

## FUZZY PREFERENCES IN KNOWLEDGE-BASED SYSTEMS

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## ABSTRACT

This paper deals with fuzzy preferences. Many different alternative definitions of strict and weak fuzzy preferences can be found in the literature. But most of them do not take into account that fuzzy preferences should allow a complete preference structure, where not only strict and weak preferences are defined, but also indifference and incomparability. The whole preference structure should be built in order to be properly considered as a basis for knowledge based systems. Taking into account a previous work of Montero, we introduce some modifications in the model recently developed by Fodor, Ovchinnikov and Roubens in such a way that consistent fuzzy preference structures are fully characterized.

*Keywords.* fuzzy sets, fuzzy preferences relations.

## 1. Introduction

Classical preference model is usually modeled by asking the decision maker to define a weak crisp preference relation

$$R: A \times A \rightarrow \{0, 1\},$$

where  $A$  is a finite set of feasible alternatives and  $R(a, b) = 1$  whenever alternative  $a$  is considered weakly better than alternative  $b$  (i.e.,  $a \geq b$ ). From such a data set, strict preference ( $a > b$ ), indifference ( $a \sim b$ ) and incomparability ( $a \parallel b$ ) relations are univocally given. We are therefore defining the whole preference structure just by defining weak preferences.

Fodor, Ovchinnikov and Roubens (see <sup>3,4,5,6,14</sup>), face such an approach in a fuzzy context, now assuming that weak preferences are given in terms of fuzzy binary preference relations

$$R: A \times A \rightarrow [0, 1],$$

$\mu(a, b)$  being the degree to which  $a > b$  holds. Among other particular solutions, they obtain axiomatic justifications for the additive model proposed by Montero <sup>7,8</sup> (see also <sup>1,2</sup>).

## 2. Basic assumptions

As pointed out in <sup>7,8</sup>, the objective should be to split the whole intensity into four basic intensity values

$$P(a, b), I(a, b), P(b, a), J(b, a)$$

representing, respectively, the degrees of strict preference  $a > b$ , indifference ( $a \sim b$ ), strict preference  $b > a$  and incomparability ( $a \parallel b$ ). In order to explain key intuitive properties, a fuzzy conjunction and a disjunction are needed. For example, the disjunction of  $P(a, b)$  and  $I(a, b)$  should give  $R(a, b)$ , and incomparability should be negation of comparability, defined as the disjunction of  $P(a, b)$ ,  $I(a, b)$ ,  $P(b, a)$ .

Following traditional fuzzy logic literature, disjunction and conjunction are modeled in <sup>3,4,5,6,14</sup> according to a t-conorm  $S$  and a t-norm  $T$ , being both continuous mappings and related by means of a strict negation function  $\pi$  in such a way that

$$\pi(S(x, y)) = T(\pi(x), \pi(y))$$

for each  $x, y \in [0, 1]$  (a strict negation  $\pi$  is a continuous, strictly increasing mapping  $\pi: [0, 1] \rightarrow [0, 1]$  such that  $\pi(0) = 1$  and  $\pi(1) = 0$ ).

Three general intuitive axioms, namely Independence of Irrelevant Alternatives, Positive Association and Symmetry can be assumed (see <sup>6</sup>):

(IA) Independence of Irrelevant Alternatives: for any two alternatives  $a, b \in A$ , the values  $P(a, b)$ ,  $I(a, b)$  and  $J(a, b)$  depend only upon the values  $R(a, b)$ ,  $R(b, a)$ .

In particular, we will assume the existence of three intensity continuous functions

$$p, i, j: [0, 1] \times [0, 1] \rightarrow [0, 1]$$

such that

- $P(a, b) = p(R(a, b), R(b, a))$ ,
- $I(a, b) = i(R(a, b), R(b, a))$  and
- $J(a, b) = j(R(a, b), R(b, a))$ .

(PA) Positive association principle:  $p(x, y)$  is nondecreasing with respect to its first argument but nonincreasing with respect to its second argument,  $i(x, y)$  is nondecreasing with respect to both arguments, and  $j(x, y)$  is nonincreasing with respect to both arguments.

(SY) Symmetry:  $i(x, y) = i(y, x)$  and  $j(x, y) = j(y, x)$ .

Moreover, the following two equations are also assumed in <sup>6</sup>:

$$(FOR.1v) S(p(x, y), i(x, y)) = x$$

$$(FOR.3v) S(p(y, x), j(y, x)) = n(x)$$

Additional conditions leads in <sup>6</sup> to particular expressions for  $p, i, j$ .

### 3. FORM model

An alternative approach to the above fuzzy preference structure has been proposed by the authors in <sup>10</sup>. Disjunction is not assumed unique neither associative, but it is imposed that every disjunction should be strictly increasing in the real range (see <sup>10</sup> for details). The negations  $n$  is also assumed to be strong, in such a way that comparability is the negation of incomparability, and viceversa. Then it follows that such a disjunction operator is in fact unique and associative, being basically additive in such a way that

$$S(x, y) = \phi^{-1}(\phi(x) + \phi(y))$$

in the real range (see <sup>10</sup>). Moreover, the automorphism  $\phi$  characterizing such a disjunction operator  $S$  turns out to simultaneously characterize the negation function (see <sup>10</sup> for other less relevant assumptions).

Under such a FORM model, it can be checked that functions  $p, i, j$  can not be arbitrarily defined (see <sup>10</sup> for a proof).

**Theorem 1** *Bring any of the functions  $i, p, j, c$  fixed, then the preference structure is fixed up to an automorphism  $\phi$ .*

Hence, with just one feasible intensity function  $p, i, c$  or  $j$  a particular solution is settled, up to some automorphism  $\phi$  in the unit interval. Next theorem gives a characterization for strict preference functions in terms of such an automorphism (see also <sup>10</sup> for a proof).

**Theorem 2** *Let  $p: [0, 1] \times [0, 1] \rightarrow [0, 1]$  be a continuous function, nondecreasing in the first coordinate and nonincreasing in the second one. Then it is a strict preference intensity function for a FORM model with automorphism  $\phi$  if and only if the following two conditions hold:*

$$(P1) p(x, y) \leq \min(x, n(y))$$

$$(P2) \phi(p(x, y)) + \phi(p(y, x)) = \phi(x) + \phi(y)$$

for all  $x, y$ , and where  $n(x) = \phi^{-1}(1 - \phi(x))$ .

Analogous conditions can be imposed in order to be characterized each solution from the indifference intensity function  $i$  (analogous results for  $c$  and  $j$  can also be obtained)

**Theorem 3** *Let  $i: [0, 1] \times [0, 1] \rightarrow [0, 1]$  be a symmetric continuous function, which is nondecreasing in both coordinates. Then it is an indifference intensity function for a FORM model with automorphism  $\phi$  if and only if the following two conditions hold:*

(I1)  $\phi(x) - \phi(i(x, y))$  is nondecreasing in  $x$ , for any fixed  $y$ .

(I2)  $\phi^{-1}(\max(\phi(x) + \phi(y) - 1, 0)) \leq i(x, y) \leq \min(x, y)$  for all  $x, y$ .

### 4. Discussion on the literature

In this section we shall check if some particular preference relations that can be found in the literature do allow the definition of the whole preference structure. Whenever it is not possible, we would expect serious difficulties when applied as a key element in a relational knowledge-based system.

Here is a list of some proposals for strict fuzzy preferences which have been used in the past in the fuzzy literature:

1.  $p(x, y) = x$  for all  $x, y$  does not hold (P1)
2.  $p(x, y) = 0$  for all  $x, y$  does not hold (P2)
3.  $p(x, y) = \phi^{-1}(\frac{\phi(x) + \phi(y)}{2})^2$  does not hold (P1).
4. Neither (P1) or (P2) are verified if

$$p(x, y) = \begin{cases} x & \text{if } x > y \\ 0 & \text{otherwise} \end{cases}$$

(see <sup>12</sup>)

5. If  $p(x, y) = \phi^{-1}(\phi(x) \cdot \phi(y))$ , then

- $i(x, y) = \phi^{-1}(\phi(x)\phi(y))$
- $c(x, y) = \phi^{-1}(\phi(x) + \phi(y) - \phi(x)\phi(y))$
- $j(x, y) = \phi^{-1}(1 - \phi(x) - \phi(y) + \phi(x)\phi(y))$

6. Montero's proposal <sup>7,8,9</sup> appears as a particular case of the following solution characterized in Fodor-Roubens <sup>6</sup>:

- $p(x, y) = \phi^{-1}(\min(\phi(x), 1 - \phi(y)))$
- $i(x, y) = \phi^{-1}(\min(\phi(x) + \phi(y) - 1, 0))$
- $c(x, y) = \phi^{-1}(\min(\phi(x) + \phi(y), 1))$
- $j(x, y) = \phi^{-1}(\max(1 - \phi(x) - \phi(y), 0))$

7.  $p(x, y) = \max(0, x - y)$  does verify both (P1) and (P2).

- $p(x, y) = \phi^{-1}(\max(\phi(x) - \phi(y), 0))$

- $i(x, y) = \min(x, y)$
- $c(x, y) = \max(x, y)$
- $j(x, y) = \min(1 - x, 1 - y)$

(see also 3,6,14).

## 5. Final remarks

It should be pointed out that (PA) condition is partially redundant in FORM model. Imposing positive association for  $p$  (or any one of the other intensity functions  $i, j, c$ ) will assure positive association of every intensity function. In fact, as shown above, once any one of these intensity functions has been defined, the whole structure is fully fixed.

Of course, not every intensity function we can think of will be consistent with a complete preference structure. It has been shown that in FORM model we are able to characterize which functions  $p$  can play the role of a strict intensity function, allowing consistent definitions of  $i, c, j$ . If the proposed strict intensity does not verify conditions (P1) and (P2), then there will be no indifference and incomparability consistently defined, and then decision making procedures based on it may lead to paradoxes.

Once a proper strict preference has been defined, additional conditions should just fix automorphism  $\phi$ , characterizing both the aggregation function and the negation function.

## 6. Acknowledgements

This research has been partially supported by Dirección General de Investigación Científica y Técnica (Spain).

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**Proceedings of FAPT '95**

# FOUNDATIONS AND APPLICATIONS OF POSSIBILITY THEORY

*Ghent, Belgium*  
*13–15 December 1995*

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## FOREWORD

### Published by

World Scientific Publishing Co Pte Ltd

P O Box 128, Farrer Road, Singapore 9128

USA office: Suite 1B, 1060 Main Street, River Edge, NJ 07661

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

### Library of Congress Cataloging-in-Publication Data

FAPT '95 (1995 : Ghent, Belgium)

Foundations and applications of possibility theory : proceedings  
of FAPT '95 : Ghent, Belgium, 13-15 December 1995 / editors, Ger de  
Cooman, Da Ruan, Etienne E. Kerre.

p. cm. -- (Advances in fuzzy systems ; vol. 8)

Includes bibliographical references and index.

ISBN 9810222890

1. Fuzzy sets -- Congresses. 2. Possibility -- Congresses.

I. Cooman, Ger de. II. Ruan, Da. III. Kerre, Etienne E.

IV. Title. V. Series.

QA248.F277 1995

511.3--dc20

95-38390

CIP

### British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

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Printed in Singapore

Any book on possibility theory must face up to the question of whether it is a distinct theory, with its own agenda and domain of applications. The papers assembled in *Foundations and Applications of Possibility Theory (FAPT)*, provide an unequivocal answer in the affirmative. In what follows, I should like to offer a comment on this contentious issue.

There is a long-standing tradition in science of viewing probability theory as the methodology of choice for dealing with problems in which uncertainty and/or imprecision play a significant role.

The validity of this tradition was called into question by the advent of fuzzy set theory, possibility theory and — in a different context — the Dempster-Shafer theory of evidence. At the base of this issue is the question: Is probability theory sufficient for dealing with what might be called "nondeterminism," that is, the realm of phenomena in which what is lacking is certainty and/or precision?

If a poll were taken today, a majority would almost surely answer the question in the affirmative. But a growing and increasingly persuasive minority would argue to the contrary. In this perspective, *FAPT* may be viewed as a convincing refutation of the majority view.

The views of the proponents of the primacy of probability theory are epitomized by the statements made by Lindley (1987) and Cheeseman (1985). To quote Lindley: "The only satisfactory description of uncertainty is probability. By this I mean that every uncertainty statement must be in the form of a probability; that several uncertainties must be combined using the rules of probability; and that the calculus of probabilities is adequate to handle all situations involving uncertainty . . . probability is the only sensible description of uncertainty and is adequate for all problems involving uncertainty. All other methods are inadequate . . . anything that can be done with fuzzy logic, belief functions, upper and lower probabilities, or any other alternative to probability can better be done with probability." In a similar vein, Cheeseman asserts that: "The numerous schemes for representing and reasoning about uncertainty that have appeared in the AI literature are unnecessary — probability is all that is needed."

The papers assembled in *FAPT* challenge these views. In essence, they show that (a) possibility theory is a distinct theory with its own agenda and domain of applicability; and that (b) in the main, probability theory and possibility theory are complementary rather than competitive.

In essence, nondeterminism has two distinct constituents: uncertainty and unsharpness, with the latter subsuming inspecificity and set-valued valuations. Probability theory is concerned mainly with uncertainty, whereas fuzzy set theory and possibility theory are focused on unsharpness.