Branched folded coverings and 3-manifolds

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ABSTRACT

Under the framework of Fox spreads and its completions a theory that generalizes coverings (folding covering theory) and a theory that generalizes branched coverings (branched folding theory) is defined and some properties are proved. Two applications to 3-manifold theory are given. A problem is stated.

Key words: knots, links, 3-manifolds, branched coverings, foldings, strings, spreads, Cantor sets.

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1. Introduction

This is mainly a didactical paper and, except for the results of the last section, there is no claim at all for originality. Perhaps the only original thing in it is just the definition of folding and branched folding. These concepts generalize, respectively, coverings and branched coverings (see [25] and [24]).

After the work of Thurston, and also the early work of Poincaré, Bianchi and others, it is clear that it is not necessary to distinguish between groups acting freely and those acting with fixed points. All of them give rise to quotient spaces, orbifolds, that generalize the concept of manifold. As a consequence it is convenient to generalize the concept of covering and branched covering and this is the content of this paper, which resumes my lectures at the Universidad Complutense along several years.

As an example take the 3-sphere S^3. There are several involutions all of them having an m-sphere Σ^m of fixed points.

If $m = 1$ the quotient space is S^3 and the induced identification map $f : S^3 \to S^3_1$ is the standard 2-fold covering branched over the trivial knot $f(\Sigma^1) = \Sigma^1_1$. The restriction g of f to $S^3 \backslash \Sigma^1$ is an ordinary covering over $S^3_1 \backslash \Sigma^1_1$. Moreover the composition of g and the natural inclusion $j : S^3_1 \backslash \Sigma^1_1 \to S^3_1$ is a spread

$$j \circ g : S^3 \backslash \Sigma^1 \to S^3_1$$

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in Fox sense, and the original map f is its Fox completion (completing the missing points over Σ^1_1). The important point here that allows the Fox completion is that Σ^1_1 fails to separate S^3_1. Similarly if we take the reflection of S^3 across a zero dimensional sphere Σ^0. The quotient space is the suspension $S(\mathbb{R}P^2)$ of the projective plane $\mathbb{R}P^2$. The the induced identification map $f : S^3 \to S(\mathbb{R}P^2)$ is a 2-fold orbifold covering and the topological underlying map f is a 2-fold covering branched over the suspension 0-sphere, denoted S^0. The restriction g of f to $S^3\backslash\Sigma^0$ is an ordinary covering over $S(\mathbb{R}P^2)\backslash S^0$. The composition of g and the natural inclusion $j : S(\mathbb{R}P^2)\backslash S^0 \to S(\mathbb{R}P^2)$ is a spread

$$j \circ g : S^3\backslash\Sigma^1 \to S(\mathbb{R}P^2)$$

in Fox sense, and the original map f is its Fox completion (completing the missing points over S^0). Here again S^0 fails to separate $S(\mathbb{R}P^2)$.

If we take the reflection of S^3 across a two dimensional sphere Σ^2 the quotient space is a 3-ball B^3 (orbifold with mirror ∂B^3) and the the induced identification map $f : S^3 \to B^3$ is a 2-fold orbifold covering. The underlying topological map is what I call a folding and generalizes the concept of covering, because here there is no branching set. The map f is a complete spread in Fox sense. Note that ∂B^3 fails to separate B^3.

Finally take the action of the Klein group G of order four, generated by both the reflection on the equator Σ^2 of S^3 and the reflection through the poles Σ^0 of S^3. The the quotient space is the cone $C(\mathbb{R}P^2)$ over the projective plane and the induced identification map $f : S^3 \to C(\mathbb{R}P^2)$ is a 4-fold orbifold covering. The underlying topological map is what I call a branched folding and generalizes the concept of branched covering. Here we have a mixture of folding and branched covering. The restriction g' of f to each component of $S^3\backslash(\Sigma^0 \cup \Sigma^2)$ is an ordinary covering. While the restriction g of f to $S^3\backslash(\Sigma^0)$ is a folding over $C(\mathbb{R}P^2)\backslash\{x\}$, where x denotes the cone apex. Moreover the composition of g and the natural inclusion

$$j : C(\mathbb{R}P^2)\backslash\{x\} \to C(\mathbb{R}P^2)$$

is a spread

$$j \circ g : S^3\backslash\Sigma^0 \to C(\mathbb{R}P^2)$$

in Fox sense, and the original map f is its Fox completion (completing the missing points over x). Note that the branching set $\{x\}$ fails to separate $C(\mathbb{R}P^2)$.

Therefore we need a theory that generalizes coverings (folding covering theory) and a theory that generalizes branched coverings (branched folding theory). Fortunately this can be done under the framework of Fox spreads and its completions.

I thank Professor Antonio Costa for his interest in this topic. He contributed to the theory with the early paper [3]. Also I thank Professor Raquel Díaz who being a student enhanced my lectures with her detailed notes.

2. Coverings and foldings

Let $g : Y \to Z$ be a map (continuous). An open nbd W of $z \in Z$ is called *elementary* if $z \in g(Y)$ and g maps each (connected) component of $g^{-1}(W)$ homeomorphically onto W.

A *covering* is a map $g : Y \to Z$, where Y and Z are connected, locally connected and each $z \in Z$ possesses an elementary nbd.

A metrizable topological space M is a (topological) *n-manifold* iff every point $x \in M$ possesses a *parametrization* $h : (X, o) \to (U, x)$, where U is an open nbd of x in M, X is either R^n or

$$R^n_+ := \{(x_1, ..., x_n) \in R^n : x_1 \geq 0\},$$

h is a homeomorphism, $o = (0, ..., 0)$, and $h(o) = x$. A point $x \in M$ belongs to the *interior* $IntM$ of M if x has a nbd homeomorphic to R^n. Otherwise, it belongs to the *boundary* ∂M of M. The n-manifold M is called *unbounded* (resp. *bounded*) iff ∂M is empty (resp. non-empty). The boundary of M is an unbounded $(n-1)$-manifold. If N is a topological n-manifold ∂N there is an open nbd U of ∂N in N and a homeomorphism $h : \partial N \times [0, 1) \to U$ such that $h((x, 0)) = x$, for each $x \in \partial N$. Such U is called a *collar*.

The following definition formalizes the idea of folding a piece of paper.

Definition 2.1. A map $g : M \to N$ between (bounded or unbounded) connected n-manifolds is a *folding* iff every point $x \in N$ possesses a *coordinate nbd* U. That is, if $x \in IntN$ the nbd U is just an elementary nbd, and if $x \in \partial N$ there is a parametrization $h : (R^n_+, o) \to (U, x)$ such that if U' is an arbitrary component of $g^{-1}(U)$ then $g(U') = U$ and, if $g|U' : U' \to U$ is not a homeomorphism, there exists a parametrization $h' : R^n \to U'$ such that

$$h^{-1} \circ g \circ h' : R^n \to R^n_+$$

is the map $(x_1, ..., x_{n-1}, x_n) \longmapsto (x_1, ..., x_{n-1}, |x_n|)$.

By definition a folding is an open map.

The points $x \in \partial N$ possessing an elementary nbd in N form obviously an open subset of ∂N. The points $x \in \partial N$ not having an elementary nbd form, according with the definition of folding, an open subset of ∂N. Call these last points *mirror points* and denote by $\partial_m N$ their union. Then $\partial_m N$ is the union of a set of connected components of ∂N. Call $\partial_m N$ the *mirror* of N. Note that if $\partial_m N$ is empty g is a covering.

Let $x \in \partial_m N$. Then a point $y = g^{-1}(x)$ possessing a nbd U' such that $g|U' : U' \to U$ is a homeomorphism is called a *mirror point in* M. The set $\partial_m M$ of mirror points of M is called the *mirror* M: it is the union of a set of connected components of ∂M. Note that $\partial_m M = \partial M \cap g^{-1}(\partial_m N)$.

Remark 2.2. A folding $g : M \to N$ where $\partial_m M$ is empty is just an ordinary covering between manifolds.

Exercise 2.3. *Fold the segment* $M = [-3, 3]$ *as a carpenter rule around the hinges* $\{-1\}$ *and* $\{1\}$ *and project it onto the segment* $N = [-1, 1]$. *This is a folding between* 1*-manifolds. The mirror boundaries are* $\partial_m M = \partial M$ *and* $\partial_m N = \partial N$.

Example 2.4. Let M be a torus represented by a rombus with opposite sides identified by translation. The reflection through a diagonal of M defines a folding $g : M \to N$ where N is a Möbius band and $\partial_m N = \partial N$.

Example 2.5. Let C be an unbounded $(n-1)$-manifold and let $q : C' \to C$ be an arbitrary covering. The map

$$g : C' \times (-1,1) \to C \times [0,1),$$

defined by $g(x,t) = (q(x), |t|)$ is a folding.

Example 2.6 (The double of a bounded manifold). Let N be a manifold with mirror $\partial_m N$. Form $N \times \{0,1\}$ and identify $(x,0)$ with $(x,1)$ for all $x \in \partial_m N$. Endow the resulting set $D_{\partial_m N}(M)$ with the topology of the identification. This is a manifold, called the *double of N along its mirror.* The map $g : D_{\partial_m N}(M) \to N$, defined by $g(x,i) = x$, for $i = 1,2$, is a folding. The mirror of $D_{\partial_m N}(M)$is empty.

Example 2.7. Let N be the Möbius band with mirror ∂N. The double of N is the Klein bottle.

Theorem 2.8. *Let $g : M \to N$ be a folding. Let the space $\{+,-\}$ have the discrete topology, Assume N is oriented. Then M is an orientable manifold iff there is a (continuous) map*

$$\varepsilon : M \backslash g^{-1}(\partial_m N) \to \{+,-\}$$

such that the points of any two connected components with common adherent points have different image.

Proof. If M is orientable select a concrete orientation. Then each component C of $M\backslash g^{-1}(\partial_m N)$ has two different orientation: one is the induced orientation as a submanifold of M, the other orientation is induced by g. Assign to C the sign $+$ (resp. $-$) iff the two orientations agree (resp. disagree). Then if $x \in \overline{C_1} \cap \overline{C_2}$ there is a nbd U' of x such that $g(U')$ is a coordinate nbd U of $g(x)$. Give to U' the orientation induced as a subset of M. Then $g|_{U'\cap C_1}$ is orientation preserving iff $g|_{U'\cap C_2}$is orientation reversing. Thus if $x_i \in U' \cap C_i$ we have $\varepsilon(x_1) \neq \varepsilon(x_2)$. Conversely, assume ε exists. Assign to an arbitrary component C of $M\backslash g^{-1}(\partial_m N)$ the orientation induced by g iff $\varepsilon(C) = +$ or the opposite orientation iff $\varepsilon(C) = -$. Then M is oriented. □

Theorem 2.9. *Let $g : Y \to Z$ be a folding (resp. a covering). Let W be an open connected set of Z, and let C be any connected component of $g^{-1}(W)$. Then $g(C) = W$ and*

$$g|_C : C \to W$$

is a folding (resp. a covering).

Proof. Since Y is locally connected and $g^{-1}(W)$ is open the component C is open in Y . Since g is open $g(C)$ is open, connected and contained in W. Let

$$z \in \overline{g(C)} \cap W.$$

Let V be a coordinate (resp. elementary) nbd of z contained in W. Since

$$V \cap g(C) \neq \emptyset,$$

there exists $x \in C$ such that $g(x) \in V$. Let D be the x-component of $g^{-1}(V)$. Then $V \subset W$ implies $D \subset C$. Since V is coordinate (resp. elementary):

$$g(D) = V \subset g(C).$$

Hence $z \in V$ lies in $g(C)$. Thus $g(C)$ is clopen in W. Since W is connected, $g(C) = W$. This proves the first part.

To see that

$$h = g|_C : C \to W$$

is a folding (resp. covering), take a coordinate (resp. elementary) nbd (with respect to g) $V \subset W$ of a point $z \in W$. Let D be any component of $h^{-1}(V)$. Since $V \subset W$, D is a component of $g^{-1}(V)$ and $D \subset C$. Then

$$h|_D = (g|_C)|_D = g|_D.$$

Thus V is a coordinate (resp. elementary) nbd (with respect to h) of the point $z \in W$. □

The definition of folding suggests that the theory of foldings is analogous to the theory of coverings. To substantiate this claim we need a proper definition ([12]) of the *fundamental group* $\pi_1^m(N, \partial_m N)$ *of* N *with mirror* $\partial_m N$. (Almost always $\partial_m N = \partial N$. In these cases we will write $\pi_1^m(N)$.)

Let N be a connected manifold with mirror $\partial_m N$. Let $*$ be some base point in $IntN$. We say that a loop $\gamma : [0,1] \to N$ with $\gamma(0) = \gamma(1) = *$ is *transversal to* $\partial_m N$ if either $\gamma([0,1]) \cap \partial_m N$ is empty or $\gamma(t_i) \in \partial_m N$ for only finitely many $t_1, ..., t_k$. Moreover if $\gamma(t_0) \in \partial_m N$ there is a parametrization

$$h : (R^n_+, 0) \to (U, \gamma(t_0))$$

and an $\varepsilon > 0$ such that

$$h^{-1} \circ (\gamma|[t_0 - \varepsilon, t_0 + \varepsilon]) : [t_0 - \varepsilon, t_0 + \varepsilon] \to R^n_+$$

is the map $t \longmapsto (0, ..., 0, |-t + t_0|)$. We will say that $\gamma([t_0 - \varepsilon, t_0 + \varepsilon])$ is a *pin*.

A homotopy

$$F : [0,1] \times [0,1] \to N$$

such that

$$F_0 = \gamma, F_1 = \gamma', F(\{1\} \times [0,1]) = F(\{0\} \times [0,1]) = *,$$

will be called *transversal to* $\partial_m N$ if F_t is transversal to $\partial_m N$ for all $t \in [0,1]$, and if $\gamma(t_0) \in \partial_m N$ then

$$F(\{t_0\} \times [0,1]) \subset \partial_m N.$$

Let $\gamma : [0,1] \to N$ be transversal to $\partial_m N$. Two pins $\gamma([t_0 - \varepsilon, t_0 + \varepsilon])$ and $\gamma([t_1 - \varepsilon', t_1 + \varepsilon'])$ are called *consecutive* iff there is no pin in $\gamma((t_0, t_1))$ and $\gamma([t_0 - \varepsilon, t_1 + \varepsilon'])$ lies in a nbd U homeomorphic to R^n_+. Define a loop γ' which is the composition of $\gamma|[0, t_0 - \varepsilon]$ with $\gamma|[t_0 + \varepsilon, t_1 - \varepsilon']$ and with $\gamma|[t_1 + \varepsilon', 1]$. Transition from γ to γ' (resp. γ' to γ) will be called *cancelation* (resp. *addition*) *of consecutive pins.*

Two transversal loops γ and γ' will be called *rel.*$\partial_m N$ *homotopic* if it possible to pass from γ to γ' by a finite application of transversal homotopies and cancelations and additions of consecutive pins.

This definition is taylored to ensure that the projection of a rel. ∂ homotopy between two paths α, α' connecting $(1, 0, ..., 0)$ to $(-1, 0, ..., 0)$ in R^n and transversal

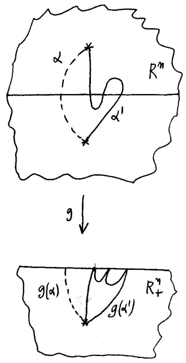

Figure 1: Projecting a homotopy

to $R^{n-1} \times \{0\}$ (see Figure 1) is a rel. ∂_m homotopy between the loops $g(\alpha)$, $g(\alpha')$ under the (local) folding $g : R^n \to R^n_+$ given by

$$(x_1, ..., x_{n-1}, x_n) \longmapsto (x_1, ..., x_{n-1}, |x_n|).$$

Of course we can define the *fundamental group* $\pi_1^m(N, *)$ *of* N *with mirror* $\partial_m N$ as the rel. $\partial_m N$ homotopy classes of transversal loops under composition.

Let α be an arc running from $*$ to some point $p \in C$, where C is a component of $\partial_m N$. Assume that the loop $\alpha * \alpha^{-1}$ is transversal to $\partial_m N$. The rel. $\partial_m N$ homotopy class of $\alpha * \alpha^{-1}$ is called a *boundary generator.* Of course boundary generators have order two in $\pi_1^m(N)$. Define

$$f : \pi_1(C, p) \to \pi_1^m(N, *)$$

by $f([\lambda]) = [\alpha\prime * \lambda' * \alpha'^{-1}]$, where α' is a subarc of α running from $*$ to a point p' very near p and λ' is a loop based on p' and running trough $IntM$ parallel to λ (use a collar of C). Then the boundary generator defined by $\alpha * \alpha^{-1}$commutes with the elements of $f(\pi_1(C, p))$ (slide the pin of $\alpha * \alpha^{-1}$ along λ').

Then it is not difficult to see that, if $\partial_m N$ has k components $C_1, ..., C_k$, the group $\pi_1^m(N, *)$ is isomorphic to the quotient

$$(\pi_1(N, *) * Z_1 * ... * Z_k)/K,$$

where Z_i is the cyclic group of two elements generated by a boundary generator c_i of C_i and K is the subgroup normally generated by the commutators of c_i with all the elements of the image of $\pi_1(C_i, p_i)$ in $\pi_1(N, *)$ for $i = 1, ..., k$.

Example 2.10. Let (N,$\partial_w N$) be the Möbius band and its boundary. Then

$$\pi_1^m(N, *) = |x, y : x^2 = 1, xy^2 = y^2 x|$$

where x is some boundary generator and y^2 generates $\pi_1(\partial N)$. The second relation says that x comes back after dragging it twice along y.

Exercise 2.11. *Let M be a connected unbounded manifold and let N be a bounded manifold. Prove that $\pi_1^m(M \times N, *)$ is isomorphic to $\pi_1(M, *) \times \pi_1^m(N, *)$.*

With these definitions it is not difficult to see that the theory of coverings can be transferred to a similar theory of foldings. To adapt the concept of path lifting to foldings we define that the lifting of the map

$$\gamma : [t_0 - \varepsilon, t_0 + \varepsilon] \to U = R^n_+, t \longmapsto (0, ..., 0, -|t| + t_0)$$

is the map

$$\gamma' : [t_0 - \varepsilon, t_0 + \varepsilon] \to U' = R^n, t \longmapsto (0, ..., 0, -t + t_0)$$

if

$$g|U' : U' = R^n \to U = R^n_+$$

is the map $(x_1, ..., x_{n-1}, x_n) \longmapsto (x_1, ..., x_{n-1}, |x_n|)$.

We will say that the folding $g : M \to N$ is a *h-sheeted folding* iff $\#g^{-1}(x) = h$ for (all) $x \in N \backslash \partial_m N$.

Given the folding $g : M \to N$ and the base point $* \in IntN$ we can define the *monodromy*

$$\omega : \pi_1^m(N, *) \to \Sigma_F.$$

This is a representation (homomorphism) into the symmetric group of the fiber $F = g^{-1}(*)$, that is, the group of bijections of F. Given $[\gamma] \in \pi_1^m(N, *)$, lift $\gamma : [0, 1] \to N$ to $\gamma' : [0, 1] \to M$ with $\gamma'(0) = p \in F$. Define $\omega([\gamma]) : F \to F$, by $\omega([\gamma])(p) = \gamma'(1) \in F$ (compare [27]).

For instance, the monodromy of the folding $g : D_{\partial_m N}(M) \to N$ is a representation $\omega : \pi_1^m(N, *) \to \Sigma_2$ sending the boundary generators to the cycle (12) and the remaining generators to (1)(2).

If $\omega : \pi_1^m(N, *) \to \Sigma_F$ is the monodromy of the folding $g : M \to N$, the set

$$\mathcal{C} = \{\omega^{-1}(Stab_p \Sigma_F) : p \in F\},$$

where $Stab_p \Sigma_F$ is the set $\{\lambda \in \Sigma_F : \lambda(p) = p\}$, is a complete class of conjugation of subgroups of $\pi_1^m(N, *)$. The group $\omega^{-1}(Stab_p \Sigma_F)$ is the image of $\pi_1^m(M, p)$ under the injective map

$$g_\# : \pi_1^m(M, p) \to \pi_1^m(N, *).$$

The class $\mathcal{C}$ (or the monodromy ω) determines g (compare [27]).

The folding g is *regular* if the class $\mathcal{C}$ has only one member (a normal subgroup), called the *folding group*. In this case the liftings of loops to the points of F either all are loops or all are (open) paths (and conversely).

The *universal folding* of N (a manifold $\widetilde{N}$ without mirror boundary) can be constructed as in the theory of coverings. Its folding group is $\{0\}$. From this construction follows the existence of the folding corresponding to a given transitive representation $\omega : \pi_1^m(N, *) \to \Sigma_F$.

Theorem 2.12. *Let N be a manifold with mirror $\partial_m N$. The universal folding of N is the universal covering of $D_{\partial_m N} N$.*

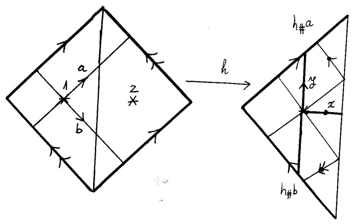

Figure 2: Folding the torus

Example 2.13. Let N be the torus $S^1 \times S^1$ minus an open 2-cell and let $\partial_m N$ be ∂N. The universal folding of N is the universal covering of the unbounded 2-manifold of genus 2. This universal covering is homeomorphic to R^2.

The group $Aut(g)$ of *automorphisms* (homeomorphism $f : M \to M$ such that $g \circ f = g$) is isomorphic to

$$\pi_1^m(N, *)/g_\#(\pi_1^m(M, p))$$

if g is regular. In general $Aut(g)$ is isomorphic to the quotient

$$N(g_\#(\pi_1^m(M, p))/g_\#(\pi_1^m(M, p),$$

where $N(g_\#(\pi_1^m(M, p))$is the normalizer $N(g_\#(\pi_1^m(M, p))$ of $g_\#(\pi_1^m(M, p)$ in $\pi_1^m(N, *)$. The action of the group $Aut(g)$ in M is proper and discontinuous and if g is regular the map g is just the quotient under this action. Of course if $\partial_m N$ is non-empty the action of g is not free.

Example 2.14. Let N be the 1-manifold $[0, 1]$ and let $\partial_m N$ be ∂N. Then $\pi_1^m(N, *)$ is the free product of two cyclic groups of order two: $C_2 * C_2$. The universal folding of N is R^1. The group of automorphisms is the group generated by the reflections through $\{0\}$ and $\{1\}$.

Exercise 2.15. *Deduce from the above example that if G is a subgroup of $C_2 * C_2$ then either $G = \{0\}$, or $G \approx C_\infty$ or $G \approx C_2 * C_2$. Moreover in the last two cases G has finite index.*

Exercise 2.16. *Let N be the* 2*-manifold* $[0, 1] \times S^1$ *(*annulus*) and let $\partial_m N$ be ∂N. Then $\pi_1^m(N, *)$ is $(C_2 * C_2) \times C_\infty$. The universal folding of N is $R \times R$. The group of automorphisms is the group generated by the reflections through* $\{0\} \times R$ *and* $\{1\} \times R$.

An interesting example is the regular 2-sheeted folding $h : M \to N$ of Example 2.4. The folding group is a normal subgroup of index two in

$$\pi_1^m(N, *) = |x, y : x^2 = 1, xy^2 = y^2 x|$$

(see Example 2.10). Let us find it geometrically (see Figure 2). Here M is a torus and

$$\pi_1(M, 1) = |a, b : ab = ba|.$$

By inspection $g_\# a = xy$ and $g_\# b = xy^{-1}$. Therefore the folding group is the subgroup $\langle xy, xy^{-1} \rangle$ of $\pi_1^m(N, *)$ normally generated by xy and xy^{-1}. The monodromy ω :

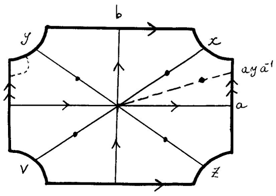

Figure 3: The punctured torus

$\pi_1^m(N,*) \to \Sigma_2$ sends x and y to the cycle (12). This can be seen directly. In fact, the lifting of x or y to the base point marked 1 finishes at 2.

Knowing the monodromy of the folding it is possible to construct it. Here is an example (Figure 3). The manifold N is a torus with a hole. Assume $\partial_m N = \partial N$. Then

$$\pi_1^m(N,*) = \left|a, b, x : x^2 = 1, [x, [a,b]] = 1\right|$$

where x is a boundary generator (marked with a point in Figure 3. Take the representation $\omega : \pi_1^m(N,*) \to \Sigma_4$ given by $\omega(a) = (123)$, $\omega(b) = (124)$ and $\omega(x) = (13)(24)$. By inspection (see Figure 3) $y = a^{-1}xa$, $z = b^{-1}xb$, $v = a^{-1}za$ and $\omega(y) = (12)(34)$, $\omega(z) = (14)(32)$, $\omega(v) = (13)(24)$. We want to construct the folding $g : M \to N$ with monodromyω. The folding is of degree 4 and the Euler characteristic $\chi(N)$ of N is -1. Therefore $\chi(M) = -4$. Since $\partial_m M$ is empty it follows that M is an unbounded 2-manifold of genus 3. To construct g effectively take four copies of the octogon depicted in Figure 3 and paste them together using the information given by ω (do this as an exercise).

3. Branched foldings

Before defining branched foldings we will need some preliminar definitions and results.

3.1. Fox´s spreads

A map $g : Y \to Z$ between T_1-spaces is a *spread* iff the connected components of inverse images of open sets of Z constitute a base for the topology of Y. That is, given a point $y \in Y$ and an open neighborhood ("nbd" for short) U of y in Y there exists an open set W in Z such that the connected component of $g^{-1}(W)$ containing y (the y-component of $g^{-1}(W)$, from now on) is an open set between y and U. It follows easily that, given a spread $g : Y \to Z$, the space Y is locally connected and that if Z_1 is an open subset of Z, then

$$g \mid g^{-1}(Z_1) : g^{-1}(Z_1) \to Z_1$$

is also a spread. If $g : Y \to Z$ is a spread and Z is a subspace of some T_1-space Z' then the natural map $i \circ g : Y \to Z'$ is a spread, where i is the inclusion.

If Z is a topological space denote by $\mathcal{E}(z)$ the set of open neighbourhoods of $z \in Z$. If $f : Y \to Z$ is a map, a *thread* y_z *over* z is a function $W \longmapsto y_z(W)$ where

$W \in \mathcal{E}(z)$ and $y_z(W)$ is a component of $f^{-1}(W)$ such that

$$y_z(W_2) \subset y_z(W_1) \text{ if } W_2 \subset W_1.$$

A spread $g : Y \to Z$ is *complete* iff, for every thread

$$y_z = \{y_z(W)\}$$

over z, and for every z, the intersection

$$\cap_{W \in \mathcal{E}(z)} y_z(W)$$

is non-empty (and consists of just one point).

Example 3.1. Coverings and foldings are complete spreads.

Lemma 3.2. *Let $g : Y \to Z$ be a complete surjective spread with discrete fibers. Assume also that Z satisfies the first axiom of numerability. Let b be an arbitrary member of Z and let $\{W_i\}_{i=1}^{\infty}$ be an arbitrary countable base of nbds of b such that $W_{i+1} \subset W_i$ for all i. Let $x \in g^{-1}(b)$ and let V_i be the connected component of $g^{-1}(W_i)$ containing x. Then there exists n such that, for all $j > i > n$, V_j is the only connected component of $g^{-1}(W_j)$ lying in V_i.*

Proof. The set $\{V_i\}_{i=1}^{\infty}$ is a base of nbds of x. Let us argue by contradiction. Then for an arbitrary V_i there is a $j > i$ such that $g^{-1}(W_j)$ contains at least two connected components, V_j and V'_j, lying in V_i . Then there is a thread y_b over b such that $y_b(W_k) \subset V'_j$ for all $k \geq j$. Since g is complete, the intersection $\cap_{i=j}^{\infty} y_b(W_i)$ is a point $x' \in g^{-1}(b)$. Then $x \neq x'$ because $x' \in V'_j$ and $V_j \cap V'_j$ is empty. Moreover $x' \in V'_j \subset V_i$. Since this is true for every nbd V_i of x this point is a limit point and the fiber $g^{-1}(x)$ is not discrete. This completes the proof. □

A space X is locally connected in Y iff Y has a base whose members intersect X in connected sets [6].

A spread $g : Y \to Z$ is a *completion* of a spread

$$f : X \to Z$$

if

(i) X is a subspace dense and locally connected in Y; and

(ii) g is complete and extends f.

The main theorem of [6] is the following:

Theorem 3.3. *Every spread f has a unique completion (called* Fox completion*).*

Let $g : Y \to Z$ be a spread. An automorphism of g is a homeomorphism $\alpha : Y \to Y$ such that $g \circ \alpha = g$. The group of all automorphisms of g is denoted by $Aut(g)$.

Theorem 3.4 (see [24])**.** *Let Z_1 be an open subset of Z and let*

$$j : Z_1 \to Z$$

be the canonical inclusion. Let

$$f : X \to Z_1$$

be a complete surjective spread. If $g : Y \to Z$ is the Fox completion of the spread

$$j \circ f : X \to Z,$$

then the restriction to X of any automorphism of Y is an automorphism of X. The map so defined between $Aut(g)$ and $Aut(f)$ is an isomorphism.

Definition 3.5. A map $g : Y \to Z$, between T_1-spaces and where Z satisfies the first axiom of numerability, is a *branched folding* (resp. *branched covering*) iff there is a closed subset B of Z such that $Z \backslash B$ is dense and locally connected in Z, the map

$$f : Y \backslash g^{-1}(B) \to Z \backslash B$$

defined by $f(x) = g(x)$ is a folding (resp. covering), and g is the Fox completion of the spread

$$g|_{Y \backslash g^{-1}(B)} : Y \backslash g^{-1}(B) \to Z.$$

If the set B is minimal with respect to these conditions we say that B is the *branching set* of g. The map f is called the *associated folding* (resp. *associated covering*) and the spread

$$g|_{Y \backslash g^{-1}(B)} : Y \backslash g^{-1}(B) \to Z$$

is called the *associated unbranched folding* (resp. *associated unbranched covering*).

Remark 3.6. The first axiom of numerability is imposed to Z to ensure that a branched folding (resp. branched covering) $g : Y \to Z$ is surjective (see [3], and compare [24]).

The following results are preliminaries to prove that if W is an open connected subset of Z, and C is a component of $g^{-1}(W)$, then $g|_C : C \to W$ is a branched folding (resp. branched covering). This implies that branched folding and branched covering are open maps (see [3], and compare [24]).

Lemma 3.7 (see [6])**.** *If X is dense and locally connected in Y, the intersection of X with any open connected set of Y is connected.*

Proof. Let V be an open, connected subset of Y. Then $U = V \cap X$ is non-empty because X is dense in Y. Suppose that $U = V \cap X$ is not connected. Then, there are non-empty disjoint open subsets A_1 and A_2 of X, such that

$$A_1 \cup A_2 = U.$$

We will construct non-empty disjoint open subsets B_1 and B_2 of Y, such that $B_1 \cup B_2 = V$. Let $y \in V$ be an arbitrary point. Using the fact that X is locally connected in Y find a nbd $N(y)$ of y in V, whose intersection with X, denoted by $M(y)$, is connected. Since

$$M(y) \subset U = A_1 \cup A_2,$$

the set $M(y)$ will be contained either in A_1 or A_2. In the first case we define y to be a member of B_1; in the second, it will define a member of B_2. Clearly

(i) $A_i \subset B_i$, $i = 1, 2$;

(ii) B_i is open: if

$$y \in B_i \Longrightarrow N(y) \subset B_i;$$

(iii) $B_1 \cap B_2 = \emptyset$;

(iv) $B_1 \cup B_2 = V$.

Therefore V is not connected. This contradiction completes the proof □

Proposition 3.8. *Let $g : Y \to Z$ be a branched folding (resp. branched covering) with branching set B. Let W an open connected subset of Z, and let C be a component of $g^{-1}(W)$. Then $C\backslash g^{-1}(B)$ is a component of $g^{-1}(W\backslash B)$ and the restriction $g|_{C\backslash g^{-1}(B)}$ is a folding (resp. covering) over $W\backslash B$. If g is regular then $g|_{C\backslash g^{-1}(B)}$ is regular.*

Proof. Since Y is locally connected, C is open (and connected) in Y. Since $g^{-1}(Z\backslash B)$ is dense and locally connected in Y, $C \cap g^{-1}(Z\backslash B) = C\backslash g^{-1}(B)$ is connected (Lemma 3.7). Also

$$C \cap g^{-1}(Z\backslash B) \subset g^{-1}(W) \cap g^{-1}(Z\backslash B) = g^{-1}(W\backslash B).$$

Hence $C\backslash g^{-1}(B)$ lies in some component D of $g^{-1}(W\backslash B)$. Next we show that $C\backslash g^{-1}(B) = D$. The set $D \subset g^{-1}(W)$ is connected and cuts the component C of $g^{-1}(W)$ then $D \subset C$. Hence $C\backslash g^{-1}(B) = D$. We can apply Theorem 2.9 to the folding (resp. covering)

$$g|_{Y\backslash g^{-1}(B)} : Y\backslash g^{-1}(B) \to Z\backslash B,$$

and to the open connected set $W\backslash B$ (Lemma 3.7) and to the connected component $C\backslash g^{-1}(B)$ of $Y\backslash g^{-1}(B)$ concluding that the restriction $g|_{C\backslash g^{-1}(B)}$ is a folding (resp. covering) over $W\backslash B$. To prove that if g is regular then $g|_{C\backslash g^{-1}(B)}$ is regular take an arbitrary loop based at some point $* \in W\backslash B$ and lift the loop to the points $g^{-1}(*) \cap (C\backslash g^{-1}(B))$. Since g is regular all these liftings will lie in $C\backslash g^{-1}(B)$ and will be closed or open in $Y\backslash g^{-1}(B)$, hence in $C\backslash g^{-1}(B)$. Thus $g|_{C\backslash g^{-1}(B)}$ is also regular. □

Lemma 3.9. *Let A be a subspace of X and let $Y \subset X$ be an open subset. Then $Y \cap A$ is locally connected in Y if A is locally connected in X. Moreover $Y \cap A$ is dense in Y if A is dense in X.*

Proof. If there is a base of X whose members when intersected with A give connected sets, take the members of that base lying in Y. Their intersection with A coincides with their intersection with $Y \cap A$ and it is, therefore, connected. If A is dense in X, every open set in Y is open in X. Hence it cuts A, and a fortiori $Y \cap A$. □

Theorem 3.10. *Let $g : Y \to Z$ be a (regular) branched folding (resp. branched covering) with branching set B. Let W an open connected subset of Z, and let C be a component of $g^{-1}(W)$. Then $g|_C : C \to W$ is a (regular) branched folding (resp. branched covering).*

Proof. Let us prove that $h = g|_C : C \to W$ is a branched folding (resp. branched covering). Since W is open $g|_{g^{-1}(W)} : g^{-1}(W) \to W$ is a complete spread and W satisfies the first axiom of numerability. Since C is open in $g^{-1}(W)$, h is also a complete spread. By Proposition 3.8 $C\backslash g^{-1}(B)$ is a component of $g^{-1}(W\backslash B)$ and the restriction $g|_{C\backslash g^{-1}(B)}$ is a folding (resp. covering) over $W\backslash B$. Then

$$h|_{C\backslash h^{-1}(B)} : C\backslash h^{-1}(B) \to W\backslash B$$

is a folding (resp. covering). Since W (resp. C) is open in Z (resp. Y) and $Z\backslash B$ (resp. $Y\backslash g^{-1}(B)$) is dense and locally connected in Z (resp. Y), Lemma 3.9 implies that $W\backslash B$ (resp. $C\backslash h^{-1}(B)$) is dense and locally connected in W (resp. C). □

Corollary 3.11. *Any branched folding (resp. branched covering) $g : Y \to Z$ is an open map.*

Proof. By Remark 3.6 any branched folding (resp. branched covering) is surjective. Therefore the map $g|_C : C \to W$ in the last Theorem is surjective. Since the components C form a base of the topology of Y then g is open. □

The fact that B is closed, and g is surjective imply (Theorem 3.4):

Theorem 3.12. *The group of automorphisms $Aut(g)$ of the branched folding (resp. branched covering) g is isomorphic (by restriction) to the group of automorphisms $Aut(f)$ of the associated folding (resp. associated covering) f.*

We say that the branched folding (resp. branched covering) $g : Y \to Z$ is *regular* iff its associated folding (resp. associated covering) f is regular.

Theorem 3.13. *If the branched folding (resp. branched covering) $g : Y \to Z$ is regular and has discrete fibers then it is topologically equivalent to the quotient of Y under the action of $Aut(g)$.*

Remark 3.14. If Y is compact the fibres of g are finite, and if Y is locally compact, the fibers are discrete [24]. We conjecture that there is an example of regular branched folding in which *Aut* fails to act transitively in the fibers.

Proof. By Remark 3.6 the map g is surjective and by Corollary 3.11 g is open. Then Z has the topology of the identification of the fibers of g to points. Since $Aut(f)$ acts transitively in the fibers of the associated folding f, we only need to show that $Aut(g)$ acts transitively in the fiber $g^{-1}(b)$, for each $b \in B$. Let $\{W_i\}_{i=1}^{\infty}$ a countable base of nbds of b in Z such that $W_{i+1} \subset W_i$. Take two points y, y' of $g^{-1}(b)$. Then $y = \cap_{i=1}^{\infty} y_b(W_i)$ (resp. $y' = \cap_{i=1}^{\infty} y'_b(W_i)$) where y_b (resp. y'_b) is a function $W_i \longmapsto y_b(W_i)$ where $y_b(W_i)$ is a component of $g^{-1}(W_i)$ such that

$$y_b(W_{i+1}) \subset y_b(W_i) \text{ if } W_{i+1} \subset W_i.$$

Since the fiber $g^{-1}(b)$ is discrete, by Lemma 3.2 there must exist N such that for all $j > i > N$ there is exactly one component (namely, $y_b(W_j)$) of $g^{-1}(W_j)$ lying in $y_b(W_i)$ and exactly one component (namely, $y'_b(W_j)$) of $g^{-1}(W_j)$ lying in $y'_b(W_i)$. Let $z \in W_{N+1}$ having an elementary nbd with respect to g. Let $x \in g^{-1}(z) \cap y_b(W_{N+1})$ (resp. $x' \in g^{-1}(z) \cap y'_b(W_{N+1})$). Let $h \in Aut(g)$ such that $h(x) = x'$. For all $j > N$ the restriction $h|_{g^{-1}(W_j)}$ acts in $g^{-1}(W_j)$ because $g \circ h = g$. Therefore h permutes the connected components of $g^{-1}(W_j)$. Then h sends the set $y_b(W_j)$ onto the set $y'_b(W_j)$ for all $j > N$ .Therefore $h(y) = \cap_{i=N+1}^{\infty} h(y_b(W_i)) = \cap_{i=N+1}^{\infty} y'_b(W_i) = y'$. This completes the proof. □

If $g : Y \to Z$ is a covering branched over B, the subset $f^{-1}(B)$ of Y is the disjoint union of the set $\widetilde{P}$ of points at which f is a local homeomorphism (*pseudo-branching cover)* and the set $\widetilde{B}$ of points at which f is not a local homeomorphism (*branching cover*).

4. Some examples of branched coverings

As we have indicated the interest of the above definition of branched folding (or covering) lies in the fact that, given the monodromy of the associated folding (or covering) f and the inclusion $j : Z \backslash B \to Z$ the branched folding (or covering) g is completely determined by the Fox completion of $j \circ f$. We illustrate this with some three dimensional examples.

Example 4.1. Let M and N orientable manifolds and let $g : M \to N$ be a folding with mirrors $\partial_m N$ and $\partial_m M$. Let

$$\varepsilon : M \backslash g^{-1}(\partial_m N) \to \{+, -\}$$

be the map granted by Theorem 2.8. Let M' (resp. N') be the quotient of $M \times [-1, 1]$ (resp. $N \times [-1, 1]$) under the equivalence relation $\equiv$ generated by $(z_1, t_1) \equiv (z_2, t_2)$ iff $z_1 = z_2$ lies in $\partial_m M$ (resp. $\partial_m N$) and $t_1 = -t_2$. The map $g' : M' \to N'$ defined by $g'(z, t) = (g(z), \varepsilon(z)t)$ is a branched covering. The branching set B is $\partial_m N \times \{0\} \subset N'$. Note that there is an epimorphism

$$f : \pi_1(N' \backslash B) \to \pi_1^m(N).$$

For instance the branched covering induced in this way by the 3-sheeted folding of Example 2.3 is a 3-sheeted branched covering of a 2-cell over a 2-cell Q with branching set two points inside Q. Crossing this branched covering with the identity $1 : B^n \to B^n$ we obtain a 3-sheeted covering of a $n+2$-cell over a $n+2$-cell $Q \times B^n$ with branching set two boundary parallel n-cells inside $Q \times B^n$ (see the ball of Figure 4 to understand the case $n = 1$).

Let (L, ω) be a *represented knot or link* in $X = S^3$ (resp. a *represented string or string-link* in $X = \mathbb{R}^3$), where ω is a simple representation (homomorphism) of $\pi_1(X \backslash L)$ onto the symmetric group S_3 of the indices $\{1, 2, 3\}$. Thus ω sends meridians of L to transpositions $(1, 2), (1, 3)$, or $(2, 3)$ of S_3, which, following a beautiful idea of Fox, will be represented by colors Red ($R = (1, 2)$), Green ($G = (1, 3)$) and Blue ($B = (2, 3)$). If the representation exists we can endow each overpass of a normal projection of L with one of the three colors R, G, B in such a way that the colors meeting in a crossing are all equal or all are different. Moreover, at least two colors are used. A knot or link L (resp. string or string-link L) in $X = S^3$ (resp. $X = \mathbb{R}^3$) with a coloration corresponding to some ω is a *colored knot or link* (resp. *colored string or string-link*).

A colored knot or link (resp. string or string-link) (L, ω) in $X = S^3$ (resp. $X = \mathbb{R}^3$) defines a complete conjugation class of subgroups of $\pi_1(X \backslash L)$. Namely the set $\{\omega^{-1}(Stab_i) : i \in \{1, 2, 3\}\}$ where $Stab_i$ is the subgroup of elements of S_3 fixing the index i. This class of subgroups determines a covering of three sheets

$$f' : Y \to X \backslash L$$

and an unbranched covering $f'' = j \circ f'$ where

$$j : X \backslash L \to X$$

is the inclusion map. A *construction* described by Neuwirth in [26] gives an extension of f'' to a branched covering $f : M(L,\omega) \to X$, branched over L, such that the associated unbranched covering of f turns out to be f''. We emphasize that f is uniquely determined by f'', not in general by f' [11]. The space $M(L,\omega)$ is a closed (resp. open), orientable 3-manifold.

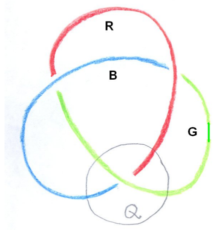

Figure 4: A colored knot

Example 4.2. Consider the colored knot (L,ω) of Figure 4. This colored knot was considered by R. H. Fox in [7]. We now prove that the closed manifold $M(L,\omega)$ is S^3 (see [17], [18] and [29]). Let $f : M(L,\omega) \to S^3$ denote the branched covering. Referring to Figure 4

$$f|_{f^{-1}(Q)} : f^{-1}(Q) \to Q$$

is a branched covering of a closed 3-cell branched over two boundary parallel arcs (and similarly $f|_{(S^3 \backslash f^{-1}(Q))}$). Now Example 4.1 shows that $f^{-1}(Q)$ is a closed 3-cell. Then $M(L,\omega)$ is the result of pasting together two closed 3-cells along their common boundary. Thus $M(L,\omega)$ is S^3.

Example 4.3. Consider the colored string (L,ω) in $\mathbb{R}^3$ of Figure 5. This colored string was first considered by R. H. Fox in the paper [5]; entitled *"A remarkable simple closed curve"*. We will call L *Fox string*.

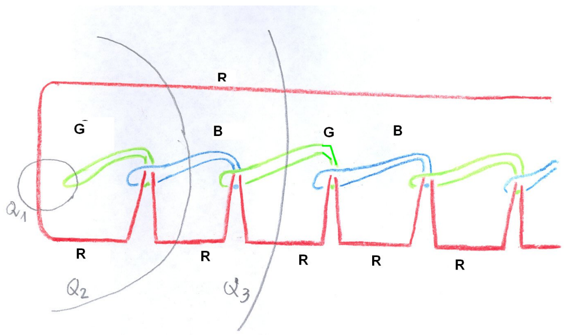

Figure 5: Fox string colored

We will prove that the space $M(L,\omega)$ is homeomorphic to $\mathbb{R}^3$. Thus *there exist a 3-fold simple covering* $\widehat{p}:\mathbb{R}^3\to\mathbb{R}^3$ *branched over the Fox string* L. In fact, select a sequence of 3-cells $\{Q_i\}_{i=1}^{\infty}$ such that $Q_i\subset Int(Q_{i+1})$ and

$$\cup_{i=1}^{\infty}Q_i=\mathbb{R}^3=S^3\backslash\{\infty\},$$

as indicated in Figure 5. Let

$$p:M(L,\omega)\to\mathbb{R}^3$$

be the simple branched covering given by the representation ω. Then, for $i\geq 1$, $p^{-1}(Q_i)$ is a 3-cell Q_i'. In fact,

$$p|p^{-1}(Q_i):p^{-1}(Q_i)\to Q_i$$

is a 3-fold simple covering of the 3-cell Q_i, branched over two properly embedded arcs; these arcs are embedded exactly as in case $i=1$ (see Figure 4). By Example 1, $p^{-1}(Q_i)$ is a 3-cell Q_i'. Then

$$M(L,\omega)=\cup_{i=1}^{\infty}Q_i'.$$

And from this follows that $M(L,\omega)$ is homeomorphic to $\mathbb{R}^3$ [1] as we wanted to prove.

5. Some examples of branched foldings

An *n-simplex*

$$\sigma=(e_0,e_1,...,e_n)$$

is the convex hull of $n+1$ points

$$\{e_0,e_1,...,e_n\},$$

of some euclidean space R^N, in general position (that is, the n vectors $\{e_0-e_1,...,e_0-e_n\}$ are linearly independent). The number n is the *dimension* $dim(\sigma)$ of σ. The *faces* of σ are the simplexes generated by arbitrary subsets of its vertices. (Therefore, the empty set is face of every simplex.) If σ' is a face of σ we write $\sigma'<\sigma$.

A *locally finite simplicial complex* is a set K whose members are simplexes (of various dimensions) of some fixed euclidean space R^N possessing the following properties:

1. Every face of an arbitrary simplex of the set K is itself an element of the set K.

2. The intersection of two members of K is a face of both.

3. Each point in $|K|=\cup_{\sigma\in K}\sigma$ has a neighbourhood in R^N which intersects only a finite number of simplexes of K.

A complex K is called *k-dimensional* (or a *k-complex*) if k is the maximum dimension of the simplexes of K. A complex K is called *finite* if the set K is finite. The i-skeletton K^i of K denotes the i-complex which is the union of all the j-simplexes of K for $j\leq i$.

The subset $|K|$ of R^N with the relative topology is called a *polyhedron* and K is called a *triangulation* of $|K|$. A complex K' is a *subdivision* of K iff every simplex of K' lies in a simplex of K. Then K' triangulates $|K|$.

The above definitions can be easily generalized using polygons instead of simplexes. A polygon p is the convex hull of m points

$$\{e_0, e_1, ..., e_m\},$$

of some euclidean space R^N. Since the points are not in general position the polygon is not necessarily a simplex. But it has faces, dimension, etc. There are polygonizations and subdivisions, etc. We leave this generalization to the reader.

Assume an n-complex K triangulates an n-manifold M (bounded or unbounded). Assume K is $(n+1)$-colorable (an $(n+1)$-*coloration* of K is a map $c : K^0 \to \{0, 1, ..., n\}$ such that $c|_{\sigma^0}$ is surjective for every n simplex $\sigma \in K$). Let τ be some arbitrarily fixed n-simplex $(e_0, e_1, ..., e_n)$. Define a map $g : M \to \tau$ by extending linearly the map $g' : K^0 \to \tau$ given by $g'(v) = e_{c(v)}$ for all $v \in K^0$. The map g is a branched folding. The branching set B is contained in the $(n-2)$-skeletton τ^{n-2} of τ. The mirror of the associated folding is $\partial\tau\backslash\tau^{n-2}$. The monodromy of the associated folding is $\omega : \pi_1^m(\tau\backslash\tau^{n-2}) \to \Sigma_S$, where S is the set of n-simplexes of K. The representation ω is defined as follows. Let $[\rho]$ be the boundary generator of $\pi_1^m(\tau\backslash\tau^{n-2})$ corresponding to the $(n-1)$-face ρ of τ. Then $\omega(\,[\rho]) : S \to S$ permutes the n-simplexes of K sharing an $(n-1)$-simplex belonging to $g^{-1}(\rho)$. Thus $\omega(\,[\rho])$ is a product of transpositions.

Let K be a triangulation (or more generally, a polygonization) of some n-manifold M. To each simplex σ (or polygon) of K associate a point $b_\sigma \in Int\sigma$ (if σ is a 0-simplex then $b_\sigma = \sigma$). Define a new complex K' declaring that $(b_{\sigma_1}, ..., b_{\sigma_k})$ is a member of K' iff $\sigma_1 < ... < \sigma_k$. Then K' is a subdivision of K and therefore K' triangulates M. The map $c : K'^0 \to \{0, 1, ..., n\}$ defined by $c(b_\sigma) = dim(\sigma)$ is then an $(n+1)$-*coloration* of K'. Therefore:

Theorem 5.1. *Every triangulated n-manifold (bounded or unbounded, orientable or not, compact or not) is a branched folding over an n-simplex σ with branching set the $(n-2)$-skeletton of σ.*

Since Moise proved that every bounded or unbounded n-manifold $n \leq 3$ is triangulated by some K [13] we have:

Theorem 5.2. *Every n-manifold (bounded or unbounded, orientable or not, compact or not) is a branched folding over an n-simplex σ for $n \leq 3$ with branching set the $(n-2)$-skeletton of σ.*

Example 5.3. Consider the polygonization K of the 2-sphere S defined by the cube. Construct K' as above and define the 48-sheeted branched folding $g : S \to \tau = (e_0, e_1, e_2)$ by extending linearly the map $g' : K'^0 \to \tau$ given by $g'(b_p) = e_{dim(p)}$ for all $p \in K$. The branched folding is regular. Therefore g is the quotient of S under $Aut(g)$. This is easily seen to be the group of isometries C of R^3 fixing the cube. Since by Theorem 3.12 $Aut(g)$ is isomorphic to the group $Aut(f)$ of the associated folding $f : S\backslash\Sigma \to \tau\backslash\tau^0$ where Σ (resp. τ^0) is the set of vertexes of K' (resp. τ), we can obtained a presentation for C as follows. The group C is isomorphic to $Aut(f)$ which is isomorphic to the quotient

$$\pi_1^m(\tau\backslash\tau^0)/f_\#(\pi_1(S\backslash\Sigma)).$$

But $\pi_1(S\backslash\Sigma)$ is generated by the meridians of Σ because S is simply connected. The image of the meridian of some b_σ when $dim(\sigma) = 0, 1, 2$ is , respectively, some conjugate of $(bc)^3$, $(ca)^2$, $(ab)^4$ in

$$\pi_1^m(\tau\backslash\tau^0) = \mid a,b,c : a^2 = b^2 = c^2 = 1 \mid$$

where a, b, c, are the boundary generators of $\tau\backslash\tau^0$. Thus

$$C = \mid a,b,c : a^2 = b^2 = c^2 = 1; (ab)^4 = (bc)^3 = (ca)^2 = 1 \mid$$

Every transitive representation ω from

$$\pi_1^m(\tau\backslash\tau^0) = \mid a,b,c : a^2 = b^2 = c^2 = 1 \mid$$

into Σ_n gives rise to a 3-colored triangulation of the total space $M(\omega)$ of the branched folding $g : M(\omega) \to \tau$ defined by the monodromy ω. The space $M(\omega)$ is an unbounded 2-manifold iff $\partial_m M(\omega)$ is empty and this happens iff the permutations $\omega(a)$, $\omega(b)$, $\omega(c)$ fail to fix any index. Hence $\omega(a)$, $\omega(b)$, $\omega(c)$ must be formed by m 2-cycles. Hence n must be even: $n = 2m$. As a geometric consequence each of $\omega(ab)$, $\omega(bc)$, $\omega(ca)$ must be the product of an even number of cycles, since powers of ab, bc, ca lift to meridians of vertexes of $M(\omega)$. Say that $\omega(ab)$, $\omega(bc)$, $\omega(ca)$ are, respectively, the product of 2α, 2β, 2γ cycles. Then the number of vertexes (edges, faces) of $M(\omega)$ is $\alpha+\beta+\gamma$ (resp. $3m$, $2m$) . Hence the Euler characteristic of $M(\omega)$ is

$$\chi(M(\omega)) = \alpha + \beta + \gamma - m.$$

Therefore $M(\omega)$ is S^2 iff $\alpha + \beta + \gamma = m + 2$.

Example 5.4. $\omega(a) = (12)(34)$, $\omega(b) = (13)(24)$, $\omega(c) = (14)(23)$. Then $\omega(ab) = (14)(23)$, $\omega(bc) = (12)(34)$, $\omega(ca) = (13)(24)$. Thus $\chi(M(\omega)) = 1$. Therefore $M(\omega)$ is the projective plane RP^2 and ω gives rise to a 4-sheeted branched folding $g : RP^2 \to \tau$ and to a 3-colored triangulation of RP^2 having four 2-simplexes. The reader can found it easily.

6. Applications to 3-manifolds

It was proved in [9] and [16], independently, that every closed, oriented 3-manifold M^3 is a 3-fold covering of S^3 branched over a knot. More concretely, given such M^3 there exist a colored knot (L,ω) such that $M(L,\omega)$ is homeomorphic to M^3. In fact, there are infinitely many such colored knots providing the same 3-manifold M^3. Among them there are also infinitely many colored knots such that at every crossing the three colors are used (see, for instance, [10]). Call such a particular colored knot or link *special.* We now consider a move (see [14] or [15]) that projects the crossing at the ball Q in Figure 4 onto the plane Π in which the knot is depicted. In this way the crossing is converted in two segments lying upon Π and crossing themselves. This move, applied at every crossing of a special colored knot or link, produces a *colored tetravalent planar graph* such that at each crossing the three colors appear and the repeated color lies on opposite edges. Since this move does not change the covering manifold, we have proved:

Theorem 6.1. *Every closed, oriented* 3*-manifold* M^3 *is a* 3*-fold covering of* S^3 *branched over a colored tetravalent planar graph.*

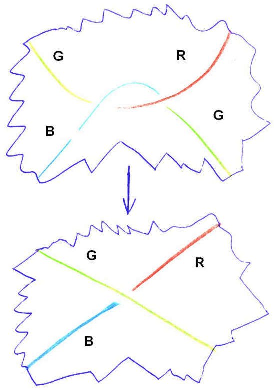

Figure 6: Move

Remark 6.2. Applying the move (see [14] or [15]) of Figure 6 we can even suppose that the regions of the tetravalent graph have three or more edges.

This can be generalized to every open orientable 3-manifold using [19]. The branching set is a colored tetravalent planar graph lying on $S^2 \backslash E$ where E is a subset of a Cantor set $c \subset S^2$. As an example, take the Fox string and project it upon a plane converting it into a colored tetravalent planar graph lying on $S^2 \backslash \{\infty\}$.

Folding S^3 along the sphere upon which lies the tetravalent graph we obtain:

Corollary 6.3. *For every closed, oriented* 3*-manifold* M^3 *there is a* 6 *to* 1 *folding* $f : M^3 \to B^3$ *over the* 3*-ball branched over a tetravalent planar graph on the boundary of* B^3*.*

Problem 6.4. *Generalize this to non-orientable* 3*-manifolds.*

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