BIJECTIVENESS OF THE NASH MAP FOR QUASI-ORDINARY HYPERSURFACE SINGULARITIES

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Abstract. In this paper we give a positive answer to a question of Nash, concerning the arc space of a singularity, for the class of quasi-ordinary hypersurface singularities extending to this case previous results and techniques of Shihoko Ishii.

Introduction

In a 1968 preprint, published later as [N], Nash introduced a map, nowadays called the *Nash map* (resp. *local Nash map*), from the set of families of arcs with origin at the singular locus of a variety X (resp. at a fixed singular point $x \in X$), to the set of essential divisors over the singular locus of X (resp. over the point x). These families of arcs are called *Nash components* (resp. *local Nash components*). Obviously, these maps coincide if the singular locus of X is reduced to an isolated singular point x. Nash showed that these maps are injective and asked if they were surjective. Ishii and Kollár have shown an affine four dimensional variety with singular locus reduced to a point, for which the Nash map is not bijective (see [I-K]). The answer of Nash question for surface and threefolds singularities is not known in general. Plenat has given sufficient conditions for the surjectivity of the Nash map for isolated surface singularities in [Pl1]. In the case of surface singularities the Nash map it is known to be bijective in the following cases: for minimal singularities by Reguera [Re1], for sandwiched singularities by the work of Lejeune and Reguera [LJ-R] and [Re2], for rational double points of type A_n, already studied by Nash [N], and of type D_n by Plenat [Pl2] (the result for rational double points in general is announced in [Pl3]). Plenat and Popescu-Pampu have shown a class varieties of dimension two and higher for which the Nash map is bijective in [Pl-PP1] and [Pl-PP2]; a similar result in the surface case is announced by Morales [Mo].

Ishii and Kollár have shown that Nash question has a positive answer in the case of normal toric varieties, see [I-K]. Ishii has generalized this result for the class pretoric algebraic varieties, which contains in particular the class of toric varieties (non necessarily normal). Petrov formulated *Nash question for pairs* (X, B), consisting of a variety X and a proper closed subvariety B containing the singular locus of X, and exhibited a positive answer in the case of pairs, (X, B), formed by a normal toric variety X and an invariant set B. He applied this result to prove the bijectiveness of the Nash map for the class of *stable toric varieties*, a class of reduced but non necessarily irreducible varieties introduced by Alexeev [Al], which generalize normal toric varieties (see [Pe]). Ishii has shown recently that the local Nash map is bijective for *analytically pretoric* singularities, a class of singularities containing toric and analytically irreducible quasi-ordinary hypersurface singularities (see [I3]). The class of quasi-ordinary singularities appears classically in Jung's strategy to obtain resolution of surface singularities from the embedded resolution of plane curves (see [J], [A] and [L1]).

2000 *Mathematics Subject Classification.* Primary 14J17; Secondary 32S10, 14M25.

Key words and phrases. Nash problem, arc space, quasi-ordinary singularities, singularities.

The authors' research was supported in part by *Programa Ramón y Cajal* and MTM2004-08080-C02-01 grants of *Ministerio de Educación y Ciencia*, Spain.

The purpose of this Note is to show that the Nash map is bijective for any reduced germ of quasi-ordinary hypersuperface singularity, Theorem 4. We show that the results and approach for the class of *pretoric singularities*, analyzed by Ishii in [I2], can be extended to the case of an analytically irreducible quasi-ordinary hypersurface germ (X, x) by showing a property, Proposition 1, deduced from Lipman's result on the structure of the singular locus (see [L2], §7). Moreover, if B is a proper closed subgerm of (X, x) containing the singular locus of X, we prove that Nash question for the pair (X, B) has a positive answer under a natural technical condition (see Hypothesis 1 and Theorem 2).

In the case of a quasi-ordinary hypersurface germ (X, x) with several irreducible components X_i, $i = 1, \dots, t$, the analysis of the pre-image of the singular locus of X by the normalization map (see Lemma 5) is essential to deduce the main result Theorem 4, the bijectiveness of the Nash map for X, from the bijectiveness of the Nash map for suitable pairs (X_i, B_i) for $i = 1, \dots, t$.

Finally, we compare the notions of pretoric singularity and analytically pretoric singularity, introduced by Ishii in [I2] and [I3] respectively, with the notion of *toric quasi-ordinary singularity* introduced [GP2] and we show that our main result holds in a slightly larger category, which we call *strongly analytically pretoric* (see Corollary 5).

The explicit description of the essential divisors over the singular locus of a quasi-ordinary hypersurface singularity, given in this paper, is applied by Hernando and the author in [GP-H] to prove that the essential divisors of an irreducible germ (X, x) of quasi-ordinary hypersurface determine, through a suitable notion of Poincaré series, the *characteristic monomials*, an analytical invariant which in the analytic case encodes the embedded topological type of the germ characterized by the work of Gau and Lipman (see [L2] and [Gau]). It should be noticed that the Poincaré series associated only to the essential divisors over the point x, which correspond by Ishii's result [I3] to the local Nash components, does not contain enough information to recover the characteristic monomials of the quasi-ordinary hypersurface (X, x) in general, see [GP-H]. This property reflects the fact that quasi-ordinary singularities are rarely isolated: X does have an isolated singularity at x if and only if the singular germ (X, x) is of dimension one or normal of dimension two. The use of all essential divisors, over the different components of the singular locus of X, and also over x, is crucial to recover the characteristic monomials of the germ (X, x), as main local invariants.

1. Basic definitions on the arc space and relative Nash problem

In this section we give basic definitions and results on the *relative Nash problem*, also called *Nash problem for a pair*. These notions are natural extensions of the corresponding ones on the classical Nash problem (see [Pe]). In this paper the scheme X is a pure dimensional reduced algebraic (resp. algebroid germ of) variety, defined over a field k, algebraically closed of characteristic zero. Let $B \subset X$ be a reduced proper k-subscheme containing the singular locus $\mathrm{Sing}(X)$ of X.

A *resolution of singularities of* (X, B) is a proper modification $\phi : Y \to X$ such that $\phi_{|Y - \phi^{-1}(B)} : Y - \phi^{-1}(B) \to X - B$ is an isomorphism. The resolution ϕ is divisorial is $\phi^{-1}(B)$ is a divisor. An *exceptional divisor* E *over* X *relative to* B is an exceptional divisor such that the center of E over X is contained in B. An exceptional divisor E over X relative to B is *essential* if for every resolution $\phi : Y \to X$ of the pair (X, B) the center of E on Y is an irreducible component of $\phi^{-1}(B)$. This center is called an *essential component* on Y.

If $k \subset K$ is a field extension an *arc* over X is a morphism $\alpha : \mathrm{Spec} K[[t]] \to X$. We denote respectively by 0 and η the closed point and generic point of $\mathrm{Spec} K[[t]]$. If $m \geq 0$ is an integer, an *m-jet* over X is a morphism $\alpha : \mathrm{Spec} K[[t]]/(t^{m+1}) \to X$. The set X_m of m-jets can be given the structure of scheme of finite type over k. We have canonical morphisms $X_{m+k} \to X_m$, corresponding to $\mathrm{Spec} K[[t]]/(t^{m+1}) \to \mathrm{Spec} K[[t]]/(t^{m+k+1})$, for all $m, k \geq 0$. The arc space $X_\infty := \varprojlim X_m$ has the structure of scheme over k, not of finite type. A point $z \in X_\infty$ corresponds to an arc $\alpha_z : \mathrm{Spec} K[[t]] \to X$ such that K is the residue field at z. We have that K-valued points of X_∞

correspond to arcs of the form Spec$K[[t]] \to X$ bijectively. We often denote the point z and the corresponding arc α_z by the same symbol. We have a canonical projection $\pi^X : X_\infty \longrightarrow X$, defined by $z \mapsto \alpha_z(0)$. A morphism of varieties (resp. of algebroid germs) $\psi : Y \to X$, corresponds to a morphism at the level of arcs: $\psi_\infty : Y_\infty \to X_\infty$. See [D-L] for the general construction of arcs spaces.

A *Nash component of* X *relative to* B is an irreducible component of $\pi_X^{-1}(B)$ which is not contained in $B_\infty \subset X_\infty$. If the field k is of characteristic zero, as it is assumed in this paper, then the proof of Lemma 2.12 [I-K] imply that the Nash components of X relative to B coincide with the irreducible components of $\pi_X^{-1}(B)$. Denote by $\bigcup_i C_i$ the union of Nash components of X relative to B. Let $\phi : Y \to X$ be a divisorial resolution of (X, B), i..e, $\phi^{-1}(B)$ is a divisor with irreducible components $E_1, \dots, E_l$. Then the restriction ϕ_∞ to $\bigcup_{j=1}^l \pi_Y^{-1}(E_j) \longrightarrow \bigcup_i C_i$. is dominant and bijective outside B_∞. For all i there exists a unique j_i such that $\pi_Y^{-1}(E_j) \to C_i$ is dominant. The analogous statement of Nash Theorem in this relative situation is that E_{j_i} is an essential divisor of X relative to B and that the *relative Nash map* $C_i \mapsto E_{j_i}$ is an injection, from the Nash components and to the set of essential divisors of X relative to B (see Lemma 2.14 [I3] for an analogous proof in this relative situation or Theorem 2.17 [Pe] for a sketch of proof in the algebraic case). See also [I-K] for a modern proof of the classical statement of Nash [N], when $B = \mathrm{Sing}(X)$). The Nash problem for X relative to B, i.e., the Nash problem for the pair (X, B), is to determine if this correspondence is bijective.

2. Basic notations on normal toric varieties

We give some basic definitions and notations (see [O] or [Fu] for proofs). If N is a lattice we denote by M the dual lattice, by $N_{\mathbf{R}}$ the real vector space spanned by N and by $\langle , \rangle$ the canonical pairing between the dual lattices N and M (resp. vector spaces $N_{\mathbf{R}}$ and $M_{\mathbf{R}}$). A *rational convex polyhedral cone* τ in $N_{\mathbf{R}}$, a *cone* in what follows, is the set $\tau := \mathrm{pos}\{a_1, \dots, a_s\}$ of non negative linear combinations of vectors $a_1, \dots, a_s \in N$. The cone τ is *strictly convex* if τ contains no linear subspace of dimension > 0. We denote by $\overset{\circ}{\tau}$ the *relative interior* of a cone σ. The *dual* cone $\tau^\vee$ (resp. *orthogonal* cone $\tau^\perp$) of τ is the set $\{w \in M_{\mathbf{R}} / \langle w, u \rangle \geq 0\}$ (resp. $\langle w, u \rangle = 0)$ $\forall u \in \tau\}$). A *fan* Σ is a family of strictly convex cones in $N_{\mathbf{R}}$ such that any face of such a cone is in the family and the intersection of any two of them is a face of each. If τ is a cone in the fan Σ, the semigroup $\tau^\vee \cap M$ is of finite type, it spans the lattice M and the variety $Z_{\tau,N} = \mathrm{Spec}\, k[\tau^\vee \cap M]$, which we denote by Z_τ when the lattice is clear from the context, is normal. The affine varieties Z_τ corresponding to cones in a fan Σ glue up to define the *toric variety* Z_Σ. The torus $T_N := Z_{\{0\}} \cong (k^*)^{\mathrm{rk}\, N}$ is embedded in Z_Σ as an open dense subset and there is an action of T_N on Z_Σ which extends the action of the torus on itself by multiplication. We have a bijection between the relative interiors of the cones of the fan and the orbits of the torus action, $\overset{\circ}{\tau} \mapsto \mathrm{orb}_{Z_\Sigma}(\tau)$, which inverses inclusions of the closures. We denote the orbit $\mathrm{orb}_{Z_\Sigma}(\tau)$ by $\mathrm{orb}(\tau)$ when the toric variety Z_Σ is clear from the context.

In this paper σ denotes a rational strictly convex cone in $N_{\mathbf{R}}$ of maximal dimension. Any non zero vector $v \in \sigma \cap N$ defines a valuation val_v of the field of fractions of $k[\sigma^\vee \cap M]$ (resp. of the m-adic completion $k[[\sigma^\vee \cap M]]$ of the localization of $k[\sigma^\vee \cap M]$ at the maximal ideal, $(\sigma^\vee \cap M) \setminus \{0\}$, defining the *origin* o_σ of the toric variety Z_σ). This valuation, called *monomial* or *toric valuation*, is defined for an element $0 \neq \phi = \sum c_u X^u \in k[[\sigma^\vee \cap M]]$, by $\mathrm{val}_v(\phi) = \min_{c_u \neq 0} \langle n, u \rangle$. If the ray $\rho := v\mathbf{R}_{\geq 0}$ belongs a fan Σ subdividing σ, the closure D_ρ of the orbit $\mathrm{orb}(\rho)$ is an invariant divisor (we denote it also by D_v if the vector v is primitive for the lattice N). We denote by val_{D_ρ} the associated divisorial valuation. If $v = qv_0$ for $q \in \mathbf{Z}_{\geq 1}$ and v_0 a primitive vector for the lattice N then we have that $\mathrm{val}_v = q\mathrm{val}_{D_\rho}$. Following Ishii we say that the valuation val_v is a *toric divisorial valuation* (see [I2]). The cone σ induces a partial order on $N_{\mathbf{R}}$, defined by $u \leq_\sigma u'$ iff $u' \in u + \sigma$ (similar definition for $\leq_{\sigma^\vee}$ on $M_{\mathbf{R}}$ holds).

3. Quasi-ordinary hypersurfaces: singular locus and normalization

A quasi-ordinary hypersurface singularity X is defined by $\mathrm{Spec}\, k[[x]][y]/(f)$ where $x = (x_1, \dots, x_d)$ and $f \in k[[x]][y]$ is a *quasi-ordinary polynomial*, i.e., a Weierstrass polynomial such that the discriminant $\Delta_y f$ with respect to y is of the form $x^\delta \epsilon$ where $\delta \in \mathbf{Z}^d_{\geq 0}$ and $\epsilon \in k[[x]]$ is a unit. By the Jung-Abhyankar Theorem the roots of quasi-ordinary polynomials are fractional power series with particular properties. Namely, if f is irreducible of degree n a root of f is of the form: $\zeta = \sum c_\lambda x^\lambda \in k[[x^{1/n}]]$, where $x^{1/n} = (x_1^{1/n}, \dots, x_d^{1/n})$, and the terms appearing in this expansion verify certain properties. In particular if f is of degree > 1 in y, certain monomials, determined by comparing the different roots of f and called *characteristic or distinguished*, appear in the expansion of ζ with non zero coefficient. The corresponding exponents, which are also called characteristic, can be relabelled in the form $\lambda_1 \leq_{\sigma^\vee} \lambda_2 \leq_{\sigma^\vee} \cdots \leq_{\sigma^\vee} \lambda_g$. These exponents determine the following nested sequence of lattices: $M_0 \subset M_1 \subset \cdots \subset M_g =: M$ where $M_0 := \mathbf{Z}^d$ and $M_j := M_{j-1} + \mathbf{Z}\lambda_j$ for $j = 1, \dots, g$ with the convention $\lambda_{g+1} = +\infty$. We have that the exponents appearing in the expansion of ζ belong to M. See [L2] and also [GP2]. We have ring extensions:

$$k[[\sigma^\vee \cap M_0]] = k[[x]] \longrightarrow \mathcal{O}_X \cong k[[x]][\zeta] \longrightarrow k[[\sigma^\vee \cap M]] \tag{1}$$

where $\sigma^\vee$ denotes the positive quadrant $\mathbf{R}^d_{\geq 0}$ and $M_0 = \mathbf{Z}^d$ (we denote by σ, N_0 and N the corresponding dual objects of $\sigma^\vee$, M_0 and M respectively). In [GP2] it is proved that the ring extension $\mathcal{O}_X \to k[[\sigma^\vee \cap M]]$ is the inclusion of $\mathcal{O}_X$ in its integral closure in its field of fractions. Geometrically, (1) corresponds to a sequence of finite maps:

$$(X_{\sigma,N}, o_\sigma) = (\bar{X}, x) \xrightarrow{\nu} (X, x) \xrightarrow{\rho} (Z, x) = (X_{\sigma,N_0}, o_\sigma) = (k^d, 0). \tag{2}$$

Since the map $\rho \circ \nu$ is equivariant, it maps the orbit $\mathrm{orb}_{\bar{X}}\tau$ to $\mathrm{orb}_Z\tau$, for each face $\tau < \sigma$.

We recall Lipman's description of the singular locus of a quasi-ordinary hypersurface, see Theorem7.3 [L2], for a precise statement (cf. also the reformulation of this result given in [PP]).

Theorem 1. *With the previous notations if (X, x) is analytically irreducible we have that the irreducible components of $\mathrm{Sing}(X)$ are of codimension one or two. The codimension one (resp. two) components are intersections of X with $x_i = 0$, (resp. with $x_i = 0$ and $x_j = 0$) for some suitable coordinate sections, in each case, determined by the characteristic monomials.* □

Proposition 1. *The set $\nu^{-1}(\mathrm{Sing}(X))$ is a germ of closed set at the origin of $\bar{X}$, which is invariant by the torus action on $\bar{X}$.*

Proof. By Lipman's theorem, the irreducible components of $\mathrm{Sing}(X)$ are the germs $\rho^{-1}(\overline{\mathrm{orb}_Z(\tau)})$ at the point x, where τ runs certain set of a one (resp. two) dimensional faces of σ. It follows from this and the previous discussion that the irreducible components of $\nu^{-1}(\mathrm{Sing}(X))$ are of codimension one or two. The codimension one (resp. two) components are of the form

$$(\rho \circ \nu)^{-1}(\overline{\mathrm{orb}_Z(\tau)}) = \overline{\mathrm{orb}_{\bar{X}}(\tau)}.$$

If $x_i = X^{u_i}$ for $i = 1, \dots, d$, in (1) then $x_i = 0$ (resp. $x_i = x_j = 0$) defines in Z the closure of the orbit $\mathrm{orb}_Z(\tau)$ where the cone τ is characterized by $\tau^\perp \cap \sigma^\vee = \mathrm{pos}_{k \neq i}(u_k)$ (resp. by $\tau^\perp \cap \sigma^\vee = \mathrm{pos}_{k \neq i,j}(u_k)$).
□

4. Relative Nash problem, the irreducible case

We follow Ishii's approach in [I2] and [I3]. Let (X, x) be an irreducible germ of quasi-ordinary hypersurface. We study the relative Nash problem for a proper closed set $B \subset X$ such that $\mathrm{Sing}(X) \subset B$. We introduce the following hypothesis on B.

Hypothesis 1. *Any irreducible component of $\nu^{-1}(B)$ is an orbit closure $\overline{\mathrm{orb}_{\bar{X}}(\tau)}$ corresponding to some face τ of σ.*

Notice that $\mathrm{Sing}(X)$ verifies this hypothesis by Proposition 1. When the hypothesis above is verified any irreducible component in the closure of the set $\nu^{-1}(B)-\mathrm{Sing}(\bar{X})$ is an orbit closure corresponding to some regular face $\tau < \sigma$ (with respect to the lattice N). We denote by $\tau_1, \dots, \tau_r$ the regular faces determined in this way and by $e_i \in N$, the barycenter of τ_i (i.e., the sum of the primitive integral vectors, for the lattice N, in the edges of τ_i), for $i = 1, \dots, r$.

Let $\{v_j\}_{j=1}^{s}$ the set of minimal elements, with respect of the partial order $\leq_\sigma$ in the set:

$$S := \bigcup_{\tau<\sigma,\tau \text{ singular}} \overset{\circ}{\tau} \cap N.$$

By [I-K] the toric divisors $\{D_{v_j}\}_{j=1,\dots,s}$ are the essential divisors of toric variety $\bar{X}$, and also the essential divisors of the germ of $\bar{X}$ at the closed orbit, by Lemma 4.9 [I3]. This characterization of essential divisors generalizes a result of Bouvier [Bo], see also [Bo-GS].

Lemma 2. *Each e_i, for $i = 1, \dots, r$ is minimal among $\{e_i, v_j\}_{i=1,\dots,r}^{j=1,\dots,s}$ for the order $\leq_\sigma$.*

Proof. See Lemma 5.7 [I2]. □

Lemma 3. *Let $\{e_i, v_j\}_{i=1,\dots,r}^{j=1,\dots,w}$, ($w \leq s$) be the set of minimal elements of $\{e_i, v_j\}_{i=1,\dots,r}^{j=1,\dots,s}$. Then there is an inclusion*

$$\{\textit{essential divisors over } X \textit{ relative to } B\} \subset \{D_{e_i}, D_{v_j}\}_{i=1,\dots,r}^{j=1,\dots,w}.$$

Proof. The statement can be translated in purely combinatorial terms in terms of the existence of resolutions of singularities of X, which are obtained by composing the normalization map with toric modifications. The precise arguments are the content of the proof of Lemma 5.7 [I2]. □

Following Ishii, [I2], we associate to a non zero vector $v \in \sigma \cap N$ a subset $T_\infty^X(v)$ of X_∞ , containing only arcs which lift to arcs with generic point in the torus T_N of $\bar{X}$:

$$T_\infty^X(v) := \{\alpha \in X_\infty \mid \alpha(\eta) \in \nu(T_N), \mathrm{ord}_t \alpha^*(x^u) = \langle v, u\rangle, \ \text{for } u \in M\}.$$

The sets $T_\infty^{\bar{X}}(v)$, defined similarly, are orbits of a natural action of $(T_N)_\infty$ on the arc space of the normal toric variety $\bar{X}$ (see [I1]).

Lemma 4. *Let $\{e_i, v_j\}_{i=1,\dots,r}^{j=1,\dots,w}$, ($w \leq s$) be the set of minimal elements of $\{e_i, v_j\}_{i=1,\dots,r}^{j=1,\dots,s}$. Then, the following closures are distinct Nash components of X_∞:*

$$\overline{T_\infty^X(e_i)},\ i = 1, \dots, r \ \text{ and } \ \overline{T_\infty^X(v_j)}, j = 1, \dots, w.$$

If $v \in \{e_i, v_j\}_{i=1,\dots,r}^{j=1,\dots,s}$ the image of the component $\overline{T_\infty^X(v)}$ by the Nash map is the divisor D_v.

Proof. The proof is analogous to the proofs of Lemma 4.6 and 4.7 in [I3]. Notice that with our hypothesis the proof holds not only for vectors $v \in \overset{\circ}{\sigma} \cap N$ but also for vectors $v \in \sigma \cap N$. See also the proof of Lemma 5.10 and Theorem 5.11 [I2]. □

Theorem 2. *Let (X, x) be a irreducible germ of quasi-ordinary hypersurface singularity. Let B a closed subscheme verifying Hypothesis 1. Then the Nash map between the set of Nash components of $\pi_X^{-1}(B)$ and the set of essential divisors of X relative to B is bijective.*

Proof. Let $u \in \{e_i, v_j\}_{i=1,\dots,r}^{j=1,\dots,w}$. Then the sequence

$$u \mapsto \overline{T_\infty^X(u)} \mapsto D_u$$

defines an injection from the set $\{e_i, v_j\}_{i=1,\dots,r}^{j=1,\dots,w}$ to the set of essential divisors over X, by Lemma 4 and by the injectivity of the Nash map. By Lemma 3 the set of essential divisors is of cardinality less or equal than $r + w$, hence it follows that this set is of cardinality equal to $r + w$ and the injection above is a bijection. This implies that the Nash map is bijective. □

Corollary 3. *If (X, x) is an analytically irreducible quasi-ordinary hypersurface the Nash map is bijective.*

Proof. By Proposition 1 the singular locus $B = \mathrm{Sing}(X)$ verifies the hypothesis 1 of Theorem 2.□

5. Nash problem for a quasi-ordinary hypersurface

Now we suppose that (X, x) is a germ of reduced quasi-ordinary hypersurface. We denote by f a *quasi-ordinary polynomial* defining (X, x). The factors f_i of the factorization of $f = f_1 \dots f_t$ as product of irreducible terms corresponds to the irreducible components of (X, x). The factors f_i are quasi-ordinary polynomials, for $i = 1, \dots, t$.

We denote by B_i the intersection:

$$B_i = X_i \cap \mathrm{Sing}(X) = \mathrm{Sing}(X_i) \cup \bigcup_{j=1,\dots,t}^{j \neq i} X_i \cap X_j.$$

We denote by $\nu_i : \bar{X}_i \to X$ the normalization of X_i, which is a toric singularity by the previous discussion.

Lemma 5. *We have that $\nu_i^{-1}(B_i)$ is a germ of invariantly closed set, at the close orbit of the toric singularity $\bar{X}_i$.*

Proof. We have already shown the statement for $\nu_i^{-1}(\mathrm{Sing}(X_i))$ by Proposition 1. If $j \neq i$ then $\nu^{-1}(X_i \cap X_j)$ is defined by $f_j(\zeta^{(i)}) = 0$ where $\zeta^{(i)}$ is a fixed root of f_i, used to define the ring extension (2) corresponding to X_i. We have that the element $f_j(\zeta^{(i)})$ is equal to the product of a monomial by a unit in the local algebra of the toric singularity $\bar{X}_i$ (this follows easily from the definition, see [GP2] for more details). This implies that $f_j(\zeta^{(i)}) = 0$ defines a germ of invariantly closed set, at the close orbit of the toric singularity $\bar{X}_i$, which is equal to $\nu_i^{-1}(X_i \cap X_j)$. □

We obtain then a generalization of Corollary 4.12 in [I3]:

Theorem 4. *Let (X, x) be a reduced germ of quasi-ordinary hypersurface singularity. Then the Nash map between the Nash components of $\pi_X^{-1}(\mathrm{Sing}(X))$ and the essential divisors of X is bijective.*

Proof. We keep the notations given above for the irreducible components of X. Notice that $\pi_X^{-1}(\mathrm{Sing}(X)) = \bigsqcup_{i=1}^t \pi_{X_i}^{-1}(B_i)$ by definition of B_i. It follows from this that:

$$\{\text{Nash components of } X\} \subset \bigsqcup_{i=1}^{t} \{\text{Nash components of } X_i \text{ relative to } B_i\}.$$

(See the proof of Lemma 4.11 [I3]). We prove that:

$$\{\text{essential divisors over } X\} \subset \bigsqcup_{i=1}^{t} \{\text{essential divisors over } X_i \text{ relative to } B_i\}.$$

Let $\phi_i : Y_i \to X_i$ be a resolution of (X_i, B_i). Then the composite $\phi : Y \to X$ defined by:

$$Y := \bigsqcup_{i=1}^{t} Y_i \stackrel{\sqcup \phi_i}{\to} \bigsqcup_{i=1}^{t} X_i \to X$$

is a resolution of the pair $(X, \mathrm{Sing}(X))$ by the definition of B_i. Let E be an essential divisor of X. The center of E in Y is an irreducible component of

$$\phi^{-1}(\mathrm{Sing}(X)) = \bigsqcup_{i=1}^{t} \phi_i^{-1}(B_i),$$

thus an irreducible component of $\phi_i^{-1}(B_i)$ for some i. This implies the assertion.

The hypothesis 1 is verified by B_i with respect to X_i for $i = 1, \dots, t$ by Lemma 5. By Theorem 2 applied to the pair (B_i, X_i) the Nash map between the set of Nash components of $\pi_{X_i}^{-1}(B_i)$ and the set of essential divisors of X_i relative to B_i is bijective. □

6. An extension of the results to a larger category

In Definition 4.1 [I3] the notion of analytically pretoric singularity is introduced in the algebroid category. A germ (X, x) is called in [I3] *analytically pretoric* if there exists a sequence of injective local homomorphisms:

$$k[[\sigma^\vee \cap M_0]] \stackrel{\rho^*}{\to} \mathcal{O}_{X,x} \stackrel{\nu^*}{\to} k[[\sigma^\vee \cap M]]$$

such that

i. $\nu^* \circ \rho^* : k[[\sigma^\vee \cap M_0]] \to k[[\sigma^\vee \cap M]]$ is the canonical injection corresponding to a finite index lattice inclusion $M_0 \subset M$,

ii. the morphism $\nu : \mathrm{Spec} k[[\sigma^\vee \cap M]] \to X$ corresponding to ν^* is the normalization map,

iii. the restriction of ν to the torus $\mathrm{Spec} k[[\sigma^\vee \cap M]][M]$ is an isomorphism onto its image.

Notice that Ishii introduced the notion of *pretoric variety* in the algebraic category in [I2] Definition 5.1. The first two conditions for a variety X to be pretoric are the formulations of axioms i. and ii. above in the algebraic category, while the third condition above is to be replaced by

iii'. The closed subset $\nu^{-1}(\mathrm{Sing}(X))$ is invariant for the torus action of $\bar{X}$.

We say that a germ of algebroid singularity is *strongly analytically pretoric* if it verifies conditions i., ii., and iii'. Notice that in the algebroid case condition iii'. implies condition iii in Ishii's definition of analytically pretoric singularity.

Corollary 5. *If (X, x) is a germ of strongly analytically pretoric singularity then the associated Nash map is bijective.*

Proof. The analysis and results done in the sections 4 and 5 extends formally for any algebroid singularity (X, x) verifying conditions i., ii., and iii'. □

We end this section by formulating some natural questions which come out from the comparison of the notions of (strongly) analytically pretoric with that of *toric quasi-ordinary singularity*, introduced in [GP2] as a suitable generalization of the notion of quasi-ordinary singularity (in [GP1] the this class of singularities was restricted to the case of relative hypersurface germ in $Z_\sigma \times \mathbf{C}$). A germ of complex analytic variety (X, x) of pure dimension d is *a toric quasi-ordinary singularity* if there exists:

a. An affine normal toric variety $X_{\sigma, N_0} = \mathrm{Spec}\ k[\sigma^\vee \cap M_0]$, defined by a d dimensional rational strictly convex cone σ for the lattice N_0 (dual of $\sigma^\vee$ and M_0 respectively).

b. A finite morphism of germs $(X, x) \stackrel{\rho}{\to} (X_{\sigma, N_0}, o_\sigma)$, where o_σ is the closed orbit, which is unramified over the torus of X_{σ, N_0}.

Condition b. means that for each representative of the morphism ρ there exists an open neighborhood of o_σ such that ρ is unramified over its intersection with the torus. If the cone σ is simplicial then (X, x) is a quasi-ordinary singularity: we reduce to this case by replacing the lattice M_0 by a sublattice M_0' of finite index such that $\sigma^\vee$ is regular for M_0' and hence $Z_{\sigma, N_0'} = \mathbf{C}^d$, for N_0' the dual lattice of M_0'. Then the normalization $(\bar{X}, x)$ is a germ of toric variety $X_{\sigma, N} = \mathrm{Spec}\ \mathbf{C}[\sigma^\vee \cap M]$ for some lattice $M \supseteq M_0$, at is closed orbit, and the composite $\rho \circ \nu$ is a germ of toric equivariant map defined by the change of lattices (see Theorem 5.1 in [PP], or see [GP2] in the hypersurface case). The definition of analytically pretoric singularity can be easily adapted to the complex analytic category. It is immediate that if (X, x) is a germ of analytically pretoric singularity then it is toric quasi-ordinary singularity, since the map ρ is then unramified over the torus by axiom i and iii. It follows from the discussion above that if the cone σ is simplicial the notions of toric quasi-ordinary singularity and that of analytically pretoric singularity coincide (at least in the complex analytic category). It seems quite reasonable that both notions coincide also in the algebroid category and without any assumption on simpliciality on the cone σ appearing in both Definitions. Is to be conjectured that for a equidimensional germ of algebroid singularity conditions i., ii., iii. and i., ii., iii'. are equivalent.

Acknowledgement. The author is grateful to Profesor Shihoko Ishii for stimulating conversations during the Hayashibara Forum on Singularities 2006 at IHÉS. During the preparation of this work the author has benefited from the hospitality of *Institut de Mathématiques de Jussieu* in Paris.

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