An extension of Whitney approximation theorem

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ABSTRACT

The classical Whitney approximation theorem states that a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ can be approximated by analytic functions. We prove that these analytic functions approximating f can be taken as *real* holomorphic functions on the whole space $\mathbb{C}^n$.

Key words: Whitney approximation theorem, approximation by analytic functions

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1. Introduction

In 1934 H. Whitney proved that a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ can be approximated by analytic functions, cf. [7]. This result is known as Whitney approximation theorem and is of fundamental importance in the study of differentiable and analytic manifolds. We refer to [1] or the survey [6] and the many references therein for applications and generalizations of it.

The aim of this paper is to extend Whitney approximation theorem proving that the analytic functions approximating f can be taken as *real* holomorphic functions on $\mathbb{C}^n$.

In other words, the classical Whitney approximation theorem asserts that the ring $\mathcal{O}(\mathbb{R}^n)$ of analytic functions on $\mathbb{R}^n$ is a dense subset of the space $C^m(\mathbb{R}^n)$ of real-valued functions on $\mathbb{R}^n$ with respect to the so called Whitney topology. In this paper, we found a strictly smaller ring which is also a dense subset of $C^m(\mathbb{R}^n)$. This is the ring

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of *real* holomorphic functions, denoted as $\mathcal{O}_{\mathbb{R}}(\mathbb{C}^n)$, whose elements are holomorphic functions on $\mathbb{C}^n$ which are invariant under conjugation or, alternatively, real analytic functions on $\mathbb{R}^n$ which can be extended analytically to $\mathbb{C}^n$.

Real holomorphic functions are of interest in the study of real analytic sets, that is, subsets of $\mathbb{R}^n$ defined as the zero set of some real analytic functions. For example, the full normalization of some real analytic sets can have complex parts, see [4], and then real analytic functions which can be extended analytically to these complex parts arise. Real holomorphic functions also appear in the solution of Hilbert 17th problem for non-coherent analytic surfaces, cf. [2].

2. Preliminaries

We consider in $\mathbb{R}^n$ the usual topology. Given a compact subset $K \subset \mathbb{R}^n$, an integer $p \in \mathbb{N}$ and a function $f \in C^m(\mathbb{R}^n)$, the seminorm $\| f \|_p^K$ of $f \in C^m(\mathbb{R}^n)$ is defined as:

$$\| f \|_p^K = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} \sup_{x \in K} |D^\alpha f(x)| ,$$

where we use the standard notations

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}, \ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, \ |\alpha| = \alpha_1 + \ldots + \alpha_n$$

and we suppose $|\alpha| \leq m \leq \infty$.

A fundamental system of neighborhoods of $f \in \mathbb{C}^m(\mathbb{R}^n)$ in the *weak topology* is the collection:

$$\{g \in C^m(\mathbb{R}^n) \mid \ \| f - g \|_p^K < \varepsilon\} ,$$

where K is a compact subset of $\mathbb{R}^n$, $p \in \mathbb{N}$ (of course, $p \leq m$) and ε is a positive real number. Intuitively, we can say that in the weak topology two functions are close if in a compact subset K the values of both functions and of their p first derivatives are close.

As $\mathbb{R}^n$ is not compact, the closeness of functions at infinity is not controlled by the weak topology. For that reason, we introduce the *Whitney topology* as a limit of the weak topology. In this topology, a fundamental system of neighborhoods of $f \in \mathbb{C}^m(\mathbb{R}^n)$ is given by the sets:

$$\{g \in C^m(\mathbb{R}^n) \mid \ \| f - g \|_{m_1}^{K_1} < \varepsilon_1, ..., \| f - g \|_{m_i}^{K_i} < \varepsilon_i, ...\} ,$$

where $\{m_i\}$ is a sequence of positive integers ($m_i \leq m, \forall i$), $\{\varepsilon_i\}$ is an arbitrary sequence of positive real numbers and the K_i are compact subsets of $\mathbb{R}^n$ verifying $K_i \subset V_i$ for some locally finite covering $\{V_i\}$ of $\mathbb{R}^n$ (that is, $\bigcup V_i = \mathbb{R}^n$ and every point of $\mathbb{R}^n$ has a neighborhood which intersects only finitely many sets in the covering). Now, we can state the

Theorem 2.1. *(Classical Whitney aproximation theorem) Consider the sequences* $\{K_i\}$, $\{m_i\}$ *and* $\{\varepsilon_i\}$ *as above, and let* $f \in \mathbb{C}^m(\mathbb{R}^n)$. *Then there exists an analytic function* $g \in \mathcal{O}(\mathbb{R}^n)$ *such that*

$$\| f - g \|_{m_i}^{K_i} < \epsilon_i, \quad \textit{for each } i \geq 1.$$

The proof can be found in the paper of H. Whitney, cf. [7], or in the book by R. Narasimhan [5]. A similar theorem is proved in [3] for analytic submanifolds of $\mathbb{R}^n$. An immediate corollary of this theorem is the following:

Corollary 2.2. *The ring $\mathcal{O}(\mathbb{R}^n)$ of analytic functions on $\mathbb{R}^n$ is a dense subset of the space $C^m(\mathbb{R}^n)$ with respect to the Whitney topology.*

3. Extended Whitney approximation theorem

First we are going to see how to approximate in the weak topology a smooth function f having compact support (that is, a function $f \in C^m(\mathbb{R}^n)$ such that the closure of $\{x \in \mathbb{R}^n \mid f(x) \neq 0\}$ is a compact set) by an analytic function. This can be achieved by convolution with a suitable exponential function. For the proof of this key lemma we will closely follow the proof of lemma 1.6.1 in [5]. A similar result can be also found in [1].

Lemma 3.1. *Let $f \in C^m(\mathbb{R}^n)$ be a function with compact support, $0 \leq m < \infty$. For $\lambda > 0$, set:*

$$g_\lambda(x) \equiv I_\lambda(f)(x) := c\lambda^{\frac{n}{2}} \int_{\mathbb{R}^n} f(y) \mathrm{e}^{-\lambda \|x-y\|^2} dy$$

where $c = \pi^{-\frac{n}{2}}$, so that $c\lambda^{\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\lambda\|x\|^2} dx = 1$. Then

$$g_\lambda \in \mathcal{O}(\mathbb{R}^n) \quad \textit{and} \quad \lim_{\lambda \to \infty} \| g_\lambda - f \|_m^{\mathbb{R}^n} = 0.$$

Proof. By a change of variables, we can write $g_\lambda(x)$ in the form:

$$g_\lambda(x) = c\lambda^{\frac{n}{2}} \int_{\mathbb{R}^n} f(x-y) \mathrm{e}^{-\lambda\|y\|^2} dy$$

and so, for any $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq m$, we have

$$D^\alpha g_\lambda(x) = c\lambda^{\frac{n}{2}} \int_{\mathbb{R}^n} (D^\alpha f)(x-y) \mathrm{e}^{-\lambda\|y\|^2} dy = c\lambda^{\frac{n}{2}} \int_{\mathbb{R}^n} (D^\alpha f)(y) \mathrm{e}^{-\lambda\|x-y\|^2} dy$$

Therefore,

$$D^\alpha g_\lambda(x) - D^\alpha f(x) = c\lambda^{\frac{n}{2}} \int_{\mathbb{R}^n} (D^\alpha f(y) - D^\alpha f(x)) \mathrm{e}^{-\lambda\|x-y\|^2} dy.$$

Now, since f has compact support and $|\alpha| \leq m < \infty$ there is $M > 0$ such that

$$|D^\alpha f(y)| < M \quad \text{for all } y.$$

Also, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|D^\alpha f(y) - D^\alpha f(x)| < \epsilon/2 \quad \text{for } \| x - y \| < \delta.$$

Thus

$$|D^\alpha g_\lambda(x) - D^\alpha f(x)| = c\lambda^{\frac{n}{2}} \left| \int_{\|x-y\|<\delta} (D^\alpha f(y) - D^\alpha f(x)) \mathrm{e}^{-\lambda\|x-y\|^2} dy \right.$$

$$\left. + \int_{\|x-y\|\geq\delta} (D^\alpha f(y) - D^\alpha f(x)) \mathrm{e}^{-\lambda\|x-y\|^2} dy \right|$$

$$\leq \frac{1}{2}\epsilon c\lambda^{\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\lambda\|x-y\|^2} dy + 2Mc\lambda^{\frac{n}{2}} \int_{\|x-y\|\geq\delta} \mathrm{e}^{-\lambda\|x-y\|^2} dy$$

As $c\lambda^{\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\lambda\|x-y\|^2} dy = 1$ and

$$c\lambda^{\frac{n}{2}} \int_{\|x-y\|\geq\delta} \mathrm{e}^{-\lambda\|x-y\|^2} dy = c\lambda^{\frac{n}{2}} \int_{\|x-y\|\geq\delta} \mathrm{e}^{-\frac{1}{2}\lambda\|x-y\|^2} \mathrm{e}^{-\frac{1}{2}\lambda\|x-y\|^2} dy$$

$$\leq \mathrm{e}^{-\frac{1}{2}\lambda\delta^2} c\lambda^{\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}\lambda\|x-y\|^2} dy = \mathrm{e}^{-\frac{1}{2}\lambda\delta^2} 2^{\frac{n}{2}},$$

we have

$$|D^\alpha g_\lambda(x) - D^\alpha f(x)| \leq \frac{1}{2}\epsilon + 2M\mathrm{e}^{-\frac{1}{2}\lambda\delta^2} 2^{\frac{n}{2}}.$$

We can choose λ as large as needed, so

$$\sup_{x\in\mathbb{R}^n} |D^\alpha g_\lambda(x) - D^\alpha f(x)| \to 0 \quad \text{as } \lambda \to \infty.$$

Finally, as f has compact support and x appears only in the analytic function $\mathrm{e}^{-\lambda\|x-y\|^2}$, then

$$g_\lambda(x) = c\lambda^{\frac{n}{2}} \int_{\text{supp } f} f(y) \mathrm{e}^{-\lambda\|x-y\|^2} dy$$

is an analytic function on $\mathbb{R}^n$, that is, $g_\lambda \in \mathcal{O}(\mathbb{R}^n)$. Even more, as the function $\mathrm{e}^{-\lambda\|x-y\|^2}$ is holomorphic on $\mathbb{C}^n$, $g_\lambda(x)$ is also holomorphic on $\mathbb{C}^n$. □

We recall that a complex-valued function $f : \mathbb{C}^n \to \mathbb{C}$ is *holomorphic* on $\mathbb{C}^n$ if each point $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ has an open neighborhood U, $w \in U$, such that the function f has a power series expansion

$$f(z) = \sum_{\nu_1,\ldots,\nu_n=0}^{\infty} a_{\nu_1\ldots\nu_n} (z_1 - w_1)^{\nu_1} \ldots (z_n - w_n)^{\nu_n},\ a_{\nu_1\ldots\nu_n} \in \mathbb{C}$$

which converges for all $z = (z_1, \ldots, z_n) \in U$. The set of all functions holomorphic on $\mathbb{C}^n$ is called the ring of holomorphic functions and is denoted by $\mathcal{O}(\mathbb{C}^n)$.

We say that a holomorphic function $f \in \mathcal{O}(\mathbb{C}^n)$ is *real* if it is invariant under complex conjugation, that is, if $f(\overline{z}) = \overline{f(z)}$ for every $z \in \mathbb{C}^n$. The set of all real holomorphic functions is denoted as $\mathcal{O}_{\mathbb{R}}(\mathbb{C}^n)$.

Remark 3.2. The ring $\mathcal{O}_{\mathbb{R}}(\mathbb{C}^n)$ is contained in the ring of analytic functions on $\mathbb{R}^n$, $\mathcal{O}(\mathbb{R}^n)$. In fact, real holomorphic functions can be identified with real analytic functions on $\mathbb{R}^n$ which can be extended analytically to $\mathbb{C}^n$. It can be checked that real holomorphic functions can be also characterized as holomorphic functions on $\mathbb{C}^n$ which take real values on $\mathbb{R}^n$.

It is easy to see that the ring $\mathcal{O}_{\mathbb{R}}(\mathbb{C}^n)$ is strictly smaller than $\mathcal{O}(\mathbb{R}^n)$. For example, the real analytic function $\frac{1}{x_1^2+1}$ is not real holomorphic.

Before proving the approximation theorem we need the following technical lemma.

Lemma 3.3. *Let U_p, $p \geq 1$, be the open (complex) subset of $\mathbb{C}^n$:*

$$U_p = \operatorname{Int}\left\{ z \in \mathbb{C}^n \;\middle|\; \operatorname{Re}\left[\sum_{k=1}^{n}(z_k - x_k)^2\right] > \frac{1}{2}\,,\ \forall x \in \mathbb{R}^n \text{ such that } \parallel x \parallel \geq p+1 \right\},$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\operatorname{Re}(\zeta)$ denotes the real part of the complex number ζ and Int *denotes the interior with respect to the usual topology in $\mathbb{C}^n$. Then*

$$\bigcup_{p \geq 1} U_p = \mathbb{C}^n.$$

Proof. To have an idea of how the U_p are, in figure 1 we can see the shape that they have in dimension 1.

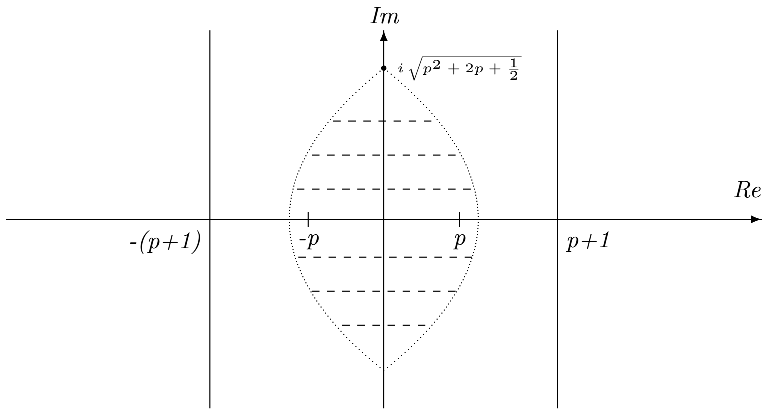

Figure 1

Take $z \in C^n$ and we will prove that $z \in U_p$ for some p. We write $z = (z_1, \ldots, z_n) = (a_1 + ib_1, \ldots, a_n + ib_n)$, $a_i, b_i \in \mathbb{R}$. Thus we have

$$\operatorname{Re}\left[\sum_{k=1}^{n}(z_k - x_k)^2\right] = \operatorname{Re}\left[\sum_{k=1}^{n}(a_k + ib_k - x_k)^2\right] = \operatorname{Re}\left[\sum_{k=1}^{n}\Big((a_k - x_k)^2 - b_k^2 + 2(a_k - x_k)b_k i\Big)\right]$$

$$= \sum_{k=1}^{n} \left((a_k - x_k)^2 - b_k^2 \right) = \sum_{k=1}^{n} (a_k - x_k)^2 - \sum_{k=1}^{n} b_k^2$$

We search for $p \in \mathbb{N}$ such that for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ verifying $\| x \| \geq p + 1$:

$$\sum_{k=1}^{n} (a_k - x_k)^2 - \sum_{k=1}^{n} b_k^2 > \frac{1}{2} \ .$$

If we consider points $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ of $\mathbb{R}^n$, the previous inequality can be rewritten as:

$$\| a - x \|^2 > \frac{1}{2} + \| b \|^2 \ . \tag{1}$$

Take p such that

$$p + 1 > \| a \| + \| b \| + 1 \ .$$

If $\| x \| \geq p + 1$ then $\| a - x \| \geq p + 1 - \| a \| > 1 + \| b \|$ and inequality (1) is satisfied. Hence for this p the point z is in U_p, so that the lemma is proved. □

Now we can prove our main theorem. Without loss of generality, in the definition of the Whitney topology we take the sequence of compact sets $K_i = \{x \in \mathbb{R}^n \mid i - 1 \leq \| x \| \leq i\}$. Our proof follows the classical one of [5], but our particular election of the sequence $\{K_i\}$ allow us to prove that the approximating function is, in fact, a real holomorphic function.

Theorem 3.4. *(Extended Whitney approximation theorem) Let $f \in C^m(\mathbb{R}^n)$, $0 \leq m \leq \infty$, and let $K_i = \{x \in \mathbb{R}^n \mid i - 1 \leq \| x \| \leq i\}$. Let $\{m_i\}$ be an arbitrary sequence of positive integers ($m_i \leq m$) and $\{\epsilon_i\}$ any sequence of strictly positive real numbers. Then there exists a real holomorphic function $g \in \mathcal{O}_{\mathbb{R}}(\mathbb{C}^n)$ verifying that*

$$\| f - g \|_{m_i}^{K_i} < \epsilon_i$$

for each $i \geq 1$.

Proof. Since for any subset $S \subset \mathbb{R}^n$ and any pair of integers $k \leq k'$ we have the inequality $\| f \|_k^S \leq \| f \|_{k'}^S$, for the proof of the theorem we can suppose $m_{p+1} \geq m_p$, for every $p \geq 1$.
The following property of seminorms will be used: if $\varphi, \psi \in C^m(\mathbb{R}^n)$ and $S \subset \mathbb{R}^n$, then we have that:

$$\| \varphi\psi \|_k^S \leq \| \varphi \|_k^S \| \psi \|_k^S \ .$$

We take a sequence of functions $\varphi_p \in C^\infty(\mathbb{R}^n)$, $p \geq 1$, with compact support, such that $\varphi_p(x) = 1$ if $x \in K_p$ and $\varphi_p(x) = 0$ if $x \notin \{y \in \mathbb{R}^n \mid p - \frac{3}{2} \leq \| y \| \leq p + \frac{1}{2}\}$. Corollary 1.2.6 of [5] assures that such functions φ_p exist.
Let

$$M_p = 1 + \| \varphi_p \|_{m_p}^{\mathbb{R}^n} \ .$$

Note that $1 < M_p < \infty$, because φ_p has compact support.
Take a sequence $\{\delta_p\}$ of positive numbers such that $\delta_{p+1} \leq \delta_p/2$ and

$$\sum_{q \geq p} \delta_q M_{q+1} \leq \frac{1}{4}\,\epsilon_p, \quad \text{for} \quad p \geq 1.$$

For example, we can take $\delta_p = \frac{1}{4}\frac{\epsilon_1 \ldots \epsilon_p}{2^p M_2 \ldots M_{p+1}}$, as we can suppose that $\epsilon_i < 1$, $\forall i \geq 1$. We define $B_p := \{x \in \mathbb{R}^n \mid \parallel x \parallel \leq p\}$, so that $B_p = \bigcup_{i=1}^p K_i$. By lemma 3.1, there exists $\lambda_1 > 0$ such that if $g_1 := I_{\lambda_1}(\varphi_1 f)$, then

$$\parallel g_1 - \varphi_1 f \parallel_{m_1}^{B_1} < \delta_1.$$

Also, by lemma 3.1, there exists $\lambda_2 > 0$ such that if $g_2 := I_{\lambda_2}(\varphi_2(f - g_1))$, then

$$\parallel g_2 - \varphi_2(f - g_1) \parallel_{m_2}^{B_2} < \delta_2.$$

By induction, we define $\lambda_p > 0$ and g_p such that if $g_p := I_{\lambda_p}(\varphi_p(f - g_1 - \ldots - g_{p-1}))$, then

$$\parallel g_p - \varphi_p(f - g_1 - \ldots - g_{p-1}) \parallel_{m_p}^{B_p} < \delta_p \;. \tag{2}$$

As $\varphi_p = 0$ in a neighbourhood of B_{p-2}, (2) implies that

$$\parallel g_p \parallel_{m_p}^{B_{p-2}} < \delta_p. \tag{3}$$

The function we are looking for is $g := \sum_{q=1}^{\infty} g_q$. First, we are going to prove that $g \in C^{m_p}(\mathbb{R}^n)$ for every p.
As $\varphi_p(x) = 1$ in a neighbourhood of K_p, from (2) we have that

$$\parallel f - g_1 - \ldots - g_p \parallel_{m_p}^{K_p} < \delta_p. \tag{4}$$

Substituting p by $p+1$ in (2), using (4) and that $m_p \leq m_{p+1}$, we have:

$$\parallel g_{p+1} \parallel_{m_p}^{K_p} \leq \parallel \varphi_{p+1}(f - \sum_{q=1}^{p} g_q) \parallel_{m_p}^{K_p} + \parallel g_{p+1} - \varphi_{p+1}(f - \sum_{q=1}^{p} g_q) \parallel_{m_p}^{K_p}$$

$$\leq \parallel \varphi_{p+1} \parallel_{m_p}^{\mathbb{R}^n} \parallel f - \sum_{q=1}^{p} g_q \parallel_{m_p}^{K_p} + \parallel g_{p+1} - \varphi_{p+1}(f - \sum_{q=1}^{p} g_q) \parallel_{m_{p+1}}^{B_{p+1}} \leq M_{p+1}\delta_p + \delta_{p+1}$$

and

$$\parallel g_{p+1} \parallel_{m_p}^{B_p} \leq \parallel g_{p+1} \parallel_{m_p}^{B_{p-1}} + \parallel g_{p+1} \parallel_{m_p}^{K_p} \leq \delta_{p+1} + (M_{p+1}\delta_p + \delta_{p+1})$$

$$= M_{p+1}\delta_p + 2\delta_{p+1} \leq M_{p+1}\delta_p + \delta_p \leq 2\delta_p M_{p+1}.$$

Therefore

$$\parallel \sum_{q>p} g_q \parallel_{m_p}^{B_p} \leq 2 \sum_{q>p} \delta_q M_{q+1} < \frac{1}{2}\epsilon_p \,, \tag{5}$$

which implies that, for every p,

$$g = \sum_{q=1}^{\infty} g_q \in C^{m_p}(\mathbb{R}^n).$$

Now, using (4) and (5) we see that g approximates f as desired:

$$\| f - g \|_{m_p}^{K_p} \leq \| f - \sum_{q=1}^{p} g_q \|_{m_p}^{K_p} + \| \sum_{q>p} g_q \|_{m_p}^{K_p} < \delta_p + \frac{1}{2}\epsilon_p < \epsilon_p\,.$$

Finally, we will see that, with an adequate choice of the λ_p, the function g is analytic over $\mathbb{C}^n$, that is, g is a real holomorphic function.
As seen in lemma 3.1 the function $g_r(x)$ is analytic on $\mathbb{R}^n$ and, moreover, it can be extended holomorphically to $\mathbb{C}^n$ by the same formula

$$g_r(z) = c\lambda_r^{n/2} \int_{\text{supp }\varphi_r} \varphi_r(y)\Big(f(y) - \sum_{q=1}^{r-1} g_q(y)\Big)\, \mathrm{e}^{-\lambda_r \sum_{i=1}^{n}(z_i-y_i)^2} dy.$$

We take open (complex) subsets U_p of $\mathbb{C}^n$ as in lemma 3.3. If $y \in \text{supp }\varphi_r$, with $r > p+2$, then $\| y \| \geq p+1$ and for $z \in U_p$ we have that

$$\text{Re}\Big[\sum_{j=1}^{n}(z_j - y_j)^2\Big] > \frac{1}{2}\,,$$

so

$$|g_r(z)| \leq c\lambda_r^{n/2}\mathrm{e}^{-\lambda_r/2} \int_{\text{supp }\varphi_r} \Big|\varphi_r(y)\Big(f(y) - \sum_{q=1}^{r-1} g_q(y)\Big)\Big| dy \leq c\lambda_r^{n/2}\,\mathrm{e}^{-\lambda_r/2}\, H_r \quad (6)$$

where H_r depends only of $\lambda_1, ..., \lambda_{r-1}$, because g_q depends only on the λ_j with $j \leq q$. Thus, we can choose inductively, the λ_q large enough to assure that

$$\sum_{q\geq 1} \lambda_q^{n/2}\mathrm{e}^{-\lambda_q/2}\, H_q < \infty.$$

With this choice of the $\{\lambda_q\}$, we have from (6) that the series

$$g(z) = \sum g_q(z)$$

converges uniformly over each U_p for every p. In fact, for any $z \in U_p$ we have

$$|g(z)| \leq \sum_{q=1}^{p+2} \sup_{z\in U_p} |g_q(z)| + \sum_{q=p+3}^{\infty} \lambda_q^{n/2}\mathrm{e}^{-\lambda_q/2}\, H_q.$$

From this it can be deduced, by the Weierstrass convergence theorem and lemma (3.3), that g is holomorphic over $\mathbb{C}^n = \bigcup U_p$.

□

Corollary 3.5. *The ring $\mathcal{O}_{\mathbb{R}}(\mathbb{C}^n)$ is dense in $\mathbb{C}^m(\mathbb{R}^n)$, $0 \leq m \leq \infty$, with respect to the Whitney topology.*

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