Lie Algebras obtained as extensions by derivations of the nilpotent algebra $\mathcal{L}_{5,3}$

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ABSTRACT

For the only quasi-filiform Lie algebra $\mathcal{L}_{5,3}$ admitting a Levi factor in its Lie algebra of derivations, the extensions by derivations are classified over $\mathbb{C}$ and $\mathbb{R}$. Moreover, the invariants of these extensions are computed.

Key words: derivation algebra, semi-direct product, solvable Lie algebra.

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1. Introduction

The problem of determining the extensions by derivations of Lie algebras is an old one, and constitutes an important technique both from the cohomological and geometrical point of view. Although it is known that any semisimple Lie algebra arises as the Levi factor of the algebra of derivations of a nilpotent algebra [1], there are still no generic criteria to determine, given a nilpotent algebra, whether the algebra of derivations is solvable or not, up to some special classes of nilpotent algebras. In [2], it was proved that a semidirect sum $\mathfrak{g} = \mathfrak{s} \oplus_R \mathfrak{r}$ of a semisimple Lie algebra $\mathfrak{s}$ and a solvable Lie algebra $\mathfrak{r}$ with respect to a representation of $\mathfrak{s}$ which does not decompose into a direct sum of ideals, cannot have filiform Lie algebra either as radical or nilradical. Therefore derivation algebras of filiform Lie algebras are solvable. For the case of quasi-filiform algebras, i.e., those having the second highest nilindex, this feature remains valid for parameterized families [3], but can fail for low dimensional representatives of the class.
The aim of this paper is to determine these exceptions and classify the corresponding

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semi-direct products arising. There are specific reason for this distinction in low dimension. The algebra considered here is actually the only among the quasi-filiform algebras to admit an associative. bilinear non-degenerated form, which is of interest in the construction of non-Abelian gauge theories [4]. In addition to the non-solvable extensions, we determine thei invariants for the coadjoint representation, which are shown to be rational functions.

2. Algebras of low dimension with $\mathcal{L}_{5,3}$ as nilradical

To any Lie algebra $\mathfrak{g}$, we can associate the lower central sequence $\mathfrak{C}^0\mathfrak{g} = \mathfrak{g} \supseteq \mathfrak{C}^1\mathfrak{g} \supseteq \cdots \supseteq \mathfrak{C}^k\mathfrak{g} \supseteq \ldots$ defined recursively by:

$$\mathfrak{C}^0\mathfrak{g} = \mathfrak{g} \qquad \mathfrak{C}^{k+1}\mathfrak{g} = \left[\mathfrak{C}^k\mathfrak{g}, \mathfrak{g}\right] \quad k \geq 0$$

If there exists $k \in \mathbb{N}$ such that $\mathfrak{C}^k\mathfrak{g} = \{0\}$ then $\mathfrak{g}$ is a nilpotent Lie algebra. The lowest value of k for which we have $\mathfrak{C}^k\mathfrak{g} = 0$ is the nilindex of $\mathfrak{g}$. If $\mathfrak{g}$ is a nilpotent Lie algebra and k its nilindex, we will say that $\mathfrak{g}$ is k-nilpotent.
Any n-dimensional Lie algebra $\mathfrak{g}$ is called filiform if

$$\dim \mathfrak{C}^k\mathfrak{g} = n - 1 - k \quad k \geq 1$$

i.e., if $\mathfrak{g}$ is $n-1$-nilpotent.
Here we shall consider the nilpotent algebra $\mathcal{L}_{5,3}$ defined over the basis $\{X_0, X_1, X_2, X_3, X_4\}$ by the brackets:

$$[X_0, X_1] = X_2, \quad [X_0, X_2] = X_3, \quad [X_1, X_2] = X_4. \tag{1}$$

Let $\mathfrak{g}$ be a real non-split Lie algebra of dimension 6 or 7 with $\mathcal{L}_{5,3}$ as nilradical. For those dimensions, $\mathfrak{g}$ must be solvable so its derived algebra is nilpotent and:

$$[\mathfrak{t}, \mathfrak{t}] \subseteq [\mathfrak{g}, \mathfrak{g}] \subseteq \mathcal{L}_{5,3} \tag{2}$$

Therefore, $\mathfrak{g}$ admits the decomposition $\mathfrak{g} = \mathcal{L}_{5,3} \vec{\oplus} \mathfrak{t}$ where $\vec{\oplus}$ denotes the semidirect sum and $[\mathfrak{t}, \mathfrak{t}] \subseteq \mathcal{L}_{5,3}$.

Proposition 2.1. *Let $\mathfrak{g}$ be a real non-split Lie algebra of dimension 6 with nilradical isomorphic to $\mathcal{L}_{5,3}$ then $\mathfrak{g}$ is solvable and is one of the algebras $\mathfrak{g}_{5,3}^{6,1}, \ldots, \mathfrak{g}_{5,3}^{6,9}$ represented by a basis $\{X_0, \ldots, X_4, Y\}$ and the Lie brackets involving Y given by:*

(i) $\mathfrak{g}_{5,3}^{6,1}$:
$[Y, X_0] = X_0$, $[Y, X_i] = (i - 1 + \lambda)X_i$ *for* $i = 1, 2, 3$, $[Y, X_4] = (1 + 2\lambda)X_4$.

(ii) $\mathfrak{g}_{5,3}^{6,2}$:
$[Y, X_i] = X_i$ *for* $i = 0, 3$, $[Y, X_1] = -X_1 + X_4$, $[Y, X_4] = -X_4$.

(iii) $\mathfrak{g}_{5,3}^{6,3}$:
$[Y, X_0] = X_0 + X_4$, $[Y, X_2] = X_2$, $[Y, X_3] = 2X_3$, $[Y, X_4] = X_4$.

(iv) $\mathfrak{g}_{5,3}^{6,4}$:
$[Y, X_0] = X_0 - X_4$, $[Y, X_2] = X_2$, $[Y, X_3] = 2X_3$, $[Y, X_4] = X_4$.

(v) $\mathfrak{g}_{5,3}^{6,5}$:
$[Y,X_0]=X_0+X_1, \quad [Y,X_i]=iX_i \quad \textit{for } i=1,2,4, \quad [Y,X_3]=3X_3+X_4.$

(vi) $\mathfrak{g}_{5,3}^{6,6}$:
$[Y,X_0]=X_0-X_1, \quad [Y,X_1]=X_0+X_1, \quad [Y,X_2]=2X_2,$
$[Y,X_3]=3X_3-X_4, \quad [Y,X_4]=X_3+3X_4.$

(vii) $\mathfrak{g}_{5,3}^{6,7}$:
$[Y,X_i]=-X_{i+1} \quad \textit{for } i=0,3, \quad [Y,X_i]=X_{i-1} \quad \textit{for } i=1,4.$

(viii) $\mathfrak{g}_{5,3}^{6,8}$:
$[Y,X_0]=-X_1+X_4, \quad [Y,X_1]=X_0, \quad [Y,X_3]=-X_4, \quad [Y,X_4]=X_3.$

(ix) $\mathfrak{g}_{5,3}^{6,9}$:
$[Y,X_0]=-X_1-X_4, \quad [Y,X_1]=X_0, \quad [Y,X_3]=-X_4, \quad [Y,X_4]=X_3.$

Proof. Following the notations above and assuming that $\mathfrak{g}$ satisfies the hypothesis, $\mathfrak{t}$ is generated by a single vector Y such that $f = ad(Y)\mid_{\mathcal{L}_{5,3}}$ is a derivation of $\mathcal{L}_{5,3}$. We consider the basis $\{X_0,X_1,X_2,X_3,X_4\}$ of (1) and we denote derivations by $f(X_j)=\sum_{i=0}^{4} f_j^i X_i$ for $j=0,\ldots,4$. Now it can be easily shown that the matrix $\{f_j^i\}_{i,j\in\{0,1,2,3,4\}}$ is given by:

$$\begin{pmatrix} f_0^0 & f_1^0 & 0 & 0 & 0 \\ f_0^1 & f_1^1 & 0 & 0 & 0 \\ f_0^2 & f_1^2 & f_0^0+f_1^1 & 0 & 0 \\ f_0^3 & f_1^3 & f_1^2 & 2f_0^0+f_1^1 & f_1^0 \\ f_0^4 & f_1^4 & -f_0^2 & f_0^1 & f_0^0+2f_1^1 \end{pmatrix}$$

The change of basis $Y \to Y - f_1^2 X_0 + f_0^2 X_1 + f_0^3 X_2$ allows us to suppose:

$$f \sim \begin{pmatrix} f_0^0 & f_1^0 & 0 & 0 & 0 \\ f_0^1 & f_1^1 & 0 & 0 & 0 \\ 0 & 0 & f_0^0+f_1^1 & 0 & 0 \\ 0 & f_1^3 & 0 & 2f_0^0+f_1^1 & f_1^0 \\ f_0^4 & f_1^4 & 0 & f_0^1 & f_0^0+2f_1^1 \end{pmatrix}$$

(i) Let $f_1^0=0$, we shall divide our investigation into subcases determined by values of f_0^0 and f_1^1.

(a) Let $f_0^0 \neq 0$, a scaling change allows us to suppose that $f_0^0=1$.

1. If $f_1^1 \neq 1$, f_0^1, f_0^4, f_1^4 can be removed and we deduce:
$f \sim diag(1, f_1^1, 1+f_1^1, 2+f_1^1, 1+2f_1^1)$

$$\begin{cases} [Y,X_0] &= X_0 \\ [Y,X_1] &= f_1^1 X_1 \\ [Y,X_2] &= (1+f_1^1)X_2 \\ [Y,X_3] &= (2+f_1^1)X_3 \\ [Y,X_4] &= (1+2f_1^1)X_4 \end{cases}$$

We obtain the family of algebras $\mathfrak{g}_{5,3}^{6,1}(\mathfrak{f}_1^1)$ described in [3].

2. If $f_1^1=-1$, we put f_0^1 and f_0^4 to zero. If $f_0^4=0$, we obtain the algebra $\mathfrak{g}_{5,3}^{6,1}(-1)$ so we can suppose that $f_0^4\neq 0$. By the change of basis $X_i\to f_1^4X_i$ for $i=1,2,3$, $X_4\to (f_1^4)^2X_4$, we put $f_0^4=1$. Therefore,
$$f\sim\begin{pmatrix}1&0&0&0&0\\0&-1&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&-1\end{pmatrix}\quad\left\{\begin{array}{lcl}[Y,X_0]&=&X_0\\ [Y,X_1]&=&-X_1+X_4\\ [Y,X_2]&=&0\\ [Y,X_3]&=&X_3\\ [Y,X_4]&=&-X_4\end{array}\right.$$
and we obtain the algebra $\mathfrak{g}_{5,3}^{6,2}$.
3. If $f_1^1=0$, we put $f_1^4=0$. When $f_0^4=0$, we reach the algebra $\mathfrak{g}_{5,3}^{6,1}(0)$ so we can suppose $f_0^4=1$, in $\mathbb{C}$ and $f_0^4=\pm 1$ in $\mathbb{R}$.
We deduce that:
$$f\sim\begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&2&0\\\delta&0&0&0&1\end{pmatrix}\quad\left\{\begin{array}{lcl}[Y,X_0]&=&X_0+\delta X_4\\ [Y,X_1]&=&0\\ [Y,X_2]&=&X_2\\ [Y,X_3]&=&2X_3\\ [Y,X_4]&=&X_4\end{array}\right.$$
with $\delta=1$ in $\mathbb{C}$ and $\delta=\pm 1$ in $\mathbb{R}$. We obtain the algebras $\mathfrak{g}_{5,3}^{6,3}$ and $\mathfrak{g}_{5,3}^{6,4}$.
4. If $f_1^1=1$, we put $f_0^4=f_1^4=0$ and we can suppose $f_0^1=1$.
$$f\sim\begin{pmatrix}1&0&0&0&0\\1&1&0&0&0\\0&0&2&0&0\\0&0&0&3&0\\0&0&0&1&3\end{pmatrix}\quad\left\{\begin{array}{lcl}[Y,X_0]&=&X_0+X_1\\ [Y,X_1]&=&X_1\\ [Y,X_2]&=&2X_2\\ [Y,X_3]&=&3X_3+X_4\\ [Y,X_4]&=&3X_4\end{array}\right.$$
We obtain the algebra $\mathfrak{g}_{5,3}^{6,5}$.

(b) Let us suppose now that $f_0^0=0$. We have $f_1^1\neq 0$ because f cannot be nilpotent, then by a scaling transformation we can suppose that $f_1^1=1$. We can also put $f_0^1=f_0^4=f_1^4=0$.
If $f_1^3=0$ then we obtain:
$$f\sim diag(0,1,1,1,2)\quad\left\{\begin{array}{lcl}[Y,X_0]&=&0\\ [Y,X_1]&=&X_1\\ [Y,X_2]&=&X_2\\ [Y,X_3]&=&X_3\\ [Y,X_4]&=&2X_4\end{array}\right.$$
If $f_1^3\neq 0$, we can make $f_1^3=1$ in $\mathbb{C}$ and $f_1^3=\pm 1$ in $\mathbb{R}$ therefore:
$$f\sim\begin{pmatrix}0&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&\delta&0&1&0\\0&0&0&0&2\end{pmatrix}\quad\left\{\begin{array}{lcl}[Y,X_0]&=&0\\ [Y,X_1]&=&X_1+\delta X_3\\ [Y,X_2]&=&X_2\\ [Y,X_3]&=&X_3\\ [Y,X_4]&=&2X_4\end{array}\right.$$

where $\delta = 1$ in $\mathbb{C}$ and $\delta = \pm 1$ in $\mathbb{R}$.
These subcases provides us the algebras $\mathfrak{g}_{5,3}^{6,1}(\mathfrak{o})$, $\mathfrak{g}_{5,3}^{6,3}$ and $\mathfrak{g}_{5,3}^{6,5}$ by the change of basis:
$X_0 \to X_1$
$X_1 \to X_0$
$X_2 \to -X_2$
$X_3 \to -X_4$
$X_4 \to -X_3$

(ii) If $f_1^0 \neq 0$, the change of basis $X_0 \to f_1^0 X_0$, $X_i \to (f_1^0)^{i-1} X_i$ for i=1,2,3 and $X_4 \to f_1^0 X_4$ allows us to suppose that $f_1^0 = 1$ and then the characteristic polynomial of f is:

$$(f_0^0 + f_1^1 - \lambda) \quad \underbrace{[(f_0^0 - \lambda)(f_1^1 - \lambda) - f_0^1]}_{p(\lambda)} \quad \underbrace{[(2f_0^0 + f_1^1 - \lambda)(f_0^0 + 2f_1^1 - \lambda) - f_0^1]}_{q(\lambda)}$$

Note that λ is a root of p if, and only if, $\lambda + f_0^0 + f_1^1$ is a root of q, therefore we just have to find the roots of p:

$$\lambda_{\pm} = \frac{\xi \pm \sqrt{\Delta}}{2} \quad \text{where} \quad \xi = f_0^0 + f_1^1 \quad \text{and} \quad \Delta = (f_0^0 + f_1^1)^2 - 4(f_0^0 f_1^1 - f_0^1)$$

When $\Delta > 0$, the eigenvalues of f are:

$$\frac{\xi - \sqrt{\Delta}}{2} \quad \frac{\xi + \sqrt{\Delta}}{2} \quad \xi \quad \frac{3\xi - \sqrt{\Delta}}{2} \quad \frac{3\xi + \sqrt{\Delta}}{2}$$

By making the change of basis:
$X_0 \to X_0 + \alpha X_1$
$X_1 \to X_1 + \beta X_0$
$X_2 \to (1 - \alpha\beta) X_2$ where $\alpha = \frac{-f_0^0 + f_1^1 - \sqrt{\Delta}}{2}$ y $\beta = \frac{2}{-f_0^0 + f_1^1 + \sqrt{\Delta}}$
$X_3 \to (1 - \alpha\beta)(X_3 + \alpha X_4)$
$X_4 \to (1 - \alpha\beta)(X_4 + \beta X_3)$
We obtain:

$$\begin{cases} [Y, X_0] &= \frac{\xi - \sqrt{\Delta}}{2} X_0 + f_0^4 X_4 \\ [Y, X_1] &= \frac{\xi + \sqrt{\Delta}}{2} X_1 + f_1^3 X_3 + f_1^4 X_4 \\ [Y, X_2] &= \xi X_2 \\ [Y, X_3] &= \frac{3\xi - \sqrt{\Delta}}{2} X_3 \\ [Y, X_4] &= \frac{3\xi + \sqrt{\Delta}}{2} X_4 \end{cases}$$

This subcase brings back to the family of algebras $\mathfrak{g}_{5,3}^{6,1}(\lambda)$.
If $\Delta = 0$, the eigenvalues of f are $\frac{\xi}{2}$, ξ, $\frac{3\xi}{2}$ and by means of the change basis $X_0 \to X_0 + \alpha X_1$, $X_3 \to X_3 + \alpha X_4$ with $\alpha = \frac{-f_0^0 + f_1^1}{2}$, we obtain:

$$f \sim \begin{pmatrix} \frac{\xi}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{\xi}{2} & 0 & 0 & 0 \\ 0 & 0 & \xi & 0 & 0 \\ 0 & f_1^3 & 0 & \frac{3\xi}{2} & 0 \\ f_0^4 & f_1^4 & 0 & 0 & \frac{3\xi}{2} \end{pmatrix}.$$

Because of the non-nilpotency of f, we can scale ξ to 1 and we can suppose that $f_0^4 = f_1^4 = f_1^3 = 0$, therefore:

$$\begin{cases} [Y, X_0] &= X_0 \\ [Y, X_1] &= X_0 + X_1 \\ [Y, X_2] &= 2X_2 \\ [Y, X_3] &= 3X_3 \\ [Y, X_4] &= X_3 + 3X_4 \end{cases}$$

The change after-specified, brings back to the algebra $\mathfrak{g}_{5,3}^{6,5}$.

$X_0 \to X_1$
$X_1 \to X_0$
$X_2 \to -X_2$
$X_3 \to -X_4$
$X_4 \to -X_3$

If $\Delta < 0$, we can't reach new algebras over $\mathbb{C}$. In $\mathbb{R}$, by making the change of basis:

$$\begin{array}{l} X_0 \to 2X_0 + \alpha X_1 \\ X_1 \to \sqrt{-\Delta}X_1 \\ X_2 \to 2\sqrt{-\Delta}X_2 \\ X_3 \to 2\sqrt{-\Delta}(2X_3 + \alpha X_4) \\ X_4 \to -2\Delta X_4 \end{array} \qquad \text{with} \quad \alpha = -f_0^0 + f_1^1,$$

we obtain:

$$f \sim \begin{pmatrix} \frac{\xi}{2} & \frac{\sqrt{-\Delta}}{2} & 0 & 0 & 0 \\ -\frac{\sqrt{-\Delta}}{2} & \frac{\xi}{2} & 0 & 0 & 0 \\ 0 & 0 & \xi & 0 & 0 \\ 0 & f_1^3 & 0 & \frac{3\xi}{2} & \frac{\sqrt{-\Delta}}{2} \\ f_0^4 & f_1^4 & 0 & -\frac{\sqrt{-\Delta}}{2} & \frac{3\xi}{2} \end{pmatrix}.$$

If ξ is nonzero, we rescale f and this give the algebra $\mathfrak{g}_{5,3}^{6,6}$.
If $\xi = 0$, we can put f_1^3 and f_1^4 to zero and we can get the algebras $\mathfrak{g}_{5,3}^{6,7}$ when $f_0^4 = 0$ and, in the opposite case, the algebras $\mathfrak{g}_{5,3}^{6,8}$ and $\mathfrak{g}_{5,3}^{6,9}$.

□

Proposition 2.2. *Let $\mathfrak{g}$ be a real non-split Lie algebra of dimension 7 with nilradical isomorphic to $\mathcal{L}_{5,3}$ then $\mathfrak{g}$ is solvable and is isomorphic to the algebra $\mathfrak{g}_{5,3}^{7,1}$ or to $\mathfrak{g}_{5,3}^{7,2}$, each one represented over the basis $\{X_0, \ldots, X_4, Y, Z\}$ and the Lie brackets involving Y and Z given by:*

(i) $\mathfrak{g}_{5,3}^{7,1}$*:* $[Y, X_i] = X_i$ *for* $i = 0, 2, 4$*,* $[Y, X_3] = 2X_3$
$[Z, X_i] = X_i$ *for* $i = 1, 2, 3$*,* $[Z, X_4] = 2X_4$

(ii) $\mathfrak{g}_{5,3}^{7,2}$*:* $[Y, X_i] = -X_{i+1}$ *for* $i = 0, 3$*,* $[Y, X_i] = X_{i-1}$ *for* $i = 1, 4$
$[Z, X_0] = X_0$*,* $[Z, X_i] = iX_i$ *for* $i = 1, 2, 3$*,* $[Z, X_4] = 3X_4$

Proof. If $\mathfrak{g}$ is a non-split Lie algebra of dimension 7 with $\mathcal{L}_{5,3}$ as nilradical, $\mathfrak{g}$ decomposes into the semi-direct sum $\mathcal{L}_{5,3} \vec{\oplus} \mathfrak{t}$ where $\mathfrak{t}$ is generated by two vectors Y and Z like $f = ad(Y) \mid_{\mathcal{L}_{5,3}}$ and $g = ad(Z) \mid_{\mathcal{L}_{5,3}}$ are derivations of $\mathcal{L}_{5,3}$.

(i) If $f_1^0 = g_1^0 = 0$, by taking linear combinations of f and g we can suppose that $f_1^1 = g_0^0 = 0$ and $f_0^0 = g_1^1 = 1$. The eigenvalues of $f + 2g$ and $f + 3g$ are all different then they are diagonalizable, so are f and g. It follows that:
$f \sim diag(1, 0, 1, 2, 1)$ $g \sim diag(0, 1, 1, 1, 2)$

$$\left\{\begin{array}{lcl}[Y, X_0] & = & X_0\\ [Y, X_1] & = & 0\\ [Y, X_2] & = & X_2\\ [Y, X_3] & = & 2X_3\\ [Y, X_4] & = & X_4\end{array}\right. \qquad \left\{\begin{array}{lcl}[Z, X_0] & = & 0\\ [Z, X_1] & = & X_1\\ [Z, X_2] & = & X_2\\ [Z, X_3] & = & X_3\\ [Z, X_4] & = & 2X_4\end{array}\right.$$

and we reach the algebra $\mathfrak{g}_{5,3}^{7,1}$.

(ii) If $f_1^0 \neq 0$ or $g_1^0 \neq 0$, without loss of generality, we can suppose that $f_1^0 = 1$ and $g_1^0 = 0$. The condition (2), implies that f and g must commute to an inner derivation then $g_1^0 = 0$ and $g_0^0 = g_1^1$. Furthermore, g_0^0 is nonzero because g cannot be nilpotent. By making the change of base $f \to f - \frac{f_0^0}{g_0^0} g$, $g \to \frac{1}{g_0^0} g$, we deduce that:

$$f \sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ f_0^1 & f_1^1 & 0 & 0 & 0\\ 0 & 0 & f_1^1 & 0 & 0\\ 0 & f_1^3 & 0 & f_1^1 & 1\\ f_0^4 & f_1^4 & 0 & f_0^1 & 2f_1^1 \end{pmatrix} \qquad g \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 2 & 0 & 0\\ 0 & g_1^3 & 0 & 3 & 0\\ g_0^4 & g_1^4 & 0 & 0 & 3 \end{pmatrix}$$

(a) If $f_0^1 = 0$, we can put $f_1^1 = 1$. Since f y g are diagonalizable, we come back to the previous case.

(b) If $f_0^1 \neq 0$, f_0^1 can be scaled to 1 over the field $\mathbb{C}$, or to ± 1 over the field $\mathbb{R}$. Preserving the form of f, it is possible to diagonalize g.

1. For $f_0^1 = 1$, f is also diagonalizable so this case brings back the algebra $\mathfrak{g}_{5,3}^{7,1}$.
2. For $f_0^1 = -1$ and $|f_1^1| < 2$, by making the change of variables:

$$\begin{array}{lcl} X_0 & \to & 2X_0 + f_1^1 X_1\\ X_1 & \to & \sqrt{-\Delta} X_1\\ X_2 & \to & 2\sqrt{-\Delta} X_2\\ X_3 & \to & 2\sqrt{-\Delta}(2X_3 + f_1^1 X_4)\\ X_4 & \to & -2\Delta X_4\\ f & \to & \frac{2}{\sqrt{-\Delta}}(f - \frac{f_1^1}{2} g) \text{ where } \Delta = (f_1^1)^2 - 4 \end{array}$$

we obtain:

$$f \sim \left(\begin{array}{c|c|c} R & 0 & 0\\ \hline 0 & 0 & 0\\ \hline 0 & 0 & R \end{array}\right), \quad \textit{with} \quad R \sim \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix} \quad g \sim diag(1, 1, 2, 3, 3)$$

$$\left\{\begin{array}{lcl}[Y, X_0] & = & -X_1\\ [Y, X_1] & = & X_0\\ [Y, X_2] & = & 0\\ [Y, X_3] & = & -X_4\\ [Y, X_4] & = & X_3\end{array}\right. \qquad \left\{\begin{array}{lcl}[Z, X_0] & = & X_0\\ [Z, X_1] & = & X_1\\ [Z, X_2] & = & 2X_2\\ [Z, X_3] & = & 3X_3\\ [Z, X_4] & = & 3X_4\end{array}\right.$$

This case provides the new algebra $\mathfrak{g}_{5,3}^{7,1}$.

3. For $f_0^1 = -1$ and $|f_1^1| > 2$, we get again the algebra $\mathfrak{g}_{5,3}^{7,1}$ because f is diagonalizable.
4. For $f_0^1 = -1$ and $|f_1^1| = \pm 2$, after the change of basis $X_0 \to X_0 \pm X_1$, $X_3 \to X_3 \pm X_4$, f and g take the form:

$$f \sim \begin{pmatrix} 1 & \pm 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \qquad g \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

This leads to a contradiction because $f - g$ should be nilpotent.

□

3. Lie algebras with $\mathcal{L}_{5,3}$ as nilradical

Once the low dimensional cases have been treated in detail, we can enumerate all Lie algebras admitting $\mathcal{L}_{5,3}$ as nilradical.

Proposition 3.1. *Let $\mathfrak{g}$ be a real non-split Lie algebra with nilradical isomorphic to $\mathcal{L}_{5,3}$.*

(i) *If $\mathfrak{g}$ is solvable then $\dim \mathfrak{g} \leq 7$ and $\mathfrak{g}$ is isomorphic to one of the following algebras:*

  (a) $\mathfrak{g}_{5,3}^{6,1}, \dots, \mathfrak{g}_{5,3}^{6,9}$ *for* $\dim \mathfrak{g} = 6$.
  (b) $\mathfrak{g}_{5,3}^{7,1}$ *or* $\mathfrak{g}_{5,3}^{7,2}$ *for* $\dim \mathfrak{g} = 7$.

(ii) *If $\mathfrak{g}$ is not solvable then $\mathfrak{g}$ is 8 or 9 dimensional and is respectively isomorphic to:*

  (a) *the algebra $\mathfrak{g}_{5,3}^{8,1}$ represented by a basis $\{X_0, \dots, X_4, Y, Z, V\}$ and the Lie brackets involving Y, Z and V given by:*
  $[Y, X_0] = X_0$, $[Y, X_1] = -X_1$, $[Y, X_3] = X_3$, $[Y, X_4] = -X_4$,
  $[Z, X_1] = X_0$, $[Z, X_4] = X_3$,
  $[V, X_0] = X_1$, $[V, X_3] = X_4$,
  $[Y, Z] = 2Z$, $[Y, V] = -2V$, $[Z, V] = Y$.
  (b) *the algebra $\mathfrak{g}_{5,3}^{9,1}$ represented by a basis $\{X_0, \dots, X_4, Y, Y', Z, V\}$ and the Lie brackets involving Y, Y', Z and V given by:*
  $[Y, X_0] = X_0$, $[Y, X_1] = -X_1$, $[Y, X_3] = X_3$, $[Y, X_4] = -X_4$,
  $[Y', X_1] = X_1$, $[Y', X_2] = X_2$, $[Y', X_3] = X_3$, $[Y', X_4] = 2X_4$,
  $[Z, X_1] = X_0$, $[Z, X_4] = X_3$,
  $[V, X_0] = X_1$, $[V, X_3] = X_4$,
  $[Y, Z] = Z$, $[Y, V] = -V$, $[Y', Z] = -Z$, $[Y', V] = V$,
  $[Z, V] = Y - Y'$.

The classification of complex algebras with nilradical $\mathcal{L}_{5,3}$ is analogous to the real one, except for the isomorphisms:

$$\begin{array}{lclcl}
\mathfrak{g}_{5,3}^{6,3}\otimes\mathbb{C} & \sim & \mathfrak{g}_{5,3}^{6,4}\otimes\mathbb{C} & & \\
\mathfrak{g}_{5,3}^{6,6}\otimes\mathbb{C} & \sim & \mathfrak{g}_{5,3}^{6,1}\left(\frac{1+i}{1-i}\right)\otimes\mathbb{C} & & \\
\mathfrak{g}_{5,3}^{6,7}\otimes\mathbb{C} & \sim & \mathfrak{g}_{5,3}^{6,1}(-1)\otimes\mathbb{C} & & \\
\mathfrak{g}_{5,3}^{6,8}\otimes\mathbb{C} & \sim & \mathfrak{g}_{5,3}^{6,9}\otimes\mathbb{C} & \sim & \mathfrak{g}_{5,3}^{6,3}\otimes\mathbb{C} \\
\mathfrak{g}_{5,3}^{7,1}\otimes\mathbb{C} & \sim & \mathfrak{g}_{5,3}^{7,2}\otimes\mathbb{C} & &
\end{array}$$

Proof. Let $\mathfrak{g}$ be a real non-split algebra with $\mathcal{L}_{5,3}$ as nilradical.If $\mathfrak{g}$ has dimension 5, then $\mathfrak{g}$ is obviously isomorphic to $\mathcal{L}_{5,3}$, otherwise $\mathfrak{g}$ has the form $\mathcal{L}_{5,3}\vec{\oplus}\mathfrak{t}$. In previous sections, we have obtain the algebras of this form when dimension of $\mathfrak{t}$ is 1 or 2. Let us suppose now that dimension of $\mathfrak{t}$ is 3, the three generators of $\mathfrak{t}$ are associated to three derivations of $\mathcal{L}_{5,3}$ called f, g and h. There are two possible cases:

(i) We suppose that f, g and h take the form:

$$f\sim\begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ f_0^1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & f_1^3 & 0 & 2 & 0\\ f_0^4 & f_1^4 & 0 & f_0^1 & 1\end{pmatrix},\quad g\sim\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ g_0^1 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & g_1^3 & 0 & 1 & 0\\ g_0^4 & g_1^4 & 0 & g_0^1 & 2\end{pmatrix},$$

$$h\sim\begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ h_0^1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & h_1^3 & 0 & 0 & 1\\ h_0^4 & h_1^4 & 0 & h_0^1 & 0\end{pmatrix}.$$

The brackets involving these derivations are:

$$[f,g]=\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ -f_0^1-g_0^1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ f_1^3g_0^1-g_1^3f_0^1 & 2g_1^3 & 0 & 0 & 0\\ f_1^4g_0^1-g_1^4f_0^1-2f_0^4 & -f_1^4+f_0^1g_1^3+g_1^4-g_0^1f_1^3 & 0 & -f_0^1-g_0^1 & 0\end{pmatrix},$$

$$[g,h]=\begin{pmatrix} -g_0^1 & -1 & 0 & 0 & 0\\ h_0^1 & g_0^1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ g_1^3h_0^1-g_0^4-h_1^3g_0^1 & -g_1^4 & 0 & -g_0^1 & -1\\ g_1^4h_0^1-h_1^4g_0^1+2h_0^4 & g_0^1h_1^3+g_0^4-g_1^3h_0^1+h_1^4 & 0 & h_0^1 & g_0^1\end{pmatrix},$$

$$[h,f]=\begin{pmatrix} f_0^1 & -1 & 0 & 0 & 0\\ h_0^1 & -f_0^1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ h_1^3f_0^1+f_0^4+f_1^3h_0^1 & f_1^4-2h_1^3 & 0 & f_0^1 & -1\\ h_1^4f_0^1-f_1^4h_0^1 & h_0^1f_1^3-f_0^4-h_1^3f_0^1-h_1^4 & 0 & h_0^1 & -f_0^1\end{pmatrix}.$$

These derivations must commute to a linear combination of them and an inner derivation of $\mathcal{L}_{5,3}$, thereby we deduce the following conditions:

$$\left\{\begin{array}{lcl} g_0^1 & = & -f_0^1\\ h_0^1 & = & (f_0^1)^2\end{array}\right.$$

If we consider the change of basis in the nilradical $X_0 \to X_0 + f_0^1 X_1$, $X_3 \to X_3 + f_0^1 X_4$ and $h \to h - f_0^1 f + f_0^1 g$, we obtain the brackets:

$$[f,g]=0, \quad [f,h]=h, \quad [g,h]=-h.$$

This case leads to a contradiction because the vector space generated by $\mathcal{L}_{5,3}$ and the vector associated to h is a nilpotent ideal.

(ii) On the other hand, if f, g and h take the form:

$$f \sim \begin{pmatrix} f_0^0 & 0 & 0 & 0 & 0 \\ 0 & f_1^1 & 0 & 0 & 0 \\ 0 & 0 & f_0^0+f_1^1 & 0 & 0 \\ 0 & f_1^3 & 0 & 2f_0^0+f_1^1 & 0 \\ f_0^4 & f_1^4 & 0 & 0 & f_0^0+2f_1^1 \end{pmatrix}, g \sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & g_1^3 & 0 & 0 & 1 \\ g_0^4 & g_1^4 & 0 & 0 & 0 \end{pmatrix},$$

$$h \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & h_1^3 & 0 & 0 & 0 \\ h_0^4 & h_1^4 & 0 & 1 & 0 \end{pmatrix}$$

we are led to the bracket:

$$[g,h] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ h_0^4+g_1^3 & h_1^4 & 0 & 1 & 0 \\ -g_1^4 & -h_0^4-g_1^3 & 0 & 0 & -1 \end{pmatrix}$$

and the condition $f_0^0 = -f_1^1 = 1$.
The others brackets involving f, g and h are:

$$[f,g] = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -f_0^4 & 2g_1^3 - f_1^4 & 0 & 0 & 2 \\ -2g_0^4 & f_0^4 & 0 & 0 & 0 \end{pmatrix}$$

$$[f,h] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ f_1^3 & 2h_1^3 & 0 & 0 & 0 \\ f_1^4-2h_0^4 & -f_1^3 & 0 & -2 & 0 \end{pmatrix}$$

From these we deduce that $f_1^3 = f_0^4 = f_1^4 = g_1^3 = g_0^4 = g_1^4 = h_1^3 = h_0^4 = h_1^4 = 0$, thereby f, g and h turns to:

$$f \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, g \sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, h \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The final commutators are:

$$[g,h]=f;\quad [f,g]=2g;\quad [f,h]=-2h$$

Thus we obtain the algebra $\mathfrak{g}_{5,3}^{8,1}$.

Finally, if the dimension of $\mathfrak{t}$ is four, we can suppose that the derivations associated to the generators of $\mathfrak{t}$ are:

$$f\sim\begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&f_1^3&0&2&0\\f_0^4&f_1^4&0&0&1\end{pmatrix},\quad f'\sim\begin{pmatrix}0&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&f_1'^3&0&1&0\\f_0'^4&f_1'^4&0&0&2\end{pmatrix},$$
$$g\sim\begin{pmatrix}0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&g_1^3&0&0&1\\g_0^4&g_1^4&0&0&0\end{pmatrix},\quad h\sim\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&h_1^3&0&0&0\\h_0^4&h_1^4&0&1&0\end{pmatrix}$$

By computing the brackets involving these derivations and by making a simple change of basis in the nilradical, we obtain that $f_1^3=f_0^4=f_1^4=f_1'^3=f_0'^4=f_1'^4=g_1^3=g_0^4=g_1^4=h_1^3=h_0^4=h_1^4=0$ and:

$$[f,f']=0,\qquad [f,g]=g,\qquad [f,h]=-h,$$
$$[g,h]=f-f',\qquad [f',g]=-g,\qquad [f',h]=h,$$

so we get the algebra $\mathfrak{g}_{5,3}^{9,1}$.

We see that the dimension of $\mathfrak{t}$ can be at most four. In fact, if dimension of $\mathfrak{t}$ is five or more, we can find an external and nilpotent derivation associated to an element of $\mathfrak{t}$ commuting with the others derivations associated to $\mathfrak{t}$. □

Remark: The classification of six dimensional solvable Lie algebras over the real and complex fields is essentially due to a series of works due to Mubarakzyanov from 1963 to 1966 [5, 6]. This result was revised and completed later by distinct authors [7, 8]. However, innacuracies have still been found, concerning either redundancies or omissions in the list. More specifically, six isomorphism classes having $\mathcal{L}_{5,3}$ as nilradical have been listed, denoted respectively by $\mathfrak{g}_{6,76},\mathfrak{g}_{6,77}\dots,\mathfrak{g}_{6,81}$. We note that the algebra $\mathfrak{g}_{6,80}$ is isomorphic to the algebra $\mathfrak{g}_{6,76}$ for $h=0$, and $\mathfrak{g}_{6,81}$ to $\mathfrak{g}_{6,77}$. Furthermore, by the preceding theorem, it follows that the algebra $\mathfrak{g}_{6,76}$ is isomorphic to $\mathfrak{g}_{5,3}^{6,1}$, $\mathfrak{g}_{6,77}$ to $\mathfrak{g}_{5,3}^{6,3}$ and $\mathfrak{g}_{5,3}^{6,4}$, $\mathfrak{g}_{6,78}$ to $\mathfrak{g}_{5,3}^{6,2}$ and $\mathfrak{g}_{6,80}$ to $\mathfrak{g}_{5,3}^{6,5}$. This means that the algebras $\mathfrak{g}_{5,3}^{6,6},\dots,\mathfrak{g}_{5,3}^{6,9}$ have been omitted in [5, 7].

4. Generalized Casimir invariants

The generalized Casimir invariants of a Lie algebra, corresponding to the invariants of the coadjoint representation, play an important role in many applications of Lie algebras. While for the class of algebraic algebras it is known that a fundamental basis can always be found among the polynomials, the situation differs radically for

solvable Lie algebras and semidirect products, where rational and even transcendental functions can appear [9]. In this section we consider the analytical approach to determine the invariants of Lie algebras [10].
Let $\mathcal{B} = \{X_1, \ldots, X_n\}$ be a basis of $\mathfrak{g}$ and consider the variables $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}^*$ associated to the dual basis of $\mathcal{B}$. If $\{C_{ij}^k\}$ denote the structure constants of $\mathfrak{g}$, we can define the differential operators

$$\widehat{X}_i = -C_{ij}^k x_k \partial_{x_j} \quad \text{for} \quad 1 \leq i \leq n.$$

A function $F \in \mathcal{C}^\infty(\mathfrak{g}^*)$ is an invariant of $\mathfrak{g}$ only if it satisfies the following system of PDEs:

$$\widehat{X}_i F(x_1, \ldots, x_n) = -C_{ij}^k x_k \partial_{x_j} F(x_1, \ldots, x_n) = 0 \tag{3}$$

A maximal system of functionally independent solution of the system is called a fundamental set of invariants. The number of independent solution $\mathcal{N}(\mathfrak{g})$ is given by the Beltrami-Blasi formula:

$$\mathcal{N}(\mathfrak{g}) = \dim \mathfrak{g} - rang(A(\mathfrak{g})) \tag{4}$$

where $A(\mathfrak{g})$ is the skew-symmetric matrix:

$$A(\mathfrak{g}) = \begin{pmatrix} 0 & C_{12}^k x_k & & \cdots & C_{1n}^k x_k \\ -C_{12}^k x_k & 0 & & \cdots & C_{2n}^k x_k \\ \vdots & & \ddots & & \\ -C_{1,n-1}^k x_k & \cdots & & 0 & C_{n-1,n}^k x_k \\ -C_{1n}^k x_k & \cdots & & -C_{n-1,n}^k x_k & 0 \end{pmatrix}.$$

For the Lie algebra $\mathcal{L}_{5,3}$, finding the invariants reduces to solve the following system of PDEs:

$$\begin{aligned} \widehat{X}_0(F) &:= x_2 \frac{\partial F}{\partial x_1} + x_3 \frac{\partial F}{\partial x_2} = 0, \\ \widehat{X}_1(F) &:= -x_2 \frac{\partial F}{\partial x_0} + x_4 \frac{\partial F}{\partial x_2} = 0, \\ \widehat{X}_2(F) &:= x_3 \frac{\partial F}{\partial x_0} + x_4 \frac{\partial F}{\partial x_1} = 0. \end{aligned} \tag{5}$$

The rank of the matrix $A(\mathcal{L}_{5,3})$ is two then by the Beltrametti-Blasi formula, the algebra have exactly three invariants. Moreover, since X_3 and X_4 are generators of the center, we can choose $I_1 = x_3$ and $I_2 = x_4$. We can easily verify that the quadratic operator

$$I_3 = 2(x_0 x_4 - x_1 x_3) + x_2^2$$

is a solution of the system (5), so $\{I_1, I_2, I_3\}$ is a fundamental set of invariants of $\mathcal{L}_{5,3}$.

Proposition 4.1. *Let $\mathfrak{g}$ be a solvable algebra whose nilradical is isomorphic to $\mathcal{L}_{5,3}$, then invariants of $\mathfrak{g}$ are functions of the variables associated to the nilradical.*

Proof. We should divide our proof into two cases , depending on the dimension of $\mathfrak{g}$.

(i) If dimension of $\mathfrak{g}$ is six, every non-nilpotent derivation acts, at least, on one central element of $\mathfrak{g}$ in a non-trivial way. We deduce that every invariant F verifies:

$$\frac{\partial F}{\partial y} = 0,$$

(ii) If dimension of $\mathfrak{g}$ is two, $\mathfrak{g}_{5,3}^{7,1}$ (respectively $\mathfrak{g}_{5,3}^{7,2}$)

$$S_1 = \begin{cases} 2x_3 \frac{\partial F}{\partial y} + x_3 \frac{\partial F}{\partial z} = 0 \\ x_4 \frac{\partial F}{\partial y} + 2x_4 \frac{\partial F}{\partial z} = 0 \end{cases},$$

respectively

$$S_2 = \begin{cases} -x_4 \frac{\partial F}{\partial y} + 3x_3 \frac{\partial F}{\partial z} = 0 \\ x_3 \frac{\partial F}{\partial y} + 3x_4 \frac{\partial F}{\partial z} = 0 \end{cases}.$$

Making elementary transformations, we deduce that the solutions must satisfy:

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0,$$

therefore, invariants of $\mathfrak{g}$ depend only on the variables associated to the nilradical.

□

Table 1 describes the invariants of the solvable algebras with nilradical $\mathcal{L}_{5,3}$ as functions of $\{I_1, I_2, I_3\}$.

Table 1: Invariants of the solvable algebras with $\mathcal{L}_{5,3}$-nilradical

$\mathfrak{g}$	Fundamental system of invariants	
$\mathfrak{g}_{5,3}^{6,1}(\lambda)$	$J_1 = I_1^{1+2\lambda} I_2^{-2-\lambda}$,	$J_2 = I_3^{2+\lambda} I_1^{-2-2\lambda}$
$\mathfrak{g}_{5,3}^{6,2}$	$J_1 = I_3 I_1^{-1} I_2^{-1} + 2\ln(I_1)$,	$J_2 = I_1 I_2$
$\mathfrak{g}_{5,3}^{6,3}$	$J_1 = I_1 I_2^{-2}$,	$J_2 = 2\ln(I_2) - I_3 I_2^{-2}$
$\mathfrak{g}_{5,3}^{6,4}$	$J_1 = I_2^2 I_1^{-1}$,	$J_2 = I_3 I_2^{-2} + 2\ln(I_2)$
$\mathfrak{g}_{5,3}^{6,5}$	$J_1 = I_3^3 I_2^{-4}$,	$J_2 = \ln(I_2) - 3I_1 I_2^{-1}$
$\mathfrak{g}_{5,3}^{6,6}$	$J_1 = -\frac{1}{2}\ln\left(\frac{I_1^2+I_2^2}{I_1^2}\right) + 3\arctan\left(I_2 I_1^{-1}\right) - \ln(I_1)$,	
	$J_2 = \ln(I_3) + 4\arctan\left(I_1 I_2^{-1}\right)$.	
$\mathfrak{g}_{5,3}^{6,7}$	$J_1 = I_3$,	$J_2 = I_1^2 + I_2^2$
$\mathfrak{g}_{5,3}^{6,8}$	$J_1 = I_3 + I_1 I_2 - \left(I_1^2 + I_2^2\right)\arctan\left(I_2 I_1^{-1}\right)$	
	$J_2 = I_1^2 + I_2^2$	
$\mathfrak{g}_{5,3}^{6,9}$	$J_1 = I_3 - I_1 I_2 + \left(I_1^2 + I_2^2\right)\arctan\left(I_2 I_1^{-1}\right)$	
	$J_2 = I_1^2 + I_2^2$	
$\mathfrak{g}_{5,3}^{7,1}$	$J_1 = I_3^3 \left(I_1 I_2\right)^{-2}$	
$\mathfrak{g}_{5,3}^{7,2}$	$J_1 = I_3^3 \left(I_1^2 + I_2^2\right)^{-2}$	

Proposition 4.2. *The algebra $\mathfrak{g}_{5,3}^{8,1}$ has exactly two independent invariants that can be chosen as:*
$J_1 = X_2^2 + 2X_0X_4 - 2X_1X_3,$
$J_2 = X_2^3 + 3X_2(X_0X_4 - X_1X_3) - 3X_3^2V + 3X_3X_4Y + 3X_4^2Z.$
The algebra $\mathfrak{g}_{5,3}^{9,1}$ has only one independent invariant which is function of J_1 and J_2 and can be chosen as $J = J_1^3 J_2^{-2}$.

Proof. In order to determine the independent invariants of $\mathfrak{g}_{5,3}^{8,1}$, we have to solve the system:

$$\begin{array}{l}
\widehat{X_0}F = (x_2\partial_{x_1} + x_3\partial_{x_2} - x_0\partial_y - x_1\partial_v)F = 0\\
\widehat{X_1}F = (-x_2\partial_{x_0} + x_4\partial_{x_2} + x_1\partial_y - x_0\partial_z)F = 0\\
\widehat{X_2}F = (-x_3\partial_{x_0} - x_4\partial_{x_1})F = 0\\
\widehat{X_3}F = (-x_3\partial_y - x_4\partial_v)F = 0\\
\widehat{X_4}F = (x_4\partial_y - x_3\partial_z)F = 0\\
\widehat{Y}F = (x_0\partial_{x_0} - x_1\partial_{x_1} + x_3\partial_{x_3} - x_4\partial_{x_4} + 2z\partial_z - 2v\partial_v)F = 0\\
\widehat{Z}F = (x_0\partial_{x_1} + x_3\partial_{x_4} - 2z\partial_y + y\partial_v)F = 0\\
\widehat{V}F = (x_1\partial_{x_0} + x_4\partial_{x_3} + 2v\partial_y - y\partial_z)F = 0
\end{array} \tag{6}$$

If we consider the new variables $x_0' = x_0x_4 - x_1x_3$ and $u = -x_3^2v + x_3x_4y + x_4^2z$, the system reduces to the following equation:

$$x_2\partial_{x_0'}F(x_0', x_2, u) - \partial_{x_2}F(x_0', x_2, u) - x_0'\partial_u F(x_0', x_2, u) = 0 \tag{7}$$

Since $x_2^2 + x_0'$ and $x_2^3 + 3x_0'x_2 + 3u$ are two particular solutions of (7), the system (6) has two solutions $J_1 = x_2^2 + 2x_0x_4 - 2x_1x_3$ and $J_2 = x_2^3 + 3x_2(x_0x_4 - x_1x_3) - 3x_3^2v + 3x_3x_4y + 3x_4^2z$, clearly functionally independent, thus we have found the fundamental set of solutions.

The system associated to the algebra $\mathfrak{g}_{5,3}^{9,1}$ is:

$$\begin{array}{l}
\widehat{X_0}F = (x_2\partial_{x_1} + x_3\partial_{x_2} - x_0\partial_y - x_1\partial_v)F = 0\\
\widehat{X_1}F = (-x_2\partial_{x_0} + x_4\partial_{x_2} + x_1\partial_y - x_0\partial_z - x_1\partial_{y'})F = 0\\
\widehat{X_2}F = (-x_3\partial_{x_0} - x_4\partial_{x_1} - x_2\partial_{y'})F = 0\\
\widehat{X_3}F = (-x_3\partial_y - x_4\partial_v - x_3\partial_{y'})F = 0\\
\widehat{X_4}F = (x_4\partial_y - x_3\partial_z - 2x_4\partial_{y'})F = 0\\
\widehat{Y}F = (x_0\partial_{x_0} - x_1\partial_{x_1} + x_3\partial_{x_3} - x_4\partial_{x_4} + 2z\partial_z - 2v\partial_v)F = 0\\
\widehat{Z}F = (x_0\partial_{x_1} + x_3\partial_{x_4} - 2z\partial_y + y\partial_v - z\partial_{y'})F = 0\\
\widehat{V}F = (x_1\partial_{x_0} + x_4\partial_{x_3} + 2v\partial_y - y\partial_z - v\partial_{y'})F = 0\\
\widehat{Y'}F = (x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3} + 2x_4\partial_{x_4} - z\partial_z + v\partial_v)F = 0
\end{array} \tag{8}$$

The rank of the matrix $A(\mathfrak{g}_{5,3}^{9,1})$ is 8 so there is only one fundamental invariant and we can verify that $J = J_1^3 J_2^{-2}$ is a solution of the system (8). □

It turns out that all invariants are of rational type, and appear as the quotient of commuting polynomials. This fact may suggest the use of these algebras in the construction of integrable systems of coadjoint orbits, as has been done for large classes of classical Lie algebras [11].

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