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Searching for the dimension of valued preference relations

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Abstract

The more information a preference structure gives, the more sophisticated representation techniques are necessary, so decision makers can have a global view of data and therefore a comprehensive understanding of the problem they are faced with. In this paper we propose to explore valued preference relations by means of a search for the number of underlying criteria allowing its representation in real space. A general representation theorem for arbitrary crisp binary relations is obtained, showing the difference in representation between incomparability—related to the intersection operator—and other inconsistencies—related to the union operator. A new concept of dimension is therefore proposed, taking into account inconsistencies in source of information. Such a result is then applied to each α-cut of valued preference relations.

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1. Introduction

Easy decision making problems are those that allow a direct and clear answer, most probably without taking into consideration any formal abstract

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model, or just a simple one. The difficulty of a decision making problem is of course relative to each particular decision maker. In fact, a frequent strategy is a search for a specialist, i.e., a person we can trust who declares such a problem as easy according to his/her knowledge or experience (decision makers, or even specialists, may be wrong, and the problem is not the way they see it, but that is another problem). Difficult decision making problems usually require a formal abstract model.

In some cases, a problem cannot be understood as easy because of the amount of data or information we should keep in mind: then we need a mathematical model showing how simple the solution is. The decision maker is overwhelmed by the amount of information, but the problem requires only an organizing data procedure.

In some other cases, the formal model has a deeper role in the decision making problem, showing its internal structure. A solution will follow after some calculation. Of course we may require a second specialist in order to find a solution for the model proposed by the first specialist (although models are usually established taking into account his/her own knowledge, in such a way that the model is at least understood by the first specialist proposing such a model and therefore the existence of a solution is expected). Classical multi-criteria American School and many optimization approaches can be allocated here (see [19]): finding out the right formal model may take some time, but once such a model is accepted, we have a complete understanding of the problem and the problem itself becomes simple, even when we cannot reach a solution. We may of course be surprised by the fact that our model cannot be implemented, due perhaps to some practical restriction, and it may be even the case that we are dealing with a yet unsolved mathematical problem. Anyway, as soon as we can propose a mathematical model fitting information and our decision problem is reduced to a formal optimization problem, we can declare that our problem has been understood (until some conflicting new information reaches us).

But too often in practice we find out that our decision making problem is complex in nature: we do not expect to get a model fully explaining such a problem. We only expect to increase our knowledge about the problem, i.e., to get a better insight into the problem structure.

Simple decision making problems are those where a complete representation model has been possible. The inner structure of the problem has been fully understood, and the problem can be analyzed by means of an appropriate software playing a decision maker role. That is not the situation when the problem is complex: in real life, most people avoid those "black boxes" telling them what to do, partially because they know how complex in nature their problem is. Users want to be the only decision makers. It is not only a claim for an interactive procedure, which indeed will help, but the advance acknowledgement that there is no such complete representation of the problem. Those decision makers are then looking for a better understanding of the problem, so they can be sure they are not forgetting essential facts or possibilities (sometimes they know the solution by heart, but they still need a formal model explaining why). Such a better knowledge of the problem will hopefully open the decision maker's mind to new alternatives or approaches (see [16,18]). In this sense, classical French School and disaggregation–aggregation approaches are closer to a position for exploiting information and improve knowledge of decision makers rather than the classical American School and the multiobjective optimization approach (see [19]).

Real complex decision making problems do need methodologies for a better understanding rather than choice proposals, i.e., aid for knowledge rather than aid for decisions. Each particular multicriteria approach can in principle be considered, in as far as each model can be showing a particular view of the problem. In this context, geometrical representation will always play a key role, as a natural way of showing elaborated information to decision makers. And it is a fact that modern multicriteria procedures give an increasing role to representation software (see, e.g., [10]).

This paper deals with such a geometrical representation. Indeed, one of the key issues in order to understand a problem is the knowledge of possible underlying criteria. Classical dimension theory [5] seems to be a natural possibility when basic information is given in terms of crisp preference relations: the number of underlying criteria may be a hint in the search of underlying criteria (to be defined later in a formal way in order to be useful). Such an approach presents well known algorithmic problems (see [23]), partially solved in [24].

When dealing with valued preference relations, searching for a representation that is comprehensible to the decision maker (or at least a Belton–Hodgkins facilitator [2]) is an absolute need. A first proposal can be found in [12] (see also [13]), by exploiting information from the *dimension function* associated with every α-cut of a given preference relation, once some assumptions have been imposed to our valued preference relation. We now generalize such an approach, taking into account a general representation for arbitrary crisp preference relations.

This paper is organized as follows: basics about classical (crisp) dimension theory are reviewed, being this dimension restricted to partial order sets (Section 2); such an approach is applied to max–min transitive valued preference relations, developing a dimension value for certain α-cuts (Section 3); a representation of arbitrary crisp preferences is then shown (Section 4), and this general result allows the definition of a dimension value for every α-cut of arbitrary valued preference relations (Section 5). Several examples are analyzed, and some particular informative indexes are proposed (Section 6).

2. Crisp dimension theory

Dimension concept has been widely developed in the context of crisp binary relations $R \subset X \times X$, i.e., mappings

$$\mu^R : X \times X \to \{0, 1\}$$

where $X = \{x_1, x_2, \ldots, x_n\}$ represents a finite set of alternatives and $\mu^R(x_i, x_j) = 1$ whenever $x_i R x_j$ and $\mu^R(x_i, x_j) = 0$ otherwise. Dimension theory was initially developed by Dushnik–Miller [5], and subsequently applied to partial orders, i.e., crisp binary relations such that the following conditions hold:

- Non-reflexivity ($\mu^R(x_i, x_i) = 0 \; \forall x_i \in X$).
- Asymmetry ($\mu^R(x_i, x_j) = 1 \Rightarrow \mu^R(x_j, x_i) = 0$).
- Transitivity ($\mu^R(x_i, x_j) = \mu^R(x_j, x_k) = 1 \Rightarrow \mu^R(x_i, x_k) = 1$).

Szpilrajn [20] proved that every partial order may be represented as intersections of linear orders. The dimension of a crisp partial order R, $\dim(R)$, is then defined by Dushnik–Miller [5] as the minimum number of linear orders (complete orders) whose intersection is R. Being R a partial order set (poset) with dimension $\dim(R) = d$, each element $x_i \in X$ can be represented in real space $(x_i^1, \ldots, x_i^d) \in \Re^d$ in such a way that

$$x_i R x_j \Longleftrightarrow x_i^k > x_j^k \quad \forall k \in \{1, \ldots, d\} \quad \forall x_i, x_j \in X$$

(see also Trotter [21]).

Within preference modeling, $x_i R x_j$ means that "alternative x_i is strictly better than alternative x_j", and it can be also denoted as $x_i > x_j$ (by $[x_i, x_j, x_k]$ we shall denote here the linear order with $x_i > x_j, x_j > x_k, x_i > x_k$). Hence, the above intersection of linear orders will be associated with the existence of incomparabilities in decision theory. The dimension $\dim(R) = d$ of a crisp poset R suggests the existence of d underlying criteria, and the coordinates of each element $x_i \in X$ represent the valuation of x_i with respect to all criteria. From this hint, the decision maker can then search for those underlying criteria.

From an algorithmic point of view, dimension theory presents well known difficulties. In particular, Yannakakis [23] proved that it is a *NP*-complete problem to determine if a poset has dimension n, whenever $n \geqslant 3$. However, the algorithm proposed by Yáñez–Montero [24] allows the evaluation of dimension for medium size posets.

In the following section we extend the dimension concept to a particular valued context: when preference relation is max–min transitive.

3. Dimension function of max–min transitive valued preference relations

Given X, a finite set of alternatives, a valued preference relation in X is (see [25]) a fuzzy subset of the cartesian product $X \times X$, being characterized by its membership function

$$\mu : X \times X \to [0, 1]$$

in such a way that $\mu(x_i, x_j)$ represents the degree to which alternative x_i is preferred to alternative x_j. We shall assume by definition that such a preference intensity is referred to as strict preference, in such a way that $\mu(x_i, x_j)$ is understood as the degree to which the assertion $x_i > x_j$ is true. Hence, by definition,

$$\mu(x_i, x_i) = 0 \quad \forall x_i \in X$$

Once $\alpha \in (0, 1]$ has been fixed, the α-cut of a valued preference relation μ is defined as a crisp binary relation R^α in X such that

$$x_i R^\alpha x_j \iff \mu(x_i, x_j) \geqslant \alpha$$

Then, as far as R^α is a poset, its dimension $\dim(R^\alpha)$ is defined. A *dimension mapping* has been in this way defined,

$$d : [0, 1] \to \mathbb{N}$$

with $d(\alpha) = \dim(R^\alpha)$ whenever such a dimension is well defined. Such a dimension mapping is translating the dimension approach into a valued preference context.

Some alternative approaches to the dimension concept of valued preference relations can be found in the literature, taking a quite different point of view. Adnadjevic [1], for example, has proposed an alternative definition of dimension for valued preference relations based upon the notion of multichain, but assuming strong consistency properties to the decision makers. On the contrary, Ovchinnikov [14] proposes a different dimension concept in terms of an underlying representation which appears to be too difficult to be managed by decision makers. Analogous criticism applies to Fodor–Roubens [6] and Doignon–Mitas [4] (both based upon a previous result of Valverde [22], valued preference relations are represented by means of valued preference relations). But managing the whole preference structure is sometimes the key difficulty for decision makers.

As pointed out in the first section, representation techniques should allow a better understanding of our valued preferences, perhaps taking advantage of informative graphics. Within these possible graphics, representation of α-cuts in real space seems to be a first proposal, to be followed of course by any other more sophisticated tool that decision makers can really deal with. Crisp dimension approach applied to all α-cuts of a valued preference relation, as proposed in [12,13], seems an useful hint for decision makers in practice, and they are in fact taken into account in [4] in order to obtain operative bounds.

However, the approach proposed in [12] requires asymmetry and transitivity for every α-cut.

In case our valued strict preference relation is max–min transitive, i.e.,

$$\mu(x_i,x_j) \geqslant \min\{\mu(x_i,x_k),\mu(x_k,x_j)\} \quad \forall x_i,x_j,x_k \in X$$

then R^α is a poset whenever asymmetry holds, i.e., meanwhile those α-cuts do not show second-order cycles. In particular (see [12]), R^α is asymmetric for all $\alpha > \alpha_2$, being

$$\alpha_2 = \max_{x_i \neq x_j} \min\{\mu(x_i,x_j),\mu(x_j,x_i)\}$$

Therefore, since μ is max–min transitive, if and only if, every α-cut R^α is transitive (see [12] but also [4]), if μ is max–min transitive, then we can consider the dimension of R^α for every $\alpha > \alpha_2$.

One problem partially addressed in [12] is how to exploit information from the dimension values

$$\{d(\alpha), \alpha \in (\alpha_2, 1]\}$$

which seems to summarize all the information about the number of underlying criteria. A graphic representation of $d(\alpha)$ versus α indeed allows a better insight into the problem, perhaps taking advantage of some appropriate location or dispersion indexes (see [12]). However, this approach shows a strong basic assumption: the valued preference relation must be max–min transitive. This is a strong restriction in practice, totally unrealistic when X is large.

So, some kind of general representation for any arbitrary α-cut is desirable, even if it is non-symmetric or non-transitive. A useful representation should allow a dimension function being defined in the whole unit interval, but in some way showing every inconsistency. Hence, we should be searching for explanatory representations of arbitrary crisp preference relations. As pointed out in [9], there is an absolute need to understand and explain decision maker inconsistencies: accepted inconsistencies are some times extremely informative. These considerations suggest a generalization of Szpilrajn [20] representation theorem, as shown in the next section.

4. Representation of a general crisp preference

Let us consider first two different situations in order to clarify our approach:

Example 4.1. Let $X = \{x_1, x_2\}$ and the empty relation R_1 on X:

$$\mu^{R_1}(x_1,x_2) = \mu^{R_1}(x_2,x_1) = 0$$

Example 4.2. Let $X = \{x_1, x_2\}$ and the complete non-reflexive preference relation R_2 on X:

$$\mu^{R_2}(x_1, x_2) = \mu^{R_2}(x_2, x_1) = 1$$

First preference relation R_1 shows two incomparable alternatives, allowing the standard representation with dimension 2, by means of the two possible linear orders

$$\mathscr{C} = \{L_1, L_2\}$$

being $L_1 = [x_1, x_2]$ and $L_2 = [x_2, x_1]$. In fact,

$$R_1 = L_1 \cap L_2$$

Second preference relation R_2 shows a cycle of two alternatives, therefore not allowing a standard representation by means of intersections of linear orders. However, linear orders, L_1 or L_2 could explain relation R_2, by using the union operator:

$$R_2 = L_1 \cup L_2$$

In both examples it can be suggested that there are two underlying simultaneous arguments, so a general concept of dimension should assign dimension 2 to both cases. But those two underlying criteria show a deeper conflict in the second case. Whenever neither $x_1 > x_2$ or $x_2 > x_1$ hold, crisp dimension theory will assure that each one appears in at least one of those underlying criteria (both preferences are added in classical representation model and intersection show this). But when both preferences $x_1 > x_2$ and $x_2 > x_1$ are simultaneously accepted by the decision maker, there is a truly deep conflict and decomposition is being done in a different way: there are cycles, but a representation is still possible by means of the union operator.

The following result shows that any strict preference relation can be represented in terms of unions and intersections of linear orders (see [8,9] but also [6]): meanwhile incomparability can be explained by means of the intersection operator, inconsistencies (i.e., symmetry and non-transitivity) require the union operator.

Theorem 4.1. *Let $X = \{x_1, \ldots, x_n\}$ be a finite set of alternatives, and let us consider*

$$\mathscr{C} = \{L/L \text{ linear order on } X\}$$

Then for every non-reflexive crisp binary relation R on X there exists a family of linear orders $\{L_{st}\}_{s,t} \subset \mathscr{C}$ such that

$$R = \bigcup_s \bigcap_t L_{st}$$

Proof. On one hand, if R is an empty binary relation (i.e., $\mu^R(x_i, x_j) = 0$ $\forall (x_i, x_j)$), R may be represented by the intersection of two linear orders

$$[x_1, x_2, \ldots, x_{n-1}, x_n] \cap [x_n, x_{n-1}, \ldots, x_2, x_1]$$

On the other hand, let R be a non-empty binary relation and let (x_i, x_j) be a pair such that $\mu^R(x_i, x_j) = 1$. We then define the poset R^{ij} such that $\mu^{R^{ij}}(x_i, x_j) = 1$ and $\mu^{R^{ij}}(x_k, x_l) = 0, \quad \forall (x_k, x_l) \neq (x_i, x_j)$. Obviously,

$$R = \bigcup_{\{(i,j)/x_i R x_j\}} R^{ij}$$

Since every poset, due to Dushnik–Miller result, can be expressed as intersection of linear orders, $R^{ij} = \cap_k L_k^{ij}$, then,

$$R = \bigcup_{ij} \bigcap_k L_k^{ij} \qquad \square$$

Now we can generalize the classical concept of dimension, initially conceived only for posets.

Definition 4.1. Let us consider X a finite set of alternatives. The *generalized dimension*, $\mathrm{Dim}(R)$, of a crisp non-reflexive binary relation R, is the minimum number of different linear orders, L_{st}, such that

$$R = \bigcup_s \bigcap_t L_{st}$$

Notice that our *generalized representation* is minimal in the sense that we search for the minimum number of different linear orders L_{st} we need (no matter if the same linear order is taken into account several times in the particular representation of some of those posets $\cap_t L_{st}$, such a linear order counts only once). Therefore, the following theorem holds.

Theorem 4.2. *Let $X = \{x_1, \ldots, x_n\}$ be a finite set of alternatives, and let R be a poset on X. Then*

$$\mathrm{Dim}(R) \leqslant \dim(R) \leqslant n/2$$

Proof. First inequality is direct from main definition of generalized dimension. $\square$

Moreover, we know (see, e.g., Trotter [21]) that $\dim(R) \leqslant n/2$ for any poset defined on a set X with $n \geqslant 4$ alternatives.

As pointed out in proof of the above Theorem 4.1, the minimal representation for the extreme case $R = \emptyset$ is obtained by means of the intersection of two linear orders, in such a way that

$$\mathrm{Dim}(\emptyset) = \dim(\emptyset) = 2$$

For general posets, however, the procedure outlined in proof of Theorem 4.1 will not necessarily produce the minimal representation covering the binary relation R. In fact, what we obtain is an upper bound: in any case we can assure that

$$\mathrm{Dim}(R) \leqslant 2n(n-1)$$

for any non-reflexive crisp binary relation R on a finite X.

Let us introduce three illustrative examples.

Example 4.3. Let us consider $X = \{x_1, x_2, x_3, x_4\}$ and the following poset on X:

$$R_3 = \{x_1 > x_2, x_3 > x_4\}$$

Following proof of Theorem 4.1,

$$R_3 = \{x_1 > x_2\} \cup \{x_3 > x_4\}$$

in such a way that such a binary relation R_3 can be covered with three different linear orders:

$$R_3 = \{[x_1, x_2, x_3, x_4] \cap [x_4, x_3, x_1, x_2]\} \cup \{[x_3, x_4, x_2, x_1] \cap [x_1, x_2, x_3, x_4]\}$$

However, the relation R_3 defines a poset

$$R = \{[x_1, x_2, x_3, x_4] \cap [x_3, x_4, x_1, x_2]\}$$

whose dimension is equal to 2:

$$\mathrm{Dim}(R_3) = \dim(R_3) = 2$$

Example 4.4. Let us consider $X = \{x_1, \ldots, x_n\}$ and the relation R_4 such that $\mu^{R_4}(x_i, x_j) = 1$, $\forall i \neq j$. Then

$$\{x_1 > x_2\} = [x_1, x_2, x_3, \ldots, x_{n-1}, x_n] \cap [x_n, x_{n-1}, \ldots, x_3, x_1, x_2]$$

and in general,

$$\{x_i > x_j\} = [x_i, x_j, x_1, \ldots, x_{n-1}, x_n] \cap [x_n, x_{n-1} \ldots, x_1, x_i, x_j] \quad \forall i \neq j$$

We can then conclude that R_4 can be covered by means of

$$4\binom{n}{2} = 2n(n-1)$$

linear orders, in such a way that we can directly assure that $\mathrm{Dim}(R_4) \leqslant n(n-1)$ (if $n = 3$ such a representation implies the use of every linear order on X). But

$$\mathrm{Dim}(R_4) = 2$$

since

$$R_4 = [x_1, x_2, x_3, \ldots, x_{n-1}, x_n] \cup [x_n, x_{n-1}, \ldots, x_3, x_2, x_1]$$

Example 4.5. Let $X = \{x_1, x_2, x_3, x_4\}$ and let R_5 be a crisp binary relation defined by the following matrix

$$\mu^{R_5} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

where, as usual, $\mu_{ij}^{R_5} = \mu^{R_5}(x_i, x_j)$. Its graph is shown in Fig. 1.

This binary relation R_5 can be represented by the union of the following three posets P_1, P_2, P_3 given respectively by the following matrices (see Fig. 2):

$$\mu^{P_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \mu^{P_2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mu^{P_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

On the other hand, each one of those three posets can be decomposed as intersection of the following linear orders:

$$P_1 = [x_1, x_4, x_2, x_3] \cap [x_4, x_3, x_1, x_2]$$
$$P_2 = [x_2, x_3, x_1, x_4] \cap [x_1, x_4, x_2, x_3]$$
$$P_3 = [x_4, x_3, x_1, x_2] \cap [x_2, x_3, x_1, x_4]$$

This representation is based upon the algorithm proposed in [9], so we get a representation in terms of three disjoint posets.

Of course, as in classical dimension theory, such a representation is not unique. The following representation, for example, takes into account three maximal posets (maximal with respect to the natural inclusion, see Fig. 3):

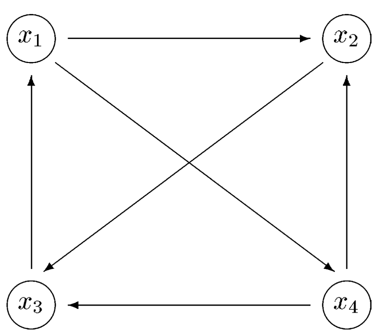

Fig. 1. Binary relation in Example 4.5.

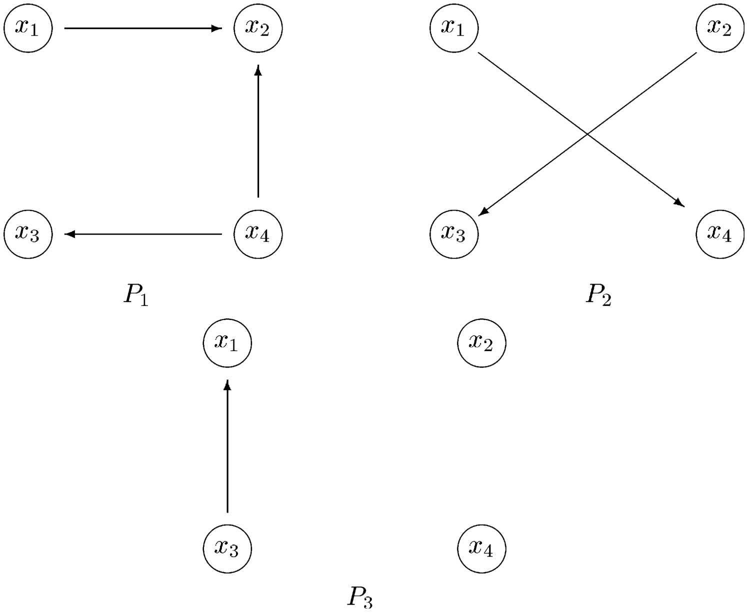

Fig. 2. Binary relation in Example 4.5 decomposed in disjoint partial orders.

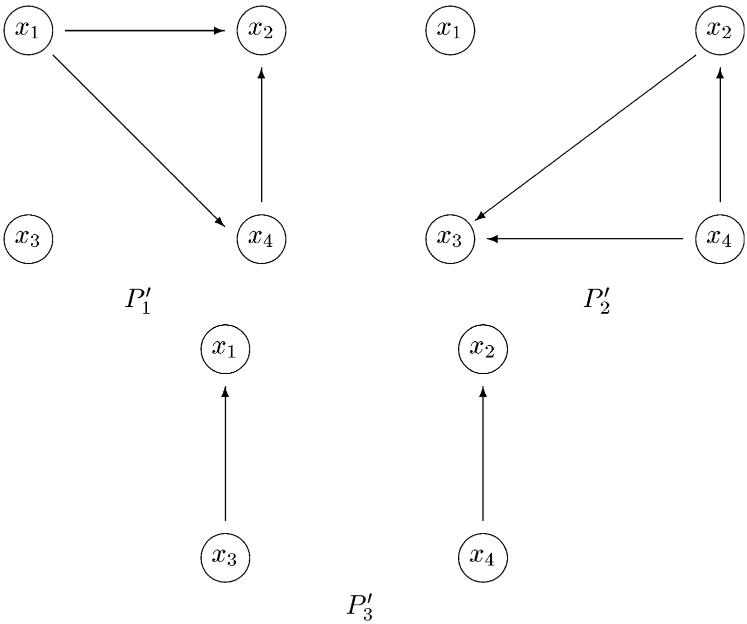

Fig. 3. Binary relation in Example 4.5 decomposed in maximal posets.

$$\mu^{P_1'} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \mu^{P_2'} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \mu^{P_3'} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

in such a way that R_5 can be also represented by the union of these three posets P_1', P_2', P_3' and

$$P_1' = [x_1, x_4, x_2, x_3] \cap [x_3, x_1, x_4, x_2]$$
$$P_2' = [x_4, x_2, x_3, x_1] \cap [x_1, x_4, x_2, x_3]$$
$$P_3' = [x_4, x_2, x_3, x_1] \cap [x_3, x_1, x_4, x_2]$$

In any case, its generalized dimension is 3:

$$\mathrm{Dim}(R_5) = 3$$

It is very interesting to point out that this new concept of *generalized dimension* is not an extension of classical dimension: if restricted to posets, the minimal generalized representation may need less linear orders than classical representation, as shown in the next example.

Example 4.6. Let $X = \{y_1, \dots, y_n, z_1, \dots, z_n\}$ be a finite family of $2n$ alternatives, $n \geqslant 6$. Let us consider R_6 the classical crown on X (see Trotter [21], p. 34): $\mu^{R_6}(y_i, z_j) = 1$ whenever $i \neq j$ and $\mu^{R_6}(x, x') = 0$ otherwise. It is well known (see, e.g., [21]) that $\dim(R_6) = n$. But $\mathrm{Dim}(R_6) < n$, as it is now shown.

Let us take k, $2 < k < n - 2$, and consider P and Q two partial order subsets of R_6 with

- $\mu^P(y_i, z_j) = 1$ whenever $i \leqslant k < j$ or $j \leqslant k < i$ ($\mu^P(x, x') = 0$ otherwise), and
- $\mu^Q(y_i, z_j) = 1$ whenever $i \neq j$ and $i, j \leqslant k$ or $i, j > k$ ($\mu^Q(x, x') = 0$ otherwise).

Then it is easy to check that $\dim(P) = 2$, while $\dim(Q) = \max(k, n - k)$. Since

$$R_6 = P \cup Q$$

we have proved that there exists a generalized representation with $\max(k, n - k) + 2 < n$ linear orders, and some of these linear orders can still be repeated or such a representation is not minimal. In any case,

$$\mathrm{Dim}(R_6) < \dim(R_6)$$

Of course, practical implementation of generalized dimension presents analogous criticism to searching classical dimension: its algorithmic complexity. However, a bound for this new concept may be obtained by a combination of algorithms presented in [9,24].

5. Generalized dimension function

Once we have fully generalized and overcome all key restrictions of classical dimension theory, the above general representation result for crisp strict relations can be therefore translated to relax the normative approach given in [12] and evaluate the generalized dimension $\mathrm{Dim}(R^\alpha)$ for each α-cut, with $\alpha \in (0, 1]$. We no longer need to impose that every α-cut defines a poset.

This approach will then lead to a *generalized dimension function* showing the generalized dimension for every α-cut, no matter our valued preference relation μ is max–min transitive or not.

Definition 5.1. Let X be a finite set of alternatives, and let $\mu : X \times X \to [0, 1]$ be a valued preference relation such that $\mu(x, x) = 0, \forall x \in X$. Then its *generalized dimension function* is a mapping

$$D : [0, 1] \to \mathbb{N}$$

where $D(\alpha) = \mathrm{Dim}(R^\alpha)$.

The following example considers a max–min transitive valued relation μ where there exists a threshold α_2 such that there is no inconsistency in any R^α, for all $\alpha > \alpha_2$, and incomparability is present.

Example 5.1. Let us consider $X = \{x_1, x_2\}$ and let us denote by $L_1 = [x_1, x_2]$ and $L_2 = [x_2, x_1]$ the two possible linear orders on X. Let R_7 be a strict valued binary relation such that $\mu^{R_7}(x_1, x_2) = 0.3$ and $\mu^{R_7}(x_2, x_1) = 0.4$. This relation is depicted in Fig. 4. In this case,

1. If $\alpha > 0.3$, R_7^α is a poset, although two cases can be distinguished:
   (a) $\alpha > 0.4$,

$$\mu^{R_7^\alpha} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

in such a way that

$$\mathrm{Dim}(R_7^\alpha) = \dim(R_7^\alpha) = 2$$

with $R_7^\alpha = L_1 \cap L_2$.

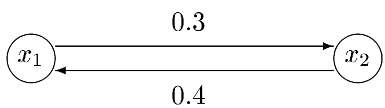

Fig. 4. Binary valued relation in Example 5.1.

(b) $0.3 < \alpha \leqslant 0.4$,

$$\mu^{R_7^\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$\mathrm{Dim}(R_7^\alpha) = \dim(R_7^\alpha) = 1$$

with $R_7^\alpha = L_1$.
2. If $\alpha \leqslant 0.3$, R_7^α is not a poset and

$$\mu^{R_7^\alpha} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in such a way that

$$\mathrm{Dim}(R_7^\alpha) = 2$$

with $R_7^\alpha = L_1 \cup L_2$.
The generalized dimension function of this example is shown in Fig. 5.

It is important to note in the above example that we have found different representations, showing different decision maker attitudes. In fact, a decision maker defining a valued preference relation μ, if forced to be crisp, can face different crisp problems depending on their exigency level: if the decision maker does not take into account low intensities (high α in the above example),

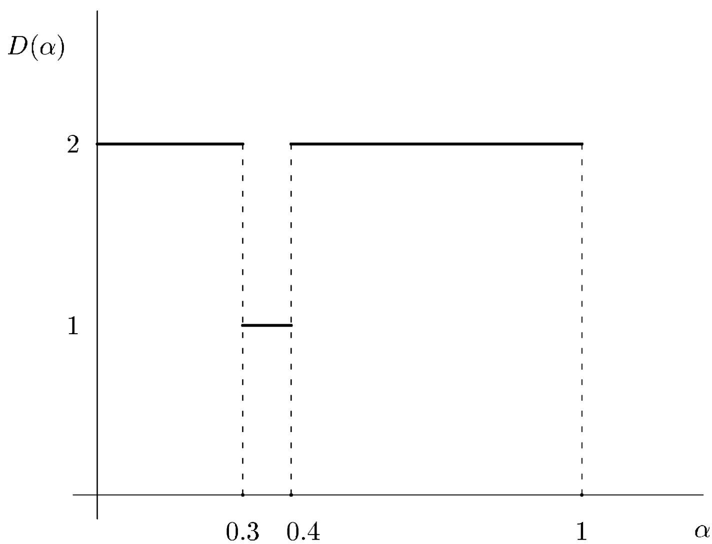

Fig. 5. Generalized dimension function in Example 5.1.

alternatives are easily incomparable (no alternative is sufficiently better than the other according to any underlying criteria); but if the decision maker is sensible to low intensities (low α in the above example), formal cycles will be frequent. Note that formal cycles will most probably introduce a special kind of stress, different from the one with incomparability (see, e.g., [15,17]). In any case, the sequence of α-cuts shows how decision makers, if forced to define crisp preferences when they are valued, may give different answers depending on the α-level they choose.

A simple case of a non-max–min transitive valued preference relation is analyzed in the following example.

Example 5.2. Let us consider $X = \{x_1, x_2, x_3\}$ and the valued preference relation R_8 such that

$$\mu^{R_8} = \begin{pmatrix} 0 & 0.4 & 0 \\ 0 & 0 & 0.6 \\ 0.7 & 0 & 0 \end{pmatrix}$$

depicted in Fig. 6. Four cases can be considered:

- When $\alpha \leqslant 0.4$, there is a three-cycle in the associated α-cut, and such an α-cut can be represented in terms of the following three crisp linear orders:

$$R_8^\alpha = \{[x_1, x_2, x_3] \cap [x_3, x_1, x_2]\} \cup \{[x_2, x_3, x_1] \cap [x_1, x_2, x_3]\} \\ \cup \{[x_3, x_1, x_2] \cap [x_2, x_3, x_1]\}$$

Hence, we have

$$\mathrm{Dim}(R_8^\alpha) = 3$$

- When $0.4 < \alpha \leqslant 0.6$, no cycle is present, but since $x_2 > x_3$ and $x_3 > x_1$ are present, it is missing $x_2 > x_1$ (R_8^α is not a poset):

$$R_8^\alpha = \{[x_2, x_3, x_1] \cap [x_1, x_2, x_3]\} \cup \{[x_3, x_1, x_2] \cap [x_2, x_3, x_1]\}$$

So, we also needed three linear orders:

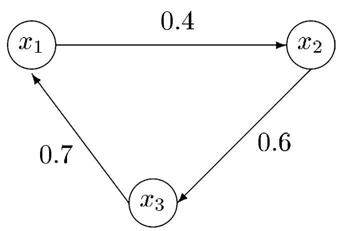

Fig. 6. Binary valued relation in Example 5.2.

$$\mathrm{Dim}(R_8^\alpha) = 3$$

- When $0.6 < \alpha \leqslant 0.7$, we have a crisp partial ordered set with only one arc ($x_3 > x_1$):

$$R_8^\alpha = [x_3, x_1, x_2] \cap [x_2, x_3, x_1]$$

This poset has dimension 2:

$$\mathrm{Dim}(R_8^\alpha) = \dim(R_8^\alpha) = 2$$

- When $\alpha > 0.7$, we have the poset with incomparability between every pair:

$$R_8^\alpha = [x_1, x_2, x_3] \cap [x_3, x_2, x_1]$$

and, consequently, two linear orders are again needed:

$$\mathrm{Dim}(R_8^\alpha) = \dim(R_8^\alpha) = 2$$

The generalized dimension function of valued binary relation of this example is depicted in Fig. 7.

Example 5.3. Let us consider $X = \{x_1, x_2, x_3\}$ and let R_9 be the strict valued preference relation depicted in Fig. 8,

$$\mu^{R_9} = \begin{pmatrix} 0 & 0.2 & 0.3 \\ 0.4 & 0 & 0.6 \\ 0.7 & 0.1 & 0 \end{pmatrix}$$

Seven different α-cuts intervals can be considered:

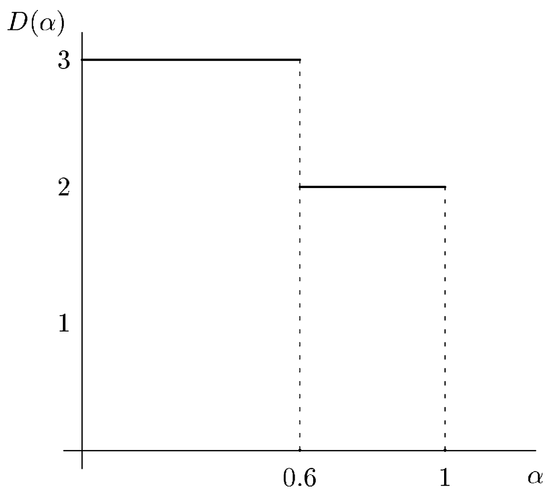

Fig. 7. Generalized dimension function in Example 5.2.

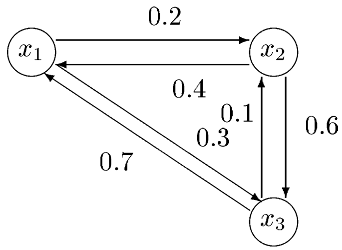

Fig. 8. Binary valued relation in Example 5.3.

1. When $\alpha \leqslant 0.1$, we have

$$\mu^{R_9^\alpha} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

This relation shows cycles (e.g., $x_1 > x_3$, $x_3 > x_1$) but it can be obtained as

$$R_9^\alpha = L_1 \cup L_2$$

where

$$\mu^{L_1} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \mu^{L_2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

That is,

$$R_9^\alpha = ([x_1, x_2, x_3]) \cup ([x_3, x_2, x_1])$$

and $\mathrm{Dim}(R_9^\alpha) = 2$.

2. When $0.1 < \alpha \leqslant 0.2$, R_9^α also shows cycles

$$\mu^{R_9^\alpha} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

In this case, R_9^α can be obtained as the union of three linear orders

$$R_9^\alpha = L_1 \cup P_1$$

where

$$\mu^{P_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

in such a way that

$$R_9^\alpha = ([x_1, x_2, x_3]) \cup ([x_3, x_2, x_1] \cap [x_2, x_3, x_1])$$

and $\mathrm{Dim}(R_9^\alpha) = 3$.

3. When $0.2 < \alpha \leqslant 0.3$, relation R_9^α still shows cycles:

$$\mu^{R_9^\alpha} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Hence,

$$R_9^\alpha = L_3 \cup L_4$$

where

$$\mu^{L_3} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \mu^{L_4} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$R_9^\alpha = ([x_2, x_3, x_1]) \cup ([x_2, x_1, x_3])$$

in such a way that $\mathrm{Dim}(R_9^\alpha) = 2$.

4. When $0.3 < \alpha \leqslant 0.4$, the α-cut is a poset:

$$R_9^\alpha = L_3 = [x_2, x_3, x_1]$$

Therefore, $\mathrm{Dim}(R_9^\alpha) = 1$.

5. When $0.4 < \alpha \leqslant 0.6$, however, the relation R_9^α becomes non-transitive:

$$\mu^{R_9^\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and the union operator is again needed:

$$R_9^\alpha = P_2 \cup P_3$$

where

$$\mu^{P_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \mu^{P_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

in such a way that

$$R_9^\alpha = ([x_2, x_3, x_1] \cap [x_3, x_1, x_2]) \cup ([x_1, x_2, x_3] \cap [x_2, x_3, x_1])$$

and $\mathrm{Dim}(R_9^\alpha) = 3$.

6. When $0.6 < \alpha \leqslant 0.7$, the relation R^α defines the previous P_2 poset:

$$\mu^{R_9^\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$R_9^\alpha = [x_2, x_3, x_1] \cap [x_3, x_1, x_2]$$

in such a way that $\mathrm{Dim}(R_9^\alpha) = 2$.
7. When $0.7 < \alpha$, the relation R_9^α is the empty relation.

$$\mu^{R_9^\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $\mathrm{Dim}(R_9^\alpha) = 2$ and

$$R_9^\alpha = [x_1, x_2, x_3] \cap [x_3, x_2, x_1]$$

The generalized dimension function of this Example 5.3 is depicted in Fig. 9.

An interesting result shown in [7] is that our generalized dimension function will not show big jumps in case arcs are being deleted or added one by one:

Theorem 5.1. *Let us assume that*

$$\mu(x_i, x_j) = \mu(x_k, x_l) \Rightarrow \mu(x_i, x_j) = \mu(x_k, x_l) = 0$$

holds for any $x_i, x_j, x_k, x_l \in X$, *and let us denote*

$$R^{\alpha^-} = \lim_{\alpha_k \uparrow \alpha} R^{\alpha_k}$$

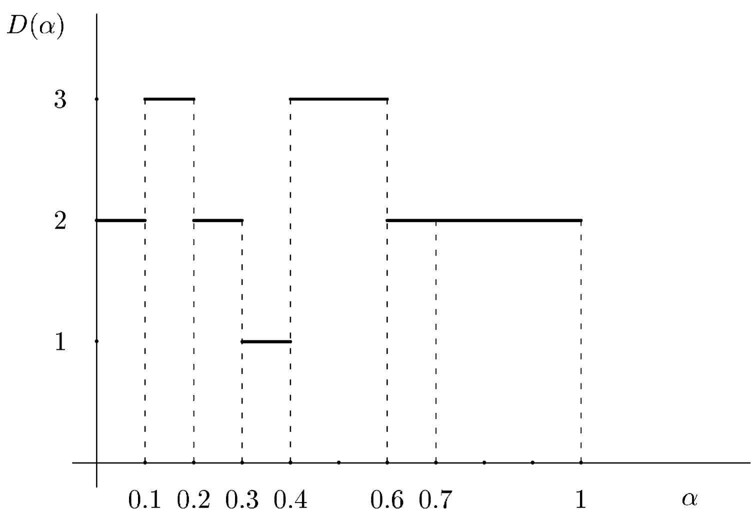

Fig. 9. Generalized dimension function in Example 5.3.

Then

(a) $\mathrm{Dim}(R^{\alpha^-}) - \mathrm{Dim}(R^{\alpha)} \leqslant 2, \quad \forall \alpha > 0$
(b) $\mathrm{Dim}(R^{\alpha)} - \mathrm{Dim}(R^{\alpha^-)} \leqslant 2, \quad \forall \alpha > 0$

Proof. Since the difference between R^{α} and R^{α^-} is in this case one pair at most, and this isolated pair can be represented as the intersection of two linear orders, it can be added by means of the union of this intersection. In case such an arc has to be deleted, we only need to note that the complementary of that arc can be represented as the union of two linear orders. □

6. Critical levels of a generalized dimension function

Generalized dimension function D is therefore well defined for every valued strict binary preference relation R, $D(\alpha) = \mathrm{Dim}(R^{\alpha})$ for all $\alpha \in (0, 1]$.

Indeed, dimension function does not capture all the information contained in the associated representation. It is obvious from previous examples that different representations may arise with the same dimension. Depending on the value of α, each α-cut of a valued preference relation can be either:

- a complete order, i.e., R^{α} is a linear order; or
- a partial order, i.e., asymmetry and transitivity hold, but some incomparability appears; or
- a non-transitive relation without cycles (i.e., some arcs implied by transitivity are missing); or
- a conflictive relation because of cycles, i.e., sequences $x_1, \ldots, x_k$ such that $x_i > x_{i+1}$, $\forall i = 1, \ldots, k-1$ and $x_k > x_1$ (notice that this definition includes symmetry, i.e., second-order cycles where both arcs $x_i > x_j$ and $x_j > x_i$ simultaneously hold).

The last two cases will require the union operator in order to get a generalized representation, although the union operator may also be present in the minimal representation of some posets.

Hence, we wish to evaluate meaningful critical levels in order to verify if α-cuts are still posets, or the degree of the shortest cycle, if it exists. On one hand, it has already been pointed out that transitivity may fail either because some arcs are missing (*weak non-transitivity*) or because a cycle appears (*strong non-transitivity*). Hence, non-transitivity region can be again divided into two parts, depending on whether there are cycles or not. *Strong non-transitivity* region can also be divided into different regions, depending on the length of its shortest cycle (following [11], the shorter the cycle is, the larger the conflict should be considered in practice). This fact leads us to introduce a family of critical values that generalizes the above α_2 critical level for second-order cycles, as defined in Section 2. The following levels can therefore be introduced:

1. The *transitivity level* α_0, representing the minimal value such that R^α is transitive for all $\alpha > \alpha_0$. As an exercise, we can check transitivity of α-cuts in Example 5.3:
   - $\alpha \in (0.0, 0.1]$: R_9^α is transitive.
   - $\alpha \in (0.1, 0.2]$: R_9^α is non-transitive, since

     $$\mu^{R_9}(x_3, x_2) = 0.1 < \min\{\mu^{R_9}(x_3, x_1) = 0.7, \mu^{R_9}(x_1, x_2) = 0.2\}$$

   - $\alpha \in (0.2, 0.4]$: R_9^α is transitive.
   - $\alpha \in (0.4, 0.6]$: R_9^α is non-transitive, since

     $$\mu^{R_9}(x_2, x_1) = 0.4 < \min\{\mu^{R_9}(x_2, x_3) = 0.6, \mu^{R_9}(x_3, x_1) = 0.7\}$$

   - $\alpha \in (0.6, 1.0]$: R_9^α is again transitive.

Hence, $\alpha_0 = 0.6$ for this relation R_9 (see Fig. 10).

2. The *k-acyclicity* level α_k, representing the minimal value such that R^α has no k-order cycles (nor has the lower order). Of course, $\alpha_1 = 0$ since we have assumed by definition that $\mu(x, x) = 0$, $\forall x \in X$, and α_2 has been already defined. It is obvious that $(\alpha_k)_{k=1}^{\infty}$ is a non-decreasing sequence whose maximum is a critical *acyclicity level* α_∞ being the minimum value such that there is no cycle in R^α, $\forall \alpha > \alpha_\infty$. Obviously, $\alpha_\infty = \alpha_n$, n being the number of elements in X.

Indeed, our generalized dimension function together with the sequence of critical values

$$\alpha_0, \alpha_2, \ldots, \alpha_n$$

gives a quite complete approach to the underlying representation and its associated inconsistencies. In particular, we can assure that α-cuts are posets whenever

$$\alpha > \max\{\alpha_0, \alpha_2\}$$

The transitivity level α_0 can be easily obtained by means of the following algorithm, with complexity $O(n^3)$:

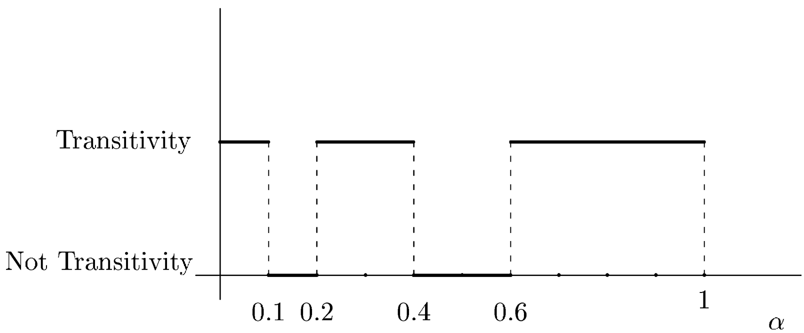

Fig. 10. Transitivity intervals in Example 5.3.

Transitivity level computation

```
α_0 = 0
do i = 1, n
   do j = 1, n (j ≠ i)
     do k = 1, n (k ≠ i, k ≠ j)
        β = min{μ_ik; μ_kj}
        if (μ_ij < β) then
           α_0 = max{α_0; β}
        endif
     enddo
   enddo
enddo
```

In order to compute the critical acyclicity level, we require to enumerate all possible cycles defined in X. Taking into account that there are

$$\binom{n}{k}(k-1)!$$

cycles of k elements in X, the total number of cycles in X are

$$\sum_{k=2}^{n}\binom{n}{k}(k-1)!$$

and the critical acyclicity level can be computed as

$$\alpha_n = \max_{C(x_{i_1},\ldots,x_{i_k})}\{\min\{\mu(x_{i_1},x_{i_2}),\ldots,\mu(x_{i_k},x_{i_1})\}\}$$

This computation has exponential complexity.

Example 6.1. In previous Example 5.3, we find three second-order cycles and two third-order cycles in R_9 ($n = 3$):
1. $C(x_1,x_2)$: $\min\{\mu^{R_9}(x_1,x_2) = 0.2;\ \mu^{R_9}(x_2,x_1) = 0.4\} = 0.2$.
2. $C(x_1,x_3)$: $\min\{\mu^{R_9}(x_1,x_3) = 0.3;\ \mu^{R_9}(x_3,x_1) = 0.7\} = 0.3$.
3. $C(x_2,x_3)$: $\min\{\mu^{R_9}(x_2,x_3) = 0.6;\ \mu^{R_9}(x_3,x_2) = 0.1\} = 0.1$.
4. $C(x_1,x_2,x_3)$: $\min\{\mu^{R_9}(x_1,x_2) = 0.2;\ \mu^{R_9}(x_2,x_3) = 0.6;\ \mu^{R_9}(x_3,x_1) = 0.7\} = 0.2$.
5. $C(x_3,x_2,x_1)$: $\min\{\mu^{R_9}(x_3,x_2) = 0.1;\ \mu^{R_9}(x_2,x_1) = 0.4;\ \mu^{R_9}(x_1,x_3) = 0.3\} = 0.1$.

Hence,

$$\alpha_3 = \max\{0.2, 0.3, 0.1, 0.2, 0.1\} = 0.3 = \alpha_2$$

The acyclicity interval $(0.3, 1]$ is depicted in Fig. 11, meanwhile a cycle appears in R_9^{α} for $\alpha \in (0.0, 0.3]$ (notice that interval $(0.4, 0.6]$ does not show cycles, but transitivity does not hold).

Poset region (i.e., the values of α for which the binary relation R_9^{α} is a poset) in Example 5.3 is depicted in Fig. 12: $(0.3, 0.4] \cup (0.6, 1.0]$. Classical dimension

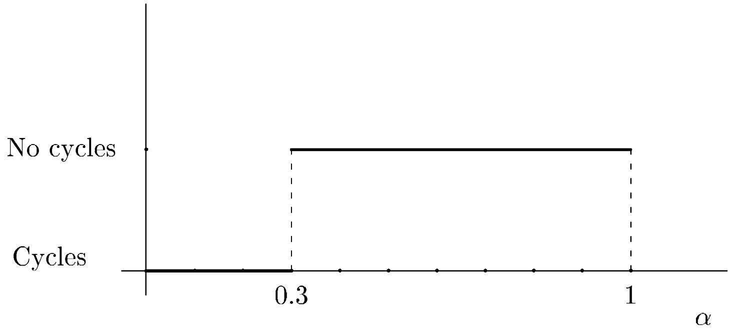

Fig. 11. Acyclicity intervals in Example 5.3.

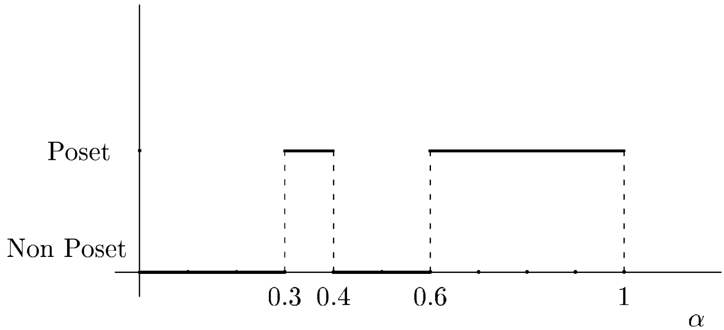

Fig. 12. Poset intervals in Example 5.3.

theory does apply to these two intervals, meanwhile our extended dimension theory also applies to $(0.0, 0.3] \cup (0.4, 0.6]$.

7. Final comments

Once a general representation theorem for crisp preference relations have been proved, it has been possible to develop a more general dimension concept, not being restricted now to partial ordered sets. The fact that this new generalized dimension is not an extension of classical dimension should not be disturbing: as we pointed out in the first section of this paper, a key issue for any multicriteria methodology, if we wish it to be a useful *knowledge aid* tool, should focus on the representation issue. It could have been expected that allowing representations by means of the union operator together with the intersection operator will give more accurate representations than representations obtained taking into account only intersections. Our main objective should not be a number ($\mathrm{Dim}(R)$), but an informative representation

($R = \cup_s \cap_t L_{st}$). Dimension, no matter how we define it, is only a hint for a attractive representation. Perhaps generalized dimension deserves as much theoretical attention as classical dimension theory has deserved in the past.

In any case, our generalized dimension is associated with the minimal number of underlying linear orders explaining preference relations in the presence of incomparability and inconsistency (other rational backgrounds, according to [3] will be considered in the future, see [7]). Such a result has been translated to every α-cut of an arbitrary valued preference relation, allowing a generalized dimension function which should in turn give a better insight into the structure of underlying criteria explaining our preferences. In this way, decision makers can take some advantage of the standard representation in the real spaces they are used to.

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