1 - \theta_j u_j + x_j R_j(x_j, u_j) \text{ for some } R_j \in \boldsymbol{C}\{x_j, u_j\}. \tag{10}$$

As a consequence of Proposition 1.11 we have the following:

Remark 1.13.

(i) If $2 \leq j \leq g$ the Newton polygons of $f(x_1, y_1)$ and of $f_j^{e_{j-1}}(x_1, y_1)$ have only one compact edge Γ_1, defined in Section 1.2, and the symbolic restrictions of f and of $f_j^{e_{j-1}}$ coincide on this edge.

(ii) If $2 < j \leq g$ similar statement holds for $f^{(2)}(x_2, y_2)$ and of $(f_j^{(2)})^{e_{j-1}}(x_2, y_2)$ and Γ_2.

DEFINITION 1.14. The *semigroup of the plane branch* $(C, 0)$ is $\Lambda_C := \{(f, h)_0 \mid h \in \boldsymbol{C}\{x, y\} - (f)\}$.

The semigroup Λ_C is generated by the elements in the sequence (9). The sequence (9) is called the *characteristic sequence of generators* of the semigroup Λ_C with respect to the line $\{x_1 = 0\}$. If the line $\{x_1 = 0\}$ is not tangent to C at the origin then the set (9) is a minimal set of generators of the semigroup Λ_C and the notation, $\bar{\beta}_j$ instead of $\bar{b}_j$, is the usual one in the literature. The semigroup Λ_C has the following properties (see [**T2**], for instance).

LEMMA 1.15. *Any $\bar{b} \in \Lambda_C$ has a unique expansion of the form*:

$$\bar{b} = \eta_0 \bar{b}_0 + \eta_1 \bar{b}_1 + \cdots + \eta_g \bar{b}_g, \tag{11}$$

where $0 \leq \eta_0$ and $0 \leq \eta_j < n_j$, for $j = 1, \ldots, g$. The image of $\bar{b}_j$ in the group $\boldsymbol{Z}/(\sum_{i=0}^{j-1} \boldsymbol{Z}\bar{b}_i)$ is of order n_j. We have that:

$$n_j \bar{b}_j \in \boldsymbol{Z}_{\geq 0} \bar{b}_0 + \cdots + \boldsymbol{Z}_{\geq 0} \bar{b}_{j-1} \text{ and } n_j \bar{b}_j < \bar{b}_{j+1}, \text{ for } j = 1, \ldots, g. \tag{12}$$

The following proposition states some numerical relations between the sequences $\{(n_j, m_j)\}_{j=1}^{g}$ and $(\bar{b}_j)_{j=0}^{g}$ (see for instance [**GP**, Section 3.4]).

PROPOSITION 1.16. *We have that*

$$\begin{aligned} \left(x_j, f^{(j)}\right)_{o_j} &= e_{j-1} = n_j e_j \quad \text{and} \\ \left(y_j, f^{(j)}\right)_{o_j} &= \bar{b}_j - n_{j-1}\bar{b}_{j-1} = m_j e_j, \text{ for } 1 \leq j \leq g. \end{aligned}$$

The following proposition shows the relations between the characteristic sequences of generators of the semigroups of the plane branch $(C, 0)$ and of its semi-root C_{j+1}.

PROPOSITION 1.17. *Let C_{j+1} be a $(j+1)^{th}$-semiroot of the plane branch $(C, 0)$, for some $j = 1, \ldots, g$ (see Definition 1.7). The characteristic sequence of the semigroup of the plane branch C_{j+1} with respect to the line $\{x_1 = 0\}$ is equal to $(1/e_j)\bar{b}_0, \ldots, (1/e_j)\bar{b}_j$, for $j = 1, \ldots, g$ (see (9)).*

The *normalization map* $(\boldsymbol{C}, 0) \to (C, 0)$ of the branch $(C, 0)$, which is of the form $\tau \mapsto (x_1(\tau), y_1(\tau))$, may be defined explicitly in terms of a Newton Puiseux parametrization of the branch. If $h(x_1, y_1) \in \boldsymbol{C}\{x_1\}[y_1]$ defines a plane curve germ, we have that $(f, h)_0 = \mathrm{ord}_\tau(h(x_1(\tau), y_1(\tau)))$, where ord_τ denotes the τ-adic valuation of the field $\boldsymbol{C}((\tau))$ of Laurent series. We abuse the notation by denoting with the same letter the functions u_j, x_j and y_j and their images $u_j(\tau), x_j(\tau)$ and $y_j(\tau)$, induced by the normalization map, in the field $\boldsymbol{C}((\tau))$.

Lemma 1.18. *We have that* $\mathrm{ord}_\tau(u_{j+1}) = 0$ *and* $\mathrm{ord}_\tau(\mathrm{Exc}(f, \pi_1 \circ \cdots \circ \pi_j)) = e_{j-1}\bar{b}_j$ *for* $1 \le j \le g$.

Proof. Notice that $\mathrm{ord}_\tau(x_1) = (x_1, f)_0 = e_1 n_1$ and $\mathrm{ord}_\tau(y_1) = (y_1, f)_0 = e_1 m_1$. We deduce from (5) that $u_2 = x_1^{m_1} y_1^{-n_1}$. It follows that $\mathrm{ord}_\tau(u_2) = 0$. The equality $\mathrm{ord}_\tau(\mathrm{Exc}(f, \pi_1)) = e_0 \bar{b}_1$, follows from formula (8). We conclude the proof by an easy induction on j using Proposition 1.16 and formula (8). □

Example 1.19. A local embedded resolution of the real plane branch singularity $(C, 0)$ defined by $F = (y_1^2 - x_1^3)^3 - x_1^{10} = 0$ is as follows. The morphism π_1 of the toric resolution is defined by

$$x_1 = u_2^1 x_2^2,$$
$$y_1 = u_2^1 x_2^3.$$

We have that $f_2 := y_1^2 - x_1^3$ is a 2^{nd}-curvette for $(C, 0)$ and $\{x_1 = 0\}$. We have $f_2 \circ \pi_1 = u_2^2 x_2^6 (1 - u_2) = u_2^2 x_2^6 y_2$, where $y_2 := 1 - u_2$ defines the strict transform function of f_2, and together with x_2 defines local coordinates at the point of intersection o_2 with the exceptional divisor $\{x_2 = 0\}$. Notice in this case that the term R_2 in (10) is zero. For F we find that:

$$F \circ \pi_1 = u_2^6 x_2^{18} \big((1 - u_2)^3 - u_2^4 x_2^2 \big).$$

Hence $\mathrm{Exc}(F, \pi_1) := y_1^6 \circ \pi_1 = u_2^6 x_2^{18}$ is the exceptional function associated to F, and $F^{(2)} = y_2^3 - (1 - y_2)^4 x_2^2$ is the strict transform function. Comparing to (7) we see that $e_2 = 1$, $n_2 = 3$, $m_2 = 2$ and the restriction to $F^{(2)}(x_2, y_2)$ to the compact edge of its local Newton polygon is equal to $y_2^3 - x_2^2$. The map $\pi_2 : Z_3 \to Z_2$ is defined by $x_2 = u_3^2 x_3^3$ and $x_3 = u_3 x_3^2$. The composition $\pi_1 \circ \pi_2$ defines a local embedded resolution of $(C, 0)$.

2. Monomials in the semi-roots from the embedded resolution.

We keep notations of the previous section (cf. Notation 1.9 and Remark 1.5). For $2 \leq j \leq g$ we consider a sequence of integers of the form

$$0 \leq i_0, \quad 0 \leq i_1 < n_1, \ \ldots, \ 0 \leq i_{j-1} < n_{j-1}, \quad 0 \leq i_j < e_{j-1}.$$

Notice that by Proposition 1.2 the term

$$\mathscr{M} = x^{i_0} f_1^{i_1} f_2^{i_2} \cdots f_j^{i_j} \tag{13}$$

may appear in the $(f_1, f_2, \ldots, f_j)$-expansion of f. For any integer $2 \leq j \leq g$ we define below a map which associates to a monomial of the form,

$$x_j^r y_j^s, \text{ with } 0 \leq r, s < e_{j-1}$$

a monomial in $x, f_1, \ldots, f_j$ of the form (13). We study conditions for a term of the form (13) to appear in the $(f_1, \ldots, f_j)$-expansion of f. We use these ideas to analyze equisingular (and non equisingular) classes of deformations of the branch $(C, 0)$ in the following sections.

Remark 2.1. To avoid cumbersome notations if $2 \leq j \leq g+1$ we denote simply by u_i the term $u_i \circ \pi_{i-1} \circ \cdots \circ \pi_{j-1}$, whenever $i < j$ and the integer j is clear from the context. The function $u_i \circ \pi_{i-1} \circ \cdots \circ \pi_{j-1}$ has an expansion as a series in $\boldsymbol{C}\{x_j, y_j\}$ with non-zero constant term (see (10) at level $i < j$).

The following lemma is an elementary observation which is useful to motivate our results:

Lemma 2.2. *Given a monomial* $\mathscr{M} = x_1^{i_0} f_1^{i_1} f_2^{i_2} \cdots f_j^{i_j}$ *of the form* (13) *there exists unique integers* $r, s = i_j$ *and* $k_2, \ldots, k_j$ *such that*

$$u_2^{k_2} \cdots u_j^{k_j} x_j^r y_j^s = (\mathscr{M} \circ \pi_1 \circ \cdots \circ \pi_{j-1}) (\mathrm{Exc}(f, \pi_1 \circ \cdots \circ \pi_{j-1}))^{-1}. \tag{14}$$

The integer r *depends only on* $\mathscr{M}$ *and the sequences* $\{(n_i, m_i)\}_{i=1}^{j-1}$ *and* $\{e_i\}_{i=0}^{j-1}$. *The term* $u_j^{k_j} \cdots u_2^{k_2}$ *is a unit in* $\boldsymbol{C}\{x_j, y_j\}$.

Proof. By formulas (8) and (5) we have that $(\mathrm{Exc}(f, \pi_1))^{-1}(\mathscr{M} \circ \pi_1) = u_2^{k_2} x_2^{i'_0} y_2^{i_2} (f_3^{(2)})^{i_3} \cdots (f_j^{(2)})^{i_j}$ for some integer k_2 where $i'_0 = n_1 i_0 + m_1(-e_0 + i_1 + n_1 i_2 + \cdots + n_1 \cdots n_{j-1} i_j)$. By Remark 2.1 the term $u_2^{k_2}$ is a unit in the ring

$\boldsymbol{C}\{x_2, y_2\}$. The result is proved if $j = 2$. If $j > 2$ we find that $(\operatorname{Exc}(f, \pi_1 \circ \pi_2))^{-1}(\mathscr{M} \circ \pi_1 \circ \pi_2) = u_2^{k_2} u_3^{k_3} x_3^{i_0''} y_3^{i_3} (f_4^{(3)})^{i_4} \cdots (f_j^{(3)})^{i_j}$ for some integer k_3 where $i_0'' = n_2 i_0' + m_2(-e_1 + i_2 + n_2 i_3 + \cdots + n_2 \cdots n_{j-1} i_j)$. The assertion follows by an easy induction on j. □

REMARK 2.3. Notice that the condition $r \geq 0$ is not guaranteed by Lemma 2.2. See Example 2.9.

The following key proposition shows that given $(r, s) \in \boldsymbol{Z}_{\geq 0}$ with $s < e_{j-1}$ there is a unique way to determine a suitable monomial $\mathscr{M}_j(r, s)$ in x_1 and the semi-roots $y_1 = f_1, f_2, \ldots, f_j$, such that the composite $\mathscr{M}_j(r, s) \circ \pi_1 \circ \cdots \circ \pi_{j-1}$ is equal to the product of the exceptional divisor function $\operatorname{Exc}(f, \pi_1 \circ \cdots \circ \pi_{j-1})$ by the monomial $x_j^r y_j^s$ times a unit in the ring $\boldsymbol{C}\{x_j, y_j\}$.

PROPOSITION 2.4. *Let us fix a real plane branch* $(C, 0)$ *together with a local toric embedded resolution* $\pi_1 \circ \cdots \circ \pi_g$ *(cf. notations of Section* 1.2*). If* $2 \leq j \leq g$ *and* $(r, s) \in \boldsymbol{Z}^2_{\geq 0}$ *with* $s < e_{j-1}$ *then there exists unique integers*

$$0 < i_0, \quad 0 \leq i_1 < n_1, \ \ldots, \ 0 \leq i_{j-1} < n_{j-1}, \quad i_j = s, \tag{15}$$

and $k_2, \ldots, k_j > 0$ *such that* (14) *holds.*

Recall that the integers c_1, d_1 are defined by (4) in terms of the pair (m_1, n_1).

LEMMA 2.5. *If* $r \geq 0, l > 0$ *are integers there exist unique integers* k, i_0, i_1 *such that* $u_2^k x_2^r = ((x_1^{i_0} y_1^{i_1}) \circ \pi_1)(y_1^{l n_1} \circ \pi_1)^{-1}$ *with* $0 < i_0, k$ *and* $0 \leq i_1 < n_1$. *We have that*:

$$k = l + [c_1 r / n_1], \quad i_0 = k m_1 - r d_1 \text{ and } i_1 = c_1 r - n_1 [c_1 r / n_1]. \tag{16}$$

In particular, $i_1 = 0$ *if and only if* $r = p n_1$ *for some integer* p.

PROOF. By (5) we deduce that $u_2 = x_1^{m_1} y_1^{-n_1}$ and $x_2 = x_1^{-d_1} y_1^{c_1}$. The term

$$\left(y_1^{l n_1} \circ \pi_1\right) u_2^k x_2^r = \left(x_1^{k m_1 - r d_1} y_1^{r c_1 + (l-k) n_1}\right) \circ \pi_1$$

is the transform of a holomorphic monomial by π_1 if and only if:

$$0 \leq i_0' := k m_1 - r d_1 \text{ and } 0 \leq i_1' := r c_1 + (l - k) n_1,$$

or equivalently, $(d_1/m_1) r \leq k \leq (c_1/n_1) r + l$. By (4) we have that $m_1 c_1 -$

$d_1 n_1 = 1$. This implies that $d_1/m_1 < c_1/n_1$, thus the interval of the real line $[(d_1/m_1)r, (c_1 n_1)r + l]$ is of length greater than $l \geq 1$. Any integer k lying on this interval is convenient to define a holomorphic monomial. The condition $i_1' < n_1$, is equivalent to $(c_1/n_1)r + l - k < 1$, and it is verified if and only if $k = [(c_1/n_1)r + l] = l + [(c_1/n_1)r] > 0$. We denote the integers i_0' and i_1' corresponding to this choice of k by i_0 and i_1 respectively. We have that:

$$\begin{aligned} i_0 &= \left(\frac{c_1}{n_1} r + l\right) m_1 - r d_1 > \left(\frac{c_1}{n_1} r + l - 1\right) m_1 - r d_1 \\ &= r m_1 \left(\frac{c_1}{n_1} - \frac{d_1}{m_1}\right) + (l-1) m_1 \geq (l-1) m_1 \geq 0. \end{aligned}$$

For the last assertion, we have that $i_1 = c_1 r - n_1 [c_1 r / n_1] = 0$ if and only if n_1 divides r, since $\gcd(c_1, n_1) = 1$ by (5). ☐

LEMMA 2.6. *If* $(r,s) \in \boldsymbol{Z}_{\geq 0}$ *with* $s < e_1$ *there exist unique integers* k, i_0, i_1 *with* $0 < k, i_0$ *and* $0 \leq i_1 < n_1$ *such that:* $u_2^k x_2{}^r y_2{}^s = ((x_1{}^{i_0} y^{i_1} f_2^s) \circ \pi_1)(\mathrm{Exc}(f, \pi_1))^{-1}$. *These integers are*

$$k = e_1 - s + [c_1 r / n_1], \quad i_0 = k m_1 - r d_1, \ \textit{and} \ \ i_1 = c_1 r - n_1 [c_1 r / n_1]. \tag{17}$$

In particular, $i_1 = 0$ *if and only if* $r = p n_1$ *for some integer* p.

PROOF. We use that $\mathrm{Exc}(f, \pi_1) = y_1^n \circ \pi_1$ by (8) and that $f_2^s \circ \pi_1 = (y_1^{s n_1} \circ \pi_1) y_2{}^s$. Hence we deduce that $\mathrm{Exc}(f, \pi_1) y_2{}^s = (y_1^{n - s n_1} f_2^s) \circ \pi_1$. Since $s < e_1$ we have that $n - s n_1 = n_1(e_1 - s)$. Then we apply Lemma 2.5 for $r \geq 0$ and $l = e_1 - s > 0$. ☐

PROOF OF PROPOSITION 2.4. We prove the result by induction on the number g of monomial maps in the local toric embedded resolution, with respect to the line $\{x_1 = 0\}$. The case $g = 1$ is proved in Lemma 2.6. By induction using (8), we have that if $(r,s) \in \boldsymbol{Z}^2_{\geq 0}$ and if $s < e_{j-1}$ there exist unique integers $k_3, \ldots, k_j, i_0', i_2, \ldots i_j$ with $0 < i_0'$, $0 \leq i_2 < n_2$, $\ldots$, $0 \leq i_{j-1} < n_j$, $i_j = s$ such that

$$\begin{aligned} u_3^{k_3} \cdots u_j^{k_j} x_j^r y_j^s = \Big(&\big(x_2^{i_0'} y_2^{i_2} (f_3^{(1)})^{i_3} \cdots (f_j^{(1)})^{i_j}\big) \circ \pi_2 \circ \cdots \circ \pi_{j-1} \Big) \\ &\cdot \big(\mathrm{Exc}(f^{(2)}, \pi_2 \circ \cdots \circ \pi_{j-1})\big)^{-1}. \end{aligned} \tag{18}$$

We show that there exist unique integers $0 < k_2, i_0$ and $0 \leq i_1 < n_1$ such that

$$u_2^{k_2} x_2^{i_0'} y_2^{i_2} \big(f_3^{(2)}\big)^{i_3} \cdots \big(f_j^{(2)}\big)^{i_j} = \big(\big(x_1^{i_0}\, y_1^{i_1}\, f_2^{i_2} \cdots f_j^{i_j}\big) \circ \pi_1\big) \big(\mathrm{Exc}(f^{(1)}, \pi_1)\big)^{-1}. \quad (19)$$

By (8) we have that: $y_2^{i_2} (f_3^{(2)})^{i_3} \cdots (f_j^{(2)})^{i_j} = ((y_1^q f_2^{i_2} \cdots f_j^{i_j}) \circ \pi_1)\, (\mathrm{Exc}(f^{(1)}, \pi_1))^{-1}$, where the integer

$$\begin{aligned} q &:= n_1(e_1 - i_2 - n_2 i_3 - n_2 \cdots n_{j-1} i_j) \\ &= n_1(n_2(\cdots(n_{j-1}(e_{j-1} - i_j) - i_{j-1})\cdots) - i_2) \end{aligned} \quad (20)$$

is a positive multiple of n_1 by the inequalities (15). Then we apply Lemma 2.5. □

REMARK 2.7. Given the integer e_{j-1} and the pairs $(n_1, m_1), \ldots, (n_{j-1}, m_{j-1})$ then a pair (r, s) with $r \geq 0$ and $s < e_{j-1}$, and the integers (15) such that (14) holds, determine each other by Lemma 2.2 and the proof of Proposition 2.4.

DEFINITION 2.8. If $0 \leq r$ and if $0 \leq s < e_{j-1}$ we define a monomial in $x, f_1, \ldots, f_j$ by:

$$\mathscr{M}_j(r, s) := x^{i_0} f_1^{i_1} \cdots f_j^{i_j} \quad (21)$$

by relation (14) in Proposition 2.4. We use the notation $\mathscr{M}_1(r, s)$ for $x_1^r y_1^s$. We denote the term $f_j^{e_{j-1}}$ by $\mathscr{M}_j(0, e_{j-1})$. We denote the term $\mathscr{M}_j(r, s)$ by $\mathscr{M}_{j,f}(r, s)$ to emphasize the dependency with the series $f(x_1, y_1)$ defining the plane branch $(C, 0)$.

EXAMPLE 2.9. The following table indicates some terms $\mathscr{M}_2(r, s)$ in the case of Example 1.19.

(r, s)	$(0, 0)$	$(0, 1)$	$(0, 2)$	$(1, 1)$	$(1, 0)$
$\mathscr{M}_2(r, s)$	x_1^9	$x_1^6 f_2$	$x_1^3 f_2^2$	$x_1^5 y_1 f_2$	$x_1^8 y_1$

For instance, we have that $\mathscr{M}_2(1, 1) = x_1^5 y_1 f_2$, since $x_1^5 y_1 f_2 \circ \pi_1 = \mathrm{Exc}(F^{(1)}, \pi_1) u_2^2 x_2 y_2$, where $\mathrm{Exc}(F^{(1)}, \pi_1) = u_2^6 x_2^{18}$ by Example 1.19. Notice also that the analytic function $x_2 y_2 \mathrm{Exc}(F, \pi_1)$ on Z_2 is equal to $(x_1^{-1} y_1^5 f_2) \circ \pi_1$, i.e., it is the transform by π_1 of a *meromorphic* function. Both of the following formulas

$$y_1^6 \circ \pi_1 = \mathrm{Exc}(F^{(1)}, \pi_1) \text{ and } x_1^9 \circ \pi_1 = \mathrm{Exc}(F^{(1)}, \pi_1) u_2^3$$

seem to correspond to (14) in the case $(r, s) = (0, 0)$, however the term y_1^6 is not of the form prescribed by the inequalities (15), hence the first formula is not the one considered by Lemma 2.4.

LEMMA 2.10. *If* $0 \leq r$ *and* $s < e_{j-1}$*, we have that*:

$$(\mathscr{M}_j(r,s), f)_0 = e_{j-2}\bar{b}_{j-1} + re_{j-1} + s\,(\bar{b}_{j+1} - n_j\bar{b}_j), \quad \textit{for } j = 2, \ldots, g+1.$$

PROOF. By Lemma 2.4 we have that:

$$\mathscr{M}_j(r,s) \circ \pi_1 \circ \cdots \circ \pi_{j-1} = \mathrm{Exc}(f, \pi_1 \circ \cdots \circ \pi_{j-1})\, u_2^{k_2} \cdots u_j^{k_j} x_j^r y_j^s.$$

By Proposition 1.16 and Lemma 1.18 we deduce that:

$$(\mathscr{M}_j(r,s), f)_0 = \mathrm{ord}_\tau(\mathrm{Exc}(f, \pi_1 \circ \cdots \circ \pi_{j-1})) + re_{j-1} + s\,(\bar{b}_{j+1} - n_j\bar{b}_j). \qquad \square$$

LEMMA 2.11. *If* $0 \leq r$ *and* $0 \leq s < e_{j-1}$ *the Newton polygon of a term* $\mathscr{M}_j(r,s)$*, with respect to the coordinates* (x_1, y_1)*, is contained in the Newton polygon* $\mathscr{N}(f(x_1,y_1))$ *for* $2 \leq j \leq g+1$*. It is contained in the interior of* $\mathscr{N}(f(x_1,y_1))$ *unless* $j = 2$*,* $r = 0$ *and* $0 \leq s < e_1$*.*

PROOF. If $j = 2$ we have that $\mathscr{M}_2(r,s) = x_1^{i_0} y_1^{i_1} f_2^s$ by Lemma 2.6. The vector $\vec{v} := (i_0 + sm_1, i_1)$ is a vertex of the Newton polygon of $\mathscr{M}_2(r,s)$ and $\vec{w} := (\bar{b}_1, 0)$ is a vertex of the only compact edge Γ_1 of $\mathscr{N}(f(x_1,y_1))$. Notice that if $s = 0$ then Newton polygon of $\mathscr{M}_2(r,s)$ has only one compact face $\{\vec{v}\}$, otherwise it has only one compact edge which is parallel to Γ_1 (see Subsection 1.2). The vector $\vec{p}_1 = (n_1, m_1)$ is orthogonal to Γ_1 hence we deduce the inequality:

$$\begin{aligned} n_1\bar{b}_1 = \langle \vec{p}_1, \vec{w} \rangle \leq \langle \vec{p}_1, \vec{v} \rangle &= n_1 i_0 + sn_1 m_1 + m_1 i_1 \\ &= e_1 n_1 m_1 + r(m_1 c_1 - n_1 d_1) \overset{(4)}{=} n_1\bar{b}_1 + r, \end{aligned}$$

using (17). Equality holds in formula above if and only if $r = 0$.

If $j > 2$ we follow the proof of Proposition 2.4: there exist integers $0 < i_1 \leq n_1$, $0 < i_0', k_2$ such that (19) holds. By Lemma 2.5 we have that $k_2 = l + [c_1 i_0'/n_1]$ where the integer l is $l := e_1 - i_2 - n_2 i_3 - \cdots - n_2 \cdots n_{j-1} i_j$. The vector $\vec{v} := (i_0 + m_1(e_1 - l), i_1)$ is a vertex of $\mathscr{N}(\mathscr{M}_j(r,s))$. By the construction the Newton polygon of $\mathscr{M}_j(r,s)$ has at most one compact edge, which is in addition parallel to Γ_1. We deduce from a simple calculation using (16) that:

$$n_1\bar{b}_1 \leq \langle \vec{p}_1, \vec{v} \rangle = n_1\bar{b}_1 + i_0'. \tag{22}$$

By the proof of Proposition 2.4 we have that $i'_0 > 0$, hence the inequality (22) is strict. □

Remark 2.12. By induction using the same arguments as in Lemma 2.11 we check that if $1 \leq i < j$, $0 \leq r$, and $0 \leq s < e_{j-1}$ that the Newton polygon of $(\mathscr{M}_j(r,s) \circ \pi_1 \circ \cdots \circ \pi_{i-1})\, (\mathrm{Exc}\ (f, \pi_1 \circ \cdots \circ \pi_{i-1}))^{-1}$ with respect to the coordinates (x_i, y_i), is contained in $\mathscr{N}(f^{(i)})$. It is contained in the interior of $\mathscr{N}(f^{(i)})$ unless $j = i+1$, $r = 0$ and $0 \leq s < e_i$.

3. Irreducibility and equisingularity criterions.

Abhyankar's irreducibility criterion gives an affirmative answer to a question of Kuo mentioned in [**Abh4**]: "Can we decide the irreducibility of a power series $F(x,y)$ without blowing up and without getting into fractional power series ?" We have revisited the Abhyankar's criterion in the light of toric geometry methods. In particular, our proof explains that if F is irreducible, some information on the Newton polygons of the strict transform of F at the infinitely near points of the toric resolution can be read from the expansions in certain semi-roots of F. See [**C-M2**] and [**C-M1**], for an extension of this criterion to the case of base field of positive characteristic. As an application we give an equisingularity criterion for an equimultiple family of plane curves to be equisingular to a plane branch (See Section 3.3).

3.1. Straight line conditions in the toric resolution.

We consider a plane branch $(C,0)$ together with its local toric resolution. We keep notations of Section 1.2 (see also Notation 1.9). We give some precisions on the $(f_1,\ldots,f_j)$-expansion of f (see Proposition 1.2). We have that the $(f_1,\ldots,f_j)$-expansion of f is of the form:

$$f = f_j^{e_{j-1}} + \sum_{I=(i_1,\ldots,i_j)} \alpha_I(x_1) f_1^{i_1} \cdots f_j^{i_j}, \text{ with } \alpha_I(x_1) \in \boldsymbol{C}\{x_1\}, \tag{23}$$

with $0 \leq i_1 < n_1, \ldots, 0 \leq i_{j-1} < n_{j-1}$, $0 \leq i_j < e_{j-1}$, for $2 \leq j \leq g$.

By expanding the coefficients of the terms in (23), as series in x_1, we obtain the following expansion

$$f = f_j^{e_{j-1}} + \sum_{J=(i_0,\ldots,i_j)} \beta_J\, x_1^{i_0}\, f_1^{i_1} \cdots f_j^{i_j} \text{ with } \beta_J \in \boldsymbol{C}, \tag{24}$$

which we call the $(x_1, f_1, \ldots, f_j)$-*expansion* of f. The main result of this section is the following (see Definition 2.8).

THEOREM 3.1. *The* $(x, f_1, \dots, f_j)$*-expansion of* f*, for* $j = 2, \dots, g$*, is of the form*:

$$f = f_j^{e_{j-1}} + \sum_{(r,s)} c_{r,s}\, \mathscr{M}_j(r,s),$$

where $c_{r,s} \in \boldsymbol{C}$ *and the pairs* $(r,s) \in \boldsymbol{Z}^2$ *verify that*

$$0 < r, \quad 0 \leq s < e_{j-1}, \quad e_{j-1}(\bar{b}_j - n_{j-1}\bar{b}_{j-1}) \leq r e_{j-1} + s(\bar{b}_j - n_{j-1}\bar{b}_{j-1}).$$

Among the terms of this expansion with minimal intersection multiplicity with f *there exist* $f_j^{e_{j-1}}$ *and* $\mathscr{M}_j(\bar{b}_{j+1} - n_j \bar{b}_j, 0)$*. Moreover, if* $j = g-1$ *these two terms are exactly the terms with minimal intersection multiplicity with* f*.*

Before entering into the proof of Theorem 3.1 we discuss the following propositions.

PROPOSITION 3.2. *If* $j > 1$ *and the coefficient* $\alpha_I(x_1)$ *in* (23) *does not vanish then the Newton polygon of the term* $\alpha_I(x_1) f_1^{i_1} \cdots f_j^{i_j}$ *is contained in the interior of the Newton polygon of* f*.*

PROOF. Since $\deg f = e_{j-1} \deg f_j$ and both are monic polynomials we have that the term $f_j^{e_{j-1}}$ appears in the $(f_1, \dots, f_j)$-expansion of f with coefficient one.

For an index $I = (i_1, \dots, i_j)$ appearing in (23) we denote by $\mathscr{M}_I$ the term $\alpha_I(x) f_1^{i_1} \cdots f_j^{i_j}$. By Remark 1.13, the Newton polygon $\mathscr{N}(\mathscr{M}_I)$ of $\mathscr{M}_I$ has only one compact face Γ_I of maximal dimension which is parallel to the compact face Γ_1 of $\mathscr{N}(f(x_1, y_1))$. The vector $\vec{p_1} = (n_1, m_1)$, which was defined in Section 1.2, is orthogonal Γ_1.

We set also the numbers

$$B_I := \min\{\langle \vec{p_1}, \vec{u} \rangle \mid \vec{u} \in \mathscr{N}(\mathscr{M}_I)\} \text{ and } q_I := \operatorname{ord}_{y_1}\left(f_1^{i_1} \cdots f_j^{i_j}\right)_{|x_1 = 0},$$

for I appearing in the expansion (23) with non-zero coefficient. The numbers q_I defined above, are all distinct by Remark 1.3 applied to $0 \leq i_1 < n_1, \dots, 0 \leq i_{j-1} < n_{j-1}$ and $0 \leq i_j < e_{j-1}$.

Suppose that there exists an index $\tilde{I} = (\tilde{\imath}_1, \dots, \tilde{\imath}_j)$ with $\alpha_{\tilde{I}} \neq 0$, such that the polygon $\mathscr{N}(\mathscr{M}_{\tilde{I}})$ is not contained in $\mathscr{N}(f(x_1, y_1))$. This holds if and only if $B_{\tilde{I}} < \min\{\langle \vec{p_1}, \vec{u} \rangle \mid \vec{u} \in \mathscr{N}(f)\}$. Hence $\mathscr{M}_{\tilde{I}}$ is not equal to $f_j^{e_{j-1}}$, since $\mathscr{N}(f) = \mathscr{N}(f_j^{e_{j-1}})$ by Remark 1.13. We can suppose in addition that $B_{\tilde{I}}$ is the minimal number of this form. Moreover, we can assume that $\tilde{I}$ has the following property: if the index $I = (i'_1, \dots, i'_j) \neq \tilde{I}$, which appears in (23) with non zero coefficient, verifies that $B_{\tilde I} = B_I$ then $q_{\tilde I} > q_I$. If $(r,s) \in \Gamma_{\tilde I} \cap \boldsymbol{Z}^2$, the sum $K_{r,s}$ of the coefficients of the term $x^r y^s$ in $\alpha_I \mathscr{M}_I$, for those indices I with $B_I = B_{\tilde I}$, must vanish. But if (r,s) is the vertex of $\Gamma_{\tilde I}$ with $s = q_{\tilde I}$ then we obtain that $K_{r,s}$ is the initial coefficient of the series $\alpha_{\tilde I}$, a contradiction. Thus, for all index I appearing in (23) we have the inclusion $\mathscr{N}(\mathscr{M}_I) \subset \mathscr{N}(f(x_1,y_1))$.

By Remark 1.13 the symbolic restrictions of f and of $f_j^{e_{j-1}}$, to the compact face Γ_1 of the Newton polygon coincide. Suppose that there exists an index I appearing in the expansion (23) with non zero coefficient such that $\mathscr{M}_I \neq f_j^{e_{j-1}}$ and $B_I = \min\{\langle \vec{p}_1, u\rangle \mid u \in \mathscr{N}(f)\}$. In this case for any $(r,s) \in \Gamma_1 \cap \boldsymbol{Z}^2$ the sum of the coefficients of the terms $x^r y^s$ in $\mathscr{M}_{I'}$, for those I' with $B_I = B_{I'}$ and $\mathscr{M}_{I'} \neq f_j^{e_{j-1}}$, must vanish. We argue as in the previous case to prove that this cannot happen. □

LEMMA 3.3. *Let $\mathscr{M}_J = x_1^{i_0} f_1^{i_1} \cdots f_j^{i_j}$ be a term in the expansion* (24) *with non-zero coefficient corresponding to the index $J = (i_0,\dots,i_j)$. Set $q_J := i_1 + n_1 i_2 + \cdots + n_1 \cdots n_{j-1} i_j$. We can factor $\mathscr{M}_J \circ \pi_1$ as*:

$$(\mathscr{M}_J \circ \pi_1)\,(\mathrm{Exc}(f,\pi_1))^{-1} = u_2^{k_2(J)}\, x_2^{i_0'(J)}\, \big(f_2^{(2)}\big)^{i_2} \cdots \big(f_j^{(2)}\big)^{i_j}, \tag{25}$$

where $i_0'(J) = n_1 i_0 - m_1(e_0 - q_J) > 0$ and $k_2(J) = c_1 i_0 - d_1(e_0 - q_J) > 0$.

PROOF. Notice that q_J is the degree in y of the term $\mathscr{M}_J$. By Proposition 3.2 the Newton polygon of the term $\mathscr{M}_J$ is contained in the interior of the Newton polygon of f. This implies that $\vec{v}_J = (i_0, q_J)$ is a vertex of the Newton polygon of $\mathscr{M}_J$ and $\langle \vec{p}_1, \vec{v}_J\rangle > e_0 m_1$. This implies that $i_0'(J) > 0$. We deduce from this that $k_2(J) > 0$ and (25) holds. □

We obtain the following expansion from (24), by factoring out $\mathrm{Exc}(f,\pi_1)$ from $f \circ \pi_1$:

$$f^{(2)} = \big(f_j^{(2)}\big)^{e_{j-1}} + \sum_{J=(i_0,\dots,i_j)} c_J\, u_2^{k_2(J)}\, x_2^{i_0'(J)}\, \big(f_2^{(2)}\big)^{i_2} \cdots \big(f_j^{(2)}\big)^{i_j}. \tag{26}$$

The following expansion is obtained from (26) by collecting the terms with the same index $I' = (i_2,\dots,i_j)$:

$$f^{(2)} = \big(f_j^{(2)}\big)^{e_{j-1}} + \sum_{I'=(i_2,\dots,i_j)} \alpha_{I'}^{(2)}(x_2,u_2)\, \big(f_2^{(2)}\big)^{i_2} \cdots \big(f_j^{(2)}\big)^{i_j}. \tag{27}$$

By (10) the coefficient $\alpha_{I'}^{(2)}(x_2,u_2)$, viewed in $\boldsymbol{C}\{x_2,y_2\}$, is of the form:

$$\alpha_{I'}^{(2)} = \epsilon_{I'}^{(2)} x_2^{r_2(I')} \text{ with } r_2(I') > 0 \text{ and } \epsilon_{I'}^{(2)} \text{ a unit in } \boldsymbol{C}\{x_2, y_2\}. \tag{28}$$

DEFINITION 3.4. We call the expansion (26) (respectively (27)) the $(u_2, x_2, f_2^{(2)}, \ldots, f_j^{(2)})$-expansion (respectively $(f_2^{(2)}, \ldots, f_j^{(2)})$-expansion) of $f^{(2)}$.

PROPOSITION 3.5. *Suppose that* $2 \leq j \leq g$. *Let us consider an index* $I' = (i_2, \ldots, i_j)$ *appearing in the expansion* (27) *with coefficient* $\alpha_{I'}^{(2)}(x_2, u_2) \neq 0$. *Denote by* $q_{I'}$ *the order in* y_2 *of the series* $((f_2^{(2)})^{i_2} \cdots (f_j^{(2)})^{i_j})_{|x_2=0}$.

- (i) *For any pair* $I_1' \neq I_2'$ *of indexes in* (27) *with* $\alpha_{I_1'}^{(2)} \alpha_{I_2'}^{(2)} \neq 0$ *we have that* $q_{I_1'} \neq q_{I_2'}$.
- (ii) *If* $j > 2$ *and* $\alpha_{I'}^{(2)}(x_2, u_2) \neq 0$ *the Newton polygon of the term* $\alpha_{I'}^{(2)}(x_2, u_2) \cdot (f_2^{(2)})^{i_2} \cdots (f_j^{(2)})^{i_j}$ *(with respect to the coordinates* (x_2, y_2)*) is contained in the interior of* $\mathscr{N}(f^{(2)}(x_2, y_2))$.

PROOF. The assertion on the orders in y_2 of the series $((f_2^{(2)})^{i_2} \cdots (f_j^{(2)})^{i_j})_{|x_2=0}$ is consequence of Remark 1.3 with respect to the integers $n_2, \ldots, n_{j-1} > 1$.

For the second assertion notice that the Newton polygon with respect to the coordinates (x_2, y_2) of a term $\mathscr{M}_{I'}^{(2)} := \alpha_{I'}^{(2)}(x_2, u_2) (f_2^{(2)})^{i_2} \cdots (f_j^{(2)})^{i_j}$, appearing in the expansion (27), has at most one compact face which is parallel to Γ_2. We deduce that the Newton polygon of $\mathscr{M}_{I'}^{(2)}$ is contained in the interior of $\mathscr{N}(f^{(2)}(x_2, y_2))$ by repeating the argument of Proposition 3.2 combined with Remark 1.13 (ii). □

PROOF OF THEOREM 3.1. Let $\mathscr{M}_I := x^{i_0} f_1^{i_1} \cdots f_j^{i_j}$ be a monomial appearing in (24). Using Proposition 3.2, Proposition 3.5 and Lemma 3.3 we deduce inductively that $(\mathscr{M}_I \circ \pi_1 \circ \cdots \circ \pi_{j-1}) (\mathrm{Exc}(f, \pi_1 \circ \cdots \circ \pi_{j-1}))^{-1} = u_2^{k_2(I)} \cdots u_j^{k_j(I)} x_j^{r(I)} y_j^{s(I)}$, where $k_2(I), \ldots, k_j(I), r(I) > 0$ and $s(I) = i_j$. It follows that we have an expansion:

$$f^{(j)} = y_j^{e_{j-1}} + \sum_I c_I \, u_2^{k_2(I)} \cdots u_j^{k_j(I)} \, x_j^{r(I)} \, y_j^{s(I)}. \tag{29}$$

By the unicity statement in Proposition 2.4 it follows that $\mathscr{M}_I = \mathscr{M}(r(I), s(I))$, hence, if $I \neq I'$ are two different indices appearing in (24), then $(r(I), s(I)) \neq (r(I'), s(I'))$. By (10) the term $u_2^{k_2(I)} \cdots u_j^{k_j(I)}$ is a unit viewed in $\boldsymbol{C}\{x_j, y_j\}$, therefore the Newton polygon of $f^{(j)}(x_j, y_j)$ is equal to the convex hull of the set, $\bigcup_I (r(I), s(I)) + \boldsymbol{R}_{\geq 0}^2$. By Proposition 1.16 this polygon has vertices $(0, e_{j-1})$ and $(\bar{b}_j - n_{j-1}\bar{b}_{j-1}, 0)$. If $j = g$, these two vertices are the unique integral points in the

Newton polygon. By Lemma 2.10, the exponents $(r, s) \in \Gamma_j$ correspond to terms $\mathscr{M}_j(r, s)$ with minimal intersection multiplicity with f at the origin. □

We deduce from Theorem 3.1 the following Corollary, where the coefficient θ_j is the same as the one appearing in formula (7) at level j.

COROLLARY 3.6. *If* $j \in \{2, \dots, g\}$, *the* $(x, f_1, \dots, f_j)$*-expansion of* f_{j+1} *is of the form* (*cf. Definition* 2.8)

$$f_{j+1} = f_j^{n_j} - \theta_j \mathscr{M}_{j,f_{j+1}}(m_j, 0) + \sum_{(r,s)} c_{r,s}\, \mathscr{M}_{j,f_{j+1}}(r, s), \tag{30}$$

where (r, s) *above verify that* $0 < r$, $0 \leq s < n_j$ *and* $n_j m_j < n_j r + m_j s$.

REMARK 3.7. In some cases it may be useful to have $\theta_1 = \cdots = \theta_g = 1$. We can reduce to this case by replacing the terms $(x_1, f_1, \dots, f_g)$ by $(\eta_0 x_1, \eta_1 f_1, \dots, \eta_g f_g)$ for some suitable constants $\eta_0, \dots, \eta_g \in \boldsymbol{C}^*$.

To see this, by a change of coordinates of this form, we can assume that the image of $x_1, f_1, \dots, f_g$ in the integral closure $\boldsymbol{C}\{\tau\}$ of the algebra of $(C, 0)$ are series with constant term equal to one. By Lemma 2.10 we have that $\bar{b}_{j+1} = \mathrm{ord}_\tau f_{j+1}(x_1(\tau), y_1(\tau)) > n_j \bar{b}_j = \mathrm{ord}_\tau(f_j^{n_j}(x_1(\tau), y_1(\tau))) = \mathrm{ord}_\tau(\mathscr{M}_{j,f_{j+1}}(m_j, 0))$ and $n_j \bar{b}_j < \mathrm{ord}_\tau(\mathscr{M}_{j,f_{j+1}}(r, s))$, for those pairs (r, s) appearing in (30). We deduce by a standard valuative argument that in this case $\theta_j = 1$.

3.2. Abyankar's generalized Newton polygons, straight line condition and irreducibility criterion.

We follow the presentation given by Assi and Barile in [**As-B**] of results in [**Abh4**].

3.2.1. Generalized Newton polygons.

Given a sequence $\underline{\bar{B}} := (\bar{B}_0, \bar{B}_1, \dots, \bar{B}_G)$ of positive integers with $\bar{B}_1 < \cdots < \bar{B}_G$, we associate to them sequences $E_j = \gcd(\bar{B}_0, \bar{B}_1, \dots, \bar{B}_j)$ and $N_0 = 1$, $N_j = E_{j-1}/E_j$, for $j = 0, \dots, G$. Notice that if $\underline{\bar{B}}$ a characteristic sequence of generators of the semigroup Λ_C associated to a plane branch $(C, 0)$, we set $g = G$ and we have with the notations of the first section that $E_j = e_j$ and $N_j = n_j$, for $j = 0, \dots, g$.

Let F be a Weierstrass polynomial of the form:

$$F = y^N + \sum_{i=2}^{N} A_i(x) y^{N-i} \in \boldsymbol{C}\{x\}[y] \tag{31}$$

We assume that y is an approximate root of F since the coefficient of y^{N-1} is equal to zero. We denote by F_j the approximate root of F of degree $N_0 \cdots N_{j-1}$, and by $\underline{F}_j$ the sequence $(F_1, \ldots, F_j)$ for $j = 1, \ldots, G+1$ and $\underline{F} = \underline{F}_{G+1}$.

Let $P \in \boldsymbol{C}\{x\}[y]$ be a monic polynomial. The $(F_1, \ldots, F_{G+1})$-expansion of P is of the form $P = \sum_I \alpha_I(x) F_1^{i_1} \cdots F_G^{i_G} F_{G+1}^{i_{G+1}}$ (see Proposition 1.2). The *formal intersection multiplicity* of P and $\underline{F}$, with respect to the sequence $\underline{\bar{B}}$ is defined as

$$\text{formal}\ _{\underline{\bar{B}}}(P, \underline{F}) := \min \left\{ \sum_{j=0}^{G} i_j \bar{B}_j \mid I = (i_1, \ldots, i_G, 0), \alpha_I(x) \neq 0 \right\}. \tag{32}$$

Notice that when this value is $< +\infty$, it is reached at only one coefficient.

Let $P, Q \in \boldsymbol{C}\{x\}[y]$ be two monic polynomials of degrees p, q with $p = mq$. We have the Q-adic expansion of P is of the form: $P = Q^m + \alpha_1 Q^{m-1} + \cdots + \alpha_m$. The *generalized Newton polygon* $\mathscr{N}(P, Q, \underline{\bar{B}}, \underline{F})$ of P with respect to Q and the sequences $\underline{\bar{B}}$ and $\underline{\bar{F}}$ is the convex hull of the set:

$$\bigcup_{k=0}^{m} \big(\text{formal}\ _{\underline{\bar{B}}}(\alpha_k, \underline{F}), (m-k)\text{formal}\ _{\underline{\bar{B}}}(Q, \underline{F})\big) + \boldsymbol{R}^2_{\geq 0}. \tag{33}$$

3.2.2. Abhyankar's irreducibility criterion.

To a monic polynomial F of the form (31) it is associated a sequence $\bar{B}$ as follows: Set $\bar{B}_0 = E_0 := N$, $F_1 = y$, $\bar{B}_1 = (F_1, F)_0$, $E_1 = \gcd(\bar{B}_0, \bar{B}_1)$ and $N_1 := E_0/E_1$. Then, for $j \geq 2$ the integers $E_0, \ldots, E_j = \gcd(\bar{B}_0, \ldots, \bar{B}_j)$ and $N_1, \ldots, N_{j-1}$ are defined by induction. We set $B_{j+1} = (F, F_j)_0$, where F_j denotes the approximate root of F of degree $N_1 \cdots N_{j-1}$.

Theorem 3.8 ([**Abh4**]). *With the above notations the polynomial $F \in \boldsymbol{C}\{x\}[y]$ is irreducible if and only if the following conditions hold*:

(i) *there exists an integer $G \in \boldsymbol{Z}_{>0}$ such that $E_G = 1$,*
(ii) *$\bar{B}_{j+1} > N_j \bar{B}_j$ for $j = 1, \ldots, G-1$,*
(iii) (straight line condition) *the generalized Newton polygon $\mathscr{N}(F_{j+1}, F_j, (1/E_j)\underline{\bar{B}}_j, \underline{F}_j)$ has only one compact edge with vertices $((1/E_j)N_j\bar{B}_j, 0)$ and $(0, (1/E_j)N_j\bar{B}_j)$.*

Proof. We prove first that if F verifies the conditions of the theorem then F is irreducible. By the straight line condition the vertices of the generalized Newton polygon, $\mathscr{N}(F_{j+1}, F_j, (1/E_j)\underline{\bar{B}}_j, \underline{F}_j)$, correspond to the terms $F_j^{n_j}$ and $\alpha_0^{(j)}(x) F_1^{\eta_1^{(j)}} \cdots F_{j-1}^{\eta_{j-1}^{(j)}}$ of the $(F_1, \ldots, F_j)$-expansion of F_{j+1}, where $\operatorname{ord}_x \alpha_0^{(j)}(x) =$

$\eta_0^{(0)} \geq 0$ and $0 \leq \eta_i^{(j)} < N_i$ for $i = 1, \dots, j-1$. The straight line condition implies that $(1/E_j)N_j\bar{B}_j = (1/E_j)(\eta_0^{(j)}\bar{B}_0 + \cdots + \eta_{j-1}^{(j)}\bar{B}_{j-1})$. It follows that $N_j\bar{B}_j$ belongs to the semigroup generated by $\bar{B}_0, \dots, \bar{B}_{j-1}$. By Lemma 1.15 this numerical condition together with (i) and (ii) guarantee that the semigroup generated by $\bar{B}_0, \dots, \bar{B}_G$ is the semigroup of a plane branch. Let $0 \neq F' \in \boldsymbol{C}\{x\}[y]$ be any polynomial of degree $< N = \deg F$. We consider the $(F_1, \dots, F_G)$-expansion of F':

$$F' = \sum_I \alpha_I(x) F_1^{i_1} \cdots F_G^{i_G} \text{ with } \alpha_I(x) \in \boldsymbol{C}\{x\}. \tag{34}$$

Set $i_0 = \mathrm{ord}_x \alpha_I(x)$. The intersection multiplicities $(F, \alpha_I(x)F_1^{i_1} \cdots F_G^{i_G})_0 = \sum_{j=0}^{G} i_j \bar{B}_j$ obtained for the different terms in the expansion (34) are all different (this reduces to an arithmetical property which can be proved similarly as Lemma 1.15). We deduce that $(F, F')_0 = \min\{\sum_{j=0}^{G} i_j\bar{B}_j \mid \alpha_I(x) \neq 0\} < +\infty$. The polynomial F is irreducible, otherwise there is an irreducible factor of F' of F of degree $< \deg F$ and then $(F, F')_0 = +\infty$, a contradiction.

Suppose now that F is irreducible. Then $\bar{B}_0, \dots, \bar{B}_G$ are the generators of the semigroup of the branch $F = 0$ with respect to the line $\{x = 0\}$. By Lemma 1.15, the first two conditions in the statement of the theorem hold automatically. By Proposition 1.17 the approximate root F_{j+1} is irreducible and define a plane branch with semigroup generated by $(1/E_j)\bar{B}_0, \dots, (1/E_j)\bar{B}_j$. By Theorem 3.1 the Newton polygon of $F_{j+1}^{(j)}(x_j, y_j)$ has only two vertices $(0, N_j)$ and $(M_j, 0)$ which correspond respectively to the terms $F_j^{N_j}$ and $\mathscr{M}_{j,F_{j+1}}(0, M_j)$ of the $(x, F_1, \dots, F_j)$-expansion of F_{j+1} (see Definition 2.8). By Proposition 1.16 and induction the vertices of the Newton polygon of the strict transform function $F^{(j)}(x_j, y_j)$ are $(\bar{B}_j - N_{j-1}\bar{B}_{j-1}, 0)$ and $(0, E_jN_j)$. It follows that $M_j = (1/E_j)(\bar{B}_j - N_{j-1}\bar{B}_{j-1})$. By Lemma 1.18 we have that $\mathrm{ord}_t(\mathrm{Exc}(F_{j+1}, \pi_1 \circ \cdots \circ \pi_{j-1})) = (E_{j-2}/E_j)(\bar{B}_{j-1}/E_j) = (1/E_j)N_{j-1}N_j\bar{B}_j$. By Lemmas 2.10 and 1.18 we deduce the equality:

$$\begin{aligned}(F_{j+1}, \mathscr{M}_{j,F_{j+1}}(0, M_j))_0 &= \frac{1}{E_j}N_{j-1}N_j\bar{B}_{j-1} + \frac{E_{j-1}}{E_j}M_j \\ &= \frac{1}{E_j}\big(N_jN_{j-1}\bar{B}_{j-1} + N_j\big(\bar{B}_j - N_{j-1}\bar{B}_{j-1}\big)\big) \\ &= \frac{1}{E_j}N_j\bar{B}_j \\ &= \big(F_{j+1}, F_j^{N_j}\big)_0.\end{aligned}$$

A similar computation using Lemmas 2.10 and 1.18 proves that if $\mathscr{M}_{j,F_{j+1}}(r,s)$ appears in the $(x, F_1, \dots, F_j)$-expansion of F_{j+1} then $(F_{j+1}, \mathscr{M}_{j,F_{j+1}}(r,s))_0 > (1/E_j)N_j\bar{B}_j$. $\square$

REMARK 3.9. We keep notations of the proof of Theorem 3.8. Suppose that F is irreducible. Let $\phi_j : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be the linear function given by $\phi_j(r,s) = (rN_j, sM_j)$. We denote by R_j the number $R_j := (1/E_j)N_{j-1}N_j\bar{B}_j$. Then we have that:

$$\mathscr{N}\left(F_{j+1}, F_j, \frac{1}{E_j}\underline{\bar{B}}_j, \underline{F}_j\right) = (R_j, R_j) + \phi_j\big(\mathscr{N}(F^{(j)}_{j+1}(x_j, y_j))\big).$$

3.3. Equisingularity criterions.

Let $F \in \boldsymbol{C}\{t,x\}[y]$ be a Weierstrass polynomial in y. We suppose that y is an approximate root of F, i.e., F is of the form:

$$F = y^N + \sum_{i=2}^{N} A_{i,t}(x) y^{N-i} \in \boldsymbol{C}\{x,t\}[y]. \tag{35}$$

Set $F_t(x,y) = F(t,x,y)$ and consider the family of germs $(C_t, 0)$ defined by $F_t = 0$. We assume that $(x, F_t)_0 = e_0 > 1$, for $0 \le |t| \ll 1$.

We give an algorithm to check whether a family of curves $(C_t, 0)$ of the form (35) is equisingular at $t = 0$ to a plane branch (irreducible and reduced). If the answer of the algorithm is no, then either $(C_0, 0)$ is not analytically irreducible or $(C_t, 0)$ is not equisingular at $t = 0$. The proof follows from the discussion in Section 3.1.

ALGORITHM 3.10.

Step 1: *Set* $\mathscr{N}_1$ *the Newton polygon of* $F = \sum \alpha^{(1)}_{r,s}(t) x^r y^s$ *(with respect to* (x,y)*).*

(1.a) *Check that* $\mathscr{N}_1$ *has only one edge* Γ_1 *with vertices* $(e_1 m_1, 0)$ *and* $(0, e_1 n_1)$ *with* $e_1 \ge 1$, $\gcd(n_1, m_1) = 1$ *and* $e_1 \ge 1$. *If* $e_1 = 1$ *answer yes, otherwise verify that* $\min\{n_1, m_1\} > 1$. *Notice that* $e_0 = e_1 n_1$.

(1.b) *Check that the polynomial* $\sum_{k=0}^{e_1} \alpha^{(1)}_{e_1 m_1 - k m_1, k n_1}(t)\, z^k \in \boldsymbol{C}\{t\}[z]$ *is of the form* $(z - \theta_1(t))^{e_1}$ *for some series* $\theta_1(t) \in \boldsymbol{C}\{t\}$ *with* $\theta_1(0) \neq 0$.

Step 2: *If* $e_1 > 1$ *and conditions* (1.a) *and* (1.b) *hold, set* F_2 *for the approximate root of* F *of degree* n_1. *Compute the* (y, F_2)*-expansion* $F = \sum_{\mathit{finite}} A^{(2)}_{i_1,i_2}(t,x)\, y^{i_1} F_2^{i_2}$. *From the data* (n_1, m_1) *and* e_0 *each triple*

$$(i_0, i_1, i_2), \ \textit{for } i_0 := \mathrm{ord}_x(A^{(2)}_{i_1,i_2}(t,x)),$$

$$\textit{determines } (r,s) \textit{ with } r > 0 \textit{ and } s < e_1 \textit{ or } (r,s) = (0, e_1) \tag{36}$$

and the converse also holds. (*This follows by Remark* 2.7 *and the method given in the proof of Lemma* 3.3). *For* (r,s) *and* (i_0, i_1, i_2) *in* (36) *denote* $x^{i_0} y^{i_1} F_2^{i_2}$ *by* $\mathscr{M}_2(r,s)$.

(2.a) *Denote by* $\mathscr{N}_2$ *for the convex hull of the set* $\bigcup_{(r,s)}\{(r,s) + \boldsymbol{R}^2_{\geq 0}\}$, *for* (r,s) *those of* (36). *Check that* $\mathscr{N}_2$ *has only one edge* Γ_2 *with vertices* $(e_2 m_2, 0)$ *and* $(0, e_2 n_2)$ *with* $\gcd(n_2, m_2) = 1$ *and* $e_2 \geq 1$. *Verify that* $n_2 > 1$ *and* $m_2 \geq 1$.

(2.b) *If* $i_0 = \mathrm{ord}_x A^{(2)}_{i_1,i_2}(t,x)$ *we denote by* $\alpha^{(2)}_{r,s}(t)$ *the coefficient of* x^{i_0} *in the expansion of* $A^{(2)}_{i_1,i_2}(t,x)$ *as a series in* x, *where* (r,s) *is determined by* (i_0, i_1, i_2) *in terms of* (36). *Set*

$$F^{\Gamma_2} := \sum_{k=0}^{e_2} \alpha^{(2)}_{e_2 m_2 - k m_2, k n_2}(t)\, \mathscr{M}_2(e_2 m_2 - k m_2, k n_2).$$

Compute the approximate root F_3 *of degree* $n_1 n_2$ *of* F. *Check that* $F_3^{e_2}$ *is of the form* $F_3^{e_2} = F^{\Gamma_2} + \sum_{n_2 r + m_2 s > e_2 n_2 m_2} \gamma^{(2)}_{r,s}(t)\, \mathscr{M}_2(r,s)$, *for some* $\gamma^{(2)}_{r,s}(t) \in \boldsymbol{C}\{t\}$.

(2.c) *Check that the polynomial* $\sum_{k=0}^{e_2} \alpha^{(2)}_{e_2 m_2 - k m_2, k n_2}(t)\, z^k \in \boldsymbol{C}\{t\}[z]$ *is of the form* $(z - \theta_2(t))^{e_2}$, *for some series* $\theta_2(t) \in \boldsymbol{C}\{t\}$ *with* $\theta_2(0) \neq 0$.

Step $j > 2$: *If the conditions* (j-1.a), (j-1.b) *and* (j-1.c), *corresponding to* (2.a), (2.b) *and* (2.c) *respectively are verified and* $e_{j-1} > 1$ *compute* $(y, F_2, \dots, F_j)$-*expansion of* F *and check the conditions* (j.a), (j.b) *and* (j.c), *corresponding to* (2.a), (2.b) *and* (2.c) *respectively.*

The algorithm stops whenever some condition is not verified, answering NO, or when all conditions are verified and $e_g = 1$ *for some integer* g, *answering then YES.*

Remark 3.11. Our criterion extends the one given by A'Campo and Oka in [**A'C-Ok**]. They assume that certain approximate roots of F_t do not depend on t. We do not need this hypothesis. We do not compute intersection numbers as in Abhyankar's irreducibility criterion [**Abh4**] nor resultants as in [**GB-G**] (see Subsection 3.3.1.).

Example 3.12. We consider F of the form (35).

$$\begin{aligned}F := {} & y^{12} + (-6x^3 + 6tx^4)y^{10} + (15x^6 - 30tx^7)y^8 + (-20x^9 + (60t-2)x^{10})y^6 \\ & + (15x^{12} + (6-60t)x^{13} + (-6t+\lambda)x^{14})y^4 - x^{16}y^3 \\ & + (-6x^{15} + (-6+30t)x^{16} + (-2\lambda + 12t)x^{17})y^2 + x^{19}y + x^{18} \\ & + (2-6t)x^{19} + (1-6t+\lambda)x^{20}.\end{aligned}$$

The approximate roots of F of degrees 2 and 6 are $F_2 := y^2 - x^3 + tx^4$ and $F_3 := F_2^3 - (15/2)t^2x^8F_2 + x^{10}(-1 + 20t^3x^2)$. Notice that both polynomials depend on the parameter t hence we cannot apply the equisingularity criterion of [**A'C-Ok**]. We check that the Newton polygon of F has only two vertices $(0,12)$ and $(18,0)$. We set $e_1 = 6$, $n_1 = 2$ and $m_1 = 3$. The conditions (1.a) and (2.a) are verified for F_2. We compute the (y, F_2)-expansion of F and we find:

$$\begin{aligned}F = {} & F_2^6 - 15t^2x^8F_2^4 + (-2 + 40t^3x^2)x^{10}F_2^3 + (\lambda - 45t^4x^2)x^{14}F_2^2 - x^{16}yF_2 \\ & + (-2\lambda t + 6t^2 + 24t^5x^2)x^{18}F_2 + tx^{20}y + (1 + (\lambda t^2 - 4t^3)x^2 + 5t^6x^4)x^{20}.\end{aligned}$$

With the notations introduced above we have that $\mathscr{N}_2$ only two vertices $(0,6)$ and $(4,0)$, hence $e_2 = 2$, $n_2 = 3$ and $m_2 = 2$. We have that $F^{\Gamma_2} = \mathscr{M}_2(0,6) - 2\mathscr{M}_2(3,2) + \mathscr{M}_2(0,4)$, where $\mathscr{M}_2(0,6) = F_2^6$, $\mathscr{M}_2(3,2) = x^{10}F_2^3$ and $\mathscr{M}_2(0,4) = x_1^{20}$. We check that the conditions (2.b) and (2.c) are satisfied. We compute the (y, F_2, F_3)-expansion of F, for F_3 the approximate root of degree 6 of F. We obtain that

$$\begin{aligned}F = {} & F_3^2 + \left(\lambda - \frac{405}{4}t^4x^2\right)x^{14}F_2^2 - x^{16}yF_2 + (-2\lambda t - 9t^2 + 324t^5x^2)x^{18}F_2 \\ & + tx^{20}y + (\lambda t^2 + 36t^3 - 405t^6x^2)x^{22}.\end{aligned}$$

In order to compute the polygon $\mathscr{N}_3$ we consider the leading terms in the expansion above and we use the method of Lemma 3.3. We have that $x^{i_0}y^{i_1}F_2^{i_2}F_3^{i_3} = \mathscr{M}_2(r_1, s_1)F_3^{i_3}$ where $s_1 = i_2$ and $r_1 = i_0n_1 + m_1(-e_0 + i_1 + n_1i_2 + n_1n_2i_3)$. We have then that $\mathscr{M}_2(r_1, s_1)F_3^{i_2} = \mathscr{M}_3(r, s)$ where $s = i_2$ and $r = n_2r_1 + m_1(-e_1 + s_1 + n_2i_3)$. For instance, we have that $x^{14}F_2^2 = \mathscr{M}_2(4,2) = \mathscr{M}_3(4,0)$ and $x^{16}yF_2 = \mathscr{M}_2(5,1) = \mathscr{M}_3(5,0)$. We distinguish two cases in terms of the constant $\lambda \in \boldsymbol{C}$.

(a) If $\lambda \neq 0$ then we check that $\mathscr{N}_3$ is a polygon with vertices $(0,2)$ and $(4,0)$. We have that $e_3 = 2$, $n_3 = 1$ and $m_3 = 2$ hence $\{F_t = 0\}$ is not equisingular at $t = 0$.

(b) If $\lambda = 0$ then $\mathscr{N}_3$ is a polygon with vertices $(0,2)$ and $(5,0)$. We have that $e_3 = 1$ and the conditions (2.a), (2.b) and (2.c) are verified, hence $\{F_t = 0\}$ is equisingular at $t = 0$.

3.3.1. Equisingularity criterion by Jacobian Newton polygons.

Let $f \in \boldsymbol{C}\{x\}[y]$ be a Weierstrass polynomial. The *Jacobian Newton polygon* of f with respect to the line $\{x = 0\}$ is the Newton polygon of $\mathscr{J}_f(s,x) := \mathrm{Res}_y(s - f, \partial f/\partial y) \in \boldsymbol{C}\{x,s\}$, where Res_y denotes the resultant with respect to y. The Jacobian Newton polygon appears in more general contexts related to invariants of equisingularity (see [**T1**]). García Barroso and Gwoździewicz have proved that if $f' \in \boldsymbol{C}\{x\}[y]$ is irreducible and $\mathscr{J}_f(s,x) = \mathscr{J}_{f'}(s,x)$ then f is irreducible. They have given two methods which characterize Jacobian polygons of plane branches among other Newton polygons by a finite number of combinatorial operations on the polygons (see [**GB-G**, Theorems 1, 2 and 3]). The following algorithm is consequence of their work.

Algorithm 3.13. *Input: A family* $F_t(x,y)$ *of the form* (35).

(a) *Compute* $\mathscr{J}_{F_t}(s,x)$.
(b) *Compute the Newton polygon* $\mathscr{N}_t$ *of* $\mathscr{J}_{F_t}(s,x)$. *Check that* $\mathscr{N}_t = \mathscr{N}_0$.
(c) *Test if* $\mathscr{N}_0$ *is a Jacobian Newton polygon of a plane branch by using Theorem* 2 *or* 3 *in* [**GB-G**].

If all the steps of the algorithm give a positive answer then $F_t = 0$ *is equisingular at* $t = 0$ *to a plane branch.*

4. Multi-semi-quasi-homogeneous deformations.

In this Section we introduce a class of (non equisingular) deformations of a plane branch $(C,0)$ and we study some of its basic properties which are essential for the applications in the real case (see [**GP-R**]).

We keep notations of Section 1.2. The resolution is described in terms of a fixed sequence $f_1, \dots, f_g$ of semi-roots. For simplicity we assume that $\theta_1, \dots, \theta_g = 1$ (see Remark 3.7). We introduce the following notations:

Notation 4.1. For $j = 1, \dots, g$ we set:

(i) $\Gamma_j = [(m_j e_j, 0), (0, n_j e_j)]$ the compact edge of the local Newton polygon of $f^{(j)}(x_j, y_j)$ (see (7)).
(ii) Δ_j the triangle bounded by the Newton polygon of $f^{(j)}(x_j, y_j)$ and the coordinate axis; we denote by Δ_j^- the set $\Delta_j^- = \Delta_j \setminus \Gamma_j$.
(iii) Let $\omega_j : \Delta_j \cap \boldsymbol{Z}^2 \to \boldsymbol{Z}$ be defined by $\omega_j(r,s) = e_j(e_j n_j m_j - r n_j - s m_j)$.

The symbol $\underline{t}_j$ denotes the parameters $(t_j, \dots, t_g)$ for any $1 \leq j \leq g$. We consider sequences of multiparametric deformations $C_{\underline{t}_1}, \dots, C_{\underline{t}_g}$ of $(C, 0)$ defined by $P_{\underline{t}_1}, \dots, P_{\underline{t}_g}$ of the form (see Definition 2.8):

$$\begin{cases} P_{\underline{t}_g} := F + \displaystyle\sum_{(r,s)\in\Delta_g^- \cap \boldsymbol{Z}^2} a_{r,s}^{(g)}(t_g)\, M_g(r,s) \\ \cdots\cdots\cdots\cdots \qquad \cdots \qquad \cdots \\ P_{\underline{t}_1} := P_{\underline{t}_2} + \displaystyle\sum_{(r,s)\in\Delta_1^- \cap \boldsymbol{Z}^2} a_{r,s}^{(1)}(t_1)\, M_1(r,s), \end{cases} \tag{37}$$

where $a_{r,s}^{(j)}(t_j) \in \boldsymbol{C}\{t_j\}$ for $(r,s) \in \Delta_j^- \cap \boldsymbol{Z}^2$ and $j = 1, \dots, g$. Notice that $P_{\underline{t}_1}$ determines any of the terms $P_{\underline{t}_j}$ for $1 < j \leq g$, by substituting $t_1 = \cdots = t_{j-1} = 0$ in $P_{\underline{t}_1}$. The multiparametric deformation $C_{\underline{t}_1}$ is *multi-semi-quasi-homogeneous* (msqh) if in addition $a_{r,s}^{(j)} = A_{r,s}^{(j)}\, t_j^{\omega_j(r,s)}$, for $1 \leq j \leq g$ and $(r,s) \in \Delta_j^- \cap \boldsymbol{Z}^2$, where $\omega_j(r,s) \in \boldsymbol{Z}_{\geq 0}$ is defined in Notation 4.1, $A_{r,s}^{(j)} \in \boldsymbol{C}$ and $A_{0,0}^{(j)} \neq 0$, for $j = 1, \dots, g$.

REMARK 4.2. In the real case we have studied the topological types of the msqh-deformations with real part with the maximal number of connected components. The hypothesis of being msqh is related in that paper to the study of the asymptotic scales of the ovals when the parameters tend to zero (joint work with Risler [**GP-R**]).

We denote by $C_{l,\underline{t}}^{(j)} \subset Z_j$ the strict transform of $C_{l,\underline{t}}$ by the composition of toric maps $\pi_1 \circ \cdots \circ \pi_{j-1}$ and by $P_{\underline{t}_l}^{(j)}(x_j, y_j)$ the polynomial defining $C_{\underline{t}_l}^{(j)}$ in the coordinates (x_j, y_j) for $2 \leq j \leq l \leq g$. These notations are analogous to those used for C in Section 1.2, see (6).

PROPOSITION 4.3. *If $1 \leq j < l \leq g$ the curves $C_{\underline{t}_l}^{(j)}$ and $C^{(j)}$ meet the exceptional divisor of $\pi_1 \circ \cdots \circ \pi_{j-1}$ only at the point $o_j \in \{x_j = 0\}$ and with the same intersection multiplicity e_{j-1}.*

PROOF. If $j = 1$ we have that f and $P_{\underline{t}_l}$ have the same Newton polygon and moreover the symbolic restrictions of these two polynomials to the compact face Γ_1 of the Newton polygon coincide by Lemma 2.11. If $j > 1$ we show the result by induction using Remark 2.12. □

PROPOSITION 4.4. *If $1 < j \leq g$ then $\{x_j = 0\}$ is the only irreducible component of the exceptional divisor of $\pi_1 \circ \cdots \circ \pi_{j-1}$ which intersects $C_{\underline{t}_j}^{(j)}$ at e_{j-1} points counted with multiplicity. More precisely, we have that*:

(i) *The symbolic restriction of $P_{\underline{t}_{j+1}}^{(j)}(x_j, y_j)$ to the edge Γ_j of its local Newton* *polygon is of the form:* $\alpha_j \prod_{s=1}^{e_j} \left(y_j^{n_j} - (1+\gamma_s^{(j)} t_{j+1}^{e_{j+1}m_{j+1}}) x_j^{m_j}\right)$, *with* $\alpha_j, \gamma_s^{(j)} \in \boldsymbol{C}^*$, *for* $s = 1, \ldots, e_j$.

(ii) *The points of intersection of* $\{x_{j+1} = 0\}$ *with* $C_{\underline{t}_{j+1}}^{(j+1)}$ *are those with coordinates* $x_{j+1} = 0$ *and*

$$u_{j+1} = \left(1 + \gamma_s^{(j)} t_{j+1}^{e_{j+1}m_{j+1}}\right)^{-1}, \text{ for } s = 1, \ldots, e_j. \tag{38}$$

Proof. If $j = 2$ we have that the terms of the expansion of $P_{\underline{t}_2}$ which have exponents on the compact face of the Newton polygon $\mathscr{N}(f)$ are f and $\mathscr{M}_2(0,s)$ for $0 \leq s < e_1$ by Lemma 2.11. By Proposition 2.4 we have that $\mathscr{M}_2(0,s) = x_1^{m_1(e_1-s)} f_2^s$. The restriction of the polynomial $f + \sum_{s=0}^{e_1-1} A_{(0,s)}^{(2)} t_2^{e_2 m_2 (e_1 - s)} \cdot x_1^{m_1(e_1-s)} f_2^s$ to the face Γ_j is equal to:

$$\left(y_1^{n_1} - x_1^{m_1}\right)^{e_1} + \sum_{s=0}^{e_1-1} A_{(0,s)}^{(2)} \left(t_2^{e_2 m_2} x_1^{m_1}\right)^{e_1 - s} \left(y_1 - x_1^{m_1}\right)^s. \tag{39}$$

Let us consider the polynomial $Q_1(V_1, V_2) := V_1^{e_1} + \sum_{s=0}^{e_1-1} A_{(0,s)}^{(2)} V_1^s V_2^{e_1-s}$. By hypothesis $A_{(0,0)}^{(2)} \neq 0$ hence the homogeneous polynomial Q_1 factors as: $Q_1(V_1, V_2) = \prod_{s=1}^{e_1} (V_1 - \gamma_s^{(1)} V_2)$ for some $\gamma_s^{(1)} \in \boldsymbol{C}^*$. The expression (39) is of the form: $Q_1(y_1^{n_1} - x_1^{m_1}, t_2^{m_2 e_2} x_1^{m_1}) = \prod_{s=1}^{e_1} (y^{n_1} - (1 + \gamma_s^{(1)} t_2^{m_2 e_2}) x_1^{m_1})$. This proves the first assertion in this case. The second follows from this by the discussion of Section 1.2.

If $j > 2$ we deduce by induction, by using Remarks 2.12 and 2.1, that the restriction of $P_{\underline{t}_{j+1}}^{(j-1)}$ to the compact face of $\mathscr{N}(f^{(j-1)})$ is of the form $(y_j^{n_j} - x_j^{m_j})^{e_j} + \sum_{s=0}^{e_j-1} A_{(0,s)}^{(j+1)} (t_{j+1}^{e_{j+1}m_{j+1}} x_j^{m_{j+1}})^{e_j - s} (y_j^{n_j} - x_j^{m_j})^s$. The result follows by the same argument. □

4.1. Milnor number and generic msqh-smoothings.

If $(D,0) \subset (\boldsymbol{C}^2, 0)$ is the germ of a plane curve singularity, defined by $h = 0$, for $h \in \boldsymbol{C}\{x,y\}$ reduced, we denote by $\mu(h)_0$ or by $\mu(D)_0$ the *Milnor number* $\dim_{\boldsymbol{C}} \boldsymbol{C}\{x,y\}/(h_x, h_y)$. We have the following formula (see [**R**] and [**Z2**]):

$$\mu(h)_0 = \left(h, \frac{\partial h}{\partial y}\right)_0 - (h, x)_0 + 1. \tag{40}$$

The Milnor number of the plane branch $(C,0)$ expresses also in terms of the generators of the semigroup Λ_C with respect to the coordinate line $\{x = 0\}$ (see [**M**], [**GB**] and [**Z2**]) by using:

$$\left(f, \frac{\partial f}{\partial y}\right)_0 = \sum_{j=1}^{g} (n_j - 1)\bar{b}_j. \tag{41}$$

DEFINITION 4.5. We say that the deformation $P_{\underline{t}_1}$ of the plane branch $(C, 0)$ is *generic* if the numbers $\{\gamma_s^{(j)}\}_{s=1}^{e_j}$ appearing in Proposition 4.4 are all distinct, for $1 < j \leq g$.

The following proposition provides a geometrical incarnation in terms of the sequence of generic msqh-deformations of the Milnor's formula $\mu(C)_0 = (1/2)\delta(C)_0$, for $\delta(C)_0$ the delta invariant of $(C, 0)$ (see [**W**], [**Ca**, Ex. 5.6], see also [**G**]).

PROPOSITION 4.6. *Let* $P_{\underline{t}_1}$ *be a generic msqh-deformation of a plane branch* $(C, 0)$*, then we have that*

$$\mu(C)_0 = \sum_{j=1}^{g} \big(\mu(C_{\underline{t}_{j+1}}^{(j)})_{o_j} + e_j - 1\big).$$

PROOF. We prove the result by induction on g. If $g = 1$ the assertion is trivial. We suppose the assertion true for branches with $g - 1$ characteristic exponents with respect to some system of coordinates. By Proposition 1.11 and the induction hypothesis it is easy to check that $\mu(C^{(2)})_{o_2} = \sum_{j=2}^{g}(\mu(C_{\underline{t}_{j+1}}^{(j)})_{o_j} + e_j - 1)$.

By Proposition 4.4 and the definition of generic msqh-deformation, we have that the curve $C_{\underline{t}_2}^{(1)}$, defined by the polynomial $P_{\underline{t}_2}^{(1)}(x_1, y_1)$, is non-degenerate with respect to its Newton polygon. By (40) we have that $\mu(C_{\underline{t}_2}^{(1)})_{o_1} = e_0 b_1 - e_0 - b_1 + 1$. By (41) we have that: $\mu(C)_{o_1} - \mu(C^{(2)})_{o_2} = (f, \partial f/\partial y)_0 - (f^{(2)}, \partial f^{(2)}/\partial y_2)_{o_2} + e_1 - e_0$. The assertion holds if and only if $(f, \partial f/\partial y)_0 - (f^{(2)}, \partial f^{(2)}/\partial y_2)_{o_2} = b_1(e_0 - 1)$. Using (41) and Lemma 1.18 we verify that $(f, \partial f/\partial y)_0 - (f^{(2)}, \partial f^{(2)}/\partial y_2)_{o_2}$ is equal to: $\sum_{j=1}^{g}(n_k - 1)\bar{b}_j - \sum_{j=2}^{g}(n_k - 1)(\bar{b}_j - \bar{b}_1 e_0/e_{j-1}) = (n_1 - 1)\bar{b}_1 + \sum_{j=2}^{g}\bar{b}_1 e_0/e_{j-1} = (n_1 - 1 + (n_2 - 1)n_1 + \cdots + (n_g - 1)n_1 \cdots n_{g-1})\bar{b}_1 = (e_0 - 1)b_1$. ☐

COROLLARY 4.7.

$$\mu(C)_0 = 2\bigg(\sum_{j=0}^{g-1} \big(\#(\mathring{\Delta}_j \cap \boldsymbol{Z}^2) + e_{j+1} - 1\big)\bigg).$$

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