Manifolds with Corners Modeled on Convenient Vector Spaces

Juan MARGALEF and Enrique OUTERELO

Instituto de Física Fundamental
Consejo Superior de Investigaciones Científicas
Calle Serrano, 123
E-28006 Madrid, Spain
margalef@imaff.cfmac.csic.es

Departamento de Geometría y Topología
Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid
E-28040 Madrid, Spain
outerelo@mat.ucm.es

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ABSTRACT

The authors of the present paper realize a quite systematic study of infinite-dimensional Banach manifolds with corners in [8]. Here, we extend some features of the manifolds with corners modeled on Banach spaces to manifolds with corners modeled on convenient vector spaces, that have arisen as important, in the last years, in Global Analysis.

Key words: Quadrants on vector spaces, Bornological space, Mackey convergence, Mackey complete spaces, Convenient vector spaces, Differentiation theory in convenient vector spaces, Manifolds with corners.

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1. Introduction

The study of manifolds with corners was originally developed by J. Cerf [1] and A. Douady [2] as a natural generalization of the concept of finite-dimensional manifold with smooth boundary.

Stability under finite products of manifolds with corners is of a great importance and one can consider enough by itself to justify the detailed study of this type of manifolds.

Another important question in this field, with pioneer contributions by the authors in the category of manifolds with corners modeled on Banach spaces (as has been recognized by Jean Pradines in [10]), is the existence of quotient manifolds of manifolds with smooth boundary or more general manifolds with corners (see [8]).

In [9] the authors survey the main features about the construction of manifolds with corners modeled on Banach spaces, normed spaces, locally convex topological vector spaces and convenient vector spaces. In this paper we present, with detailed proofs of the principal results, the construction of manifolds with corners modeled on convenient vector spaces with the necessary differential calculus.

2. Quadrants

The local models to construct the manifolds with corners are open subsets of quadrants of topological vector spaces. Thus in this paragraph we introduce the quadrants in vector spaces, topological vector spaces and convenient vector spaces (Mackey complete vector spaces) and we survey the main properties of them.
All the vector spaces will be real vector spaces (*rvs*). However, we remark that the results established here can be extended to complex vector spaces by considering its real restrictions and carrying out adequate adjustments of the notions considered.

Quadrants in vector spaces

Definition 2.1. Let E be a real vector space (*rvs*). A subset Q of E is called quadrant of E if there exist a basis $B = \{u_i\}_{i\in I}$ of E and a subset K of I such that

$$Q = L\{u_k : k \in K\} + \left\{ \sum_{i\in I\setminus K} a_i u_i : \sum_{i\in I\setminus K} a_i u_i \text{ has positive finite support} \right\}$$

($L\{u_k : k \in K\}$ is the vector subspace of E generated by $\{u_k : k \in K\}$ and positive finite support means: There exists a finite subset F of $I \setminus K$ such that $a_i > 0$ for all $i \in F$ and $a_i = 0$ for all $i \in (I \setminus K) \setminus F$).
In this case, we say that the pair (B, K) is adapted to the quadrant Q.

Note that B is a subset of Q and

$$L\{u_k : k \in K\} \bigcap \left\{ \sum_{i\in I\setminus K} a_i u_i : \sum_{i\in I\setminus K} a_i u_i \text{ has positive finite support} \right\} = \{\bar{0}\}$$

Note that E is a quadrant of E, $(K = I)$, and if E is non-trivial, then $\{\bar{0}\}$ is not a quadrant of E.

The following result has an easy proof by the usual techniques of Linear Algebra.

Proposition 2.2. *Let Q be a quadrant of a rvs E and let $(B = \{u_i\}_{i\in I}, K \subset I)$ and $(B' = \{u'_m\}_{m\in M}, N \subset M)$ be pairs adapted to Q. Then:*

(i) $L\{u_k : k \in K\} = L\{u'_n : n \in N\}$.

(ii) $card(I) = card(M), card(K) = card(N), card(I \setminus K) = card(M \setminus N)$.

(iii) *There exists a bijective map $\sigma : I \setminus K \to M \setminus N$ such that $u'_{\sigma(i)} = r_i u_i + x_i$ for all $i \in I \setminus K$, where $x_i \in L\{u_k : k \in K\}$ and r_i is a positive real number.*

(iv) *If $K = \emptyset$, there exists a bijective map $\sigma : I \to M$ such that $u'_{\sigma(i)} = r_i u_i$ for all $i \in I$, where r_i is a positive real number.*

This result proves the consistence of the following definitions.

Definition 2.3. Let Q be a quadrant of a *rvs* E and $(B = \{u_i\}_{i\in I}, K \subset I)$ a pair adapted to Q. Then:

(i) $L\{u_k : k \in K\}$ is called the *kernel* of Q and is denoted by Q^0.

(ii) $card(I \setminus K)$ is called the *index* of Q and is denoted by $index(Q)$. Finally $card(K)$ (i.e. the dimension of Q^0) is called *coindex* of Q and is denoted by $coindex(Q)$.

Proposition 2.4. *Let Q and Q' be quadrants of a rvs E such that $index(Q) = index(Q')$ and $coindex(Q) = coindex(Q')$. Then there exists a linear isomorphism $\alpha : E \to E$ such that $\alpha(Q) = Q'$ and $\alpha(Q^0) = Q'^0$.*

Proposition 2.5. *Let E be a rvs. We have:*

(i) *If Q is a quadrant of E and $(B = \{u_i : i \in I\}, K \subset I)$ is a pair adapted to Q, then there exists a linear isomorphism*

$$\delta : E \to Q^0 \times \mathbb{R}^{(I \setminus K)}$$

such that $\delta(Q) = Q^0 \times \left(\mathbb{R}^{(I \setminus K)}\right)^+$, where

$$\mathbb{R}^{(I \setminus K)} = \left\{x \in \mathbb{R}^{I \setminus K} : \text{ there is } F_x \text{ finite subset of } I \setminus K \text{ with } x_i = 0 \text{ for all } i \notin F_x\right\}$$
$$\text{and } \left(\mathbb{R}^{(I \setminus K)}\right)^+ = \left\{x \in \mathbb{R}^{(I \setminus K)} : x_i \geqslant 0 \text{ for all } i \in I \setminus K\right\},$$

and $\delta(Q^0) = Q^0 \times \{\bar{0}\}$.
($\delta^{-1}\left(\left(x_0, (x_i)_{i \in I \setminus K}\right)\right) = x_0 + \sum_{i \in I \setminus K} x_i u_i$, $x_0 \in Q^0$, $(x_i)_{i \in I \setminus K} \in \mathbb{R}^{(I \setminus K)}$).

(ii) *Let Q and Q' be quadrants of E with finite indexes. Suppose that $index(Q) = index(Q')$. Then $coindex(Q) = coindex(Q')$, (Q^0 and Q'^0 have the same dimension).*

The quadrants in a *rvs* E can be described by linear maps on E.

Proposition 2.6. *Let Q be a quadrant of a rvs E. Then there exists $\Lambda = \{\lambda_m : m \in M\}$, a linearly independent system of elements of*

$$E^* = L(E, \mathbb{R}) = \{\lambda : \lambda \text{ is a linear map from } E \text{ into } \mathbb{R}\},$$

such that

$$Q = E^+_\Lambda = \{x \in E : \lambda_m(x) \geqslant 0 \text{ for all } m \in M\},$$
$$Q^0 = E^0_\Lambda = \{x \in E : \lambda_m(x) = 0 \text{ for all } m \in M\} \text{ and } card(M) = index(Q).$$

Therefore Q is a convex set of E (i.e. for every $x, y \in Q$ and all real number t with $0 \leqslant t \leqslant 1$, one has $tx + (1-t)y \in Q$) and $tQ \subset Q$ for every real number $t \geqslant 0$, i.e. Q is a wedge set of E.

Proof. Let $(B = \{u_i : i \in I\}, K \subset I)$ be a pair adapted to Q. For every $j \in I \setminus K$, let λ_j be the element of E^* defined by: $\lambda_j(u_k) = 0$ for all $k \in I \setminus \{j\}$ and $\lambda_j(u_j) = 1$. We take $M = I \setminus K$ and all the statements are easily proved. □

Remark 2.7. If $index(Q) = 0$, (i.e. $K = I$), then $Q = Q^0 = E$ and, obviously, $E_\Lambda^+ = E_\Lambda^0 = E$ with $\Lambda = \emptyset$.

Proposition 2.8. *Let E be a rvs and $\Lambda = \{\lambda_1, ..., \lambda_p\}$ a finite set of $L(E, \mathbb{R})$. Then,*

(i) *$\{\lambda_1, ..., \lambda_p\}$ is a linearly independent system of elements of $L(E, \mathbb{R})$ if and only if there exists a finite subset $\{x_1, ..., x_p\}$ of E such that $\lambda_i(x_j) = \delta_{ij}$, $i, j = 1, ..., p$ (δ_{ij} Kronecker's index). In this case, $E = E_\Lambda^0 \bigoplus L\{x_1, ..., x_p\}$.*

(ii) *If $\{\lambda_1, ..., \lambda_p\}$ is a linearly independent system of elements of $L(E, \mathbb{R})$, we have that E_Λ^+ is a quadrant of E such that index $\left(E_\Lambda^+\right) = card\,(\Lambda) = p$ and $\left(E_\Lambda^+\right)^0 = E_\Lambda^0$, (By (i), we have $E = E_\Lambda^0 \oplus L\{x_1, ..., x_p\}$, $(\lambda_i(x_j) = \delta_{ij})$).*

Proposition 2.9. *Let Q be a quadrant of finite index on a rvs E. Then, if $\Lambda = \{\lambda_m : m \in M\}$ and $\Lambda' = \{\mu_p : p \in P\}$ are linearly independent systems of elements of $L(E, \mathbb{R})$ such that $Q = E_\Lambda^+ = E_{\Lambda'}^+$, one verifies:*

(i) *$Q^0 = E_\Lambda^0 = E_{\Lambda'}^0$.*

(ii) *$card(M) = card(P) = index(Q)$.*

(iii) *There exists a bijective map $\sigma : M \longrightarrow P$ such that $\mu_{\sigma(m)} = r_m \lambda_m$, where r_m is a positive real number, for all $m \in M$.*

Remark 2.10. In the preceding result, the statement "*index* (Q) is finite" is an essential condition, as proves the example of page 986 of [9].

Proposition 2.11. *Let E be a rvs.*

(a). *Let Q be a quadrant of E and $(\{u_i : i \in I\}, K \subset I)$ a pair adapted to Q. Then:*

(i) *We know that $Q^0 = L\{u_k : k \in K\}$ is an intrinsic subset of Q (2.2 and 2.3). If $K = I$, then $Q^0 = E = Q$ and if $K = \emptyset$, then $Q^0 = \{\overline{0}\}$.*

(ii) *For all $n \in \mathbb{N}$, let Q^n be the set*

$$\{x \in Q : \text{ there exist } x_0 \in Q^0, i_1, \ldots, i_n \in I \setminus K \text{ and } a_{i_1}, \ldots, a_{i_n} \in \mathbb{R} \text{ such that } a_{i_1} > 0, \ldots, a_{i_n} > 0, \text{ and } x = x_0 + a_{i_1} u_{i_1} + \cdots + a_{i_n} u_{i_n}\}.$$

This definition of Q^n does not depend of the pair adapted to Q considered, (2.2), and it is clear that Q^n can be empty, but if $Q^n \neq \emptyset$ then $Q^m \neq \emptyset$ for all $m \in \mathbb{N}$ with $m < n$.

(b). *Let Q be a quadrant of E. Then:*

(i) *If $index(Q)$ is infinite, then $\{Q^n : n \in \mathbb{N} \cup \{0\}\}$ is a partition of the quadrant Q.*

(ii) *If $index\,(Q) = n$, $n \in \mathbb{N} \cup \{0\}$, then $Q^m = \emptyset$ for all $m > n$ and $\left\{Q^0, Q^1, ..., Q^n\right\}$ is a partition of Q.*

(c). *Let Q be a quadrant of E. Then:*

(i) *If $index(Q)$ is infinite and $x \in Q^n$, $(n \in \mathbb{N} \cup \{0\})$, we say that x has coindex n ($coindex(x) = n$).*

(ii) *If $index(Q)$ is finite ($index(Q) = n$) and $x \in Q^m$, $0 \leqslant m \leqslant n$, we say that x has coindex m and index $n - m$ ($index(x) = n - m$). In this case (of $index(Q) = n$), $\{x \in Q : x \text{ has index } n\} = Q^0$.*

(d). *Let Q be a quadrant of E with finite index n. Let $\Lambda = \{\nu_1, ..., \nu_n\}$ be a linearly independent system of elements of $L(E, \mathbb{R})$ such that $Q = E^+_\Lambda$, (consequently $Q^0 = E^0_\Lambda$ (2.9)). Then for $x \in Q$ we have:*

$$coindex(x) = m \text{ if and only if } card\{i : \nu_i(x) \neq 0\} = m$$
$$\text{and } index(x) = p \text{ if and only if } card\{i : \nu_i(x) = 0\} = p.$$

Quadrants in Hausdorff real topological vector spaces

In Hausdorff real topological vector spaces we are only interested in quadrants with non-empty interior.

Lemma 2.12. *Let $(E, \mathcal{T})$ be a Hausdorff real topological vector space (Hrtvs) (i.e. E is a rvs, $\mathcal{T}$ is a Hausdorff topology in E and the addition of vectors, $+ : (E, \mathcal{T}) \times (E, \mathcal{T}) \to (E, \mathcal{T})$, and the product of vectors by scalars, $\cdot : (\mathbb{R}, T_u) \times (E, \mathcal{T}) \to (E, \mathcal{T})$ are continuous maps) and Q a quadrant of E. Then:*

- *If $index(Q)$ is infinite, $int(Q) = \emptyset$, ($int(Q)$ is the interior of Q in the topological space $(E, \mathcal{T})$).*
- *If $index(Q)$ is finite and Q^0 is not closed in $(E, \mathcal{T})$, $int(Q) = \emptyset$.*

Proof. Let $(B = \{u_i : i \in I\}, K \subset I)$ be a pair adapted to Q. Suppose $index(Q) = card(I \setminus K)$ infinite. Let x be an element of Q and U an open neighborhood of x. Then $-x + U(= V)$ is an open set of E with $0 \in V$. The point x can be uniquely expressed by $x = x_0 + a_1 u_{j_1} + ... + a_r u_{j_r}$, $x_0 \in Q^0$, j_1,...,j_r elements of $I \setminus K$, $a_1 > 0$,...,$a_r > 0$. If j_0 is an element of $I \setminus K$ with $j_0 \notin \{j_1, ..., j_r\}$ and $-\frac{1}{n_0} u_{j_0} \in V$, then $x - \frac{1}{n_0} u_{j_0} \in U$ and $x - \frac{1}{n_0} u_{j_0} = x_0 + a_1 u_{j_1} + ... + a_r u_{j_r} - \frac{1}{n_0} u_{j_0} \notin Q$. Thus $int(Q) = \emptyset$.

Suppose now that $index(Q)$ is finite and Q^0 is not closed in $(E, \mathcal{T})$. Then $I \setminus K = \{i_1, ..., i_n\}$. One considers the linearly independent system of elements of $E^* = L(E, \mathbb{R})$, $\Lambda = \{\lambda_i\}_{i \in \{i_1, ..., i_n\}}$ given by $\lambda_i(u_j) = \delta_{ij}$ (δ_{ij} Kronecker's index), $i \in \{i_1, ..., i_n\}$, $j \in I$. Then, by 2.6, $Q = E^+_\Lambda$ and $Q^0 = \bigcap_{j=1}^{j=n} ker(\lambda_{i_j})$. Since Q^0 is not closed in $(E, \mathcal{T})$, there exists $j \in \{1, ..., n\}$ such that $ker(\lambda_{i_j})$ is dense in $(E, \mathcal{T})$ (5.4, page 37, of [5]), (suppose $j = 1$). Let x be an element of Q and V an open neighborhood of x. Then, there exists $y \in ker(\lambda_{i_1}) \cap V$ and therefore $y = y_0 + a_2 u_{i_2} + ... + a_n u_{i_n}$ with $y_0 \in Q^0$. Thus, if $-\frac{1}{n_0} u_{i_1} \in V - y$ (a such element always exists), $y - \frac{1}{n_0} u_{i_1} = y_0 - \frac{1}{n_0} u_{i_1} + a_2 u_{i_2} + ... + a_n u_{i_n}$ is an element of $V \cap (E \setminus Q)$ and $int(Q) = \emptyset$. □

Proposition 2.13. *Let Q be a quadrant of a Hrtvs $(E, \mathcal{T})$. Then the following statements are equivalent:*

(i) *$int(Q) \neq \emptyset$ ($int(Q)$ is the interior of Q in the topological space $(E, \mathcal{T})$).*

(ii) *$index(Q)$ is finite and Q^0 is closed in $(E, \mathcal{T})$ (we remark that if $index(Q) = 0$, then $Q^0 = Q = E$).*

Proof. The step "$(i) \Longrightarrow (ii)$" follows from the above lemma.
Step "$(ii) \Longrightarrow (i)$". The case $index(Q) = 0$ is trivial ($Q = E = Q^0$). Suppose $index(Q) = n \geqslant 1$ and let $(\{u_i : i \in I\}, K \subset I)$ be a pair adapted to Q. Then, $I \setminus K = \{i_1, ..., i_n\}$ and by 2.6 the linearly independent system $\Lambda = \{\lambda_1, ..., \lambda_n\}$ of elements of $E^* = L(E, \mathbb{R})$, where $\lambda_k(u_j) = \delta_{kj}$ for all $j \in I$ and $k = 1, ..., n$ verifies that $Q = E^+_\Lambda$ and $Q^0 = E^0_\Lambda$. Moreover

$$\alpha : (Q^0, \mathcal{T}|_{Q^0}) \times (\mathbb{R}^n, \mathcal{T}^n_u) \to (Q^0, \mathcal{T}|_{Q^0}) \times (L\{u_{i_1}, ..., u_{i_n}\}, \mathcal{T}|_{L\{u_{i_1}, ..., u_{i_n}\}}) \xrightarrow{+} (E, \mathcal{T}),$$
$$(x_0, (r_1, ..., r_n)) \mapsto (x_0, r_1 u_1 + ... + r_n u_n) \mapsto x_0 + r_1 u_1 + ... + r_n u_n,$$

is a linear homeomorphism, since we apply the general results: (1) *Let $(E, \mathcal{T})$ be a topological vector space and F a closed vector subspace of finite codimension of $(E, \mathcal{T})$. Then every algebraic supplement G of F in E is a topological supplement of F in $(E, \mathcal{T})$*; (2) *A linear subspace G of finite dimension n of a Hrtvs $(E, \mathcal{T})$ is linearly homeomorphic to $(\mathbb{R}^n, \mathcal{T}^n_u)$ (therefore G is complete and closed in $(E, \mathcal{T})$).* Consequently, for all $i \in \{1, ..., n\}$, $\lambda_i = p_i \circ \alpha^{-1}$, (where $p_i : Q^0 \times \mathbb{R}^n \to \mathbb{R}$, $p_i(x_0, (r_1, ..., r_n)) = r_i$), is a continuous mapping from $(E, \mathcal{T})$ into $(\mathbb{R}, \mathcal{T}_u)$. Thus,

$$\Lambda \subset \mathcal{L}((E, \mathcal{T}), (\mathbb{R}, \mathcal{T}_u)) = \{\lambda : \ \lambda \text{ is a continuous linear map of } (E, \mathcal{T}) \text{ into } (\mathbb{R}, \mathcal{T}_u)\}.$$

Furthermore, $\alpha^{-1}(Q) = \{(x_0, (r_1, ..., r_n)) \in Q^0 \times \mathbb{R}^n : r_1 \geqslant 0, ..., r_n \geqslant 0\}$ and therefore

$$int(Q) = \left\{x_0 + r_1 u_{i_1} + ... + r_n u_{i_n} : \ x_0 \in Q^0, \ r_1 > 0, ..., r_n > 0\right\} \neq \emptyset.$$

□

Proposition 2.14. *Let Q be a quadrant of a Hrtvs $(E, \mathcal{T})$. Then the following statements are equivalent:*

(i) *$index(Q)$ is finite and Q^0 is closed in $(E, \mathcal{T})$.*

(ii) *There exists a finite linearly independent system Λ of elements of*

$$\mathcal{L}\left((E, \mathcal{T}), (\mathbb{R}, \mathcal{T}_u)\right)) = \{\lambda : \lambda \text{ is a continuous linear map from } (E, \mathcal{T}) \text{ in } (\mathbb{R}, \mathcal{T}_u)\}$$

such that $Q = E^+_\Lambda$. (See 2.8(ii))

Moreover if (ii) is fulfilled, $cardinal(\Lambda) = index(Q)$ and $Q^0 = E^0_\Lambda$.

Proposition 2.15. *Let Q and P be quadrants of a Hrtvs $(E, \mathcal{T})$ such that Q^0 and P^0 are closed in $(E, \mathcal{T})$, and $index(Q) = index(P)$, finite. Then, there exists a linear homeomorphism $\alpha : E \to E$ such that $\alpha(P) = Q$ and $\alpha(P^0) = Q^0$.*

Proof. It is an easy consequence of the general result: *Let $(E, \mathcal{T})$ be a Hrtvs and F, G closed vector subspaces of E with the same finite codimension. Then F and G are linearly homeomorphic.* □

Quadrants in Hausdorff locally convex real topological vector spaces

Now we consider a special class of Hausdorff topological vector spaces, namely the class of the Hausdorff locally convex real topological vector spaces (*Hlcrtvs*) (i.e. Hausdorff topological vector spaces such that the zero vector $\bar{0}$ has a basis of neighborhoods consisting of open convex sets). For a detailed study of the *Hlcrtv* spaces the reader can consult the books [4] and [7], where one can find the omitted proofs of some results established in the present paper.

Definition 2.16. (i) Let E be a *rvs*. A subset C of E is called *absolutely convex* if for every $x, y \in C$ and real numbers r, s with $|r| + |s| \leqslant 1$, one has $rx + sy \in C$. A subset R of E is called *radial* if $[0,1]R \subset R$.
A subset A of E is called *absorbing* if $\bigcup_{r>0} rA = E$.

(ii) Let $(E, \mathcal{T})$ be a *rtvs*. A subset B of E is called *bounded* if for all neighborhood U of $\bar{0}$ there exists a real number r such that $B \subset rU$.
A radial subset V of E is called *bornivorous* if it absorbs each bounded set (i.e. for every bounded set B of E there exists a real number $r > 0$ such that $B \subset [0, r]V$).

Proposition 2.17. *Let $(E, \mathcal{T})$ be a Hausdorff topological real vector space. Then, the following results are equivalent:*

(i) *$(E, \mathcal{T})$ is a Hlcrtvs.*

(ii) *The zero vector $\bar{0}$ has a basis of neighborhoods consisting on closed absolutely convex sets.*

(iii) *The zero vector $\bar{0}$ has a basis of neighborhoods consisting on open absolutely convex sets.*

(iv) *The topology $\mathcal{T}$ is described by a family of semi-norms $\{p_i = \|\cdot\|_i\}_{i\in I}$, (i.e. the family of finite intersections of elements of $\{p_i^{-1}([0,\varepsilon)) : \ i \in I, \ \varepsilon > 0\}$ is a basis of neighborhoods of the zero vector $\bar{0}$).*

(v) *The topology $\mathcal{T}$ is described by a family of continuous semi-norms $\{p'_i = \|\cdot\|'_i\}_{i\in I}$, (i.e. the family of finite intersections of elements of $\{(p'_i)^{-1}([0,\varepsilon)) : \ i \in I, \ \varepsilon > 0\}$ is a basis of neighborhoods of the zero vector $\bar{0}$).*

(A semi-norm p in a *rvs*, E, is a map $p : E \to \mathbb{R}$ such that: $p(x) \geqslant 0$, $p(x+y) \leqslant p(x) + p(y)$, $p(rx) = |r|p(x)$, $x, y \in E$, $r \in \mathbb{R}$. If B is an absolutely convex absorbing subset of E, then $p_B : E \to \mathbb{R}$, $x \mapsto \inf\{r \in \mathbb{R} : \ x \in rB\}$, is a semi-norm in E such that $\{x : p_B(x) < 1\} \subset B \subset \{x : p_B(x) \leqslant 1\}$ (Minkowski functional). In the proof of the step "(iii)$\Longrightarrow$(iv)" of the above proposition consider the Minkowski functionals p_U, U open absolutely convex neighborhood of $\bar{0}$ (U is clearly absorbing)).

Proposition 2.18. *Let $(E, \mathcal{T})$ be a Hausdorff locally convex rtvs (Hlcrtvs) and*

$$\mathcal{V}(0) = \{V \subset E : V \text{ is a bornivorous absolutely convex subset of } E\}.$$

Then:

(i) $\mathcal{V}(0)$ *is a* 0*-neighborhood basis of a (unique) Hausdorff locally convex topology on* E*, denoted by* $\mathcal{T}_{born}$ *and called bornologification of* $(E, \mathcal{T})$*. This topology,* $\mathcal{T}_{born}$*, is finer than* $\mathcal{T}$ $(\mathcal{T} \subset \mathcal{T}_{born})$.

(ii) $(E, \mathcal{T})$ *and* $(E, \mathcal{T}_{born})$ *have the same collection of bounded sets, and* $(E, \mathcal{T}_{born})$ *is the finest Hausdorff locally convex topology on* E *with this property.*

Proposition 2.19. *Let* $(E, \mathcal{T})$ *be a Hlcrtvs. Then:*

(i) $(E, \mathcal{T}_{born})$ *is a bornological Hlcrtvs (A Hlcrtvs* $(E', \mathcal{T}')$ *is said to be a bornological space, if every bornivorous absolutely convex subset in* $(E', \mathcal{T}')$ *is a neighborhood of* 0 *in* $(E', \mathcal{T}')$*).*

(ii) $(E, \mathcal{T})$ *is bornological if and only if* $\mathcal{T} = \mathcal{T}_{born}$.

(iii) *For any Hlcrtvs* $(F, \mathcal{T}')$,

$$\begin{aligned}&\mathcal{L}((E, \mathcal{T}_{born}), (F, \mathcal{T}')) = \\ &\quad = \{h : (E, \mathcal{T}_{born}) \to (F, \mathcal{T}') : \ h \text{ is a linear continuous map}\} = \\ &= \{f : (E, \mathcal{T}) \to (F, \mathcal{T}') : \ f \text{ is a bounded linear map}\} (= L_b((E, \mathcal{T}), (F, \mathcal{T}'))).\end{aligned}$$

(iv) *Let* A *be an absolutely convex set in* $(E, \mathcal{T})$*. Then* A *is a* 0*-neighborhood in* $(E, \mathcal{T}_{born})$ *if and only if* A *is a bornivorous subset.*

(v) *The continuous semi-norms on* $(E, \mathcal{T}_{born})$ *are exactly the bounded semi-norms on* $(E, \mathcal{T})$.

(vi) *If* F *is a vector subspace of* $(E, \mathcal{T})$ *of finite codimension, then* $(\mathcal{T}|_F)_{born} = (\mathcal{T}_{born})|_F$.

(vii) *If* $(E, \mathcal{T})$ *is metrizable,* E *is bornological. In particular, if* E *is a Fréchet (or normable) space, then* $(E, \mathcal{T})$ *is bornological.*

(viii) *The topological product of at most countably many bornological spaces is again bornological.*

Definition 2.20. Let $(E, \mathcal{T})$ be a *Hlcrtvs*, G an open set of $(\mathbb{R}, \mathcal{T}_u)$ and let $c : G \longrightarrow E$ be a map. Then:

(i) c is called differentiable if the derivative, $c'(t) = \lim_{s \to 0} \frac{c(t+s)-c(t)}{s}$ at t, exists for all $t \in G$, (consequently c is continuous).

(ii) We say that c is C^0 if c is continuous on G. We say that c is C^1 if $c'(t)$ exists for all $t \in G$ and c' is continuous on G. We say that c is C^2 if c' and $(c')'$ $(= c'')$ exist and c'' is continuous. In general, we say that c is C^n if c', c'', ..., $c^{(n}$ exist and $c^{(n}$ is continuous.

(iii) c is called smooth or C^∞ if all the iterated derivatives exist.

The set of C^p maps of G into E will be denoted by $C^p(G, E)$, $(p \in \{0\} \cup \mathbb{N} \cup \{\infty\})$.

The above definition also has meaning for maps defined on arbitrary intervals of $\mathbb{R}$ considering lateral limits in the extreme points (if they exist) of the interval.

Definition 2.21. Let $(E, \mathcal{T})$ be a Hlcrtvs.

(i) We denote by $C^\infty\mathcal{T}$, (which will be called C^∞-topology), the final topology in E induced by the family

$$C^\infty(\mathbb{R}, E) = \{c : \mathbb{R} \longrightarrow E : c \text{ is a smooth map (curve) from } \mathbb{R} \text{ into } (E, \mathcal{T})\}.$$

The topological space $(E, C^\infty\mathcal{T})$, will be also denoted by $C^\infty E$.

(ii) Let X be a subset of E. Let us consider the set

$$\mathcal{C}_X^\infty = \{c : \mathbb{R} \longrightarrow E : \ c \text{ is a smooth curve in } (E, \mathcal{T}) \text{ with } im(c) \subset X\}.$$

The final topology in X generated by the family $\mathcal{C}_X^\infty$ will be called $C^\infty-$topology of X and the corresponding topological space will be denoted by $C^\infty X$.
This topology contains the topology $(C^\infty\mathcal{T})|_X$, but, in general, they do not coincide.
Note that, if X is open in $(E, C^\infty\mathcal{T})$ or X is convex and locally closed in $(E, C^\infty\mathcal{T})$, then the C^∞-topology of X is equal to $(C^\infty\mathcal{T})\,|_X$.

Proposition 2.22. *Let $(E, \mathcal{T})$ be a Hlcrtvs. Then $C^\infty\mathcal{T}$ is the final topology $\mathcal{T}^\mathcal{F}$ in E induced by the family*

$$\mathcal{F} = \{c : G \to E : \ G \text{ open in } (\mathbb{R}, \mathcal{T}_u) \ \text{ and } c \text{ is a smooth map}\}.$$

Proof. Since $C^\infty(\mathbb{R}, E) \subset \mathcal{F}$, we have that $\mathcal{T}^\mathcal{F} \subset C^\infty\mathcal{T}$.
On the other hand, if G is an open set of $(\mathbb{R}, T_u)$ and $c : G \to E$ is a smooth map, one has that $G = \bigcup_{i \in N}(a_i, b_i)$ where N is a countable set (finite or infinite) and $(a_i, b_i) \cap (a_j, b_j) = \emptyset$, $i, j \in N$, $i \neq j$, and for all $i \in N$ there is a bijective map $\varphi_i : (a_i, b_i) \to \mathbb{R}$ such that φ_i and φ_i^{-1} are C^∞-maps. Thus if $A \in C^\infty\mathcal{T}$, then

$$c^{-1}(A) = \bigcup_{i \in N}(c|_{(a_i,b_i)})^{-1}(A) = \bigcup_{i \in N}\varphi_i^{-1}((c|_{(a_i,b_i)}(\varphi_i)^{-1})^{-1}(A))$$

is open in $(\mathbb{R}, T_u)$ since $c|_{(a_i,b_i)}(\varphi_i)^{-1} : \mathbb{R} \to E$ is a smooth map for all $i \in N$. Thus, $C^\infty\mathcal{T} \subset \mathcal{T}^\mathcal{F}$ □

Proposition 2.23. *Let $(E, \mathcal{T})$ and $(F, \mathcal{T}')$ be Hlcrtv spaces and $\lambda : (E, \mathcal{T}) \to (F, \mathcal{T}')$ a continuous linear mapping. Then,*

(i) *If G is an open set of $(\mathbb{R}, \mathcal{T}_u)$ and $c : G \to E$ is a smooth map, one has that $\lambda c : G \to F$ is a smooth map and $(\lambda c)^{(n} = \lambda c^{(n}$ for all $n \in \mathbb{N}$.*

(ii) *λ is a continuous mapping from $(E, C^\infty\mathcal{T})$ into $(F, C^\infty\mathcal{T}')$.*

Proposition 2.24. *Let $(E, \mathcal{T})$ be a Hlcrtvs.*

(i) *$\mathcal{T} \subset \mathcal{T}_{born} \subset C^\infty\mathcal{T}$.*

(ii) *The spaces $(E, \mathcal{T})$ and $(E, \mathcal{T}_{born})$ have the same smooth curves, (see [7]).*

(iii) *$C^\infty(\mathcal{T}_{born}) = C^\infty\mathcal{T}$.*

(iv) *If* $(E,\mathcal{T})$ *is metrizable, then,* $\mathcal{T} = \mathcal{T}_{born} = C^\infty\mathcal{T}$*. In particular, this takes place if* $(E,\mathcal{T})$ *is a Fréchet space, and in the special case of* $(\mathbb{R}^n, T_u^n)$*,* $C^\infty T_u^n = (\mathcal{T}_u^n)_{born} = T_u^n$ *(the usual topology of* $\mathbb{R}^n$*).*

Note that $(E, C^\infty\mathcal{T})$ is not, in general, a *rtvs* ($C^\infty\mathbb{R}^I$, where $card(I) \geqslant 2^{\aleph_0}$, is not completely regular (see [7], page 46)).

Remark 2.25. In general, the C^∞-topology of a product of two *Hlcrtv* spaces is not the product of the C^∞-topologies of the factor spaces. However, if $(E,\mathcal{T})$ and $(F,\mathcal{T}')$ are *Hlcrtv* spaces, then $C^\infty\mathcal{T} \times C^\infty\mathcal{T}' \subset C^\infty(\mathcal{T} \times \mathcal{T}')$.

Lemma 2.26. *([7], page 41). Let* $(E,\mathcal{T})$ *be a Hlcrtvs,* U *an open set in* $C^\infty(E\times\mathbb{R})$ *and* K *a compact subset of* $(\mathbb{R}, \mathcal{T}_u)$*. Then,* $(U_0 =)\{x \in E : \{x\} \times K \subset U\}$ *is open in* $C^\infty E$*.*

Proposition 2.27. *Let* $(E,\mathcal{T})$ *be Hlcrtvs and* $n \in \mathbb{N}$*. Then,*

(i) *The* C^∞*-topology of* $E \times \mathbb{R}^n$ *is the product topology of* $C^\infty\mathcal{T}$ *by* $\mathcal{T}_u^n$*.*

(ii) *The* C^∞*-topology of* $E\times[0,+\infty)^n$ *is the product topology of* $C^\infty\mathcal{T}$ *and* $\mathcal{T}_u^n|_{[0,+\infty)^n}$*, i.e. the topology induced by the* C^∞*-topology of* $E \times \mathbb{R}^n$ *in* $E \times [0,+\infty)^n$*.*

Proof. (i) See the page 42 of [7].
(ii) Let $\mathcal{T}^*$ be the C^∞-topology of $E \times [0,+\infty)^n$. By 2.21(ii), we know that $C^\infty\mathcal{T} \times \mathcal{T}_u^n|_{E\times[0,+\infty)^n} \subset \mathcal{T}^*$.
The other inclusion follows recursively from the special case $E \times [0,+\infty)$, for which we can proceed as follows. Take an open neighborhood U of an arbitrary point $(x,t) \in E \times [0,+\infty)$ in the topology $\mathcal{T}^*$. Since the map $c(s) = (x,s^2)$ from $\mathbb{R}$ into $E \times [0,+\infty)$ is a smooth curve, there exists $\varepsilon > 0$ such that $c(s) \in U$ for all $s \in (\sqrt{t}-\varepsilon, \sqrt{t}+\varepsilon)$. Then $K = c([\sqrt{t}-\varepsilon/2, \sqrt{t}+\varepsilon/2])$ is a compact neighborhood of t in $([0,+\infty), \mathcal{T}_u|_{[0,+\infty)})$. Then, by the above lemma (2.26), $U_0 = \{y \in E : \{y\} \times K \subset U\}$ is open in $C^\infty E$, $x \in U_0$ and $U_0 \times K \subset U$. Thus U is an neighborhood of (x,t) in $(E, C^\infty\mathcal{T}) \times ([0,+\infty), \mathcal{T}_u|_{[0,+\infty)})$, and $\mathcal{T}^* \subset C^\infty\mathcal{T} \times \mathcal{T}_u|_{[0,+\infty)}$. □

Proposition 2.28. *Let* $(E,\mathcal{T})$ *be a Hlcrtvs and* F *a linear subspace of* E *closed in* $(E, C^\infty\mathcal{T})$ *(in particular, in* $(E,\mathcal{T})$*). Then,*

(i) *A curve into* F *is smooth if and only if it is smooth in* E*, ([7], page 28).*

(ii) *The topology* $C^\infty\mathcal{T}$ *induces in* F *the topology* $C^\infty(\mathcal{T}|_F)$*. If* F *is not closed this result is not true in general, ([7], page 47).*

Proposition 2.29. *Let* $(E,\mathcal{T})$ *be a bornological Hlcrtvs (*$\mathcal{T} = \mathcal{T}_{born}$*). One has:*

(i) *If* V *is an absolutely convex subset of* E*, then* V *is a* 0*-neighborhood of* $(E,\mathcal{T})$ *if and only if* V *is a* 0*-neighborhood in* $C^\infty E = (E, C^\infty\mathcal{T})$*.*

(ii) *If* U *is a convex subset of* E*, then* U *is* C^∞*-open if and only if* U *is open in* $(E,\mathcal{T})$*.*

Lemma 2.30. *Let $(E,\mathcal{T})$ be a Hlcrtvs, x_0 and x_1 elements of E and $r \in \mathbb{R} \setminus \{0\}$. Then the map*

$$\tau : (E, C^\infty\mathcal{T}) \to (E, C^\infty\mathcal{T}),\ x \mapsto x_0 + r(x - x_1)$$

is a homeomorphism ($\tau^{-1}(y) = x_1 + r^{-1}(y - x_0)$).

Proposition 2.31. *([7], page 247). Let $(E,\mathcal{T})$ be a Hlcrtvs and K a convex set (in particular, a quadrant Q) of $(E,\mathcal{T})$ with non-empty C^∞-interior, that is, $int_{C^\infty\mathcal{T}}(K) \neq \emptyset$. Then,*

(i) *The segment $(x,y] = \{x + t(y-x) :\ 0 < t \leqslant 1\} \subset int_{C^\infty\mathcal{T}}(K)$ for all $x \in K$ and all element y of $int_{C^\infty\mathcal{T}}(K)$.*

(ii) *The C^∞-interior of K is convex and open even in $(E,\mathcal{T}_{born})$, ($\mathcal{T}_{born} \subset C^\infty\mathcal{T}$). Furthermore, $int_{C^\infty\mathcal{T}}(K) = int_{\mathcal{T}_{born}}(K) \neq \emptyset$*

(iii) *K is closed in $(E,\mathcal{T}_{born})$ if and only if it is closed in $(E, C^\infty\mathcal{T})$.*

Theorem 2.32. *Let $(E,\mathcal{T})$ be a Hlcrtvs, and let Q be a quadrant in E. Then, the following statements are equivalent:*

(i) *The C^∞-interior of Q is non-empty, (that is $int_{C^\infty\mathcal{T}}Q \neq \emptyset$).*

(ii) *The index of Q is finite and Q^0 is closed in $(E,\mathcal{T}_{born})$.*

(iii) *There exists a finite linearly independent system Λ of elements of $L_b(E,\mathbb{R})$ (the space of bounded linear maps from $(E,\mathcal{T})$ into $\mathbb{R}$) such that $Q = E^+_\Lambda = \{x \in E :\ \lambda(x) \geqslant 0$ for all $\lambda \in \Lambda\}$.*

Moreover if (iii) is fulfilled, $cardinal(\Lambda) = index(Q)$ and $Q^0 = E^0_\Lambda$.

Proof. (i)$\Longrightarrow$ (ii). By 2.31(ii), $int_{\mathcal{T}_{born}}Q = int_{C^\infty\mathcal{T}}Q \neq \emptyset$. Then, by 2.13, the index of Q is finite and Q^0 is closed in $(E,\mathcal{T}_{born})$.
(ii)$\Longrightarrow$ (iii). By 2.14 there exists a linearly independent system Λ of elements of $\mathcal{L}((E,\mathcal{T}_{born}),(\mathbb{R},\mathcal{T}_u))$ such that $Q = E^+_\Lambda$ (moreover $cardinal(\Lambda) = index(Q)$ and $Q^0 = E^0_\Lambda$). Then, (iii) follows of 2.19 (iii).
(iii)$\Longrightarrow$ (i). Since $L_b(E,\mathbb{R}) = \mathcal{L}((E,\mathcal{T}_{born}),(\mathbb{R},\mathcal{T}_u))$, (2.19(iii)), by 2.14 and 2.13 one has that $int_{born}Q \neq \emptyset$, and (i) follows of 2.31(ii). □

Proposition 2.33. *Let $(E,\mathcal{T})$ be a Hlcrtvs and let Λ be a finite linearly independent system of elements of $L_b((E,\mathcal{T}),\mathbb{R}) = \mathcal{L}((E,\mathcal{T}_{born}),(\mathbb{R},\mathcal{T}_u))$ (2.19(iii)). Then:*

(i) *$Q = E^+_\Lambda = \{x \in E :\ \lambda(x) \geqslant 0$ for all $\lambda \in \Lambda\}$ is a quadrant in E such that $index(Q) = cardinal(\Lambda)$. Furthermore, $Q^0 = E^0_\Lambda = \{x \in E :\ \lambda(x) = 0$ for all $\lambda \in \Lambda\}$. Finally, if $\Lambda = \emptyset$, one has that $Q = Q^0 = E$.*

(ii) *$Q = E^+_\Lambda$ and $Q^0 = E^0_\Lambda$ are closed in $(E,\mathcal{T}_{born})$ and therefore in $(E, C^\infty\mathcal{T})$.*

(iii) *$int_{\mathcal{T}_{born}}(Q) = int_{C^\infty\mathcal{T}}Q \neq \emptyset$.*

(iv) *If* $\Lambda = \{\lambda_1, ..., \lambda_n\} \neq \emptyset$ *and* $\{x_1, ..., x_n\}$ *is a subset of* E *(a such subset always exists) such that* $\lambda_i(x_j) = \delta_{ij}$, $i, j = 1, ..., n$, *one has that,*

$$\alpha : Q^0 \times \mathbb{R}^n \to E, \;\; (x_0, (r_1, ..., r_n)) \mapsto x_0 + r_1x_1 + ... + r_nx_n.$$

is a linear homeomorphism from $C^\infty Q^0 \times (\mathbb{R}^n, \mathcal{T}_u^n)$ *into* $C^\infty E = (E, C^\infty\mathcal{T})$.

(v) *The* C^∞*-topology on* Q *coincides with* $C^\infty\mathcal{T}|_Q$, *and the* C^∞*-topology of* Q^0 *coincides with* $C^\infty\mathcal{T}|_{Q^0}$.

(vi) *Let* U *be a non-void element of* $(C^\infty\mathcal{T})|_Q$. *Then* $int_{C^\infty\mathcal{T}}(U)$ *is also non-empty.*

Proof. (i). If $\Lambda \neq \emptyset$, the result follows from 2.8(ii). The case $\Lambda = \emptyset$ is a trivial verification.
(ii). It follows from (i).
(iii). By the above theorem $int_{C^\infty E}Q \neq \emptyset$ and by 2.31 one has that $int_{C^\infty E}Q = int_{\mathcal{T}_{born}}Q$.
(iv). By (ii), we have that

$$(Q^0, \mathcal{T}_{born}|_{Q^0}) \times (L\{x_1, ..., x_n\}, \mathcal{T}_{born}|_{L\{x_1,...,x_n\}}) \to (E, \mathcal{T}_{born}),$$
$$(x_0, a_1x_1 + ... + a_nx_n) \mapsto x_0 + a_1x_1 + ... + a_nx_n,$$

is a linear homeomorphism. Consequently, by 2.24(iv),

$$\alpha : (Q^0, \mathcal{T}_{born}|_{Q^0}) \times (\mathbb{R}^n, \mathcal{T}_u^n) \to (E, \mathcal{T}_{born}), (x_0, (a_1, ..., a_n)) \mapsto x_0 + a_1x_1 + ... + a_nx_n,$$

is a linear homeomorphism and by 2.23(ii), 2.24(iii) and 2.27(i)

$$\alpha : (Q^0, C^\infty(\mathcal{T}_{born}|_{Q^0})) \times (\mathbb{R}^n, \mathcal{T}_u^n) \to (E, C^\infty\mathcal{T})$$

is also a linear homeomorphism. Finally, since Q^0 is a subspace of $(E, \mathcal{T})$ of finite codimension (n), by 2.19(vi) one has that $(\mathcal{T}|_{Q^0})_{born} = \mathcal{T}_{born}|_{Q^0}$ and therefore $C^\infty(\mathcal{T}_{born}|_{Q^0}) = C^\infty((\mathcal{T}|_{Q^0})_{born})) = C^\infty((\mathcal{T}|_{Q^0}) = C^\infty\mathcal{T}|_{Q^0}$.
(v) It follows from (iv) and 2.27(ii).
(vi). It is a consequence of (iv). □

Corollary 2.34. *Let* $(E, \mathcal{T})$ *be a Hlcrtvs, and* Q *a quadrant in* E *with* $int_{C^\infty E}Q \neq \emptyset$. *Then, for all non-empty* C^∞*-open set* U *of* Q, $(C^\infty(T|_Q) = C^\infty T|_Q)$, *one verifies that* $int_{C^\infty E}U = U \cap int_{C^\infty E}Q \neq \emptyset$.

Proof. By the above proposition, we have that

$$\alpha : C^\infty Q^0 \times (\mathbb{R}^n, \mathcal{T}_u^n) \to (E, C^\infty\mathcal{T}), \;\; (x_0, (r_1, ..., r_n)) \mapsto x_0 + r_1x_1 + ... + r_nx_n$$

is a linear homeomorphism. Thus, if U is a non-empty C^∞-open set of Q, $\alpha^{-1}(U)$ is a non-empty open set of $C^\infty Q^0 \times ([0, +\infty)^n, \mathcal{T}_u^n|_{[0,+\infty)^n})$, and $\alpha^{-1}(U) \cap (Q^0 \times (0, +\infty)^n) \neq \emptyset$ which proves that $int_{C^\infty E}U = U \cap int_{C^\infty E}Q \neq \emptyset$. □

3. Differentiation theory in convenient vector spaces

For a detailed study of the differentiation theory of maps defined on C^∞-open subsets of convenient vector spaces the reader can consult the books [3] and [7].

Mackey-convergence

Definition 3.1. (Mackey-convergence). Let E be a *Hlcrtvs*. We say that a net $\{x_\gamma\}_{\gamma\in\Gamma}$ in E Mackey-converges (or M-converges) to $x \in E$, if there exists a closed bounded absolutely convex subset B of E such that

$$\{x_\gamma :\ \gamma \in \Gamma\} \cup \{x\} \subset \langle B\rangle = L(B) = \bigcup_{t>0} t\cdot B$$

and the net $\{x_\gamma\}_{\gamma\in\Gamma}$ converges to x in the normed space $E_B = (\langle B\rangle, p_B)$ where $p_B(y) = \inf\{r > 0 :\ y \in r\cdot B\}$ for all $y \in \langle B\rangle$.

Proposition 3.2. *Let E be a Hlcrtvs, $\{x_\gamma\}_{\gamma\in\Gamma}$ a net in E and x a point of E. Then $\{x_\gamma\}_{\gamma\in\Gamma}$ Mackey-converges to $x\in E$ if and only if there exists a bounded absolutely convex subset B' of E such that $\{x_\gamma : \gamma\in\Gamma\}\cup\{x\} \subset \langle B'\rangle$ and the net $\{x_\gamma\}_{\gamma\in\Gamma}$ converges to x in the normed space $E_{B'} = (\langle B'\rangle, p_{B'})$.*

Remarks 3.3. **(1).** A net $\{x_\gamma\}_{\gamma\in\Gamma}$ in E Mackey-converges to $x \in E$ if and only if $\{x_\gamma - x\}_{\gamma\in\Gamma}$ Mackey-converges to $\bar{0} \in E$.

(2). If the net $\{x_\gamma\}_{\gamma\in\Gamma}$ in E Mackey-converges to $x \in E$, then there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ in Γ such that the sequence $\{x_{\gamma_n}\}_{n\in\mathbb{N}}$ Mackey-converges to $x \in E$.

(3). Let $(E,\mathcal{T})$ be a *Hlcrtvs* and B a bounded absolutely convex subset of E. Since

$$\langle B\rangle = \bigcup_{t>0} t\cdot B \text{ and } \{x\in\langle B\rangle : p_B(x) < 1\} \subset B \subset \{x \in \langle B\rangle : p_B(x) \leqslant 1\},$$

we have that the inclusion $i_B : E_B \hookrightarrow E$ is a continuous map, which is equivalent to $\mathcal{T}|_{\langle B\rangle} \subset \mathcal{T}_{p_B}$ ($E_B = (\langle B\rangle, p_B)$). Therefore, if $\{x_\gamma\}_{\gamma\in\Gamma}$ is a net in E that Mackey-converges to $x \in E$, then $\{x_\gamma\}_{\gamma\in\Gamma}$ converges to x in $(E,\mathcal{T})$.

(4). Let $(E,\mathcal{T})$ be a *Hlcrtvs*. Then $(E,\mathcal{T}_{born})$ has the same Mackey converging sequences as $(E,\mathcal{T})$ and $\mathcal{T}_{born}$ is the final topology in E with respect to the inclusions $i_B : E_B \hookrightarrow E$ for all B bounded and absolutely convex in E. Moreover, if $\widetilde{\mathcal{T}}$ is a locally convex topology on E and it has the same Mackey converging sequences as $(E,\mathcal{T})$, then $\widetilde{\mathcal{T}} \subset \mathcal{T}_{born}$.

(5). Let $(E,\mathcal{T})$ be a *Hlcrtvs*. Then, $\mathcal{T}_{born}$ is the final topology in E with respect to the inclusions $i_B : E_B \to E$, for all B closed bounded and absolutely convex in $(E,\mathcal{T})$.

Proposition 3.4. *Let E be a Hlcrtvs, B a bounded absolutely convex subset of E, $\{x_\gamma\}_{\gamma\in\Gamma}$ a net in $\langle B\rangle$ and $x\in\langle B\rangle$. Then the following conditions are equivalent:*

(i) $\{x_\gamma\}_{\gamma\in\Gamma}$ *converges to* $x \in E_B$.

(ii) *There exists a net $\{\mu_\gamma\}_{\gamma\in\Gamma}$ in $\mathbb{R}$ which converges to 0 such that $(x_\gamma - x) \in \mu_\gamma\cdot B$, for all $\gamma\in\Gamma$.*

Proposition 3.5. *Let E be a Hlcrtvs. Then:*

(i) *Let $c : \mathbb{R} \to E$ be a C^1-curve and $\{t_n\}_{n\in\mathbb{N}}$ a sequence of real numbers that converge to 0. Then, the sequence $\{c(t_n)\}_{n\in\mathbb{N}}$ Mackey-converge to $c(0) \in E$ (use the Mean value Theorem).*

(ii) *If $c : \mathbb{R} \to E$ is a C^2-curve in E, then the curve $t \longmapsto \frac{1}{t}(\frac{1}{t}(c(t) - c(0)) - c'(0))$ is bounded on bounded subsets of $\mathbb{R} \setminus \{0\}$. Therefore, if the sequence $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R} \setminus \{0\}$ converges to 0, the sequence $\{\frac{c(t_n)-c(0)}{t_n}\}_{n\in\mathbb{N}}$ Mackey-converges to $c'(0) \in E$.*

Proposition 3.6. *Let $(E, \mathcal{T})$ be a Hlcrtvs. Then:*

(i) *A subset A of E is closed in $C^\infty E = (E, C^\infty\mathcal{T})$ if and only if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in A, which Mackey-converges to $x \in E$, the point x belongs to A.*

(ii) *If U is a subset of E such that $U \cap \langle B\rangle$ is open in E_B for all B bounded and absolutely convex in E, then U is C^∞-open in $(E, \mathcal{T})$, i.e. $U \in C^\infty\mathcal{T}$.*

Mackey complete spaces or convenient vector spaces

Definition 3.7. (Mackey-Cauchy net). Let $\{x_\gamma, \gamma \in \Gamma, \leqslant\}$ be a net in a *Hlcrtvs E*. This net will be called Mackey-Cauchy net if there exist a bounded absolutely convex subset B of E and a net $\{\mu_{\gamma,\gamma'}, (\gamma, \gamma') \in \Gamma \times \Gamma, \leqslant \times \leqslant\}$ in $\mathbb{R}$ converging to 0 such that $x_\gamma - x_{\gamma'} \in \mu_{\gamma,\gamma'} \cdot B$, for all $(\gamma, \gamma') \in \Gamma \times \Gamma$.

Let B be a bounded absolutely convex subset of a *Hlcrtvs E* and $\{x_\gamma\}_{\gamma\in\Gamma}$ a net in $\langle B\rangle$. Then $\{x_\gamma\}_{\gamma\in\Gamma}$ is a Cauchy net in the normed space E_B if and only if there exists a net $\{\mu_{\gamma,\gamma'}, (\gamma, \gamma') \in \Gamma \times \Gamma, \leqslant \times \leqslant\}$ in $\mathbb{R}$ converging to 0 such that $x_\gamma - x_{\gamma'} \in \mu_{\gamma,\gamma'} \cdot B$, for all $(\gamma, \gamma') \in \Gamma \times \Gamma$ (i.e., $\{x_\gamma\}_{\gamma\in\Gamma}$ is a Mackey-Cauchy net in E). Therefore, if $\{y_\gamma\}_{\gamma\in\Gamma}$ Mackey-converges to y in E, then $\{y_\gamma\}_{\gamma\in\Gamma}$ is a Mackey-Cauchy net.

Proposition 3.8. *Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a Mackey-Cauchy net in a Hlcrtvs $(E, \mathcal{T})$ and $x \in E$. Then, $\{x_\gamma\}_{\gamma\in\Gamma}$ converges to x in $(E, \mathcal{T})$ if and only if $\{x_\gamma\}_{\gamma\in\Gamma}$ Mackey converges to $x \in E$.*

Definition 3.9. (Mackey complete space). The *Hlcrtvs* $(E, \mathcal{T})$ is called Mackey complete (or convenient) if every Mackey-Cauchy net in E converges in $(E, \mathcal{T})$.

Theorem 3.10. *([7]). Let E be a Hlcrtvs. The following conditions are equivalent:*
(i) E is Mackey-complete.
(ii) Every Mackey-Cauchy net in E, Mackey converges in E.
(iii) Every Mackey-Cauchy sequence in E converges in E.
(iv) Every Mackey-Cauchy sequence in E, Mackey converges in E.
(v) For all absolutely convex bounded subset B of E, the normed space E_B is complete.
(vi) For every absolutely convex closed bounded subset B of E, the normed space E_B is complete, i.e. E_B is a Banach space.
(vii) For every bounded subset B of E there exists an absolutely convex closed bounded subset B' of E such that $B \subset B'$ and $E_{B'}$ is complete.
(viii) For every bounded subset B of E there exists an absolutely convex bounded subset B' of E such that $B \subset B'$ and $E_{B'}$ is complete.
(ix) Any Lipschitz curve in E is locally Riemann integrable.
(x) For any $c_1 \in C^\infty(\mathbb{R}, E)$ there is $c_2 \in C^\infty(\mathbb{R}, E)$ such that $c_2' = c_1$.

(xi) E is closed in the C^∞-topology of any Hlcrtvs $\widetilde{E}$, where E is a topological vector subspace of $\widetilde{E}$ (Recall that the C^∞-topology on $\widetilde{E}$ is the final topology with respect to all smooth curves $c : \mathbb{R} \to \widetilde{E}$).
(xii) If $c : \mathbb{R} \to E$ is a curve such that $\lambda c : \mathbb{R} \to \mathbb{R}$ is smooth for all $\lambda \in \mathcal{L}(E, \mathbb{R})$, then c is smooth.
(xiii) Any continuous linear mapping from a normed space F into E has a continuous extension to the completion of the normed space F.

Recall that $c : [a, b] \to E$ is Riemann integrable (by definition) if the net of Riemann sums converges in E. Moreover if c is continuous and E is sequentially complete (hence convenient), then c is Riemann integrable (in fact, the net of Riemann sums is a Cauchy net that has a subnet that is a sequence, and on the other hand an agglomeration point of a Cauchy net is a point of convergence of this net).

Proposition 3.11. *Let $(E, \mathcal{T})$ be a Hlcrtvs. Then we have:*

(i) *If E is complete (every Cauchy net in E converges in E), then E is sequentially complete (every Cauchy sequence in E converges in E).*

(ii) *If E is sequentially complete, then E is Mackey complete.*

(iii) *If E is metrizable, then: the statements "E is complete", "E is sequentially complete", and "E is Mackey complete", are equivalent.*

(iv) *$(E, \mathcal{T})$ is Mackey complete if and only if $(E, \mathcal{T}_{born})$ is Mackey complete (use the preceding theorem and 3.3(4).*

Proposition 3.12. *Let E, F be Hlcrtv spaces, and let $l : E \to F$ be a linear map. Then l is bounded (it maps bounded sets to bounded sets) if and only if it maps smooth curves in E to smooth curves in F.*

Smooth maps on C^∞-open sets of *Hlcrtv* spaces

Definition 3.13. ([7], page 30). Let E, F be Hlcrtv spaces, U a C^∞-open subset of E and $f : U \to F$ a map. Then f is called smooth if it maps smooth curves in U to smooth curves in F.

Note that in this case $f : C^\infty U \to C^\infty F$ is a continuous map, (since U is C^∞-open, we have that $C^\infty U$ is a subspace of $C^\infty E$, (2.21(ii))).

Remark 3.14. If $U = E = \mathbb{R}$, then $f : \mathbb{R} \to F$ is smooth (with the preceding definition) if and only if f is a smooth curve.

Proposition 3.15. *([7], page 28). Let $(E, \mathcal{T})$ be a Hlcrtvs. Then, the set of smooth curves $C^\infty(\mathbb{R}, E)$ is a Hlcrtvs with the pointwise vector operations ($(c_1 + c_2)(x) = c_1(x) + c_2(x)$, $(rc)(x) = r(c(x))$, $x \in \mathbb{R}$) and the topology of uniform convergence on compact sets of each derivative separately. This space is Mackey complete if and only if $(E, \mathcal{T})$ is Mackey complete.*

Proposition 3.16. *([7]) Let E, F be Hlcrtv spaces and U a C^∞-open subset of E. Then,*

$$(C^\infty(U, F) =)\{f : U \to F : \ f \text{ is a smooth map}\}$$

is a Hlcrtvs with pointwise linear structure ($(f+g)(x) = f(x) + g(x)$, $(rf)(x) = r(f(x))$, $x \in U$) *and the inicial topology with respect all mappings, (3.15),*

$$c^* : C^\infty(U, F) \to C^\infty(\mathbb{R}, F), \ \ f \mapsto fc, \ \ c : \mathbb{R} \to U \ \ \textit{smooth curve.}$$

Theorem 3.17. *([7], page 30). Let* $(E_1, \mathcal{T}_1)$, $(E_2, \mathcal{T}_2)$, $(F, \mathcal{T}')$ *be Hlcrtv spaces,* U_1 *a* C^∞*-open subset of* $(E_1, \mathcal{T}_1)$ *and* U_2 *a* C^∞*-open subset of* $(E_2, \mathcal{T}_2)$*. Then, a mapping* $f : U_1 \times U_2 \to F$ *is smooth if and only if the canonical associated mapping* $f^\vee : U_1 \to C^\infty(U_2, F)$, $x_1 \mapsto f(x_1, \cdot)$, *exists and is smooth.*

Corollary 3.18. *([7], page 31). Let* $(E, \mathcal{T})$, $(F, \mathcal{T}')$, $(G, \mathcal{T}'')$ *be Hlcrtv spaces and* U *a* C^∞*-open subset of* $(E, \mathcal{T})$*. Then, the following canonical mappings are smooth:*

(i) $ev : C^\infty(U, F) \times U \longrightarrow F$, $(f, x) \mapsto f(x)$

(ii) $ins : E \longrightarrow C^\infty(F, E \times F)$, $x \mapsto (y \mapsto (x, y))$.

(iii) $\circ : C^\infty(U, F) \times C^\infty(F, G) \longrightarrow C^\infty(U, G)$, $(f, g) \mapsto gf$.

Corollary 3.19. *(Boman theorem, [7] page 31) Let* $f : A \to F$ *be a map, where* A *is an open set of* $\mathbb{R}^n$ *and* F *is a Hlcrtvs. Then* f *is smooth (with the preceding definition) if and only if* f *is an usual* C^∞*-map, i.e. there exist all partial derivatives of any order of* f *and all of them are continuous maps. In this case* $df : A \times \mathbb{R}^n \to F$, $(x, v) \mapsto d_v f(x)$, *is smooth and therefore* $d.f(x) : \mathbb{R}^n \to F$ *is smooth for all* $x \in A$.

Theorem 3.20. *([7], page 33)*
Let $(E, \mathcal{T})$ *and* $(F, \mathcal{T}')$ *be Hlcrtvs. Then:*

(i) $L_b((E, \mathcal{T}), (F, \mathcal{T}')) = \mathcal{L}((E, \mathcal{T}_{born}), (F, \mathcal{T}'))$ *is a closed linear subspace of the Hlcrtvs* $C^\infty(E, F)$ *and therefore is a Hlcrtvs. Furthermore, if* W *is a* C^∞*-open subset of a Hlcrtvs* G*, then a mapping* $f : U \to L_b(E, F)$ *is smooth if and only if composite map* $U \xrightarrow{f} L_b(E, F) \overset{i}{\hookrightarrow} C^\infty(E, F)$ *is smooth.*

(ii) *If* $(E, \mathcal{T})$ *is a convenient vector space, one has that* $L_b(E, F)$ *is a bornological space and the bornology on this space* $L_b(E, F)$ *consists of all pointwise bounded sets. So a mapping into* $L_b(E, F)$ *is smooth if and only if all composites with evaluations at points in* E *are smooth. Finally,* $L_b(E, F)$ *is a convenient vector space.*

Proposition 3.21. *Let* $(E, \mathcal{T})$ *be a convenient vector space,* $(F, \mathcal{T}')$ *a Hlcrtv space,* U *a* C^∞*-open subset of* E*, (*U *is open in* $C^\infty E = (E, C^\infty \mathcal{T})$*), and let* $f : U \to F$ *be a smooth map. Then:*

(i) *For all* $x \in U$ *and all* $v \in E$ *there exists* $\lim_{t \to 0} \frac{f(x+tv) - f(x)}{t} (= d_v f(x))$.

(ii) *For all* $x \in U$, $d.f(x) (= df(x)) : E \to F$, $v \mapsto d_v f(x)$, *is a linear map.*

(iii) *The map* $d.f(\cdot) (= df) : U \times E \to F$, $(x, v) \mapsto d_v f(x)$, *is smooth.*

(iv) *For all* $x \in U$ *the linear map* $df(x) : E \to F$ *is a smooth map* ($\Longleftrightarrow$ *bounded map, (3.12)) and* $df(x)$ *is a continuous linear map of* $(E, \mathcal{T}_{born})$ *into* $(F, \mathcal{T}')$.

Proof. (i). The map $c : \mathbb{R} \to E$, $t \mapsto x + tv$, is a smooth curve. Then there exists $\varepsilon > 0$ such that $c(t) \in U$ for all $t \in (-\varepsilon, \varepsilon)$. Let $\sigma : \mathbb{R} \to (-\varepsilon, \varepsilon)$ be a smooth curve such that $\sigma(t) = t$ for all $t \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$. Then $c\sigma : \mathbb{R} \to U \hookrightarrow E$ is a smooth curve and $fc\sigma : \mathbb{R} \to F$ is a smooth curve. Consequently

$$(fc\sigma)'(0) = \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t}.$$

(ii). Let x be a element of U, let v be an element of E and let r be a real number with $r \neq 0$. Then:

$$d_{rv}f(x) = \lim_{t\to 0} \frac{f(x+trv) - f(x)}{t} = r \lim_{t\to 0} \frac{f(x+trv) - f(x)}{rt} = \\ = r \lim_{s\to 0} \frac{f(x+sv) - f(x)}{s} = r d_v f(x)$$

$(s = rt)$. If $r = 0$, then $d_{0\cdot v}f(x) = 0 = 0 d_v f(x)$. Thus, $df(x) : E \to F$ is a homogeneous map.

Let x be an element of U and let u, v be elements of E. The map $h : \mathbb{R}^2 \to E$, $(t, s) \mapsto x + tu + sv$, is smooth, (3.19), and therefore is a continuous map of $C^\infty\mathbb{R}^2 = (\mathbb{R}^2, T_u^2)$, (2.24(iv)), in $C^\infty E$. Thus, $h^{-1}(U)$ is an open set in $(\mathbb{R}^2, T_u^2)$ that contains $(0, 0)$. Therefore, there exists $\delta > 0$ such that $x + tu + sv \in U$ for all $(t, s) \in (-\delta, \delta) \times (-\delta, \delta)$. Hence, we have the smooth map $g = f(h|_{(-\delta,\delta)\times(-\delta,\delta)}) : (-\delta, \delta) \times (-\delta, \delta) \to F$, $(t, s) \mapsto f(x + tu + sv)$. Then, by the Boman's theorem (3.19), g is continuously differentiable and $Dg(0, 0)(1, 1) = 1 \cdot D_1 g(0, 0) + 1 \cdot D_2 g(0, 0)$, i.e.

$$Dg(0,0)(1,1) = \lim_{t\to 0} \frac{g(t,t) - g(0,0)}{t} = \lim_{t\to 0} \frac{f(x + t(u+v)) - f(x)}{t} = \\ = d_{u+v}f(x) = \lim_{\tau\to 0} \frac{g(\tau,0) - g(0,0)}{\tau} + \lim_{\sigma\to 0} \frac{g(0,\sigma) - g(0,0)}{\sigma} = \\ \lim_{\tau\to 0} \frac{f(x+\tau u) - f(x)}{\tau} + \lim_{\sigma\to 0} \frac{f(x+\sigma v) - f(x)}{\sigma} = d_u f(x) + d_v f(x).$$

Thus, $df(x) : E \to F$ is a linear map for all $x \in U$.

(iii). The map $df : U \times E \to F$, $(x, v) \mapsto d_v f(x)$, is smooth. In fact:
First, we remark that $U \times E$ is open $C^\infty(E \times E)$ since by 2.25, $C^\infty\mathcal{T} \times C^\infty\mathcal{T} \subset C^\infty(\mathcal{T} \times \mathcal{T})$. Thus it has meaning the smoothness of df on $U \times E$.
Let $c : \mathbb{R} \to U \times E$ be a smooth curve. Since $p_1 : U \times E \to U$, $(x, v) \mapsto x$, and $p_2 : U \times E \to E$, $(x, v) \mapsto v$, are smooth maps, we have the smooth curves $c_1 = p_1 c : \mathbb{R} \to U$ and $c_2 = p_2 c : \mathbb{R} \to E$ such that $c = (c_1, c_2) : \mathbb{R} \to U \times E$.
We consider the smooth map $\alpha : \mathbb{R}^2 \xrightarrow{\beta} U \times E \xrightarrow{+} E$, where $\beta(t, s) = (c_1(t), sc_2(t))$, $(t, s) \in \mathbb{R}^2$, and therefore $\alpha(t, s) = c_1(t) + sc_2(t)$ (the map $+ : U \times E \to E$ is smooth and therefore is a continuous map of $C^\infty(U \times E)$ into $C^\infty E$).
Then, $\alpha^{-1}(U)$ is open in $C^\infty\mathbb{R}^2 = (\mathbb{R}^2, T_u^2)$ and $\alpha^{-1}(U) \supset \mathbb{R} \times \{0\}$. Thus for every $r > 0$ there exists $\delta_r > 0$ such that $\alpha^{-1}(U) \supset (-r, r) \times (-\delta_r, \delta_r)$. Therefore for all $r > 0$, the map $h : (-r, r) \times (-\delta_r, \delta_r) \to F$, defined by $h(t, s) = f(c_1(t) + sc_2(t))$ is

smooth and

$$\partial_2 h(t,0) = \lim_{s\to 0} \frac{f(c_1(t)+sc_2(t)) - f(c_1(t))}{s} = d_{c_2(t)} f(c_1(t)) = ((df)c)(t) \text{ for all } t \in (-r,r).$$

This proves that $(df)c$ is a smooth curve and $df : U \times E \to F$ is smooth.

(iv). As a consequence of (iii), for all $x \in U$, $df(x) : E \to F$ is smooth since $df(x) = (df)i_x$, where $i_x(v) = (x,v)$, $v \in E$, and this map is smooth. Therefore, for all $x \in U$, $df(x) : E \to F$ is a bounded linear map and $df(x) : (E, \mathcal{T}_{born}) \to (F, \mathcal{T}')$ is a linear continuous map (see, 2.19(iii)). □

Proposition 3.22. *(chain rule) Let E, F, G be Hlcrtv spaces, U a C^∞-open subset in E, V a C^∞-open subset in F, $f : U \to F$ a map with $f(U) \subset V$ and $g : V \to G$ a map. Suppose that f, g are smooth maps. Then $gf : U \to G$ is a smooth map and $d(gf)(x,v) = dg(f(x), df(x,v))$, for all $(x,v) \in U \times E$. Consequently $d(gf)(x,\cdot) = dg(f(x),\cdot)df(x,\cdot) : E \to G$ for all $x \in U$. (See page 33 of [7]).*

Smooth maps on quadrants of convenient vector spaces

Definition 3.23. Let E, F be convenient vector spaces (i.e., Mackey complete spaces), Q a quadrant of E with $int_{C^\infty E}(Q) \neq \emptyset$, U a C^∞-open subset of Q and $f : U \to F$ a map. We say that f is smooth if for all smooth curve $c : \mathbb{R} \to U$, $fc : \mathbb{R} \to F$ is a smooth curve (recall that if $U \neq \emptyset$, then $int_{C^\infty E}(U) = U \cap int_{C^\infty E} Q \neq \emptyset$, (2.34)), (if $Q = E$ this definition is the given in 3.13).

Proposition 3.24. *Let $f : U \to V$ be a smooth map, where U is a C^∞-open set of a quadrant Q of a convenient vector space E with $int_{C^\infty E} Q \neq \emptyset$, and V is a C^∞-open set of a quadrant Q' of a convenient vector space F with $int_{C^\infty F} Q' \neq \emptyset$, and let $g : V \to G$ a smooth map with values in a convenient vector space G. Then $gf : U \to G$ is a smooth map.*

Proof. Let $c : \mathbb{R} \to U$ be a smooth curve. Since $f : U \to V$ is a smooth map, we have that $fc : \mathbb{R} \to V$ is a smooth curve. Thus, since $g : V \to G$ is a smooth map, $g(fc) = (gf)c : U \to G$ is a smooth curve. □

The result 3.12 can be drafted as follows:

Proposition 3.25. *Let $(E, \mathcal{T})$ and $(F, \mathcal{T}')$ convenient vector spaces, and let $\lambda : E \to F$ be a linear mapping. Then, λ is smooth if and only if λ is bounded (or bornological) ($\iff \lambda : (E, \mathcal{T}_{born}) \to (F, \mathcal{T}')$ is a continuous map).*

Proposition 3.26. *Let $(E, \mathcal{T})$, $(F, \mathcal{T}')$ be convenient vector spaces, Q a quadrant of E with $int_{C^\infty E}(Q) \neq \emptyset$, U a C^∞-open subset of Q and $f : U \to F$ a smooth map. Then, f is continuous of $C^\infty U = (U, C^\infty \mathcal{T}|_U)$ into $C^\infty F = (F, C^\infty \mathcal{T}')$. Therefore, f is a continuos map of $C^\infty U$ into $(F, \mathcal{T}')$, (by 2.24(i), $\mathcal{T}' \subset \mathcal{T}'_{born} \subset C^\infty \mathcal{T}'$).*

Proof. Let A be an open set of $C^\infty F$. For every smooth curve $c : \mathbb{R} \to U$, since f is a smooth map, $fc : \mathbb{R} \to F$ is a smooth curve, and therefore $(fc)^{-1}(A) = c^{-1}((f^{-1})(A))$ is open in $(\mathbb{R}, T_u)$. This proves that $f^{-1}(A)$ is open in the C^∞-topology of U. □

Theorem 3.27. *(Derivative of smooth maps on quadrants)*
Let $(E,\mathcal{T})$, $(F,\mathcal{T}')$ be convenient vector spaces, Q a quadrant of E with $int_{C^\infty E}(Q) \neq \emptyset$, U a C^∞-open subset of Q and $f: U \to F$ a smooth map (preceding definition). Then the map $(\rho{=})f|_{int_{C^\infty E}(U)} : int_{C^\infty E}(U) \to F$ is smooth (3.13) and the map, $x \in int_{C^\infty E}(U) \mapsto d.\rho(x) = d\rho(x) \in \{\lambda : E \to F | \lambda$ is linear and bounded$\} = L_b(E,F)$, extends to a unique smooth map (preceding definition) $df : U \to L_b(E,F)$, and $df(x)$ is linear and continuous from $(E,\mathcal{T}_{born})$ into $(F,\mathcal{T}')$ for all $x \in U$, (2.19(iii). Note that the map, $(x,v) \in int_{C^\infty E}(U) \times E \mapsto d_v\rho(x) \in F$, is smooth (3.21(iii)). Moreover, for all $x \in int_{C^\infty E}(U)$, $d\rho(x) : E \to F$ is smooth 3.13 and continuous from $(E,\mathcal{T}_{born})$ into $(F,\mathcal{T}')$, (3.21(iv)).

Proof. By 2.34, $int_{C^\infty E}U = U \cap int_{C^\infty E}Q$ is a non-empty C^∞-open set of E.
Since a smooth curve $c : \mathbb{R} \to int_{C^\infty E}U$ can be considered as a smooth curve with values in U, it is obvious that $f|_{int_{C^\infty E}} : int_{C^\infty E}U \to F$ is a smooth mapping. Thus only the extension property should be shown.
Let us first try to find a candidate to the value of $d_v f(x)$ for $x \in U$ and $v \in E$ with $x + v \in int_{C^\infty E}Q = int_{\mathcal{T}_{born}}Q$ (2.33(iii)). By convexity the smooth curve $c_{x,v} : t \mapsto x + t^2 v$ takes for $0 < |t| \leqslant 1$ values in $int_{C^\infty E}Q \subset Q$ (see 2.31, $(x + t^2 v = (1-t^2)x + t^2(x+v))))$ and $c_{x,v}(0) = x \in U \subset Q$. By the continuity of $c_{x,v} : ((-1,1), T_u|_{(-1,1)}) \to (Q, C^\infty(\mathcal{T}|_Q))$, $(C^\infty(\mathcal{T}|_Q) = (C^\infty\mathcal{T})|_Q$ is the final topology in Q induced by the family of all smooth curves $\tau : \mathbb{R} \to Q)$, there exists ε such that $0 < \varepsilon < 1$ and $c_{x,v} : (-\varepsilon,\varepsilon) \to U$, $t \mapsto x + t^2 v$, is a smooth curve, and consequently $fc_{x,v} : (-\varepsilon,\varepsilon) \to F$ is a smooth curve. In the special case where $x \in int_{C^\infty E}U = U \cap int_{C^\infty E}Q$ we have that $c_{x,v}((-\varepsilon,\varepsilon)) \subset int_{C^\infty E}U$ and by the chain rule (3.21 and 3.22)

$$(\rho c_{x,v})'(t) = d_{c'_{x,v}(t)}\rho(c_{x,v}(t)) = 2t d_v\rho(x + t^2 v), \text{ for all } t \in (-\varepsilon,\varepsilon),$$

$$(\rho c_{x,v})''(0) = \lim_{t\to 0} \frac{2t d_v\rho(x+t^2v) - 0}{t} = 2d_v\rho(x), \ (3.21(iii)).$$

Thus we define $d_v f(x) = (1/2)(fc_{x,v})''(0)$ for $x \in U$ and $v \in int_{C^\infty E}Q - x$.

Note that for $x \in U$, $v \in int_{C^\infty E}U - x$ and $0 < \delta < 1$ we have $d_{\delta v} f(x) = \delta d_v f(x)$, since $c_{x,\delta v}(t) = x + t^2\delta v = c_{x,v}(t\sqrt{\delta})$ and we can apply twice the chain rule to the composition of the smooth curves $fc_{x,v}$ and $\alpha : (-\varepsilon,\varepsilon) \to (-\varepsilon,\varepsilon)$, $t \mapsto t\sqrt{\delta}$.

Next let us show that

$$d_v f(\cdot) : (U_v =) \{x \in U : x + v \in int_{C^\infty E}Q\} \longrightarrow F$$

is smooth, $(U_v = \{x \in U : x \in int_{C^\infty E}Q - v\} = U \cap (int_{C^\infty E}Q - v)$ is C^∞-open in U, (2.30), and therefore C^∞-open in Q), so let $s \mapsto x(s)$ be a smooth curve in U_v. Then, $v \in int_{C^\infty E}Q - x(0)$, $x(s) + v \in int_{C^\infty E}Q$ for all s, $x(0) \in U_v$ and thus the map $(s,t) \mapsto c_{x(s),v}(t) = x(s) + t^2 v = (1-t^2)x(s) + t^2(x(s)+v) \in Q$ is smooth from some neighborhood of $(0,0)$ into $U_v \subset U$ $(c_{x(s),v} : \mathbb{R}^2 \to E$, $(s,t) \mapsto x(s) + t^2 v$, is a smooth map). Hence, $(s,t) \mapsto f(c_{x(s),v}(t))$ is smooth, (3.24), and also its second derivative $s \mapsto (fc_{x(s),v})''(0) = 2d_v f(x(s))$. Thus, $d_v f(\cdot) : U_v \to F$ is a smooth map. In particular, let $x_0 \in U$ and $v_0 \in E$ such that $v_0 \in int_{C^\infty E}Q - x_0$ and $x(s) =$

$x_0 + s^2 v_0$ $(x(s) + v_0 = x_0 + (1 + s^2)v_0 \in int_{C^\infty E}Q$ for sufficiently small s, since $x_0 + v_0 \in int_{C^\infty E}Q)$. Then

$$2d_{v_0}f(x_0) = (fc_{x_0,v_0})''(0) = \lim_{s\to 0}(\rho c_{x(s),v_0})''(0) = \lim_{s\to 0} 2d_{v_0}\rho(x(s))$$

with $x(s) \in int_{C^\infty E}Q$ for $0 < |s| < \varepsilon$. Obviously this shows that the given definition of $d_{v_0}f(x_0)$ is the only possible smooth extension of $d_{v_0}\rho(\cdot)$ to $\{x_0\} \cup int_{C^\infty E}U$.
Now let $v \in E$ arbitrary. Choose a $v_0 \in int_{C^\infty E}Q - x_0$. Since the set $int_{C^\infty E}Q - x_0 - v_0$ is a C^∞-open convex neighborhood of $\bar{0}$, hence absorbing (see 2.31 and 2.29), there exists some $\varepsilon > 0$ such that $v_0 + \varepsilon v \in int_{C^\infty E}Q - x_0$. For all $x \in int_{C^\infty E}U$ one has

$$d_v\rho(x) = \frac{1}{\varepsilon}d_{\varepsilon v}\rho(x) = \frac{1}{\varepsilon}\left(d_{v_0+\varepsilon v}\rho(x) - d_{v_0}\rho(x)\right).$$

By what we have proven above, the right side extends smoothly to $\{x_0\} \cup int_{c^\infty E}U$. The same is true for the left side. i.e. we define $d_vf(x_0) = \lim_{s\to 0} d_vf(x(s))$ for some smooth curve $x : (-\varepsilon, \varepsilon) \to U$ with $x(s) \in int_{C^\infty E}U$, $x(0) = x_0$, for $0 < |s| < \varepsilon$. Then $df(x)$ is linear as pointwise limit of $df(x(s)) \in L_b(E, F)$ and is bounded by the Banach-Steinhaus theorem $((E, \mathcal{T}_{born})$ is a bornological convenient vector space and $L_b((E, \mathcal{T}), (F, \mathcal{T}')) = \mathcal{L}((E, \mathcal{T}_{born}), (F, \mathcal{T}')))$.
This proves at the same time, that the definition does not depend on the smooth curve x, since for $v \in x_0 + int_{C^\infty E}Q$ it is the unique extension.
In order to show that $df : U \to L_b(E, F)$ is smooth it is, by 3.20, enough to show that the map

$$\mathbb{R} \xrightarrow{x} U \xrightarrow{df} L_b(E, F) \xrightarrow{ev_v} F,\ s \mapsto d_vf(x(s))$$

is smooth for all $v \in E$ and all smooth curves $x : \mathbb{R} \to U$. For $v \in int_{C^\infty E}Q$ this was shown above (in this case, if $x : \mathbb{R} \to U$ is a smooth curve, then $x(t) \in U_v$ for all $t \in \mathbb{R}$ and x is a smooth curve in U_v, (2.33(v)). For general $v \in E$, this follows since $d_vf(x(s))$ is a linear combination of $d_{v_0}f(x(s))$ for two $v_0 \in int_{C^\infty E}Q$. □

Lemma 3.28. *(Chain rule for curves) Let Q be a quadrant of a convenient vector space $(E, \mathcal{T})$ with $int_{C^\infty E}Q \neq \emptyset$, U a non-empty C^∞-open set of Q, $(F, \mathcal{T}')$ a convenient vector space and $f : U \to F$ a map such that $f|_{int_{C^\infty E}U} = \rho$ is a smooth mapping and there exists $df : U \to L_b(E, F)$ an extension of $d(f|_{int_{C^\infty E}}U) = d\rho$ which is continuous for the C^∞-topology of U. Then, if $c : \mathbb{R} \to U$ is a smooth curve, $(fc)'(t) = d_{c'(t)}f(c(t))$ for all $t \in \mathbb{R}$.*

Proof. We choose $y \in int_{C^\infty E}U = U \cap int_{C^\infty E}Q \subset Q$ and for all $s \in (-1, 1)$ let $c_s(t) = c(t) + s^2(y - c(t)) \in Q$, (2.31). Since, $\alpha : \mathbb{R} \times (-1, 1) \to Q$, $(t, s) \mapsto c_s(t)$, is a smooth map, one has that $\alpha^{-1}(U)$ is open in $(\mathbb{R}, \mathcal{T}_u) \times ((-1, 1), \mathcal{T}_u|_{(-1,1)})$ and further $\alpha^{-1}(U) \supset \mathbb{R} \times \{0\}$. Thus, there exists ε with $0 < \varepsilon < 1$ and such that $c_s(t) \in U$ for all $s \in (-\varepsilon, \varepsilon)$ and all $t \in [-1, 1]$, and by 2.31 $c_s(t) \in int_{C^\infty E}U$ for all $s \in (-\varepsilon, \varepsilon) \setminus \{0\}$ and $t \in [-1, 1]$. Furthermore, $c_0 = c$, $(s, t) \mapsto c_s(t)$ is smooth and $c_s'(t) = (1 - s^2)c'(t)$. Then, by 3.10(ix), 3.22 and 3.29, for all $s \in (-\varepsilon, \varepsilon) \setminus \{0\}$,

$$\frac{\rho(c_s(t)) - \rho(c_s(0))}{t} = \int_0^1 (\rho c_s)'(t\tau)d\tau = (1 - s^2)\int_0^1 d_{c'(t\tau)}\rho(c_s(t\tau))d\tau.$$

Now consider the specific case where $c(t) = x + tv$ with $x, x + v \in U$. Then, there exists $\delta > 0$ such that $x + tv \in U$ for all $t \in (-\delta, \delta)$, and since f is continuous along

$(t,s) \mapsto c_s(t) = x+tv+s^2(y-x-tv)$, the left side of the above equation converges to $(f(c(t))-f(c(0)))/t$ for $s \to 0$ and since $d_v f(\cdot)$ is continuous along $(t,\tau,s) \mapsto c_s(t\tau)$ we have that $d_{c'(t\tau)}\rho(c_s(t\tau))$ converges to $d_v f(c(t\tau))$ uniformly with respect to $0 \leqslant \tau \leqslant 1$ for $s \to 0$. Thus, the right side of the above equation converges to $\int_0^1 d_v f(c(t\tau))d\tau$. Hence, we have for $t \to 0$

$$\frac{f(c(t))-f(c(0))}{t} = \int_0^1 d_v f(x+t\tau v)d\tau \to \int_0^1 d_v f(c(0)) = d_{c'(0)}f(c(0)).$$

Now let $c : \mathbb{R} \to U$ be an arbitrary curve. Then, $(s,t) \mapsto c(0)+s(c(t)-c(0))$ is smooth and has values in U for $0 \leqslant s \leqslant \varepsilon$. By the above consideration we have for $x = c(0)$ and $v = (c(t)-c(0))/t$ that

$$\frac{f(c(t))-f(c(0))}{t} = \int_0^1 d_{\frac{c(t)-c(0)}{t}} f(c(0)+\tau(c(t)-c(0))d\tau,$$

which converges to $d_{c'(0)}f(c(0))$ for $t \to 0$, since df is continuous along smooth curves in U and thus $df(c(0)+\tau(c(t)-c(0))) \to df(c(0))$ uniformly on the bounded set $\{(c(t)-c(0))/t : \ t \text{ near } 0\}$.
Thus fc is differentiable with derivative $(fc)'(t) = d_{c'(t)}f(c(t))$. □

Proposition 3.29. *(Chain rule) Let E, F, G convenient vector spaces, Q a quadrant of E with with $int_{C^\infty E}(Q) \neq \emptyset$, U a C^∞-open subset of Q, P a quadrant of F with $int_{C^\infty F}(P) \neq \emptyset$ and V a C^∞-open subset of P. One has:*
If $f : U \to F$ is a smooth map and $g : V \to G$ is a smooth map with $f(U) \subset V$, then $gf : U \to G$ is a smooth map and $d(gf)(x) = dg(f(x))df(x)$, for all $x \in U$.

Proof. Let x be an element of $int_{C^\infty E}U$ and $v \in E$. We consider the smooth curve $c(t) = x+tv$ in U $(t \in (-\varepsilon,+\varepsilon))$. Then, $e(t) = (fc)(t) = f(x+tv)$ is a smooth curve in V. By the above lemma (3.28), $(ge)'(t) = d_{e'(t)}g(e(t))$ and in particular $d_v(gf)(x) = (ge)'(0) = d_{e'(0)}g(e(0)) = d_{d_vf(x)}g(f(x))$. Thus for all $x \in int_{C^\infty E}U$, $d(gf)(x) = dg(f(x))df(x)$.
On the other hand by 3.27, 3.20 and 3.18, the map

$$\begin{gathered} U \to L_b(E,F)\times V \to L_b(E,F)\times L_b(F,G) \hookrightarrow C^\infty(E,F)\times C^\infty(F,G) \to C^\infty(E,G),\\ x \mapsto (df(x), f(x)) \mapsto (df(x), dg(f(x))) \mapsto (df(x), dg(f(x))) \mapsto dg(f(x))df(x), \end{gathered}$$

is smooth, which proves by 3.27 that $d(gf)(x) = dg(f(x))df(x)$ for all $x \in U$. □

Proposition 3.30. *Let E, F, convenient vector spaces, Q a quadrant of E with $int_{C^\infty E}(Q) \neq \emptyset$ and U a C^∞-open subset of Q. One has:*

(i) *(Restriction to open sets). If $f : U \to F$ is a smooth map and W is a C^∞-open subset of U, ($C^\infty U$ is a topological subspace of $C^\infty E$), then $f|_W : W \to F$ is a smooth map (of course W is a C^∞-open subset of Q) and $d(f|_W)(x) = df(x)$ for all $x \in W$.*

(ii) *(Restriction to vector subspaces). Let $f : U \to F$ be a smooth map, H a closed linear subspace of E (then a convenient vector space), Q_H a quadrant of H with $int_{C^\infty H}(Q_H) \neq \emptyset$. Suppose that $U \cap H$ is a C^∞-open subset of Q_H. Then $f|_{U\cap H} : U \cap H \to F$ is a smooth map and $d(f|_{U\cap H})(x) = df(x)|_H$, for all $x \in U \cap H$.*

(iii) *(Open covering property). If $f : U \to F$ is a map and $\{V_j | j \in J\}$ be a C^∞-open covering of U, then $f : U \to F$ is smooth if and only if $f|_{V_j} : V_j \to F$ is smooth for all $j \in J$.*

Proof. (i). $f|_W$ is a smooth map since every smooth curve in W is a smooth curve in U. On the other hand, $f|_W = fj$ where $j : W \hookrightarrow U$ is the inclusion map. Obviously, j is a smooth map and $d_v j(x) = v$ for all $x \in W$ and $v \in E$, and therefore by the chain rule, (3.29), $df(x) = d(f|_W)(x)$ for all $x \in W$.
(ii). $f|_{U\cap H} : U \cap H \to F$ is a smooth map since every smooth curve in $U \cap H$ can be considered as a smooth curve in U. Furthermore, $f|_{U\cap H} = fi$, where $i : U \cap H \hookrightarrow U$ is the inclusion map which is smooth and $d_v i(x) = v$ for all $x \in U \cap H$ and $v \in H$. Thus by the chain rule, (3.29), $df(x)|_H = d(f|_{U\cap H})(x)$, for all $x \in U \cap H$.
(iii). If $f : U \to F$ is a smooth map, then for all $j \in J$, $f|_{V_j} : V_j \to F$ is a smooth map by (i). Conversely, suppose that $f|_{V_j} : V_j \to F$ is a smooth map for all $j \in J$ and let $c : \mathbb{R} \to U$ be a smooth curve. Then for all $j \in J$, $A_j = c^{-1}(V_j)$ is an open set in $(\mathbb{R}, T_u)$ and $f(c|_{A_j})$ is a smooth map. Thus, since $\bigcup_{j\in J} A_j = \mathbb{R}$, we have that fc is a smooth curve and $f : U \to F$ is a smooth map. □

The above results give the essential tools (together with the theorem of invariance of the boundary) to construct manifolds with corners modeled on convenient vector spaces.

Proposition 3.31. *Let $(E, \mathcal{T})$ and $(F, \mathcal{T}')$ be convenient vector spaces, Q a quadrant of $(E, \mathcal{T})$ with $int_{C^\infty E}Q \neq \emptyset$, U a C^∞-open subset of Q and let $f : U \to F$ be a map. Then $f : U \to F$ is a smooth map if and only if $\lambda f : U \to \mathbb{R}$ is a smooth map for all $\lambda \in \mathcal{L}((F, \mathcal{T}'_{born}), (\mathbb{R}, \mathcal{T}_u))$ (3.23), (in particular F may be Fréchet or Banach).*

Definition 3.32. Let $(E, \mathcal{T})$ and $(F, \mathcal{T}')$ be convenient vector spaces, Q a quadrant of $(E, \mathcal{T})$ with $int_{C^\infty E}Q \neq \emptyset$, U a C^∞-open subset of Q, P a quadrant of $(F, \mathcal{T}')$ with $int_{C^\infty F}P \neq \emptyset$, V a C^∞-open subset of P, and let $f : U \to V$ be a map. We say that $f : U \to V$ is a smooth diffeomorphism if $f : U \to V$ is a bijective map and f, f^{-1} are smooth maps (3.23). Note that in this case $f^{-1} : V \to U$ is also a smooth diffeomorphism and one says that U and V are *sth*-diffeomorphic.

Proposition 3.33. (i) *The composition of a finite number of smooth diffeomorphisms is again a smooth diffeomorphism, and the identity map is a smooth diffeomorphism.*

(ii) *Let $f : U \to V$ be a smooth diffeomorphism. Then by 3.27, we have the smooth maps $df : U \to L_b(E, F)$ and $d(f^{-1}) : V \to L_b(F, E)$ and we know that $df(x) \in \mathcal{L}((E, T_{born}), (F, \mathcal{T}'))$ and $d(f^{-1})(f(x)) \in \mathcal{L}((F, T'_{born}), (E, \mathcal{T}))$ for all $x \in U$. By 3.29, $df(x)$ is an isomorphism and $(df(x))^{-1} = d(f^{-1})(f(x))$ for all $x \in U$. Finally $df(x)$ is a linear homeomorphism from (E, T_{born}) onto (F, T'_{born}) for all $x \in U$.*

Invariance of the boundary for smooth diffeomorphisms

Lemma 3.34. *Let $(E, \mathcal{T})$, $(F, \mathcal{T}')$ be convenient vector spaces, U a C^∞-open subset of E, $\lambda : F \to \mathbb{R}$ a bounded linear mapping with $\lambda \neq 0$, $x \in U$ and $f : U \to F^+_\lambda = \{v \in F : \lambda(v) \geqslant 0\}$ a smooth map such that $f(x) \in F^0_\lambda = \{v \in F : \lambda(v) = 0\}$. Then $df(x)(E) \subset F^0_\lambda$, (3.27).*

Proof. Let $\mathcal{T}'_{born}$ the bornologification of F. Then, $\lambda : (F, \mathcal{T}'_{born}) \to \mathbb{R}$ is continuous. Let $\{\| \ \|_i\}_{i\in I}$ be a collection of semi-norms on $(F, \mathcal{T}'_{born})$ that describes the locally convex topology of $(F, \mathcal{T}'_{born})$, (2.17(iv)). By 2.24(ii), $f : U \to (F, \mathcal{T}'_{born})$ is smooth. Since $\lambda : (F, \mathcal{T}'_{born}) \to \mathbb{R}$ is continuous and $\lambda(\bar{0}) = 0$, there exists a neighborhood $V^{\bar{0}}$ of $\bar{0}$ in $(F, \mathcal{T}'_{born})$ such that $\lambda(V^{\bar{0}}) \subset [-1, 1]$.
Then, there exist $\delta > 0$ and $i_1, ..., i_n \in I$ such that

$$\bigcap_{j=1}^{n} B_{\delta}^{\| \ \|_{i_j}}(\bar{0}) \subset V^{\bar{0}}$$

and therefore for all $z \in F$ with $\|z\|_{i_j} < \delta$ for all $j \in \{1, ..., n\}$, $|\lambda(z)| \leqslant 1$. Thus for all $w \in F$ with $\|w\|_{i_j} < 1$ for all $j = 1, ..., n$, one has that $\|\delta w\|_{i_j} = \delta \|w\|_{i_j} < \delta$ for all $j = 1, ..., n$ and $|\lambda(\delta w)| = \delta |\lambda(w)| \leqslant 1$, which implies $|\lambda(w)| \leqslant 1/\delta$.
Let $\varepsilon > 0$ and $u \in E$. Since, by 3.21(i),

$$d_u f(x) = \lim_{t\to 0} \frac{f(x + tu) - f(x)}{t} \ \ (\text{in}(F, \mathcal{T}'_{born}))$$

and for all $j \in \{1, ..., n\}$, $B_{\varepsilon}^{\| \ \|_{i_j}}(d_u f(x))$ is a neighborhood of $d_u f(x)$ in (F, T_{born}), there exists $\eta_j > 0$ such that for all $t \in (-\eta_j, +\eta_j)$ with $t \neq 0$, $x + tu \in V^{\bar{0}}$ and

$$\|d_u f(x) - \frac{f(x + tu) - f(x)}{t}\|_{i_j} < \varepsilon, \ (j \in \{1, ..., n\}).$$

Let η be the positive real number $\min\{\eta_1, ..., \eta_n\}$. Thus, for all $t \in (-\eta, \eta)$ with $t \neq 0$

$$\|d_u f(x) - \frac{f(x + tu) - f(x)}{t}\|_{i_j} < \varepsilon, \ \text{ for all } \ j \in \{1, ..., n\}.$$

For all $t \in (-\eta, +\eta)$ with $t \neq 0$, let

$$u_t = d_u f(x) - \frac{f(x + tu) - f(x)}{t} \ \text{ and } \ v_t = \frac{u_t}{\varepsilon}.$$

Then $\|v_t\|_{i_j} < 1$ for all $j \in \{1, ..., n\}$, and

$$\varepsilon t v_t = t d_u f(x) - f(x + tu) + f(x) \ \text{ or } \ f(x + tu) - f(x) = t(d_u f(x) - \varepsilon v_t).$$

Since $f(x) \in F^0_\lambda$, $-f(x) \in F^0_\lambda$ and $f(x + tu) - f(x) \in F^+_\lambda$. Thus,

$$\lambda(f(x + tu) - f(x)) = \lambda(t(d_u f(x) - \varepsilon v_t)) = t(\lambda(d_u f(x)) - \varepsilon \lambda(v_t)) \geqslant 0.$$

Then one follows that:
(1^{st}) If $0 < t_1 < \eta$, then $\lambda(d_u f(x)) \geqslant \varepsilon \lambda(v_{t_1}) \geqslant -\varepsilon \cdot (1/\delta)$.
(2^{d}) If $-\eta < t_2 < 0$, then $\lambda(d_u f(x)) \leqslant \varepsilon \lambda(v_{t_2}) \leqslant \varepsilon \cdot (1/\delta)$.
Since ε is arbitrary, $\lambda(d_u f(x)) = 0$ and $d_u f(x) \in F^0_\lambda$. □

Theorem 3.35. *(Invariance of the boundary). Let $(E, \mathcal{T})$, $(F, \mathcal{T}')$ be convenient vector spaces, Q a quadrant of E with $int_{C^\infty E} Q \neq \emptyset$ (by 2.32, index(Q) is finite and Q^0 is closed in (E, T_{born})), P a quadrant of F with $int_{C^\infty F} P \neq \emptyset$ (by 2.32, index(P) is finite and P^0 is closed in (F, T'_{born})), U a C^∞-open set of Q, V a C^∞-open set of P and let $f : U \to V$ be a smooth diffeomorphism (3.32). Then we have:*

(i) *index*$(x) =$ *index*$(f(x))$ *for all* $x \in U$, *(2.11).*

(ii) $\{x \in U | index(x) \geqslant 1\} \neq \emptyset$ *if and only if* $\{y \in V | index(y) \geqslant 1\} \neq \emptyset$. *(Note that* $int_{C^\infty E}Q = int_{T_{born}}Q \neq \emptyset$ *and* $int_{C^\infty F}P = int_{T'_{born}}P \neq \emptyset$, *(2.32 and 2.33(iii)).*

(iii) *If* $U \neq \emptyset$, $int_{C^\infty E}(U) \neq \emptyset$ *and* $int_{C^\infty F}(V) \neq \emptyset$.

(iv) $f(\{x \in U | index(x) \geqslant k\}) = \{y \in V | index(y) \geqslant k\}$, *for all* $0 \leqslant k \leqslant index(Q)$.

(v) $f(\{x \in U | index(x) = k\}) = \{y \in V | index(y) = k\}$, *for all* $0 \leqslant k \leqslant index(Q)$.

(vi) $f(int_{C^\infty E}(U)) = int_{C^\infty F}(V)$.

(vii) $f|_{int_{C^\infty E}(U)} : int_{C^\infty E}(U) \to int_{C^\infty F}(V)$ *is a smooth diffeomorphism (3.32) and* $d(f|_{int_{C^\infty E}(U)})(x) = df(x)$ *for all* $x \in int_{C^\infty E}(U)$.

Proof. By 2.32 and 2.33 there exist a linear independent system of elements of $L_b(E, \mathbb{R})$, $\Lambda = \{\lambda_1, ..., \lambda_n\}$, and a linearly independent system of elements of $L_b(F, \mathbb{R})$, $M = \{\eta_1, ..., \eta_n\}$, such that $Q = E^+_\Lambda$ and $P = F^+_M$.
(i). If $x \in U$, $k = index(x)$ and $k' = index(f(x))$, then, exactly k elements of Λ, λ_{i_1},...,λ_{i_k}, vanish at x and exactly k' elements of M, η_{i_1},..., $\eta_{i_{k'}}$, vanish at $f(x)$. We consider $J = \{i_1, ..., i_k\}$, $E_J = \bigcap_{j \in J} E^0_{\lambda_j}$ and $\Lambda_J = \{\lambda_i|_{E_J} : i \in I \setminus J\}$. Then, Λ_J is a linearly independent system of elements of $L_b(E_J, \mathbb{R})$ and $H = (E_J)_{\Lambda^+_J} = E^+_\Lambda \cap E_J$. Hence, by 2.33(v), $U \cap H$ is an C^∞-open set of H and $int_{C^\infty E_J}(U \cap H)$ is a C^∞-open set of E_J which contains the point x. We denote by V^x the C^∞-open set $int_{C^\infty E_J}(U \cap H)$. On the other hand, since E_J is C^∞-closed, $f|_{V^x} : V^x \to V \subset F^+_M$ is a smooth map and $(f|_{V^x})(x) \in \bigcap_{j=1}^{k'} F^0_{\eta_{i_j}}$. Hence by the above lemma (3.34) $d(f|_{V^x})(x, \cdot)(E_J) \subset \bigcap_{j=1}^{k'} F_{\eta^0_{i_j}}$ and

$$codim(df(x, \cdot)(E_J) = codim(E_J) = k \geqslant codim(\bigcap_{j=1}^{k'} F^0_{\eta_{i_j}}) = k'$$

Similarly, using f^{-1}, we obtain $k' \geqslant k$ and hence $k = k'$.
(ii), (iv) and (v) follow from (i), and (iii) follows from (i) and 2.34. Finally, (vi) and (vii) are consequence from (i) and 3.30. □

Proposition 3.36. *Let E be a convenient vector space and Q, Q' quadrants in E with non-empty C^∞-interior. Then Q, Q' are smooth diffeomorphic (3.32) if and only if* $index(Q) = index(Q')$, *(2.3(ii)).*

Proof. If $index(Q) = index(Q')$, by 2.32 and 2.15 there exists a linear homeomorphism $\alpha : (E, T_{born}) \to (E, T_{born})$ such that $\alpha(Q) = Q'$ and $\alpha(Q^0) = Q'^0$. By 3.25, $\alpha|_Q : Q \to Q'$ is a smooth diffeomorphism (3.32). For the converse, apply 3.35(i).
□

On constructing smooth manifolds with corners modeled over convenient vector spaces, the C^∞-open subsets of quadrants, with non-empty C^∞-interior will be the local models.

4. Manifolds with corners modeled on convenient real vector spaces

Let X be a non-empty set. We say that $(U, \varphi, (E, Q))$ is a chart on X if: U is a subset of X, E is a convenient vector space (i.e. a *Hlcrtvs* which is Mackey complete (3.9)), Q is a quadrant on E with $int_{C^\infty \mathcal{T}} Q \neq \emptyset$ (where, $\mathcal{T}$ is the topology of E), $\varphi : U \to Q$ is an injective map, and $\varphi(U)$ is a C^∞-open subset of Q (note that the C^∞-topology of Q (2.33(v)) is $C^\infty \mathcal{T}|_Q$). In this case, U will be called the domain, φ the morphism and E (or Q) the model of the chart.
Recall that (2.32 and 2.33) $index(Q)$ is finite, Q^0 is closed in (E, T_{born}) and $int_{T_{born}} Q$ is non-empty. Consequently if $index(Q) = n \in \mathbb{N}$, by 2.32(iii), there exists a linearly independent system $\Lambda = \{\lambda_1, ..., \lambda_n\}$ of elements of $L_b(E, \mathbb{R})$ such that $Q = E^+_\Lambda$, (hence $Q^0 = E^0_\Lambda$, Q and Q^0 are closed in (E, T_{born}), and $int_{C^\infty T}(Q) = int_{T_{born}}(Q)$, (2.33(iii))).

Let $(U, \varphi, (E, Q))$, $(U', \varphi', (E', Q'))$ be charts on X. We say that they are smooth-compatible (*sth*-compatible) if: $\varphi(U \cap U')$ and $\varphi'(U \cap U')$ are C^∞-open subsets of Q and Q', respectively, and $\varphi'\varphi^{-1} : \varphi(U \cap U') \to \varphi'(U \cap U')$, $\varphi\varphi'^{-1} : \varphi'(U \cap U') \to \varphi(U \cap U')$ are smooth maps (3.23) (and hence inverse homeomorphisms for the C^∞-topology, (3.26)). In this case we shall write $(U, \varphi, (E, Q)) \sim (U', \varphi', (E', Q'))$.

A collection $\mathcal{A}$ of charts on X is called a smooth-atlas on X (*sth*-atlas) if the domains of the charts of $\mathcal{A}$ cover X and any two of them are *sth*-compatible.

Two *sth*-atlases $\mathcal{A}$, $\mathcal{A}'$ on X are called smooth-equivalent (*sth*-equivalent) if $\mathcal{A} \cup \mathcal{A}'$ is a *sth*-atlas on X. In this case we shall write $\mathcal{A} \sim \mathcal{A}'$.

Proposition 4.1. *Let X be a non-empty set. Then the preceding relation binary $\sim$ is an equivalence relation over the sth-atlases on X.*

Proof. Reflexive and symmetric properties are obvious.
$\sim$ is transitive: Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ be *sth*-atlases on X such that $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{D}$. It should be proven that $A \sim \mathcal{D}$, which is equivalent to prove that $\mathcal{A} \bigcup \mathcal{D}$ is a *sth*-atlas on X.
It is clear that the elements of $\mathcal{A} \bigcup \mathcal{D}$ are charts on X and its domains cover X. Thus, it remains to prove that every element $(U, \varphi, (E, Q))$ of $\mathcal{A}$ is smooth compatible with every element $(U', \varphi', (E', Q'))$ of $\mathcal{D}$. We have that

$$\varphi(U \cap U') = \bigcup_{(c''=)(U'', \varphi'', (E'', Q'')) \in \mathcal{B}} \varphi(U \cap U' \cap U'') = \bigcup_{c'' \in \mathcal{B}} \varphi\varphi''^{-1}\varphi''(U \cap U' \cap U'')$$
$$= \bigcup_{(U'', \varphi'', (E'', Q'')) \in \mathcal{B}} \varphi\varphi''^{-1}(\varphi''(U \cap U'') \cap \varphi''(U' \cap U'')),$$
$$(U \cap U' = \bigcup_{(U'', \varphi'', (E'', Q'')) \in \mathcal{B}} (U \cap U' \cap U''), \ \ (U \cap U' \cap U'' = (U \cap U'') \cap (U' \cap U''))),$$

is C^∞-open in the quadrant Q, since $(U, \varphi, (E, Q)) \sim (U'', \varphi'', (E'', Q''))$ $(\mathcal{A} \sim \mathcal{B})$, $(U', \varphi', (E', Q')) \sim (U'', \varphi'', (E'', Q''))$ $(\mathcal{B} \sim \mathcal{D})$ and $\varphi\varphi''^{-1} : \varphi''(U \cap U'') \to \varphi(U \cap U'')$

is a C^∞-homeomorphism. Analogously $\varphi'(U \cap U')$ is C^∞-open in the quadrant Q'. Since

$$\varphi'\varphi^{-1}|_{\varphi(U\cap U'\cap U'')} = (\varphi'\varphi''^{-1})(\varphi''\varphi^{-1})|_{\varphi(U\cap U'\cap U'')}$$

is a smooth map (by the chain rule (3.29) and the restriction of a smooth map to an C^∞-open set (3.30(i))), we have that $\varphi'\varphi^{-1} : \varphi(U \cap U') \to \varphi'(U \cap U')$ is a smooth map by the open covering property of the smooth maps (3.30(iii)). Analogously, $\varphi\varphi'^{-1} : \varphi'(U \cap U') \to \varphi(U \cap U')$ is a smooth map. □

Definition 4.2. If $\mathcal{A}$ is a *sth*-atlas on X, the equivalence class $[\mathcal{A}]$ is called *sth*-structure on X and the pair $(X, [\mathcal{A}])$ is called *sth*-manifold with corners (or *sth*-manifold).
If $(X, [\mathcal{A}])$ is a *sth*-manifold with corners, we shall say that $\mathcal{B}$ is an atlas of $(X, [\mathcal{A}])$ if $\mathcal{B} \in [\mathcal{A}]$, and we shall say that a chart $(U, \varphi, (E, Q))$ of X is a chart of $(X, [\mathcal{A}])$ if there exists $\mathcal{B} \in [\mathcal{A}]$ such that $(U, \varphi, (E, Q)) \in \mathcal{B}$.

Remark 4.3. For all *Hlcrtvs* $(E, \mathcal{T})$ we have: $C^\infty(\mathcal{T}_{born}) = C^\infty\mathcal{T}$ (2.24(iii)), $(E, \mathcal{T}_{born})$ is a bornological *Hlcrtvs* (2.19(i)), and: $(E, \mathcal{T})$ *is convenient if and only if* $(E, \mathcal{T}_{born})$ *is convenient* (3.11(iv)). Moreover, if $(E, \mathcal{T})$ is a convenient vector space, then the identity map, $1_E : (E, \mathcal{T}) \to (E, \mathcal{T}_{born})$, is a smooth diffeomorphism by 2.24(ii). In this way, we can consider that the *sth*-manifolds are modeled on bornological convenient vector spaces (see 2.19).

Remarks 4.4. (i) Two charts of a *sth*-manifold with corners are smooth compatible.

(ii) Let $(X, [\mathcal{A}])$ be a *sth*-manifold with corners and $(U, \varphi, (E, Q))$ a chart on X. Then $(U, \varphi, (E, Q))$ is a chart of $(X, [\mathcal{A}])$ if and only if there exists an atlas $\mathcal{B}$ of $(X, [\mathcal{A}])$ such that $(U, \varphi, (E, Q))$ is smooth compatible with all chart of $\mathcal{B}$.

(iii) Let $\mathcal{A}$, $\mathcal{B}$ be smooth atlas on X. Then, $\mathcal{A}$, $\mathcal{B}$ are smooth equivalent if and only if for all $x \in X$ there exists $(U, \varphi, (E, Q))$ and $(U', \varphi', (E', Q'))$ charts of $\mathcal{A}$ and $\mathcal{B}$, respectively, such that $x \in U \cap U'$ and these charts are smooth compatible.

(iv) Let $(X, [\mathcal{A}])$ be a *sth*-manifold with corners and $x \in X$. If $(U, \varphi, (E, Q))$ and $(U', \varphi', (E', Q'))$ are charts of $(X, [\mathcal{A}])$ with $x \in U \cap U'$, by chain rule (3.29), E and E' are linearly C^∞-diffeomorphic and $dim(E)$ is called dimension of $(X, [\mathcal{A}])$ at x and denoted $dim_x X$.

(v) If in the above definition of a *sth*-manifold with corners the charts are modeled only on Fréchet (Banach) spaces, we have the concept of Fréchet (Banach) manifold with corners (see [8]).

(vi) The topological manifolds with border modeled on convenient vector spaces can be defined analogously as we defined the *sth*-manifolds.

(vii) Let X be a set and $\mathcal{A}_\infty$ the collection of all smooth atlases on X. We have the equivalence relation $\sim$ on $\mathcal{A}_\infty$ and the ordering relation $\prec$ on $\mathcal{A}_\infty$ defined by: $\mathcal{A} \prec \mathcal{B}$ if and only if $\mathcal{A} \subset \mathcal{B}$.
Then, if $\mathcal{A} \in \mathcal{A}_\infty$, $\mathcal{A}$ is a maximal element if and only if $\mathcal{A} = \bigcup\{\mathcal{B} : \mathcal{B} \in [\mathcal{A}]\}$.

(viii) Let $(X, [\mathcal{A}])$ be a *sth*-manifold with corners. Then:
(1). $\bigcup\{\mathcal{B} : \mathcal{B} \in [\mathcal{A}]\} \sim \mathcal{A}$.
(2). $\bigcup\{\mathcal{B} : \mathcal{B} \in [\mathcal{A}]\}$ is a maximal element.

The smooth manifolds with corners are endowed with a natural topology induced by the differentiable structure.

Proposition 4.5. *Let $(X, [\mathcal{A}])$ be is a sth-manifold with corners. Then, the set $\mathcal{B}_{[\mathcal{A}]} = \{U \subset X : U$ is a domain of a chart of $(X, [\mathcal{A}])\}$ is a basis of a topology $\mathcal{T}_{[\mathcal{A}]}$ on X (called the natural topology induced by $[\mathcal{A}]$).*

Proof. (1). $\bigcup_{U \in \mathcal{B}_{[\mathcal{A}]}} U = X$, since the domains of all charts that belong to the atlas $\mathcal{A}$ is a covering of X.
(2). For all $U_1, U_2 \in \mathcal{B}_{[\mathcal{A}]}$ and all $x \in U_1 \cap U_2$ there is $U_3 \in \mathcal{B}_{[\mathcal{A}]}$ such that $x \in U_3 \subset U_1 \cap U_2$. In fact: If $(U_1, \varphi_1, (E_1, Q_1))$ and $(U_2, \varphi_2, (E_2, Q_2))$ are chart of $(X, [\mathcal{A}])$, then $(U_1 \cap U_2, \varphi_1|_{U_1 \cap U_2}, (E_1, Q_1))$ and $(U_1 \cap U_2, \varphi_2|_{U_1 \cap U_2}, (E_2, Q_2))$ are charts of $(X, [\mathcal{A}])$. Thus, $U_3 = U_1 \cap U_2 \in \mathcal{B}_{[\mathcal{A}]}$. □

Remark 4.6. If $(U, \varphi, (E, Q))$ is a chart of the *sth*-manifold $(X, [\mathcal{A}])$ and V is an open set of $(X, \mathcal{T}_{[\mathcal{A}]})$, then $(U \cap V, \varphi|_{U \cap V}, (E, Q))$ is a chart of $(X, [\mathcal{A}])$.

Proposition 4.7. *Let $(X, [\mathcal{A}])$ be a sth-manifold with corners and $(U, \varphi, (E, Q))$ a chart of this manifold. Then, $\varphi : (U, \mathcal{T}_{[\mathcal{A}]}) \to (\varphi(U), \mathcal{T}^\infty \varphi(U))$, where $\mathcal{T}^\infty \varphi(U)$ is the C^∞-topology induced in $\varphi(U)$ by the C^∞-topology of Q (therefore by the C^∞-topology of E, (2.33(v)), is a homeomorphism.*

Proof. (1). $\varphi : U \to \varphi(U)$ is a bijective map.
(2). Let $(U', \varphi', (E', Q'))$ be a chart of $(X, [\mathcal{A}])$. Then, $\varphi(U \cap U')$ is C^∞-open in Q and therefore is an element of $\mathcal{T}^\infty \varphi(U)$, and φ is an open map.
(3). Let A be an element of $\mathcal{T}^\infty \varphi(U)$. Then, $(\varphi^{-1}(A), \varphi|_{\varphi^{-1}(A)}, (E, Q))$ is a chart of $(X, [\mathcal{A}])$. Thus $\varphi^{-1}(A) \in \mathcal{B}_{[\mathcal{A}]} \subset \mathcal{T}_{[\mathcal{A}]}$ and φ is a continuous map. □

The next proposition gives us a criterion to know when a topology on a *sth*-manifold with corners is the associated topology of the manifold.

Proposition 4.8. *Let $(X, [\mathcal{A}])$ be a sth-manifold with corners and $\mathcal{T}$ a topology on X. Then the following statements are equivalent:*

(i) $\mathcal{T}_{[\mathcal{A}]} = \mathcal{T}$.

(ii) *There is a sth-atlas $\{(U_j, \varphi_j, (E_j, Q_j)) : j \in J\}$ of $(X, [\mathcal{A}])$ such that $U_j \in \mathcal{T}$ and $\varphi_j : (U_j, \mathcal{T}|_{U_j}) \to (\varphi_j(U_j), \mathcal{T}^\infty \varphi_j(U_j))$ is a homeomorphism, for every $j \in J$.*

Proof. The result follows from the above proposition (4.7) and the general statement: *If two topologies on a set agree on every element of a common open covering, then they must be equal.* □

Definition 4.9. Let $(X, [\mathcal{A}])$ be a *sth*-manifold and $c : \mathbb{R} \to X$ a map. We say that c is a *sth*-curve on $(X, [\mathcal{A}])$ if:

(i) $c : (\mathbb{R}, T_u) \to (X, \mathcal{T}_{[\mathcal{A}]})$ is a continuous map.

(ii) For all chart $(U, \varphi, (E, Q))$ of $(X, [\mathcal{A}])$ such that $U \cap c(\mathbb{R}) \neq \emptyset$, the map

$$i\varphi(c|_{c^{-1}(U)}) : \ c^{-1}(U) \overset{c|_{c^{-1}(U)}}{\longrightarrow} U \overset{\varphi}{\longrightarrow} \varphi(U) \overset{i}{\hookrightarrow} Q$$

is a C^∞ map (2.20), (by (i), $c^{-1}(U)$ is a non-empty open set of $(\mathbb{R}, \mathcal{T}_u)$).

Proposition 4.10. *Let $(X, [\mathcal{A}])$ be a sth-manifold with corners. Then, $T_{[\mathcal{A}]}$ is the final topology $\mathcal{T}^\infty_{[\mathcal{A}]}$ in X respect to the family $\{c : \mathbb{R} \to X | c$ is a sth-curve on $(X, [\mathcal{A}])\}$.*

Proof. Since every *sth*-curve on $(X, [\mathcal{A}])$ is a continuous map of $(\mathbb{R}, \mathcal{T}_u)$ into $(X, \mathcal{T}_{[\mathcal{A}]})$, we have the inclusion $\mathcal{T}_{[\mathcal{A}]} \subset \mathcal{T}^\infty_{[\mathcal{A}]}$.
On the other hand, if $(U, \varphi, (E, Q))$ is a chart of $(X, [\mathcal{A}])$, then φ maps *sth*-curves in U into *sht*-curves in $\varphi(U)$ and conversely. Thus, $\varphi : C^\infty U \to C^\infty \varphi(U)$ is a homeomorphism and $C^\infty U = (U, \mathcal{T}^\infty_{[\mathcal{A}]}|_U) = (U, \mathcal{T}_{[\mathcal{A}]}|_U)$. Then by 4.8, $\mathcal{T}^\infty_{[\mathcal{A}]} = \mathcal{T}_{[\mathcal{A}]}$. □

Moreover, the topology of a *sth*-manifold verifies the T_1 axiom (the C^∞-topology of a *Hlcrtvs* is Hausdorff), and finally a *sth*-manifold fulfils the first axiom of countability if and only if it is a Fréchet manifold (see the proof of 4.19 in [7]).

Boundary of a *sth*-manifold with corners

Theorem 4.11. *Let $(X, [\mathcal{A}])$ be a sth-manifold with corners and $x \in X$. Then, if $(U, \varphi, (E, Q))$ and $(U', \varphi', (E', Q'))$ are charts of $(X, [\mathcal{A}])$ with $x \in U \cap U'$, we have that $index_Q(\varphi(x)) = index_{Q'}(\varphi'(x'))$.*

Proof. Since $\varphi'\varphi^{-1} : \ \varphi(U \cap U') \to \varphi'(U \cap U')$ is a *sth*-difeomorphism, the result follows from the theorem 3.35(i). □

Definition 4.12. Let $(X, [\mathcal{A}])$ be a *sth*-manifold with corners. Then:

(i) Let x be a point of X. The index of $\varphi(x)$ in Q, where $(U, \varphi, (E, Q))$ is a chart of $(X, [\mathcal{A}])$ with $x \in U$, will be called index of x and will be denoted by $index(x)$ (see the above theorem).

(ii) For all $k \in \mathbb{N} \cup \{0\}$, the set $\{x \in X | index(x) \geqslant k\} (= \partial^k X)$ is called k-boundary of $(X, [\mathcal{A}])$ and the set $\partial^1 X (= \partial X)$ the boundary of $(X, [\mathcal{A}])$, $(\partial^0 X = X)$.

(iii) For all $k \in \mathbb{N} \cup \{0\}$, the set $\{x \in X | index(x) = k\}$ is denoted by $B_k X$, (note that $\partial^k X = \bigcup_{k' \geqslant k} B_{k'} X$). The set $B_0 X = \{x \in X : \ index(x) = 0\} = X \setminus \partial X$ will be called the interior of X and denoted by Int(X).

Proposition 4.13. *Let $(X, [\mathcal{A}])$ be a sth-manifold with corners and $x \in X$. Then, there is a chart $(U, \varphi, (E, Q))$ of $(X, [\mathcal{A}])$ such that $x \in U$ and $\varphi(x) = 0$, and hence $index(x) = index(Q)$. Such a chart will be called centered at the point x.*

Proof. Let $(U', \varphi', (E', Q'))$ be a chart of $(X, [\mathcal{A}])$ such that $x \in U'$. By 2.32(iii) there is a linearly independent system $\Lambda' = \{\lambda'_1, ..., \lambda'_n\}$ of elements of $L_b(E', \mathbb{R})$ such that $Q' = E'^{+}_{\Lambda'}$. If $\lambda'_1(\varphi'(x))) = 0$,...,$\lambda'_n(\varphi'(x)) = 0$, take the chart $(U', t_{\varphi'(x)}\varphi', (E', Q'))$ of $(X, [\mathcal{A}])$, where $t_{\varphi'(x)}(y) = y - \varphi'(x)$, $y \in Q'$ (see 2.30).
If $\lambda'_1(\varphi'(x))) = 0$,...,$\lambda'_{i_r}(\varphi'(x)) = 0$ and $\lambda'_{i_{r+1}}(\varphi'(x))) > 0$,...,$\lambda'_n(\varphi'(x)) > 0$, take a

C^∞-open set W of $\varphi'(U')$ such that $\varphi'(x) \in W$ and $W \cap E'^{0}_{\lambda_i} = \emptyset$, $i = r+1, ..., n$ ($E'^{0}_{\lambda_i}$ is closed in $(E', \mathcal{T}'_{born})$ and therefore is closed in $C^\infty E'$), and take the chart $(\varphi'^{-1}(W), t_{\varphi'(x)}(\varphi'|_{\varphi'^{-1}(W)}), (E', Q'))$. □

Proposition 4.14. *Let $(X, [\mathcal{A}])$ be a sth-manifold with corners and $k \in \mathbb{N}$. Then:*

(i) *$\partial^k X$ is a closed set of $(X, \mathcal{T}_{[\mathcal{A}]})$.*

(ii) *$Int(X)$ is a dense open set of $(X, \mathcal{T}_{[\mathcal{A}]})$.*

Proof. It is an easy consequence of the above proposition and 2.11(d). □

Proposition 4.15. *Let $(X, [\mathcal{A}])$ be a sth-manifold and $k \in \mathbb{N} \cup \{0\}$. Then, there exists a unique smooth differentiable structure on $B_k X$ such that for all $x \in B_k X$ and all chart $(U, \varphi, (E, Q))$ with $x \in U$ and $\varphi(x) = 0$ (4.13), the triplet $(U \cap B_k X, \varphi|_{U \cap B_k X}, Q^0)$ is a chart of that structure. Furthermore, $\partial B_k X = \emptyset$ and the topology of this manifold $B_k X$ is the topology induced by $T_{[\mathcal{A}]}$.*

Proof. If $x \in B_k X$ and $(U, \varphi, (E, Q))$ is chart of $(X, [\mathcal{A}])$ such that $x \in U$ and $\varphi(x) = 0$, we have $\varphi(U \cap B_k X) = \varphi(U) \cap Q^0$. From this equality and since that Q^0 is closed in $(E, \mathcal{T}_{born})$ (and therefore in $C^\infty E$) it is easy to deduce the assertions in the statement. □

Corollary 4.16. *Let $(X, [\mathcal{A}])$ be a sth-manifold. Then we have:*

(i) *There is a unique smooth differentiable structure on $Int(X)$ such that for all $x \in Int(X)$ and all chart $(U, \varphi, (E, Q))$ of $(X, [\mathcal{A}])$ with $x \in U$ and $\varphi(x) = 0$ (hence $Q = Q^0 = E$), the triplet (U, φ, E) is a chart of $Int(X)$. Moreover $Int(X)$ has no boundary ($\partial(Int(X)) = \emptyset$) and its topology is the topology induced by $\mathcal{T}_{[\mathcal{A}]}$.*

(ii) *If $\partial^2 X = \emptyset$, there is a unique smooth differentiable structure on ∂X such that for all $x \in \partial X$ an all chart $(U, \varphi, (E, Q))$ of $(X, [\mathcal{A}])$ with $x \in U$ and $\varphi(x) = 0$ (hence $Q = E^+_{\{\lambda\}}$ it happens that $(U \cap \partial X, \varphi|_{U \cap \partial X}, E^0_{\{\lambda\}})$ is a chart of ∂X. Furthermore $\partial(\partial X) = \emptyset$ and its topology is the topology induced by $\mathcal{T}_{[\mathcal{A}]}$.*

Examples 4.17. (i) Let $(E, \mathcal{T})$ be a convenient vector space, and let Q be a quadrant of $(E, \mathcal{T})$ with $int_{C^\infty E} Q \neq \emptyset$. Then $\mathcal{A}_E = \{(Q, Id_Q, (E, Q))\}$ is a smooth atlas on Q. Thus $(Q, [\mathcal{A}_E])$ is a *sth*-manifold with corners, called the usual differentiable structure of Q. If U is an C^∞-open subset of Q, then $(U, i, (E, Q))$ (i the inclusion map of U into Q) is a chart of $(Q, [\mathcal{A}_E])$. One has that $B_k((Q, [\mathcal{A}_E])) = \{x \in Q : \ index_Q(x) = k\}$ and $\mathcal{T}_{[\mathcal{A}_E]} = C^\infty(\mathcal{T}|_Q) = C^\infty \mathcal{T}|_Q$. Particularly, if $Q = E$, then $(E, [\mathcal{A}_E])$ is a *sth*-manifold without boundary.

(ii) Let $(X, [\mathcal{A}])$ be a *sth*-manifold and let G be an open set of $(X, \mathcal{T}_{[\mathcal{A}]})$. Then $\mathcal{A}_G = \{(U, \varphi, (E, Q)) : \ (U, \varphi, (E, Q))$ is a chart of $(X, [\mathcal{A}])$, $U \subset G\}$ is a smooth atlas on G and $(G, [\mathcal{A}_G])$ is a *sth*-manifold, called open submanifold of $(X, [\mathcal{A}])$. One has $B_k((G, [\mathcal{A}_G])) = G \cap B_k X$ and $\mathcal{T}_{[\mathcal{A}_G]} = \mathcal{T}_{[\mathcal{A}]}|_G$.

If the quadrants are omitted, the *sth*-manifolds that we obtain, in fact they were already defined in [7], p. 264, will be called smooth manifold without boundary (or corners).

Differentiable maps

The differential calculus of smooth maps defined on open sets of quadrants of convenient vector spaces and with values into Hausdorff locally convex vector spaces admits the following generalization to manifolds.

Definition 4.18. Let $(X, [\mathcal{A}])$ be a *sth*-manifold with corners, let $(F, \mathcal{T}')$ be a convenient vector space and let $f : X \to F$ be a map. We say that f is a smooth map, if for every $x \in X$ there exists a chart $c = (U, \varphi, (E, Q))$ of $(X, [\mathcal{A}])$ such that $x \in U$ and $f\varphi^{-1} : \varphi(U) \to F$ is a smooth map.

Remarks 4.19. (1) If $(X, [\mathcal{A}])$ is a *sth*-manifold with corners, $(F, \mathcal{T}')$ is a convenient vector space and $f : X \to F$ is a smooth map, then $f : (X, \mathcal{T}_{[\mathcal{A}]}) \to C^\infty F = (F, C^\infty \mathcal{T}')$ is a continuous map.
(2) Also, it is easy to prove, from the basic properties established in 3.30 and the chain rule (3.29), that: If $f : (X, [\mathcal{A}]) \to F$ is a smooth map and $(V, \psi, (E_1, Q_1))$ is a chart of $(X, [\mathcal{A}])$, then $f\psi^{-1} : \psi(V) \to F$ is a smooth map.
(3) Finally if $f : U \to F$ is a map, where U is an C^∞-open set of a quadrant Q of a convenient vector space E, with $int_{C^\infty E} Q \neq \emptyset$, and F is a convenient vector space, then the notion of smooth map given via 3.23 coincide with the notion of smooth map define above (4.18), considering U as an open *sth*-manifold with corners of Q with the usual differentiable structure (4.17)(i).

The smooth maps between *sth*-manifolds are defined as usually by localization.

Definition 4.20. Let $(X, [\mathcal{A}])$, $(X', [\mathcal{A}'])$ be smooth manifolds with corners and $f : X \to X'$ a map. We say that f is a smooth map, if for every $x' \in X'$ there exists a chart $(U', \varphi', (E', Q'))$ of $(X', [\mathcal{A}'])$ with $x' \in U'$, $f^{-1}(U')$ open in $(X, \mathcal{T}_{[\mathcal{A}]})$ and

$$j\varphi'(f|_{f^{-1}(U')}) : f^{-1}(U') \xrightarrow{f|_{f^{-1}(U')}} U' \xrightarrow{\varphi'} \varphi'(U') \overset{j}{\hookrightarrow} Q'$$

smooth map according to 4.18 ($f^{-1}(U')$ is an open submanifold of $(X, [\mathcal{A}])$).

Proposition 4.21. *Let $(X, [\mathcal{A}])$, $(X', [\mathcal{A}'])$ be smooth manifolds with corners and $f : X \to X'$ a map. Then,*

(i) *The map f is smooth if and only if for every $x \in X$ there exists a chart $(U, \varphi, (E, Q))$ of $(X, [\mathcal{A}])$ with $x \in U$ and there exists a chart $(U', \varphi', (E', Q'))$ such that $f(U) \subset U'$ and the map $\varphi' f \varphi^{-1} : \varphi(U) \to \varphi'(U')$ is a smooth map.*

(ii) *If $f : (X, [\mathcal{A}]) \to (X', [\mathcal{A}'])$ is a smooth map, one has that $f : (X, \mathcal{T}_{[\mathcal{A}]}) \to (X', \mathcal{T}_{[\mathcal{A}']})$ is a continuous map.*

Also, it is easy to prove, from the basic properties established in 3.30 and the chain rule (3.29), that if $f : (X, \mathcal{T}_{[\mathcal{A}]}) \to (X', \mathcal{T}_{[\mathcal{A}']})$ is a smooth map, $(U_1, \varphi_1, (E_1, Q_1))$ is a chart of $(X, [\mathcal{A}])$ and $(U_1', \varphi_1', (E_1', Q_1'))$ is a chart of $(X', [\mathcal{A}'])$ such that $f(U_1) \subset U_1'$, then $\varphi_1' f \varphi_1^{-1} : \varphi_1(U_1) \to \varphi_1'(U_1')$ is a smooth map.

Proposition 4.22. (i) *If $(X, [\mathcal{A}])$ is a smooth manifold, then the identity map $Id_X : (X, [\mathcal{A}]) \to (X, [\mathcal{A}])$ is smooth.*

(ii) *The finite composition of smooth maps is a smooth map.*

(iii) *Let $(X, [\mathcal{A}])$ be a smooth manifold with corners and U an open set of $(X, \mathcal{T}_{[\mathcal{A}]})$. Then the inclusion map $j : (U, [\mathcal{A}_U]) \hookrightarrow (X, [\mathcal{A}])$, (4.17(ii)), is a smooth map.*

(iv) *If $f : (X, [\mathcal{A}]) \to (X', [\mathcal{A}'])$ is a smooth map and U is an open set of $(X, \mathcal{T}_{[\mathcal{A}]})$, then $f|_U : (U, [\mathcal{A}_U]) \to (X', [\mathcal{A}'])$, (4.17(ii)), is a smooth map.*

(v) *If $(X, [\mathcal{A}])$ and $(X', [\mathcal{A}'])$ are smooth manifolds with corners, $f : X \to X'$ is a mapping and $\mathcal{U}$ is an open covering of the topological space $(X, \mathcal{T}_{[\mathcal{A}]})$ such that $f|_U : (U, [\mathcal{A}_U]) \to (X', [\mathcal{A}'])$, (4.17(ii)), is a smooth map for all $U \in \mathcal{U}$, then $f : (X, [\mathcal{A}]) \to (X', [\mathcal{A}'])$ is a smooth map.*

(vi) *If $(X, [\mathcal{A}])$ is a smooth manifold and $c : \mathbb{R} \to X$ is a smooth curve on $(X, [\mathcal{A}])$, (4.9), then $c : (\mathbb{R}, [\mathcal{A}_\mathbb{R}]) \to (X, [\mathcal{A}])$ is a smooth map (4.20).*

Definition 4.23. Let $(X, [\mathcal{A}])$, $(X', [\mathcal{A}'])$ be smooth manifolds with corners and let $f : X \to X'$ be a map. Then f is called a smooth diffeomorphism if f is a bijective map and f, f^{-1} are smooth maps.

Note that every smooth diffeomorphism is an homeomorphism. Moreover the composition of smooth diffeomorphisms is a smooth diffeomorphism. On the other hand, if $[\mathcal{A}]$ and $[\mathcal{A}']$ are smooth structures with corners over a set X, it follows that Id_X is a smooth diffeomorphism from $(X, [\mathcal{A}])$ onto $(X, [\mathcal{A}'])$ if and only if $[\mathcal{A}] = [\mathcal{A}']$.

In order to prove that the smooth diffeomorphisms preserve the index of points and hence the boundary, we need the following lemma whose proof is not difficult.

Lemma 4.24. *Let $(X, [\mathcal{A}])$ be a smooth manifold with corners, U an open set of $(X, \mathcal{T}_{[\mathcal{A}]})$, F a convenient vector space, Q a quadrant of F with $int_{C^\infty F} Q \neq \emptyset$ and $\varphi : U \to F$ a map such that $\varphi(U)$ is a C^∞-open set of Q. Then the following statements are equivalent:*

(i) *$(U, \varphi, (F, Q))$ is a chart of $(X, [\mathcal{A}])$.*

(ii) *$\varphi : (U, [\mathcal{A}_U]) \to (\varphi(U), [(\mathcal{A}_F)_{\varphi(U)}])$ is a smooth diffemorphism, (4.17).*

The following result is of a great importance to the theory of *sth*-manifolds with corners.

Theorem 4.25. *Let $(X, [\mathcal{A}])$, $(X', [\mathcal{A}'])$ be smooth manifolds with corners and let $f : (X, [\mathcal{A}]) \to (X', [\mathcal{A}'])$ be a smooth diffeomorphism. Then, we have that:*

(i) *$index(x) = index(f(x))$ for all $x \in X$.*

(ii) *$f(\partial^k(X)) = \partial^k X'$ and $f(B_k(X)) = B_k(X')$ for all $k \in \mathbb{N} \cup \{0\}$.*

(iii) *$f|_{B_k(X)} : B_k(X) \to B_k(X')$, $k \in \mathbb{N} \cup \{0\}$, is a smooth diffeomorphism, where $B_k(X)$ and $B_k(X')$ are the manifolds described in 4.15. In particular, if $\partial^2 X = \emptyset$, f is smooth diffeomorphism of $\partial X = B_1(X)$ onto $\partial X' = B_1(X')$.*

Proof. (i). For $x \in X$, let $(U, \varphi, (E, Q))$ be a chart of $(X, [\mathcal{A}])$ with $x \in U$ and $\varphi(x) = 0$. Then by 4.24, $(f(U), \varphi f^{-1}, (E, Q))$ is a chart of $(X', [\mathcal{A}'])$ such that $f(x) \in f(U)$ and $\varphi f^{-1}(f(x)) = 0$. Hence, $index(x) = index(f(x)) = card(\Lambda)$.
(ii). It follows from (i).
(iii). Let $x \in B_k(X)$ and consider a chart $(U, \varphi, (E, Q))$ of $(X, [\mathcal{A}])$ such that $x \in U$ and $\varphi(x) = 0$. Then $(U \cap B_k(X), \varphi|_{U \cap B_k X}, Q^0)$ is a chart of $B_k(X)$. Since f is a smooth diffeomorphism, $(f(U), \psi = \varphi f^{-1}, (E, Q))$ is a chart of $(X', [\mathcal{A}'])$, $f(x) \in f(U)$ and $\varphi f^{-1}(f(x)) = 0$. Hence $(f(U) \cap B_k X', \psi|_{f(U) \cap B_k X'}, Q^0)$ is a chart of $B_k X'$. Then, it is clear that $f(U \cap B_k X) = f(U) \cap B_k X'$ and the map $\psi f(\varphi|_{U \cap B_k X})^{-1} : \varphi(U) \cap Q^0 \to \varphi(U) \cap Q^0$ is the identity map. Therefore $f|_{B_k X}$ and $(f|_{B_k X})^{-1}$ are smooth maps. □

Proposition 4.26. *Let* $(X, [\mathcal{A}])$, $(X', [\mathcal{A}'])$ *be smooth manifolds with corners and* $f : X \to X'$ *a map. Then,* $f : (X, [\mathcal{A}]) \to (X', [\mathcal{A}'])$ *is a smooth map if and only if* f *maps sth-curves in* $(X, [\mathcal{A}])$ *to sth-curves in* $(X', [\mathcal{A}'])$.

Proof. If $f : (X, [\mathcal{A}]) \to (X', [\mathcal{A}'])$ is a smooth map, then by 4.22(ii) $fc : (\mathbb{R}, [\mathcal{A}_{\mathbb{R}}]) \to (X', [\mathcal{A}'])$ is a smooth curve for each smooth curve $c : (\mathbb{R}, [\mathcal{A}_{\mathbb{R}}]) \to (X, [\mathcal{A}])$.
Conversely, by the above proposition 4.21(ii), $f : (X, \mathcal{T}_{[\mathcal{A}]}) \to (X', \mathcal{T}_{[\mathcal{A}']})$ is a continuous map. Then, for all $x \in X$ there exist $(U, \varphi, (E, Q))$ and $(U', \varphi', (E', Q'))$ charts of $(X, [\mathcal{A}])$ and $(X', [\mathcal{A}'])$, respectively, such that $x \in U$ and $f(U) \subset U'$. Then, by the hypothesis and the lemma 4.24 $\varphi' f \varphi^{-1}$ is a smooth map and by 4.21(i) $f : (X, [\mathcal{A}]) \to (X', [\mathcal{A}'])$ is a smooth map. □

The reader can study, for *sth*-manifolds with corners, the *general properties* (when they have meaning) established by the authors in [8] for C^r-manifolds with corners modeled on Banach spaces.

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