Uniform Bounds on Complexity and Transfer of Global Properties of Nash Functions

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Abstract

We show that the *complexity* of semialgebraic sets and mappings can be used to parametrize Nash sets and mappings by *Nash families*. From this we deduce *uniform bounds* on the complexity of Nash functions that lead to first-order descriptions of many properties of Nash functions and a good behaviour under real closed field extension (e.g. primary decomposition). As a distinguished application, we derive the solution of the *extension* and *global equations* problems over *arbitrary real closed fields*, in particular over the field of real algebraic numbers. This last fact and a technique of change of base are used to prove that the *Artin-Mazur description* holds for abstract Nash functions on the real spectrum of any commutative ring, and solve extension and global equations in that abstract setting. To complete the view, we prove the *idempotency of the real spectrum* and an abstract version of the *separation* problem. We also discuss the conditions for the rings of abstract Nash functions to be *noetherian* .

Keywords: Nash functions, real spectrum.

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1 Introduction

Let Ω be a Nash submanifold of an affine space $\mathbb{R}^n$. Several global results concerning Nash functions on Ω were obtained (for compact Ω) in [CRS1] and (for noncompact Ω) in [CRS2, CS]. Before stating these results, we introduce some notations and a definition. Let $\mathcal{N}_\Omega$ denote the sheaf of Nash functions on Ω, $\mathcal{N}(\Omega)$ (resp. $\mathcal{O}(\Omega)$) the ring of Nash (resp. real analytic) functions on Ω. A sheaf $\mathcal{I}$ of ideals of $\mathcal{N}_\Omega$ is said to be *finite* if there is a finite open semialgebraic covering $\Omega = \bigcup_{i=1}^{\ell} U_i$ such that, for each i, $\mathcal{I}|_{U_i}$ is generated by finitely many Nash functions on U_i. (Concerning this, note that $\mathcal{N}(U_i)$ is noetherian.) This notion is stronger than *local finite generation*, by the finiteness of the covering; of course, both things are the same if Ω is compact.

Theorem 1 (Separation) *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{N}(\Omega)$. Then $\mathfrak{p}\mathcal{O}(\Omega)$ is prime.*

Theorem 2 (Global Equations) *Let $\mathcal{I}$ be a finite sheaf of ideals of $\mathcal{N}_\Omega$. Then $\mathcal{I}$ is generated by its global sections.*

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Theorem 3 (Extension) *Let I be an ideal of $\mathcal{N}(\Omega)$. Then every global section of the quotient sheaf $\mathcal{N}_\Omega/I\mathcal{N}_\Omega$ can be lifted to a Nash function on Ω.*

These results were obtained by transcendental methods. It is then classical to ask: what happens if $\mathbb{R}$ is replaced by an arbitrary real closed field R? It is also a general fact that one can use the Tarski-Seidenberg principle to transfer a statement from a real closed field to another if we can prove uniform bounds for the complexity of the semialgebraic objects whose existence is asserted by the statement, in function of the complexity of the data. In Section 2 we make precise this notion of complexity, which is strongly related to semialgebraic families. We also study in this section families of Nash manifolds and Nash mappings.

The first statement (Separation) has no immediate translation over a real closed field, since analytic functions are missing. Thus, for the moment, instead of looking for a generalization of the separation property, we study the following situation: Ω is an affine Nash manifold over a real closed field R, L is a real closed extension of R and Ω_L is the extension of Ω over L. We prove for the canonical homomorphism $\mathcal{N}(\Omega) \to \mathcal{N}(\Omega_L)$ some approximation results which look very much like the ones obtained for the morphism $\mathcal{N}(\Omega) \to \mathcal{O}(\Omega)$ in the case that Ω is over $\mathbb{R}$. For instance, if $\mathfrak{p}$ is a prime ideal of $\mathcal{N}(\Omega)$, then $\mathfrak{p}\mathcal{N}(\Omega_L)$ is prime. These results are explained in Section 3. They rely on the fact that, if $f_1, \ldots, f_n, g$ are Nash functions on Ω such that g belongs to the ideal generated by $f_1, \ldots, f_n$, then one can find Nash functions $h_1, \ldots, h_n$, with complexity bounded in function of the complexity of $f_1, \ldots, f_n, g$, such that $g = \sum_{i=1}^n h_i f_i$.

Concerning the other two statements (Global Equations and Extension), we have to understand what is the sheaf of Nash functions on a Nash submanifold Ω of R^n. We can define the ring of Nash functions $\mathcal{N}(U)$ on an open semialgebraic subset U of Ω as the ring of semialgebraic $\mathcal{C}^\infty$ functions (see [BCR, 2.2.9]), but these $\mathcal{N}(U)$'s do not form a sheaf for the usual topology on Ω. Let for instance $R = \mathbb{R}_{\mathrm{alg}}$ be the field of real algebraic numbers. The function with values 0 on $\mathbb{R}_{\mathrm{alg}} \cap (-\infty, \pi)$ and 1 on $\mathbb{R}_{\mathrm{alg}} \cap (\pi, +\infty)$ is locally Nash (for the usual topology on $\mathbb{R}_{\mathrm{alg}}$), but it is not a Nash function on $\mathbb{R}_{\mathrm{alg}}$. Instead, the rings $\mathcal{N}(U)$ obviously form a sheaf for the Grothendieck topology on the lattice of open semialgebraic subsets of Ω which is generated by the finite open semialgebraic coverings. This is the topology which is used for the definition of the semialgebraic spaces in [DK]. We shall call this topology the *semialgebraic topology*. We consider now the sheaf of Nash functions as a sheaf for the semialgebraic topology. The finite sheaves of ideals of $\mathcal{N}_\Omega$ correspond exactly to the sheaves of ideals which are locally (for the semialgebraic topology) generated by finitely many sections. Then the statements Global Equations and Extension make sense over any real closed field R. We prove them in Section 4. We also explain in more detail in the beginning of this section the relationship between the sheaves of Nash functions with respect to the usual topology and the semialgebraic topology, in the case $R = \mathbb{R}$.

The case of an arbitrary real closed field is not the ultimate generalization of Nash functions. One can define a sheaf $\mathcal{N}_A$ of abstract Nash functions over the real spectrum of any commutative ring A. The ringed space $(\mathrm{Spec}_{\mathrm{r}}(A), \mathcal{N}_A)$ obtained in this way can be regarded as the "real étale topos" of $\mathrm{Spec}(A)$ (see [Ro] or [Sc]). The definition of abstract Nash functions mimics the Artin-Mazur description of Nash functions. Let U be an open subset of $\mathrm{Spec_r}A$. Consider the diagram whose objects are the couples (B, σ), where B is an étale A-algebra and $\sigma : U \to \mathrm{Spec_r}(B)$ is a section of the local homeomorphism $\mathrm{Spec_r}(B) \to \mathrm{Spec_r}(A)$, and whose arrows from (B, σ) to (C, τ) are the morphisms of A-algebras $f : B \to C$ such that $\mathrm{Spec_r}(f) \circ \tau = \sigma$. Set $\mathcal{A}(U)$ to be the inductive limit of the étale A-algebras B over this filtered diagram. When $A = R[\mathtt{x}_1, \dots, \mathtt{x}_n]$ and U is the constructible open subset of $\mathrm{Spec_r}A$ associated to a semialgebraic open subset Ω of R^n, we have $\mathcal{A}(U) = \mathcal{N}(\Omega)$ (this is the Artin-Mazur description, cf. [BCR, 8.4.4]). In the general case, the sheaf $\mathcal{N}_A$ of abstract Nash functions on $\mathrm{Spec_r}A$ is defined to be the sheaf associated to the presheaf $\mathcal{A}$. It has been a question for a long time whether A has the *Artin-Mazur property*, that is, whether $\mathcal{N}_A(U) = \mathcal{A}(U)$ for every constructible open subset U of $\mathrm{Spec_r}(A)$. In Section 5 we prove that this holds true for every commutative ring. Indeed, we deduce it from the Extension theorem over the field of real algebraic numbers, using a technique of change of base. Thus, we have a very good description of the sheaf $\mathcal{N}_A$. The same technique of change of base is used in Section 6 to show Global Equations and Extension for *locally finitely generated* sheaves of ideals of $\mathcal{N}_A$. Since the real spectrum is compact, these are the abstract countepart of the finite sheaves on Nash manifolds. All these properties are related to another one known as the *idempotency of the real spectrum*. This property asserts that, if B is the ring of global sections of $\mathcal{N}_A$ (i.e. the ring of global abstract Nash functions on $\mathrm{Spec_r}(A)$), then the canonical morphism of ringed spaces $(\mathrm{Spec_r}(B), \mathcal{N}_B) \to (\mathrm{Spec_r}(A), \mathcal{N}_A)$ is an isomorphism (see also an extension of the idempotency problem for partially ordered rings in [SM]). It was claimed in [Ro] that the idempotency was true for every commutative ring A, but the proof contained a gap. On the other hand, it was proved in [Qu1] that the idempotency would give a rather direct proof of the Extension theorem for Nash manifolds over the reals, both compact and noncompact. In this discussion, [Qu1] shows also that the Artin-Mazur property implies the idempotency for arbitrary rings, and consequently we can now conclude (Section 7) that idempotency holds for every commutative ring. This is a big detour to prove the idempotency of the real spectrum, and in some sense goes in the unexpected direction, but so far there is no other approach that we know.

Let us finally say a word concerning separation. In [Qu2] there is an abstract version of separation for rings that satisfy some sensible conditions: A must be a noetherian excellent ring, and all constructible sets in $\mathrm{Spec_r}(A)$ must have finitely many connected components. However, once the preceding questions are settled, we realize that such conditions are needed for the ring B of global Nash functions to be noetherian, rather than for the formulation of separation. We review this formulation in Section 8, prove separation for an arbitrary commutative ring, and clarify when rings of abstract Nash functions are noetherian.

2 Complexity and families of Nash functions and manifolds

First of all, we make precise what we call the complexity of a semialgebraic object.

We say that *a semialgebraic subset S of R^k has complexity* $\leq n$ if it has a description of the following form:

$$S = \bigcup_{i=1}^{n} \bigcap_{j=1}^{n} \{x \in R^k \ ; \ f_{i,j}(x) ?_{i,j} 0\} \ ,$$

where the polynomials $f_{i,j}$ have degrees $\leq n$ and the symbols $?_{i,j}$ are either $=$ or $>$. Note that for every semialgebraic subset S of R^k there is a positive integer n such that S has complexity $\leq n$. The *complexity* of a semialgebraic subset S of R^k is the smallest positive integer n such that S is of complexity $\leq n$. This notion of complexity is strongly related to semialgebraic families. A *semialgebraic family* $\mathbf{F}$ *of subsets of* R^k *parametrized by the semialgebraic set* S is a semialgebraic subset of $S \times R^k$. For $t \in S$ we denote by $\mathbf{F}_t$ the *fiber*

$$\mathbf{F}_t = \{x \in R^k \ ; \ (t, x) \in \mathbf{F}\} \subset R^k \ .$$

If T is a semialgebraic subset of S, we denote by $\mathbf{F}|_T = \mathbf{F} \cap (T \times R^k)$ the restricted family parametrized by T.

The complexity defined above enjoys the following three properties.

1. The semialgebraic subsets of R^k of complexity $\leq n$ can be organized into a semialgebraic family. More precisely, there is a semialgebraic family $\mathbf{SA}_k^{\leq n}$ of subsets of R^k, parametrized by a semialgebraic set $M_k^{\leq n}$, whose fibers are all the semialgebraic subsets of complexity $\leq n$.

2. For every semialgebraic family $\mathbf{X}$ of subsets of R^k parametrized by $S \subset R^p$, there is a positive integer n such that every fiber $\mathbf{X}_t$ is of complexity $\leq n$.

3. If L is a real closed extension of R, then the extension $(\mathbf{SA}_k^{\leq n})_L \to (M_k^{\leq n})_L$ of the family in Property 1 is such that its fibers are all the semialgebraic subsets of L^k of complexity $\leq n$.

Proof of Properties 1 and 3. Given k and n, there are 2^{n^2} possible choices for the symbols $?_{i,j}$. For each choice, the coefficients of the n^2 polynomials $f_{i,j}$ lie in an affine space of dimension $N(k,n) = n^2 \times \binom{k+n}{n}$. We take the semialgebraic family $\mathbf{SA}_k^{\leq n}$ to be parametrized by the semialgebraic set

$$M_k^{\leq n} = \{0, 1, \ldots, 2^{n^2} - 1\} \times R^{N(k,n)} \subset R^{1+N(k,n)} \ .$$

Property 3 is clear from the construction of $\mathbf{SA}_k^{\leq n} \to M_k^{\leq n}$. □

Proof of Property 2. The family $\mathbf{X}$ has a complexity n as a subset of R^{p+k}. It is then clear that every fiber has complexity $\leq n$. □

We can actually improve the family $\mathbf{SA}_k^{\leq n} \to M_k^{\leq n}$ in order to turn it into the *universal* semialgebraic family of subsets of R^k of complexity $\leq n$. The equivalence relation $s \sim t$ defined by $\left(\mathbf{SA}_k^{\leq n}\right)_s = \left(\mathbf{SA}_k^{\leq n}\right)_t$ is a semialgebraic subset of $M_k^{\leq n} \times M_k^{\leq n}$. Hence (see [vdD] p. 94), $M_k^{\leq n}$ has a semialgebraic subset of representatives $UM_k^{\leq n}$, and the map

$M_k^{\leq n} \to UM_k^{\leq n}$ which sends each element to the representative of its class is semialgebraic. The restriction $\mathbf{SA}_k^{\leq n}|UM_k^{\leq n}$ has the following universal property: *for every semialgebraic family $\mathbf{X} \to S$ of subsets of R^k of complexities $\leq n$, there exists a unique semialgebraic map $f : S \to UM_k^{\leq n}$ such that $\mathbf{X}$ is the pullback of $\mathbf{SA}_k^{\leq n}|UM_k^{\leq n}$ along f.*

The definition of the complexity of semialgebraic subsets of R^k is somewhat arbitrary. The important point is that this complexity satisfies the three properties above.

Proposition 4 *Any two measures of complexity c_1 and c_2 satisfying these three properties are essentially equivalent, in the following sense: Given k, there are two functions φ_1 and φ_2 from positive integers to positive integers such that, for every real closed field R and every semialgebraic subset S of R^k, one has $c_1(S) \leq \varphi_1(c_2(S))$ and $c_2(S) \leq \varphi_2(c_1(S))$.*

Proof. Let $\mathbb{R}_{\mathrm{alg}}$ be the field of real algebraic numbers. By Property 1, given n and k, there are semialgebraic families $\mathbf{SA}_k^{c_i \leq n} \to M_k^{c_i \leq n}$ whose fibers are all the semialgebraic subsets S of $\mathbb{R}_{\mathrm{alg}}^k$ such that $c_i(S) \leq n$, for $i = 1, 2$. By Property 2, for every positive integer n there is a positive integer $\varphi_1(n)$ such that, for every $s \in M_k^{c_2 \leq n}$ there is $t \in M_k^{c_1 \leq \varphi_1(n)}$ such that the fibers at s and t coincide:

$$\left(\mathbf{SA}_k^{c_2 \leq n}\right)_s = \left(\mathbf{SA}_k^{c_1 \leq \varphi_1(n)}\right)_t .$$

Moreover, t can be chosen as a semialgebraic function of s, i.e. $t = \gamma(s)$, where $\gamma : M_k^{c_2 \leq n} \to M_k^{c_1 \leq \varphi_1(n)}$ is a semialgebraic map (see [vdD, p. 94]). The family $\mathbf{SA}_k^{c_1 \leq \varphi_1(n)}$ is the pullback of the family $\mathbf{SA}_k^{c_2 \leq n}$ along γ, and this remains true after extension from $\mathbb{R}_{\mathrm{alg}}$ to every other real closed field R. By Property 3, we deduce that for every semialgebraic subset S of R^k such that $c_2(S) \leq n$, we have $c_1(S) \leq \varphi_1(n)$. $\square$

We identify a semialgebraic map f from a semialgebraic subset of R^k to R^p with its graph $\Gamma(f)$, which is a semialgebraic subset of $R^k \times R^p$. We say that f has complexity $\leq n$ if its graph $\Gamma(f)$ has complexity $\leq n$. If $\mathbf{X}$ (resp. $\mathbf{Y}$) is a semialgebraic family of subsets of R^k (resp. R^p) parametrized by S, a semialgebraic map $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ such that the composition $\mathbf{X} \to \mathbf{Y} \to S$ is equal to the projection $\mathbf{X} \to S$ induces a semialgebraic map $\mathbf{f}_t : \mathbf{X}_t \to \mathbf{Y}_t$ for every $t \in S$. We then call $\mathbf{f}$ *a semialgebraic family of maps parametrized by* S, and $\mathbf{f}_t$ is the *fiber* of the family at t. If T is a subset of S, we denote by $\mathbf{f}|_T : \mathbf{X}|_T \to \mathbf{Y}|_T$ the semialgebraic family of maps parametrized by T which is the restriction of $\mathbf{f}$. Clearly, there is a uniform bound for the complexity of the fibers of a semialgebraic family of maps.

In [BCR] there is another notion of complexity for continuous semialgebraic functions on an open semialgebraic subset U of R^k, which we will call here "algebraic complexity". A function $f : U \to R$ *has algebraic complexity* $\leq q$ if there is a nonzero polynomial P in $k+1$ variables of total degree $\leq q$ such that $P(x, f(x)) = 0$ for every $x \in U$. It is shown in [BCR] that the continuous semialgebraic functions $U \to \mathbb{R}$ (with U fixed) with algebraic complexity not greater than a given n are the fibers of a semialgebraic family. On the other hand, the fibers $\mathbf{f}_t : U \to R$ of a semialgebraic family $\mathbf{f} : S \times U \to S \times R$ of continuous functions have a bounded algebraic complexity. Hence, algebraic complexity is essentially equivalent to our complexity (for a fixed U, or even for all open semialgebraic subsets U of R^k of complexity $\leq p$, p fixed). See [Ra1, Ra2] for interesting uniform bounds in terms of the algebraic complexity of Nash functions.

Now we turn to families of Nash manifolds and Nash maps. All we need to know can be summarized in the following result.

Theorem 5 *Let $\mathbf{\Omega} \subset S \times R^k$, $\mathbf{\Lambda} \subset S \times R^p$ and $\mathbf{g} : \mathbf{\Omega} \to \mathbf{\Lambda}$ be semialgebraic families parametrized by S. The set T of all parameters $t \in S$ such that $\mathbf{\Omega}_t$ is a Nash submanifold of R^k, $\mathbf{\Lambda}_t$ is a Nash submanifold of R^p and $\mathbf{g}_t$ is a Nash map from $\mathbf{\Omega}_t$ to $\mathbf{\Lambda}_t$ is a semialgebraic subset of S. Moreover, there is a finite Nash stratification $T = \bigcup_{i=1}^{\ell} T_i$ such that, for $i = 1, \ldots, \ell$, $\mathbf{\Omega}|_{T_i}$ is a Nash submanifold of $T_i \times R^k$, $\mathbf{\Lambda}|_{T_i}$ is a Nash submanifold of $T_i \times R^p$, $\mathbf{g}|_{T_i} : \mathbf{\Omega}|_{T_i} \to \mathbf{\Lambda}|_{T_i}$ is a Nash map and the projections $\mathbf{\Omega}|_{T_i} \to T_i$ and $\mathbf{\Lambda}|_{T_i} \to T_i$ are submersions.*

Proof. We use in this proof the constructible set $\widetilde{S}$ of the real spectrum, and the fibers of semialgebraic families at points $\alpha \in \widetilde{S}$. For all these notions, we refer to [BCR, Chap. 7].

The fact that $\mathbf{\Omega}_t$, $\mathbf{\Lambda}_t$ and $\mathbf{g}_t$ are of class $\mathcal{C}^r$ can be expressed by a first-order formula of the language of ordered fields with parameters in R. For instance, in order to say that $\mathbf{\Omega}_t$ is a $\mathcal{C}^r$ submanifold of R^k, one has to say that, for every $a = (a_1, \ldots, a_k)$ in $\mathbf{\Omega}_t$, there exists $\epsilon > 0$ such that, for some $d \leq k$ and up to a permutation of the coordinates in R^k, $\mathbf{\Omega}_t \cap \prod_{i=1}^{k}(a_i - \epsilon, a_i + \epsilon)$ is the graph of a $\mathcal{C}^r$ function from $\prod_{i=1}^{d}(a_i - \epsilon, a_i + \epsilon)$ to R^{k-d}. Hence, the subset $S_r \subset S$ of those t's such that $\mathbf{\Omega}_t$, $\mathbf{\Lambda}_t$ and $\mathbf{g}_t$ are of class $\mathcal{C}^r$ is semialgebraic, and $\mathbf{\Omega}_\alpha$, $\mathbf{\Lambda}_\alpha$ and $\mathbf{g}_\alpha$ are Nash if and only if α belongs to the intersection F of all $\widetilde{S_r}$, which is closed for the constructible topology on $\widetilde{S}$.

Now take $\alpha \in F$. There is a finite open semialgebraic covering $\mathbf{\Omega}_\alpha = \bigcup_{i=1}^{\ell}(\mathbf{U}_i)_\alpha$ and, for each i, there is an open semialgebraic subset $(\mathbf{V}_i)_\alpha \subset k(\alpha)^d$, a Nash map $\varphi_i : (\mathbf{V}_i)_\alpha \to k(\alpha)^{k-d}$ and a linear automorphism $\sigma_{k(\alpha)}$ of $k(\alpha)^k$ induced by a permutation of the coordinates such that $(\mathbf{U}_i)_\alpha$ is the image by $\sigma_{k(\alpha)}$ of the graph of φ_i ([BCR, 9.3.10]). We have a similar situation for $\mathbf{\Lambda}_\alpha$, and, for each i, the composition $\mathbf{g}_\alpha \circ \sigma_{k(\alpha)} \circ (\mathrm{Id}, \varphi_i)$ is a Nash map from $(\mathbf{V}_i)_\alpha$ to $k(\alpha)^p$. Applying [BCR, 8.10.3], we obtain a Nash submanifold $T^\alpha \subset S$ such that:

- $\alpha \in \widetilde{T^\alpha}$,
- the sets $\mathbf{U}_i|_{T^\alpha}$ form an open semialgebraic covering of $\mathbf{\Omega}|_{T^\alpha}$,
- for each i, $\mathbf{V}_i|_{T^\alpha}$ is a semialgebraic open subset of $T^\alpha \times R^d$,
- for each i, there is a Nash map $\Phi_i : \mathbf{V}_i|_{T^\alpha} \to R^{k-d}$ such that the fiber $(\Pi_i, \Phi_i)_\alpha$ is equal to φ_i, where $\Pi_i : \mathbf{V}_i|_{T^\alpha} \to T^\alpha$ is the projection,
- for each i, $\mathbf{U}_i|_{T^\alpha}$ is the image of the graph of Φ_i by $\mathrm{Id} \times \sigma : T^\alpha \times R^k \to T^\alpha \times R^k$, where $\sigma : R^k \to R^k$ is the linear automorphism defined by the same permutation of coordinates as $\sigma_{k(\alpha)}$.

Moreover, we can have a similar situation for $\mathbf{\Lambda}|_{T^\alpha}$, and also:

- for each i, $\mathbf{g}|_{T^\alpha} \circ (\mathrm{Id} \times \sigma) \circ (\Pi_i, \Phi_i)$ is a Nash map from $\mathbf{V}_i|_{T^\alpha}$ to $T^\alpha \times R^p$.

All of this implies that $\mathbf{\Omega}|_{T^\alpha}$, $\mathbf{\Lambda}|_{T^\alpha}$ and $\mathbf{g}|_{T^\alpha}$ are Nash, and that the projections $\mathbf{\Omega}|_{T^\alpha} \to T^\alpha$ and $\mathbf{\Lambda}|_{T^\alpha} \to T^\alpha$ are submersions.

It follows that the neighborhood $\widetilde{T^\alpha}$ of α for the constructible topology is contained in F. Hence, F is constructible, and $F = \widetilde{S_r}$ for r large enough. We deduce that T is semialgebraic and $F = \widetilde{T}$. By compactness, F can be covered by finitely many $\widetilde{T^\alpha}$'s, and T can be covered by finitely many T^α's. We can modify these T^α's to obtain a Nash stratification of T. $\square$

Corollary 6 (Uniform Artin-Mazur description) *Given positive integers p, c, ℓ, there exist positive integers k, d satisfying the following property. Let $\Omega \subset R^p$ be an affine Nash manifold of complexity $\leq c$ and $f_1, \ldots, f_\ell$ Nash functions on Ω of complexities $\leq c$. Then there exist an algebraic subset $V \subset R^k$ whose ideal is generated by $\leq d$ polynomials of degrees $\leq d$, a Nash embedding $\sigma : \Omega \to \mathrm{Reg}(V)$ of complexity $\leq d$ and polynomial functions $P_1, \ldots, P_\ell : R^k \to R$ of degrees $\leq d$ such that $f_i = P_i \circ \sigma$ for $i = 1, \ldots, \ell$.*

Proof. By Theorem 5, we can put in a semialgebraic family the Nash data Ω and $f_1, \ldots, f_\ell$ satisfying the bounds on the complexity. So we have semialgebraic families $\mathbf{\Omega}$ and $\mathbf{f}_i$ for $i = 1, \ldots, \ell$, parametrized by a semialgebraic set $T \subset R^q$. Moreover, there is a finite Nash stratification $T = \bigcup_{j=1}^r T_j$ such that, for each j, $\mathbf{\Omega}|_{T_j}$ is a Nash submanifold of $T_j \times R^p$ and the $\mathbf{f}_i|_{T_j} : \mathbf{\Omega}|_{T_j} \to T_j \times R$ are Nash maps. Applying the Artin-Mazur theorem ([BCR, 8.4.4]) for each j, we obtain an algebraic subset $\mathbf{V}_j \subset R^q \times R^k$, an open Nash imbedding $\mathbf{\Sigma}_j : \mathbf{\Omega}|_{T_j} \to \mathrm{Reg}(\mathbf{V}_j)$ and polynomial mappings $\mathbf{P}_{j,i} : R^q \times R^k \to R^q \times R$, such that the following diagram commutes,

$$\begin{tikzcd}
R^q \times R \arrow[ddd] & & & \\
& \mathrm{Reg}(\mathbf{V}_j) & \subset \mathbf{V}_j \subset & R^q \times R^k \arrow[ulll, "\mathbf{P}_{j,i}"'] \arrow[d, "\Pi"] \\
& \mathbf{\Omega}|_{T_j} \arrow[u, "\mathbf{\Sigma}_j"'] \arrow[uul, "\mathbf{f}_i|_{T_j}"] \arrow[rr, hook] & & R^q \times R^p \arrow[dlll] \\
R^q & & &
\end{tikzcd}$$

where $\Pi : R^q \times R^k = (R^q \times R^p) \times R^{k-p} \to R^q \times R^p$ is the projection on the first $q + p$ coordinates. Note that we can take the same k for all j. Since the projection $\mathbf{\Omega}|_{T_j} \to T_j$ is a submersion, for every $t \in T_j$ the map $(\mathbf{\Sigma}_j)_t$ is an open imbedding into the regular locus of $(\mathbf{V}_j)_t$. Let $E_{j,1}, \ldots, E_{j,d_j}$ be a system of generators of the ideal of $\mathbf{V}_j$ in $R^q \times R^k$. We can take d to be the maximum of the d_j's, the degrees of the $E_{j,k}$'s, the degrees of the $\mathbf{P}_{j,i}$'s and the complexities of the $\mathbf{\Sigma}_j$'s. $\square$

3 Rings of Nash functions under base field extension

We will use the following well known facts about polynomials. Let $\underline{P}_1, \dots, \underline{P}_n$ and $\underline{Q}$ be elements of $R[\mathtt{x}_1, \dots, \mathtt{x}_k]^q$ with components of degrees $\leq d$.

Fact 1. There is a generating system for the ideal $((\underline{P}_1, \dots, \underline{P}_n) : \underline{Q})$ consisting of polynomials in $R[\mathtt{x}_1, \dots, \mathtt{x}_k]$ of degrees $\leq k(qd)^{2^{k-1}}$ ([Se, § 55]).
Fact 2. If $\underline{Q}$ belongs to the submodule generated by $\underline{P}_1, \dots, \underline{P}_n$, we can find polynomials $H_1, \dots, H_n$ in $R[\mathtt{x}_1, \dots, \mathtt{x}_k]$ of degrees $\leq k(qd)^{2^{k-1}}$ such that $\underline{Q} = H_1\underline{P}_1 + \cdots + H_n\underline{P}_n$ ([Se, § 57]).

Theorem 7 *Given positive integers p, c, n, q there exists a positive integer e satisfying the following property. Let $\Omega \subset R^p$ be an affine Nash manifold of complexity $\leq c$ and $\underline{f}_1, \dots, \underline{f}_n, \underline{g}$ elements of the free module of finite type $\mathcal{N}(\Omega)^q$ whose components have complexities $\leq c$. If $\underline{g}$ belongs to the submodule generated by $\underline{f}_1, \dots, \underline{f}_n$, there are Nash functions $h_1, \dots, h_n$ of complexities $\leq e$ such that $\underline{g} = \sum_{i=1}^n h_i \underline{f}_i$.*

Proof. We have an algebraic subset $V \subset R^k$ and an imbedding $\sigma : \Omega \to \mathrm{Reg}(V)$, together with tuples $\underline{P}_1, \dots, \underline{P}_n, \underline{Q} \in R[\mathtt{x}_1, \dots, \mathtt{x}_k]^q$ such that $\underline{g} = \underline{Q} \circ \sigma$ and $\underline{f}_i = \underline{P}_i \circ \sigma$ (this is Artin-Mazur). Moreover, applying Corollary 6, we can bound k, the number and the degrees of the equations $E_1, \dots, E_r$ for V, the complexity of σ, and the degrees of the polynomials $\underline{P}_1, \dots, \underline{P}_n, \underline{Q}$ by functions of p, c, n, q. Let A be the sum of the submodule of $R[\mathtt{x}_1, \dots, \mathtt{x}_k]^q$ generated by $\underline{P}_1, \dots, \underline{P}_n$ and $E_1 R[\mathtt{x}_1, \dots, \mathtt{x}_k]^q + \cdots + E_r R[\mathtt{x}_1, \dots, \mathtt{x}_k]^q$. Applying Fact 1 above, we get a bound for the degree of the sum of squares H of a system of generators of the ideal $(A : \underline{Q})$. Applying Fact 2, we get bounds for the degrees of polynomials $H_1, \dots, H_n$ such that

$$H\underline{Q} = H_1\underline{P}_1 + \cdots + H_n\underline{P}_n \qquad \mathrm{mod}(E_1, \dots, E_r)\ .$$

We claim that $H \circ \sigma$ is invertible on Ω if $\underline{g}$ belongs to the submodule generated by $\underline{f}_1, \dots, \underline{f}_n$. Indeed, for $x \in \Omega$, the imbedding σ induces an isomorphism from the henselization of the local ring of germs of regular functions $\mathcal{R}_{V,\sigma(x)}$ to the ring of germs of Nash functions $\mathcal{N}_{\Omega,x}$. Hence, by faithful flatness, the image of $\underline{Q}$ in $\mathcal{R}^q_{V,\sigma(x)}$ belongs to the submodule generated by $\underline{P}_1, \dots, \underline{P}_n$. Thus, $(A : \underline{Q})\mathcal{R}_{V,\sigma(x)} = \mathcal{R}_{V,\sigma(x)}$, and some generator of $(A : \underline{Q})$ is a unit in $\mathcal{R}_{V,\sigma(x)}$, that is, does not vanish at $\sigma(x)$, so that H does not vanish either. This proves the claim.

Consequently, we have

$$\underline{g} = \sum_{i=1}^{n} \left(\frac{H_i}{H} \circ \sigma \right) \underline{f}_i\ .$$

We conclude by noting that we have a bound for the complexity of $h_i = (H_i/H) \circ \sigma$ in function of p, c, n, q. □

Theorem 7 is useful because it allows us to write "g belongs to the ideal generated by $f_1, \dots, f_n$", where $f_1, \dots, f_n$ are given and the complexity of the Nash function g is bounded, as a first-order formula of the language of real closed fields. One can then use the Tarski-Seidenberg principle to transfer properties concerning ideals of Nash functions up and down extensions of real closed fields.

The next results concern the relations between the ring of Nash functions on a Nash manifold Ω over R and the ring of Nash functions on the extension Ω_L of Ω over a real closed extension L of R. There is a canonical morphism $\mathcal{N}(\Omega) \to \mathcal{N}(\Omega_L)$ which sends a Nash function $f : \Omega \to R$ to its extension $f_L : \Omega_L \to L$. The ring $\mathcal{N}(\Omega)$ can be regarded as a subring of $\mathcal{N}(\Omega_L)$ via this morphism.

Proposition 8 *The ring $\mathcal{N}(\Omega_L)$ is faithfully flat over $\mathcal{N}(\Omega)$.*

Proof. Set $A = \mathcal{N}(\Omega)$ and $B = \mathcal{N}(\Omega_L)$. Let $f_1, \ldots, f_q$ be Nash functions on Ω. Let $M \subset A^q$ be the module of relations between $f_1, \ldots, f_q$. Let $\underline{G}_1, \ldots, \underline{G}_s$ be a generating family of the module M. By Theorem 7, given a positive integer d, there is a positive integer $c(d)$ such that, for every relation $\underline{H}$ between $f_1, \ldots, f_q$ whose coordinates have complexity $\leq d$, there are Nash functions $\lambda_1, \ldots, \lambda_s$ on Ω, of complexity $\leq c(d)$, such that $\underline{H} = \sum_{i=1}^{s} \lambda_i \underline{G}_i$. By Tarski-Seidenberg, it follows that the same property holds over L: for every relation $\underline{H}$ between $f_{1,L}, \ldots, f_{q,L}$ whose coordinates have complexity $\leq d$, there are Nash functions $\lambda_1, \ldots, \lambda_s$ on Ω_L, of complexity $\leq c(d)$, such that $\underline{H} = \sum_{i=1}^{s} \lambda_i \underline{G}_{i,L}$. This shows that the morphism $A \to B$ is flat. The maximal ideals of A correspond to points of Ω. Hence, they are contractions of maximal ideals of B. It follows that $A \to B$ is faithfully flat. $\square$

Actually, it is true that the morphism $\mathcal{N}(\Omega) \to \mathcal{N}(\Omega_L)$ is a regular morphism of excellent rings. We omit the argument as this is not needed in the sequel.

Proposition 9 *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{N}(\Omega)$. Then $\mathfrak{p}\mathcal{N}(\Omega_L)$ is a prime ideal of $\mathcal{N}(\Omega_L)$.*

Proof. Set again $A = \mathcal{N}(\Omega)$ and $B = \mathcal{N}(\Omega_L)$. Assume $\mathfrak{p} = (f_1, \ldots, f_n)$ and suppose that $\mathfrak{p}B$ is not prime. Then there are Nash functions g' and h' on Ω_L which do not belong to the ideal $(f_{1,L}, \ldots, f_{n,L})$ such that their product $g'h'$ belongs to $(f_{1,L}, \ldots, f_{n,L})$. Let d be the maximum of the complexities of g' and h'. By Theorem 7 and Tarski-Seidenberg, there are Nash functions g and h on Ω of complexities $\leq d$ which do not belong to $\mathfrak{p}$ such that their product gh belongs to $\mathfrak{p}$. This contradicts the fact that $\mathfrak{p}$ is prime. $\square$

We can generalize the preceding proposition to primary decompositions. The proof is quite the same as the proof of the corresponding result for $\mathcal{N}(\Omega) \to \mathcal{O}(\Omega)$ in [CRS1], so we do not repeat it.

Proposition 10 *Let M be a finitely generated $\mathcal{N}(\Omega)$-module, $N \subset M$ a submodule and $N = N_1 \cap \cdots \cap N_r$ a primary decomposition of N in M with associated prime ideals $\mathfrak{p}_i = \sqrt{N_i} \subset \mathcal{N}(\Omega)$. Then*

$$\mathcal{N}(\Omega_L) \otimes_{\mathcal{N}(\Omega)} N = (\mathcal{N}(\Omega_L) \otimes_{\mathcal{N}(\Omega)} N_1) \cap \cdots \cap (\mathcal{N}(\Omega_L) \otimes_{\mathcal{N}(\Omega)} N_r)$$

is a primary decomposition of $\mathcal{N}(\Omega_L) \otimes_{\mathcal{N}(\Omega)} N$ *in* $\mathcal{N}(\Omega_L) \otimes_{\mathcal{N}(\Omega)} M$*, with associated prime ideals*

$$\mathfrak{p}_i \mathcal{N}(\Omega_L) = \sqrt{\mathcal{N}(\Omega_L) \otimes_{\mathcal{N}(\Omega)} N_i}\ .$$

We have also a result of approximation of Nash maps between Nash manifolds.

Proposition 11 *Let* $\Lambda \subset R^\ell$ *be another Nash submanifold. Let*

$$\begin{array}{rcl} I_1, \ldots, I_m, I_1^*, \ldots, I_n^* & \subset & \mathcal{N}(\Omega) \\ J_1, \ldots, J_m, J_1^*, \ldots, J_n^* & \subset & \mathcal{N}(\Lambda) \end{array}$$

be ideals of Nash functions. For every Nash mapping $\sigma : \Omega_L \to \Lambda_L$*, there is a Nash mapping* $\tau : \Omega \to \Lambda$ *such that*

$$\begin{array}{rcl} I_k \mathcal{N}(\Omega_L) \supset J_k \mathcal{N}(\Lambda_L) \circ \sigma & \Leftrightarrow & I_k \supset J_k \circ \tau\ ,\ 1 \le k \le m\ ; \\ I_k^* \mathcal{N}(\Omega_L) \subset J_k^* \mathcal{N}(\Lambda_L) \circ \sigma & \Leftrightarrow & I_k^* \subset J_k^* \circ \tau\ ,\ 1 \le k \le n\ . \end{array}$$

Proof. Tarski-Seidenberg and Theorem 7. □

4 Transfer of Global Equations and Extension

We recall that the semialgebraic topology on a Nash submanifold Ω of R^n is the Grothendieck topology on the lattice of open semialgebraic subsets of Ω whose coverings are generated by finite open semialgebraic coverings. We regard the sheaf of Nash functions on Ω as a sheaf for the semialgebraic topology. We denote this sheaf by $\mathcal{N}_\Omega^{\mathrm{sa}}$. Equivalently, we can regard $\mathcal{N}_\Omega^{\mathrm{sa}}$ as a sheaf on the space $\widetilde{\Omega} = \mathrm{Spec}_\mathrm{r}(\mathcal{N}(\Omega))$. Indeed, we have an isomorphism $U \mapsto \widetilde{U}$ between the lattice of open semialgebraic subsets of Ω and the lattice of constructible open subsets of the spectral space $\widetilde{\Omega}$. Hence, the category of sheaves over $\widetilde{\Omega}$ is equivalent to the category of sheaves for the semialgebraic topology. See [BCR, 8.8] for the description of the sheaf of Nash functions on $\widetilde{\Omega}$.

In the case of $R = \mathbb{R}$ we have two different sheaves of Nash functions: $\mathcal{N}_\Omega$ with respect to the usual euclidean topology, and $\mathcal{N}_\Omega^{\mathrm{sa}}$ with respect to the semialgebraic topology. Let us examine the relations between these two sheaves. The homomorphic inclusion j of the lattice of open semialgebraic subsets of Ω into the lattice of all open subsets of Ω induces a direct image functor j_* from the category of sheaves with respect to the euclidean topology to the category of sheaves with respect to the semialgebraic topology, and an inverse image functor j^* in the other direction, left adjoint to j_*. The following facts are easy to prove:

- $j^* \circ j_*$ is isomorphic to the identity, since the open semialgebraic sets form a basis of the usual topology.
- $j_*(\mathcal{N}_\Omega) = \mathcal{N}_\Omega^{\mathrm{sa}}$.
- The functor j_* induces a bijection between finite sheaves of ideals of $\mathcal{N}_\Omega$ and sheaves of ideals of $\mathcal{N}_\Omega^{\mathrm{sa}}$ which are locally (with respect to the semialgebraic topology) finitely generated.

Taking into account this last fact, a sheaf $\mathcal{I}$ of ideals of $\mathcal{N}_\Omega^{\mathrm{sa}}$ will be called *finite* if it is locally (with respect to the semialgebraic topology) generated by finitely many Nash functions. This means that there is a finite open semialgebraic covering $\Omega = \bigcup_{i=1}^{\ell} U_i$ and finitely generated ideals $I_i \subset \mathcal{N}(U_i)$ such that $\mathcal{I}|_{U_i}$ is generated by I_i, for $i = 1, \ldots, \ell$. Since the rings of Nash functions on Nash manifolds over arbitrary real closed fields are noetherian ([Cu]), every ideal of $\mathcal{N}(U_i)$ is actually finitely generated.

The next fact is more difficult to prove.

- If $\mathcal{I}$ is a finite sheaf of ideals of $\mathcal{N}_\Omega$, the canonical morphism $\mathcal{N}_\Omega^{\mathrm{sa}}/j_*(\mathcal{I}) \to j_*(\mathcal{N}_\Omega/\mathcal{I})$ is an isomorphism.

In other words, every global section of $\mathcal{N}_\Omega/\mathcal{I}$ can be represented by Nash functions using a *finite* open semialgebraic covering. This is [CRS2, 2.3] in the case that $\mathcal{I}$ is radical. In the general case, the only proof we know is to apply Global Equations and Extension. Hence, it may seem a treachery to replace the euclidean topology by the semialgebraic topology in the formulation of the Extension property. But actually the crucial case for both Global Equations and Extension is that of a radical finite sheaf of ideals, as the general case follows rather formally.

Theorem 12 (Global Equations with bounds) *Given positive integers n and ℓ, there is a positive integer m such that the following property holds. Let*

- *Ω be a Nash submanifold of $\mathbb{R}^n$ of complexity $\leq \ell$,*
- *$\Omega = \bigcup_{i=1}^{\ell} U_i$ be an open semialgebraic covering, where the complexity of each U_i is $\leq \ell$,*
- *$f_{i,1}, \ldots, f_{i,\ell}$ $(1 \leq i \leq \ell)$ be Nash functions on U_i of complexities $\leq \ell$.*

If, for every $j \neq i$,

$$(f_{i,1}, \ldots, f_{i,\ell})\mathcal{N}(U_i \cap U_j) = (f_{j,1}, \ldots, f_{j,\ell})\mathcal{N}(U_i \cap U_j) ,$$

then there exist global Nash functions $g_1, \ldots, g_m$ on Ω of complexities $\leq m$ such that, for each i,

$$(f_{i,1}, \ldots, f_{i,\ell}) = (g_1, \ldots, g_m)\mathcal{N}(U_i) .$$

Proof. We can put the data Ω, U_i and $f_{i,\mu}$ satisfying the complexity bounds in semialgebraic families $\mathbf{\Omega}$, $\mathbf{U}_i$ and $\mathbf{f}_{i,\mu}$ parametrized by a semialgebraic set S. By Theorem 7, the subset T of all $t \in S$ such that, for every $i \neq j$,

$$((\mathbf{f}_{i,1})_t, \ldots, (\mathbf{f}_{i,\ell})_t)\mathcal{N}((\mathbf{U}_i)_t \cap (\mathbf{U}_j)_t) = ((\mathbf{f}_{j,1})_t, \ldots, (\mathbf{f}_{j,\ell})_t)\mathcal{N}((\mathbf{U}_i)_t \cap (\mathbf{U}_j)_t)$$

is semialgebraic. Moreover, there is a positive integer p such that, for every t in T and every $j \neq i$, there is a matrix $A_{i,j}$ of size $\ell \times \ell$ whose entries are Nash functions on $(\mathbf{U}_i)_t \cap (\mathbf{U}_j)_t$ with complexities $\leq p$ and such that

$$\begin{pmatrix} (\mathbf{f}_{i,1})_t \\ \vdots \\ (\mathbf{f}_{i,\ell})_t \end{pmatrix} = A_{i,j} \begin{pmatrix} (\mathbf{f}_{j,1})_t \\ \vdots \\ (\mathbf{f}_{j,\ell})_t \end{pmatrix} \quad \text{on } (\mathbf{U}_i)_t \cap (\mathbf{U}_j)_t .$$

We can choose the $A_{i,j}$'s in such a way that they form a semialgebraic family $\mathbf{A}_{i,j}$ parametrized by T. By Theorem 5, we can find a finite Nash stratification $T = \bigcup T_\nu$ such that, in restriction to each T_ν, $\mathbf{\Omega}$ is a Nash manifold, the $\mathbf{U}_i$'s are open, and the $\mathbf{f}_{i,\mu}$'s and the $\mathbf{A}_{i,j}$'s are Nash. Hence, these data define a finite sheaf of Nash ideals $\mathcal{I}_\nu$ on $\mathbf{\Omega}|_{T_\nu}$, for each ν. Since we are over the reals, we can apply Global Equations (Theorem 2) to get global Nash functions $\mathbf{g}_{\nu,1}, \ldots, \mathbf{g}_{\nu,m_\nu}$ on $\mathbf{\Omega}|_{T_\nu}$ which generate $\mathcal{I}_\nu$. We can take for m the maximum of the m_ν's and the complexities of all $\mathbf{g}_{\nu,\rho}$'s. □

Corollary 13 (Global Equations over an arbitrary real closed field) *Let Ω be a Nash submanifold of R^n. Every finite sheaf of ideals of $\mathcal{N}_\Omega^{\mathrm{sa}}$ is generated by its global sections.*

Proof. There is a first-order formula of the language of ordered fields which expresses the following fact:

> *Given a Nash submanifold Ω of complexity $\leq \ell$ of the affine n-space; given an open semialgebraic covering $\Omega = \bigcup_{i=1}^{\ell} U_i$, where the complexity of each U_i is $\leq \ell$; given, for $i = 1, \ldots, \ell$, Nash functions $f_{i,1}, \ldots, f_{i,\ell}$ on U_i of complexities $\leq \ell$ such that, for every $j \neq i$,*
>
> $$(f_{i,1}, \ldots, f_{i,\ell})\mathcal{N}(U_i \cap U_j) = (f_{j,1}, \ldots, f_{j,\ell})\mathcal{N}(U_i \cap U_j) ,$$
>
> *then there exist Nash functions $g_1, \ldots, g_m$ on Ω of complexities $\leq m$ such that, for each i,*
>
> $$(f_{i,1}, \ldots, f_{i,\ell}) = (g_1, \ldots, g_m)\mathcal{N}(U_i) .$$

Recall that the equality of ideals can be expressed by a first order formula since the complexity of the generators is bounded (Theorem 7).

By Theorem 12, for every n and ℓ there is an m such that the first order formula above holds true over $\mathbb{R}$. By Tarski-Seidenberg, it holds true over every real closed field R. This proves Global Equations for an arbitrary real closed field R. □

Applying the same method as for Global Equations, we obtain the corresponding results for Extension. We omit the proofs to avoid repetition.

Theorem 14 (Extension with bounds) *Given positive integers n and ℓ, there is a positive integer m such that the following property holds. Let*

- *Ω be a Nash submanifold of $\mathbb{R}^n$ of complexity $\leq \ell$,*
- *$f_1, \ldots, f_\ell : \Omega \to R$ be Nash functions of complexities $\leq \ell$,*
- *$\Omega = \bigcup_{i=1}^{\ell} U_i$ be an open semialgebraic covering, where the complexity of each U_i is $\leq \ell$,*
- *$s_i : U_i \to R$ $(1 \leq i \leq \ell)$ be Nash functions of complexities $\leq \ell$.*

If, for every $j \neq i$, $s_i - s_j$ belongs to the ideal $(f_1, \dots, f_\ell)\mathcal{N}(U_i \cap U_j)$, then there exists a global Nash function φ on Ω of complexity $\leq m$ such that, for each i, $\varphi - s_i$ belongs to the ideal $(f_1, \dots, f_\ell)\mathcal{N}(U_i)$.

Corollary 15 (Extension over an arbitrary real closed field) *Let Ω be a Nash submanifold of R^n. Let I be an ideal of $\mathcal{N}(\Omega)$. The canonical morphism*

$$\mathcal{N}(\Omega) = H^0(\Omega, \mathcal{N}_\Omega^{\mathrm{sa}}) \longrightarrow H^0(\Omega, \mathcal{N}_\Omega^{\mathrm{sa}}/I\mathcal{N}_\Omega^{\mathrm{sa}})$$

is surjective: every global section of the quotient sheaf $\mathcal{N}_\Omega^{\mathrm{sa}}/I\mathcal{N}_\Omega^{\mathrm{sa}}$ lifts to a global Nash function.

5 The Artin-Mazur property

In this section A is a commutative ring. We have briefly introduced the presheaf $\mathcal{A}$ in the introduction. Let us be more explicit now. Let U be an open subset of the real spectrum $\mathrm{Spec_r}A$. An element of $\mathcal{A}(U)$ is represented by a triple (B, σ, b), where B is an étale A-algebra, $\sigma : U \to \mathrm{Spec_r}B$ is a continuous section of the local homeomorphism $\mathrm{Spec_r}B \to \mathrm{Spec_r}A$, and $b \in B$. (For a proof that $\mathrm{Spec_r}(B) \to \mathrm{Spec_r}(A)$ is a local homeomorphism, see [Ro, 2.1], or [AR, 4.4.1], or [Sc, 1.8].) We notice that σ is then a homeomorphism onto the open subset $\sigma(U)$ of $\mathrm{Spec_r}(B)$; furthermore, if U is constructible, so is $\sigma(U)$. Two triples (B, σ, b) and (B', σ', b') represent the same element of $\mathcal{A}(U)$ if and only if there are a triple (C, τ, c) and A-algebra morphisms $u : B \to C$ and $u' : B' \to C$ such that $(\mathrm{Spec_r}u) \circ \tau = \sigma$, $(\mathrm{Spec_r}u') \circ \tau = \sigma'$ and $u(b) = u'(b') = c$. Obviously, $\mathcal{A}$ is a presheaf over $\mathrm{Spec_r}A$. The sheaf $\mathcal{N}_A$ of abstract Nash functions on $\mathrm{Spec_r}A$ is the sheaf associated to $\mathcal{A}$.

Theorem 16 (Artin-Mazur property for A) *For every open constructible subset U of $\mathrm{Spec_r}(A)$, we have $\mathcal{N}_A(U) = \mathcal{A}(U)$.*

The Artin-Mazur property can be formulated in another way. The category of sheaves over $\mathrm{Spec_r}(A)$ is equivalent to the category of sheaves over the lattice of open constructible subsets of $\mathrm{Spec_r}(A)$, with respect to the Grothendieck topology whose coverings are the open constructible coverings in $\mathrm{Spec_r}(A)$. The Artin-Mazur property says that the restriction of $\mathcal{A}$ to the lattice of open constructible subsets is a sheaf for this topology.

It is proved in [Ro, 2.8], that the presheaf $\mathcal{A}$ is separated. In order to establish the Artin-Mazur property, we have to prove the following fact. Given an open constructible subset $U \subset \mathrm{Spec_r}A$ and a finite open constructible covering $U = \bigcup_{i=1}^{\ell} U_i$, given elements φ_i of $\mathcal{A}(U_i)$ for $i = 1, \dots, \ell$ such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$, there is $\varphi \in \mathcal{A}(U)$ such that $\varphi|_{U_i} = \varphi_i$. The idea of the proof is simple. First, we show that the data $U = \bigcup_{i=1}^{\ell} U_i$ and $(\varphi_i)_{i=1,\dots,\ell}$ come by change of base from similar data over a finitely presented commutative ring $A^0 = \mathbb{Z}[\mathtt{x}_1, \dots, \mathtt{x}_p]/I$. Tensoring with $\mathbb{R}_{\mathrm{alg}}$, we obtain a section of the quotient sheaf $\mathcal{N}^{\mathrm{sa}}_{\mathbb{R}^p_{\mathrm{alg}}}/I$ over an open semialgebraic subset Ω of $\mathbb{R}^p_{\mathrm{alg}}$. Applying Extension (Corollary 15), this section lifts to a Nash function ψ on Ω. The Nash function ψ has an Artin-Mazur description. By change of base, this Artin-Mazur description gives the desired element φ of $\mathcal{A}(U)$. The technical realization of this simple idea is unfortunately rather long.

It will be useful to have more information on the sheaf $\mathcal{N}_A$ of abstract Nash functions on $\mathrm{Spec}_\mathrm{r}(A)$. We refer to [Ro] or [AR] for details. The stalk of the sheaf $\mathcal{N}_A$ at $\alpha \in \mathrm{Spec}_\mathrm{r}(A)$ is called the *real strict localization of A at α*, and denoted by A_α. It is a henselian local ring, ind-étale over A, and with residue field the real closed field $k(\alpha)$. The construction of the real spectrum with its sheaf of abstract Nash functions is functorial: a morphism $f : A \to A'$ of commutative rings induces a morphism $\mathrm{Spec}_\mathrm{r}(f) : (\mathrm{Spec}_\mathrm{r}(A'), \mathcal{N}_{A'}) \to (\mathrm{Spec}_\mathrm{r}(A), \mathcal{N}_A)$ of locally ringed spaces. In particular, we have a morphism $\mathrm{Spec}_\mathrm{r}(f)^*(\mathcal{N}_A) \to \mathcal{N}_{A'}$ whose stalk at $\beta \in \mathrm{Spec}_\mathrm{r}(A')$ is the canonical morphism of real strict localizations $A_{\mathrm{Spec}_\mathrm{r}(f)(\beta)} \to A'_\beta$. Let us consider the particular case $A' = A/I$. We denote by $Z(I)$ the set of $\alpha \in \mathrm{Spec}_\mathrm{r}(A)$ such that $I \subset \mathrm{supp}(\alpha)$. The closed subset $Z(I) \subset \mathrm{Spec}_\mathrm{r}(A)$ can be identified with the real spectrum of A/I. If $\alpha \in Z(I)$, the canonical morphism $A_\alpha/IA_\alpha \to (A/I)_\alpha$ is an isomorphism. Hence, the ringed space $(\mathrm{Spec}_\mathrm{r}(A/I), \mathcal{N}_{A/I})$ is isomorphic to $Z(I)$ equipped with the restriction of the quotient sheaf $\mathcal{N}_A/I\mathcal{N}_A$.

In order to manipulate the kind of data which represent a section of the presheaf $\mathcal{A}$, we introduce the category of *AM-data* attached to a commutative ring A. An *AM-datum for A* is a triple (U, B, σ), where U is an open constructible subset of $\mathrm{Spec}_\mathrm{r}(A)$, B an étale A-algebra and $\sigma : U \to \mathrm{Spec}_\mathrm{r}(B)$ a continuous section of the local homeomorphism $\mathrm{Spec}_\mathrm{r}(B) \to \mathrm{Spec}_\mathrm{r}(A)$. Now we define the morphisms between two AM-data $\mathfrak{B} = (U, B, \sigma)$ and $\mathfrak{C} = (V, C, \tau)$ for A. If U does not contain V, there is no morphism from $\mathfrak{B}$ to $\mathfrak{C}$. If U contains V, a morphism $u : \mathfrak{B} \to \mathfrak{C}$ is a morphism of A-algebras $u : B \to C$ such that $\sigma|_V = \mathrm{Spec}_\mathrm{r}(u) \circ \tau$. A section of the presheaf $\mathcal{A}$ over U is represented by an AM-datum $\mathfrak{B} = (U, B, \sigma)$ and an element $b \in B$. Given a morphism $u : \mathfrak{B} \to \mathfrak{C}$, the restriction to V of the section of $\mathcal{A}$ represented by $\mathfrak{B}$ and b is equal to the section represented by $\mathfrak{C}$ and $u(b)$.

Let $f : A \to A'$ be a morphism of commutative rings. Then f induces a functor f_* (change of base along f) from the category of AM-data for A to the category of AM-data for A'. Before making precise the definition of f_*, we record the following fact:

Lemma 17 *Let B be an étale A-algebra. Then:*

(i) *The canonical map* $\mathrm{Spec}_\mathrm{r}(B \otimes_A A') \to \mathrm{Spec}_\mathrm{r}(B) \times_{\mathrm{Spec}_\mathrm{r}(A)} \mathrm{Spec}_\mathrm{r}(A')$ *is a homeomorphism.*

(ii) *The diagonal* $\mathrm{Spec}_\mathrm{r}(B) \hookrightarrow \mathrm{Spec}_\mathrm{r}(B) \times_{\mathrm{Spec}_\mathrm{r}(A)} \mathrm{Spec}_\mathrm{r}(B)$ *is closed.*

Proof. (i) See [Sc, 1.7]. (ii) follows from (i) by taking $A' = B$ and using the fact that B is a quotient of $B \otimes_A B$. □

If $\mathfrak{B} = (U, B, \sigma)$, we define $f_*(\mathfrak{B})$ to be (U', B', σ'), where $U' = \mathrm{Spec}_\mathrm{r}(f)^{-1}(U)$, $B' = B \otimes_A A'$, and

$$\sigma' : U' \to \mathrm{Spec}_\mathrm{r}(B') \simeq \mathrm{Spec}_\mathrm{r}(B) \times_{\mathrm{Spec}_\mathrm{r}(A)} \mathrm{Spec}_\mathrm{r}(A')$$

is the continuous section induced by $\sigma \circ \mathrm{Spec}_\mathrm{r}(f) : U' \to \mathrm{Spec}_\mathrm{r}(B)$ via the universal property of the fiber product. If $u : \mathfrak{B} \to \mathfrak{C}$ is a morphism of AM-data for A, $f_*(u)$ is defined to be $u \otimes_A A'$. Note that, if $\varphi \in \mathcal{N}_A(U)$ is the section represented by $\mathfrak{B}$ and $b \in B$, then $f_*(\mathcal{B})$ and $b' = b \otimes 1 \in B'$ represent the section $\varphi' \in \mathcal{N}_{A'}(\mathrm{Spec_r}(f)^{-1}(U))$ which is the composition of $\mathrm{Spec_r}(f)^*(\varphi) : \mathrm{Spec_r}(f)^{-1}(U) \to \mathrm{Spec_r}(f)^*(\mathcal{N}_A)$ and $\mathrm{Spec_r}(f)^*(\mathcal{N}_A) \to \mathcal{N}_{A'}$.

The first step in the proof of the Artin-Mazur property is the following.

Proposition 18 *Let $\mathfrak{B}$ be an AM-datum for A. Then there are a finitely presented commutative ring A^0, a morphism $f : A^0 \to A$ and an AM-datum $\mathfrak{B}^0$ for A^0 such that $\mathfrak{B}$ is isomorphic to $f_*(\mathfrak{B}^0)$.*

We first prove a lemma.

Lemma 19 *Let $f : A^0 \to A$ be a morphism of commutative rings, where A^0 is finitely presented.*

(i) *Let U be an open constructible subset of* $\mathrm{Spec_r}(A)$. *Then there exist a factorization*

$$f : A^0 \xrightarrow{g} A' \xrightarrow{f'} A ,$$

where A' is a finitely presented commutative ring, and an open constructible subset U^0 of $\mathrm{Spec_r}(A')$ *such that* $\mathrm{Spec_r}(f')^{-1}(U^0) = U$.

(ii) *Let C^0 be a constructible subset of* $\mathrm{Spec_r}(A^0)$ *such that* $\mathrm{Spec_r}(f)^{-1}(C^0) = \emptyset$. *Then there exists a factorization*

$$f : A^0 \xrightarrow{g} A' \xrightarrow{f'} A ,$$

where A' is a finitely presented commutative ring, such that $\mathrm{Spec_r}(g)^{-1}(C^0) = \emptyset$.

Proof. (i) The open constructible subset U is a finite union of basic open subsets:

$$U = \bigcup_{i=1}^{\ell} \{\alpha \in \mathrm{Spec_r}(A) \; ; \; a_{i,1}(\alpha) > 0, \ldots, a_{i,m}(\alpha) > 0\} ,$$

where the $a_{i,j}$'s belong to A. We set $A' = A^0[\underline{\mathtt{t}}]$, where $\underline{\mathtt{t}} = (\mathtt{t}_{i,j})_{i=1,\ldots,\ell \,;\, j=1,\ldots,m}$ is a finite collection of indeterminates. We factorize $f : A^0 \to A$ through the morphism $f' : A' \to A$ which sends each $\mathtt{t}_{i,j}$ to $a_{i,j}$. We now have an open constructible subset

$$U^0 = \bigcup_{i=1}^{\ell} \{\alpha \in \mathrm{Spec_r}(A') \; ; \; \mathtt{t}_{i,1}(\alpha) > 0, \ldots, \mathtt{t}_{i,m}(\alpha) > 0\}$$

such that $\mathrm{Spec_r}(f')^{-1}(U^0) = U$.

(ii) The constructible subset C^0 is a finite union of constructible subsets of the form

$$\{\beta \in \mathrm{Spec_r}(A^0) \; ; \; a_1(\beta) = \cdots = a_\ell(\beta) = 0, \; b_1(\beta) > 0, \ldots, b_m(\beta) > 0\} ,$$

where the a_i's and the b_j's are elements of A^0. Hence, it suffices to prove (ii) for a constructible subset of this form. By assumption, we have

$$\{\alpha \in \mathrm{Spec_r}(A) \; ; \; f(a_1)(\alpha) = \cdots = f(a_\ell)(\alpha) = 0, \; f(b_1)(\alpha) > 0, \ldots, f(b_m)(\alpha) > 0\} = \emptyset .$$

By the formal Positivstellensatz ([BCR, 4.4.1]), there are elements $c_1, \dots, c_p$ in A and a polynomial $P(\underline{\mathtt{s}}, \underline{\mathtt{t}}, \underline{\mathtt{u}}) \in \mathbb{Z}[\underline{\mathtt{s}}, \underline{\mathtt{t}}, \underline{\mathtt{u}}]$, where $\underline{\mathtt{s}} = (\mathtt{s}_1, \dots, \mathtt{s}_\ell)$, $\underline{\mathtt{t}} = (\mathtt{t}_1, \dots, \mathtt{t}_m)$ and $\underline{\mathtt{u}} = (\mathtt{u}_1, \dots, \mathtt{u}_p)$, such that:

- $P(f(a_1), \dots, f(a_\ell), f(b_1), \dots, f(b_m), c_1, \dots, c_p) = 0$.
- For every real closed field R and every $\underline{s} \in R^\ell$, $\underline{t} \in R^m$ and $\underline{u} \in R^p$, if $s_1 = \cdots = s_\ell = 0$ and $t_1 > 0, \dots, t_m > 0$, then $P(\underline{s}, \underline{t}, \underline{u}) > 0$.

We set $A' = A^0[\underline{\mathtt{u}}]/P(\underline{a}, \underline{b}, \underline{\mathtt{u}})$ and we factorize f as

$$f : A^0 \xrightarrow{g} A' \xrightarrow{f'} A \ ,$$

where g is the canonical morphism and f' sends $\underline{\mathtt{u}}$ to $\underline{c}$. Then $\mathrm{Spec_r}(g)^{-1}(C^0) = \emptyset$. $\square$

Proof of Proposition 18. Set $\mathfrak{B} = (U, B, \sigma)$. It is well known (see [EGA, 17.7.8]) that we can find a finitely presented commutative ring A^0, an étale A^0-algebra B^0 and a morphism $f : A^0 \to A$ such that there is an isomorphism of A-algebras $B^0 \otimes_{A^0} A \to B$.

Now we proceed to the construction of a continuous section $\sigma^0 : U^0 \to \mathrm{Spec_r}(B^0)$ which gives σ by change of base.

First, we fix $\alpha \in U$. We set $\alpha^0 = \mathrm{Spec_r}(f)(\alpha) \in \mathrm{Spec_r}(A^0)$. By Lemma 17 (i), we can identify the fibers of $\mathrm{Spec_r}(B)$ over α and of $\mathrm{Spec_r}(B^0)$ over α^0. Hence, $\sigma(\alpha)$ determines an element γ in the fiber of $\mathrm{Spec_r}(B^0)$ over α^0. Since $\mathrm{Spec_r}(B^0) \to \mathrm{Spec_r}(A^0)$ is a local homeomorphism, we can choose an open constructible neighborhood Ω_{α^0} of α^0 in $\mathrm{Spec_r}(A^0)$ and a continuous section $\tau^0 : \Omega_{\alpha^0} \to \mathrm{Spec_r}(B^0)$ such that $\tau^0(\alpha^0) = \gamma$. We set $\Omega_\alpha = \mathrm{Spec_r}(f)^{-1}(\Omega_{\alpha^0})$; this is an open constructible neighborhood of α in $\mathrm{Spec_r}(A)$. By change of base along f, we obtain from τ^0 a continuous section $\tau : \Omega_\alpha \to \mathrm{Spec_r}(B)$ such that $\tau(\alpha) = \sigma(\alpha)$. Hence, τ and σ coincide on an open constructible neighborhood W_α of α in U.

When we let α range over U, the W_α's cover U. Since U is compact, we can find finitely many α's, say $\alpha_1, \dots, \alpha_n$, such that $U = W_{\alpha_1} \cup \ldots \cup W_{\alpha_n}$. By Lemma 19 (i), changing of base to a new finitely presented A^0, we can assume that there are open constructible subsets $W_1^0, \dots, W_n^0$ of $\mathrm{Spec_r}(A^0)$ such that $\mathrm{Spec_r}(f)^{-1}(W_i^0) = W_{\alpha_i}$. Since $\mathrm{Spec_r}(f)^{-1}(\Omega_{\alpha_i^0})$ contains W_{α_i}, we can take each W_i^0 contained in $\Omega_{\alpha_i^0}$. Hence, we have for each W_i^0 a continuous section $\tau_i : W_i^0 \to \mathrm{Spec_r}(B^0)$ which induces $\sigma|_{W_{\alpha_i}}$ by change of base along f. For each (i,j) with $1 \le i < j \le n$, define $\Delta_{i,j}$ to be the set of $\beta \in W_i^0 \cap W_j^0$ such that $\tau_i(\beta) \neq \tau_j(\beta)$. This $\Delta_{i,j}$ is closed and, by Lemma 17 (ii), also open in $W_i^0 \cap W_j^0$, hence constructible. Moreover, we have $\mathrm{Spec_r}(f)^{-1}(\Delta_{i,j}) = \emptyset$. It follows by Lemma 19 (ii) that we can assume $\Delta_{i,j} = \emptyset$, up to a change of base from A^0 to another finitely presented commutative ring. Hence, the sections τ_i can be glued together to give a continuous section $\sigma^0 : \bigcup_{i=1}^n W_i^0 \to \mathrm{Spec_r}(B^0)$ which induces σ by change of base along f. We set $U^0 = \bigcup_{i=1}^n W_i^0$. $\square$

Now that we have shown that every AM-datum comes by change of base from an AM-datum for a finitely presented commutative ring, we do the same for morphisms of

AM-data.

Proposition 20 *Let $f : A^0 \to A$ be a morphism of commutative rings, where A^0 is finitely presented. Let $\mathfrak{B}^0$ and $\mathfrak{C}^0$ be AM-data for A^0, and let $u : f_*(\mathfrak{B}^0) \to f_*(\mathfrak{C}^0)$ be a morphism of AM-data for A. Then there exist a factorization*

$$f : A^0 \xrightarrow{g} A' \xrightarrow{f'} A ,$$

where A' is a finitely presented commutative ring, and a morphism $u' : g_*(\mathfrak{B}^0) \to g_*(\mathfrak{C}^0)$ of AM-data for A' such that $f'_*(u') = u$.

Proof. Let $\mathfrak{B}^0 = (U^0, B^0, \sigma^0)$ and $\mathfrak{C}^0 = (V^0, C^0, \tau^0)$. Up to a change of base to a new finitely presented A^0, we can assume that there is a morphism of A^0-algebras $u^0 : B^0 \to C^0$ such that $u^0 \otimes_{A^0} A = u$. Let Δ be the constructible subset of $\alpha \in U^0 \cap V^0$ such that $\mathrm{Spec}_\mathrm{r}(u^0)(\tau^0(\alpha)) \neq \sigma^0(\alpha)$. Applying Lemma 19 (ii), we obtain a factorization

$$f : A^0 \xrightarrow{g} A' \xrightarrow{f'} A ,$$

where A' is a finitely presented commutative ring, such that $\mathrm{Spec}_\mathrm{r}(g)^{-1}(V^0)$ is contained in $\mathrm{Spec}_\mathrm{r}(g)^{-1}(U^0)$ and $\mathrm{Spec}_\mathrm{r}(g)^{-1}(\Delta) = \emptyset$. Then $u' = u^0 \otimes_{A^0} A'$ is a morphism of AM-data from $g_*(\mathfrak{B}^0)$ to $g_*(\mathfrak{C}^0)$. □

Now we have the tools needed for the reduction of the Artin-Mazur property to the Extension over the field of real algebraic numbers.

Proof of Theorem 16. We are given a finite open constructible covering $U = \bigcup_{i=1}^{\ell} U_i$ in $\mathrm{Spec}_\mathrm{r}(A)$ and elements $\varphi_i \in \mathcal{A}(U_i)$ for $i = 1, \ldots, \ell$ such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$. By the construction of $\mathcal{A}$, the φ_i's are represented by AM-data $\mathfrak{B}_i = (U_i, B_i, \sigma_i)$ and elements $b_i \in B_i$, and there are, for every i, j with $1 \leq i < j \leq \ell$, AM-data $\mathfrak{C}_{i,j} = (U_i \cap U_j, C_{i,j}, \tau_{i,j})$ with morphisms $u_{i,j} : \mathfrak{B}_i \to \mathfrak{C}_{i,j}$ and $u_{j,i} : \mathfrak{B}_j \to \mathfrak{C}_{i,j}$ such that $u_{i,j}(b_i) = u_{j,i}(b_j)$. Applying Propositions 18 and 20, we obtain

- a finitely presented commutative ring $A^0 = \mathbb{Z}[\mathtt{x}]/I$, $\mathtt{x} = (\mathtt{x}_1, \ldots, \mathtt{x}_p)$,
- a morphism $f : A^0 \to A$,
- for $i = 1 \ldots, \ell$, AM-data $\mathfrak{B}_i^0 = (U_i^0, B_i^0, \sigma_i^0)$ for A^0 such that $f_*(\mathfrak{B}_i^0)$ is isomorphic to $\mathfrak{B}_i$,
- for $1 \leq i < j \leq \ell$, AM-data $\mathfrak{C}_{i,j}^0 = (V_{i,j}^0, C_{i,j}^0, \tau_{i,j}^0)$ for A^0 such that $f_*(\mathfrak{C}_{i,j}^0)$ is isomorphic to $\mathfrak{C}_{i,j}$,
- for $1 \leq i < j \leq \ell$, morphisms of AM-data $u_{i,j}^0 : \mathfrak{B}_i^0 \to \mathfrak{C}_{i,j}^0$ and $v_{j,i}^0 : \mathfrak{B}_j^0 \to \mathfrak{C}_{i,j}^0$ such that $f_*(u_{i,j}^0)$ and $f_*(v_{i,j}^0)$ are respectively

$$f_*(\mathfrak{B}_i^0) \simeq \mathfrak{B}_i \xrightarrow{u_{i,j}} \mathfrak{C}_{i,j} \simeq f_*(\mathfrak{C}_{i,j}^0) \qquad \text{and} \qquad f_*(\mathfrak{B}_j^0) \simeq \mathfrak{B}_j \xrightarrow{v_{i,j}} \mathfrak{C}_{i,j} \simeq f_*(\mathfrak{C}_{i,j}^0) .$$

.

By Lemma 19 (ii), we can assume that $V^0_{i,j} = U^0_i \cap U^0_j$. We set $U^0 = \bigcup_{i=1}^{\ell} U^0_i$. Moreover, since A is a filtered limit of finitely presented commutative rings and filtered limits commute with tensor product, we can assume that there are elements $b^0_i \in B_i$ whose image by $B^0_i \to B^0_i \otimes_{A^0} A \simeq B_i$ is b_i and that $u^0_{i,j}(b^0_i) = u^0_{j,i}(b^0_j)$. The AM-data $\mathfrak{B}^0_i$ and the b^0_i represent sections of $\mathcal{N}_{A^0}$ over U^0_i which agree on the intersections $U^0_i \cap U^0_j$. Hence, we obtain a section $\varphi^0 \in \mathcal{N}_{A^0}(U^0)$.

We recall that $\mathbb{R}_{\mathrm{alg}}$ stands for the field of real algebraic numbers. Now, consider $A^1 = \mathbb{R}_{\mathrm{alg}}[\mathtt{x}]/I\mathbb{R}_{\mathrm{alg}}[\mathtt{x}]$. The canonical morphism $h : A^0 \to A^1$ is ind-étale, and induces an isomorphism of ringed spaces

$$\mathrm{Spec}_{\mathrm{r}}(h) : (\mathrm{Spec}_{\mathrm{r}}(A^1), \mathcal{N}_{A^1}) \simeq (\mathrm{Spec}_{\mathrm{r}}(A^0), \mathcal{N}_{A^0}) \ .$$

Indeed, $\mathrm{Spec}_{\mathrm{r}}(h)$ is a homeomorphism from $\mathrm{Spec}_{\mathrm{r}}(A^1)$ to $\mathrm{Spec}_{\mathrm{r}}(A^0)$ and, for every $\alpha \in \mathrm{Spec}_{\mathrm{r}}(A^1)$, we have $A^0_{\mathrm{Spec}_{\mathrm{r}}(h)(\alpha)} \simeq A^1_\alpha$. In the following we identify $(\mathrm{Spec}_{\mathrm{r}}(A^1), \mathcal{N}_{A^1})$ with $(\mathrm{Spec}_{\mathrm{r}}(A^0), \mathcal{N}_{A^0})$. On the other hand, let $Z \subset \mathbb{R}^p_{\mathrm{alg}}$ be the zero set of I, so that

$$\widetilde{Z} = Z(I\mathbb{R}_{\mathrm{alg}}[\mathtt{x}]) \subset \mathrm{Spec}_{\mathrm{r}}(\mathbb{R}_{\mathrm{alg}}[\mathtt{x}]) = \widetilde{\mathbb{R}^p_{\mathrm{alg}}}$$

(we identify $\mathbb{R}^p_{\mathrm{alg}}$ with the subspace of $\widetilde{\mathbb{R}^p_{\mathrm{alg}}}$ whose points are supported by the real maximal ideals). We have seen that $(\mathrm{Spec}_{\mathrm{r}}(A^1), \mathcal{N}_{A^1})$ can be identified with $\widetilde{Z}$ equipped with the restriction of the quotient sheaf $\mathcal{N}_{\mathbb{R}_{\mathrm{alg}}[\mathtt{x}]}/I\mathcal{N}_{\mathbb{R}_{\mathrm{alg}}[\mathtt{x}]}$. With all these identifications, we regard φ^0 as a section of $\mathcal{N}_{\mathbb{R}_{\mathrm{alg}}[\mathtt{x}]}/I\mathcal{N}_{\mathbb{R}_{\mathrm{alg}}[\mathtt{x}]}$ over $U^0 \subset \widetilde{Z}$. This is the same as a section of that sheaf over $\widetilde{\Omega} = (\widetilde{\mathbb{R}^p_{\mathrm{alg}}} \setminus \widetilde{Z}) \cup U^0$. Let $\Omega = \mathbb{R}^p_{\mathrm{alg}} \cap \widetilde{\Omega}$; this is an open semialgebraic subset of $\mathbb{R}^p_{\mathrm{alg}}$. Finally, we can regard φ^0 as a global section of the quotient sheaf $\mathcal{N}^{\mathrm{sa}}_\Omega / I\mathcal{N}^{\mathrm{sa}}_\Omega$. We can now apply Extension over the real closed field $\mathbb{R}_{\mathrm{alg}}$ (Corollary 15): the section φ^0 can be lifted to a Nash function ψ on Ω. Using the Artin-Mazur description of the Nash function ψ and changing of base along $\mathbb{R}_{\mathrm{alg}}[\mathtt{x}] \to \mathbb{R}_{\mathrm{alg}}[\mathtt{x}]/I\mathbb{R}_{\mathrm{alg}}[\mathtt{x}] = A^1$, we obtain an AM-datum $\mathfrak{D}^1 = (U^0, D^1, \nu^1)$ for A^1 and $d^1 \in D^1$ representing φ^0 as an element of $\mathcal{N}_{A^1}(U^0)$.

Our next task is to obtain an AM-datum $\mathfrak{D}^0 = (U^0, D^0, \nu^0)$ *for* A^0 and $d^0 \in D^0$ representing φ^0. Since A^1 is ind-étale over A^0, we obtain, using [EGA, 17.7.8], a factorization

$$h : A^0 \xrightarrow{g} A' \xrightarrow{h'} A^1 \ ,$$

where A' is an étale A^0-algebra, an étale A'-algebra D^0 and $d^0 \in D^0$ such that $D^0 \otimes_{A'} A^1$ is isomorphic to D^1, the isomorphism carrying $d^0 \otimes 1$ to d^1. Note that, by composition with g, D^0 is an étale A^0-algebra. We denote $u : D^0 \to D^0 \otimes_{A'} A^1 \simeq D^1$, and consider the composition $\nu^0 = \mathrm{Spec}_{\mathrm{r}}(u) \circ \nu^1 : U^0 \to \mathrm{Spec}_{\mathrm{r}}(D^0)$. We claim that ν^0 is a continuous section of the local homeomorphism $\mathrm{Spec}_{\mathrm{r}}(D^0) \to \mathrm{Spec}_{\mathrm{r}}(A^0)$. Indeed, the composition of ν^0 and this local homeomorphism is equal to the restriction of $\mathrm{Spec}_{\mathrm{r}}(h)$, and we identified $\mathrm{Spec}_{\mathrm{r}}(A^1)$ with $\mathrm{Spec}_{\mathrm{r}}(A^0)$ via $\mathrm{Spec}_{\mathrm{r}}(h)$. In this way we obtain an AM-datum $\mathfrak{D}^0 = (U^0, D^0, \nu^0)$ for A^0. In order to check that $\mathfrak{D}^0$ and d^0 represent $\varphi^0 \in \mathcal{N}_{A^0}(U^0)$, it suffices to exhibit a morphism of AM-data $v : h_*(\mathfrak{D}^0) \to \mathfrak{D}^1$ such that $v(d^0 \otimes 1) = d^1$. We take v to be the morphism of A^1-algebras

$$v : D^0 \otimes_{A^0} A^1 \to D^0 \otimes_{A'} A^1 \simeq D^1$$

such that the composition of v with $D^0 \to D^0 \otimes_{A^0} A^1$ is equal to u. The composition of $\mathrm{Spec_r}(v) \circ \nu^1$ with $\mathrm{Spec_r}(D^0 \otimes_{A^0} A^1) \to \mathrm{Spec_r}(D^0)$ is equal to $\mathrm{Spec_r}(u) \circ \nu^1 = \nu^0$. This proves that v is a morphism of AM-data from $h_*(\mathfrak{D}^0)$ to $\mathfrak{D}^1$. Obviously $v(d^0 \otimes 1) = d^1$.

By change of base along $f : A^0 \to A$, we obtain finally an AM-datum $f_*(\mathfrak{D}^0)$ for A and $d^0 \otimes 1 \in D^0 \otimes_{A^0} A$ which represent the section $\varphi \in \mathcal{N}_A(U)$ such that $\varphi|_{U_i} = \varphi_i$. This concludes the proof. □

Corollary 21 *For every open constructible subset U of $\mathrm{Spec_r}(A)$, $\mathcal{N}_A(U)$ is an ind-étale A-algebra.*

It is quite easy now to understand the change of base for abstract Nash functions corresponding to a morphism $f : A \to A'$: the functor f_* from AM-data for A to AM-data for A' defines the morphism $f_* : \mathcal{N}_A(U) \to \mathcal{N}'_A(U')$ for $U \subset \mathrm{Spec_r}(A)$ open constructible and $U' = \mathrm{Spec_r}(f)^{-1}(U)$.

6 Global Equations and Extension for abstract Nash functions

We remain in the same abstract setting of the preceding section. Let A be a commutative ring, and $\mathcal{N}_A$ the sheaf of (abstract) Nash functions on the real spectrum $\mathrm{Spec_r}(A)$. Let $U \subset \mathrm{Spec_r}(A)$ be an open constructible set and denote $\mathcal{N}_U = \mathcal{N}_A|_U$; we have the ringed space $(U, \mathcal{N}_U)$. A sheaf $\mathcal{I}$ of ideals of $\mathcal{N}_U$ is called *finite* if there is a finite covering $U = \bigcup_{i=1}^{\ell} U_i$ by open constructible sets such that, for each i, $\mathcal{I}|_{U_i}$ is generated by finitely many Nash functions on U_i. Notice that, since the real spectrum is compact with respect to the constructible topology, this is equivalent to say that $\mathcal{I}$ is locally finitely generated. We will show that Global Equations and Extension are true in this most general setting:

Theorem 22 *Let $\mathcal{I}$ be a finite sheaf of ideals of $\mathcal{N}_U$. Then:*

(i) (Global equations) *$\mathcal{I}$ is generated by finitely many global sections $h_1, \ldots, h_r \in \mathcal{N}_A(U)$.*

(ii) (Extension) *Every section on U of the quotient sheaf $\mathcal{N}_U/(h_1, \ldots, h_r)\mathcal{N}_U$ can be lifted to a Nash function on U.*

The proofs of these two statements follow the pattern of the preceding section, with some simplifications due to the now available Artin-Mazur property. We start with a particular case.

Proof of the finitely presented case. Suppose that $A = \mathbb{Z}[\underline{\mathrm{x}}]/I$, $\underline{\mathrm{x}} = (\mathrm{x}_1, \ldots, \mathrm{x}_p)$. We recall that setting $A' = \mathbb{R}_{\mathrm{alg}}[\underline{\mathrm{x}}]/I\,\mathbb{R}_{\mathrm{alg}}[\underline{\mathrm{x}}]$, there is an isomorphism of ringed spaces

$$(\mathrm{Spec_r}(A'), \mathcal{N}_{A'}) \simeq (\mathrm{Spec_r}(A), \mathcal{N}_A) \ .$$

On the other hand, we know that, concerning finite sheaves of Nash functions, we have an equivalence between the geometric ringed space $(\mathbb{R}^p_{\mathrm{alg}}, \mathcal{N}_{\mathbb{R}^p_{\mathrm{alg}}})$ and $(\mathrm{Spec_r}(\mathbb{R}_{\mathrm{alg}}[\underline{\mathrm{x}}]), \mathcal{N}_{\mathbb{R}_{\mathrm{alg}}[\underline{\mathrm{x}}]})$. Thus, since Global Equations and Extension hold over the real algebraic numbers, they hold for any closed subspace of the form $(\mathrm{Spec}_{\mathrm{r}}(\mathbb{R}_{\mathrm{alg}}[\underline{\mathtt{x}}]/J), \mathcal{N}_{\mathbb{R}_{\mathrm{alg}}[\underline{\mathtt{x}}]/J})$. Since $(\mathrm{Spec}_{\mathrm{r}}(A'), \mathcal{N}_{A'})$ is such a subspace, we have Global Equations and Extension for it. $\square$

Now we turn to the general case.

Proof of Global equations. The sheaf $\mathcal{I}$ is given by

- a finite covering $U = \bigcup_{i=1}^{\ell} U_i$ by open constructible sets,
- Nash functions $f_{i,\mu} \in \mathcal{N}_A(U_i)$, $1 \le i, \mu \le \ell$,
- Nash functions $g_{i,j}^{\mu,\nu} \in \mathcal{N}_A(U_i \cap U_j)$, $1 \le i, j, \mu, \nu \le \ell$,

such that, for every $i \neq j$,

$$f_{i,\mu} = \sum_{1 \le \nu \le \ell} g_{i,j}^{\mu,\nu} f_{j,\nu} \ .$$

Now, by the Artin-Mazur description, we find

- AM-data $\mathfrak{B}_i = (U_i, B_i, \sigma_i)$ and elements $b_{i,\mu} \in B_i$ representing $f_{i,\mu}$,
- AM-data $\mathfrak{C}_{i,j} = (U_i \cap U_j, C_{i,j}, \tau_{i,j})$ and elements $c_{i,j}^{\mu,\nu} \in C_{i,j}$ representing $g_{i,j}^{\mu,\nu}$,
- morphisms of AM-data $u_{i,j} : \mathfrak{B}_i \to \mathfrak{C}_{i,j}$ and $v_{i,j} : \mathfrak{B}_j \to \mathfrak{C}_{i,j}$ such that

$$u_{i,j}(b_{i,\mu}) = \sum_{1 \le \nu \le \ell} c_{i,j}^{\mu,\nu} v_{i,j}(b_{j,\nu}) \ .$$

Then, from Propositions 18 and 20, we get

- a finitely presented commutative ring $A^0 = \mathbb{Z}[\mathtt{x}]/I$, $\mathtt{x} = (\mathtt{x}_1, \ldots, \mathtt{x}_p)$,
- a morphism $f : A^0 \to A$,
- AM-data $\mathfrak{B}_i^0 = (U_i^0, B_i^0, \sigma_i^0)$ for A^0 such that $f_*(\mathfrak{B}_i^0)$ is isomorphic to $\mathfrak{B}_i$,
- AM-data $\mathfrak{C}_{i,j}^0 = (V_{i,j}^0, C_{i,j}^0, \tau_{i,j}^0)$ for A^0 such that $f_*(\mathfrak{C}_{i,j}^0)$ is isomorphic to $\mathfrak{C}_{i,j}$,
- morphisms of AM-data $u_{i,j}^0 : \mathfrak{B}_i^0 \to \mathfrak{C}_{i,j}^0$ and $v_{i,j}^0 : \mathfrak{B}_j^0 \to \mathfrak{C}_{i,j}^0$ such that $f_*(u_{i,j}^0)$ and $f_*(v_{i,j}^0)$ are respectively

$$f_*(\mathfrak{B}_i^0) \simeq \mathfrak{B}_i \xrightarrow{u_{i,j}} \mathfrak{C}_{i,j} \simeq f_*(\mathfrak{C}_{i,j}^0) \qquad \text{and} \qquad f_*(\mathfrak{B}_j^0) \simeq \mathfrak{B}_j \xrightarrow{v_{i,j}} \mathfrak{C}_{i,j} \simeq f_*(\mathfrak{C}_{i,j}^0) \ .$$

By Lemma 19 (ii), we can assume that $V_{i,j}^0 = U_i^0 \cap U_j^0$. We set $U^0 = \bigcup_{i=1}^{\ell} U_i^0$. Moreover, we can assume that there are elements $b_{i,\mu}^0 \in B_i^0$ and $c_{i,j}^{\mu,\nu,0} \in C_{i,j}^0$ which are carried respectively to $b_{i,\mu}$ and $c_{i,j}^{\mu,\nu}$, and that

$$u_{i,j}^0(b_{i,\mu}^0) = \sum_{1 \le \nu \le \ell} c_{i,j}^{\mu,\nu,0} v_{i,j}^0(b_{j,\nu}^0) \ .$$

For each i, the AM-datum $\mathfrak{B}_i^0$ and the $b_{i,\mu}^0$'s represent Nash functions $f_{i,\mu}^0 \in \mathcal{N}_{A^0}(U_i^0)$, which define a finite sheaf of ideals $\mathcal{I}_i^0$ of $\mathcal{N}_A|_{U_i^0}$, and these $\mathcal{I}_i^0$'s glue to a well defined finite sheaf of ideals $\mathcal{I}^0$ of $\mathcal{N}_{U^0} = \mathcal{N}_{A^0}|_{U^0}$. By the particular case previously considered, we know that Global Equations hold for A^0. Hence the sheaf $\mathcal{I}^0$ is generated by finitely many elements $h_1^0, \dots, h_r^0$ of $\mathcal{N}_{A^0}(U^0)$. It follows that $\mathcal{I} = \mathrm{Spec_r}(f)^*(\mathcal{I}^0)\mathcal{N}_U$ is generated by the images $h_1, \dots, h_r$ in $\mathcal{N}_A(U)$. □

Proof of Extension. We only sketch it. If Φ is a section on U of the quotient sheaf $\mathcal{N}_U/(h_1, \dots, h_r)\mathcal{N}_U$, there are a finite open constructible covering $U = \bigcup_{i=1}^{\ell}$ and Nash functions $\varphi_i \in \mathcal{N}_A(U_i)$ such that on $U_i \cap U_j$

$$\varphi_i - \varphi_j = \theta_{i,j}^1 h_1 + \cdots + \theta_{i,j}^r h_r$$

for some $\theta_{i,j}^m \in \mathcal{N}_A(U_i \cap U_j)$. As done before, using AM-data, we can show that all these Nash functions and equations come by change of base from a similar situation over a finitely presented commutative ring A^0. We find $\varphi_i^0 \in \mathcal{N}_{A^0}(U_i^0)$ which extend to φ_i via $A^0 \to A$ and define a section Φ^0 on U^0 of $\mathcal{N}_{U^0}/(h_1^0, \dots, h_r^0)\mathcal{N}_{U^0}$. Since Extension holds for A^0, there is a Nash function $\varphi^0 \in \mathcal{N}_{U^0}$ which represents Φ^0, and gives rise to another $\varphi \in \mathcal{N}_A(U)$ that represents Φ. □

7 Idempotency

As usual, let A be a commutative ring and $\mathcal{N}_A$ the sheaf of Nash functions on the real spectrum $\mathrm{Spec_r}(A)$. We fix an open constructible set $U \subset \mathrm{Spec_r}(A)$, and consider the ring $B = B_U = \mathcal{N}_A(U)$ of Nash functions on U. Then B is an A-algebra, and we have:

Theorem 23 (Idempotency of the real spectrum) *The morphism $A \to B$ induces an isomorphism of locally ringed spaces*

$$(\mathrm{Spec_r}(B), \mathcal{N}_B) \longrightarrow (U, \mathcal{N}_A|_U).$$

Proof. Denote $p : \mathrm{Spec_r}(B) \to \mathrm{Spec_r}(A)$ the canonical map. Let $\beta \in \mathrm{Spec_r}(B)$ and $\alpha = p(\beta)$. Since $A \to B$ is ind-étale, $A_\alpha \to B_\beta$ is local ind-étale morphism of henselian rings with the same residue fields. By [Rd, VIII Prop.1], $A_\alpha \to B_\beta$ is an isomorphism, and this reduces the statement to the topological fact that p is a homeomorphism from $\mathrm{Spec_r}(B)$ onto U. To show this we first look at the commutative square

$$\begin{array}{ccc} \mathrm{Spec_r}(B_\beta) & \to & G^\beta = \{\beta' \in \mathrm{Spec_r}(B);\, \beta' \to \beta\} \\ \downarrow & & \downarrow \\ \mathrm{Spec_r}(A_\alpha) & \to & G^\alpha = \{\alpha' \in \mathrm{Spec_r}(B);\, \alpha' \to \alpha\} \end{array}$$

whose left vertical arrow is a homeomorphism. But the horizontal arrows are also homeomorphisms ([ABR, II.7.11]), and consequently so is the right vertical one. Whence, p is bijective from G^β onto G^α.

Next, we define a section $s : U \to \operatorname{Spec_r}(B)$ of p. For $\alpha \in U$, let $f \mapsto f(\alpha)$ denote the canonical morphism

$$B = \mathcal{N}_A(U) \to \mathcal{N}_{A,\alpha} = A_\alpha \to k(\alpha).$$

Then $s(\alpha) \in \operatorname{Spec_r}(B)$ is the prime cone $B \to \kappa(\alpha) : f \mapsto f(\alpha)$. Notice that the Artin-Mazur property gives an easy way to compute $f(\alpha)$: pick any AM-datum (U, C, τ) and $c \in C$ that represent f and then $f(\alpha) = c(\tau(\alpha))$.

It is clear that $p \circ s = \mathrm{Id}$. Moreover, s is continuous: any $f_i \in B$ $(1 \le i \le r)$ can be represented by an AM-datum (U, C, τ) and elements c_i $(1 \le i \le r)$ so that

$$s^{-1}(\{f_1 > 0, \dots, f_r > 0\}) = \pi(\{c_1 > 0, \dots, c_r > 0\} \cap \tau(U)),$$

where $\pi : \operatorname{Spec_r}(C) \to \operatorname{Spec_r}(A)$ is the local homeomorphism associated to $A \to C$. Thus, it only remains to see that $p(\beta) \in U$ and $\beta = s(p(\beta))$ for every $\beta \in \operatorname{Spec_r}(B)$. The key fact now is [Ro, 5.6]:

$$\bigcap_{f(\beta) \ge 0} s^{-1}(\{f \ge 0\}) \cap U \ne \emptyset.$$

In other words, there is $\alpha \in U$ such that $\beta \to s(\alpha)$. But then $p(\beta) \to p(s(\alpha)) = \alpha \in U$, so that $p(\beta) \in G^\alpha \subset U$. Finally, since p induces a bijection from $G^{s(\alpha)}$ onto G^α and $s(p(\beta)) \to s(\alpha)$, we conclude $\beta = s(p(\beta))$. □

An easy remark after idempotency is that every maximal ideal $\mathfrak{m}$ of B is real: the support of some closed point of $U = \operatorname{Spec_r}(B)$. Indeed, the argument is standard. Firstly, $\bigcap_{f \in \mathfrak{m}} \{f = 0\} \ne \emptyset$, because otherwise, by compactness, there would be $f_1, \dots, f_r \in \mathfrak{m}$ such that $\{f_1 = \dots = f_r = 0\} = \emptyset$; hence, also $f = f_1^2 + \dots + f_r^2 \in \mathfrak{m}$ and $\{f = 0\} = \emptyset$. But this latter condition would define well a section $1/f$ of $\mathcal{N}_A$ over U, that is, $1/f \in B$, which is impossible. Thus, there is $\alpha \in U$ such that $f(\alpha) = 0$ for every $f \in \mathfrak{m}$, so that $\mathfrak{m}$ is contained in the support of α, and by maximality, the two ideals must coincide.

8 Separation and noetherian rings of Nash functions

Once again, let A be a commutative ring and $\mathcal{N}_A$ the sheaf of Nash functions on the real spectrum $\operatorname{Spec_r}(A)$. For every open constructible set $U \subset \operatorname{Spec_r}(A)$ we have the ring $B_U = \mathcal{N}_A(U)$ of abstract Nash functions on U.

Once idempotency is settled, we are ready for separation. What follows comes from the formulation in terms of invariant complex Nash germs given in [CRS2] for manifolds over the reals, and its generalization proposed in [Qu2]. Here we present it in a modified form, discarding technical difficulties related to noetherian hypotheses, which we will discuss afterwards.

We denote by $\mathcal{G}_U$ the set of all pairs $(\alpha, \mathfrak{p})$, where $\alpha \in U$ and $\mathfrak{p}$ is a prime ideal of the real strict localization A_α. Then we define two operators $\mathfrak{X}_U$ and $\mathcal{J}_U$,

$$I \mapsto \mathfrak{X}_U(I), \quad X \mapsto \mathcal{J}_U(X),$$

between ideals I of B_U and subsets X of $\mathcal{G}_U$ as follows:

$$\mathfrak{X}_U(I) = \{(\alpha, \mathfrak{p}) \in \mathcal{G}_U;\ \mathfrak{p} \supset IA_\alpha\}, \quad \mathcal{J}_U(X) = \{f \in B_U;\ f_\alpha \in \mathfrak{p} \text{ for all } (\alpha, \mathfrak{p}) \in X\}.$$

The usual properties of *zero sets* and *ideals of zeros* can be easily checked, and we have a topology in $\mathcal{G}_U$ whose closed sets are those zero sets. Also, we say that a closed set is *irreducible* if it is not the union of two other different closed sets. Of course, we cannot discuss *irreducible decompositions*, as $\mathcal{G}_U$ need not be noetherian. However, we do have a Nullstellensatz:

Proposition 24 *For any ideal $I \subset B_U$, it holds $\mathcal{J}_U\mathfrak{X}_U(I) = \sqrt{I}$.*

Proof. (Compare [Qu2, 4.4].) It is clear from the definitions that $\mathcal{J}_U\mathfrak{X}_U(I) \supset \sqrt{I}$. Conversely, $f \in \mathcal{J}_U\mathfrak{X}_U(I)$ means that $f_\alpha \in \mathfrak{p}$ for every $\alpha \in U$ and every prime ideal $\mathfrak{p}$ of A_α such that $I_\alpha \subset \mathfrak{p}$, so that $f_\alpha \in \sqrt{IA_\alpha}$ for every $\alpha \in U$. Now, let $\mathfrak{m}$ be a maximal ideal of B_U, and α any prime cone supported on $\mathfrak{m}$. Since the morphism $B_{U,\mathfrak{m}} \to A_\alpha$ is local ind-étale, we deduce that $f \in \sqrt{IB_{U,\mathfrak{m}}}$. This shows that $f \in \sqrt{I}$. $\square$

From this, we deduce the following result, which is the abstract version of separation:

Corollary 25 (Separation for U) *Let $\mathfrak{p}$ be a prime ideal of B_U. Then $\mathfrak{X}_U(\mathfrak{p})$ is irreducible.*

Proof. Suppose $\mathfrak{X}_U(\mathfrak{p}) \subset \mathfrak{X}_U(I) \cup \mathfrak{X}_U(J)$. Then from the Nullstellensatz we get $\mathfrak{p} \supset \sqrt{I \cap J}$, and $\mathfrak{p}$ being prime, either $\mathfrak{p} \supset I$ or $\mathfrak{p} \supset J$. Hence either $\mathfrak{X}_U(\mathfrak{p}) \subset \mathfrak{X}_U(I)$ or $\mathfrak{X}_U(\mathfrak{p}) \subset \mathfrak{X}_U(J)$. $\square$

Although this latter corollary can be seen formally as a generalization of separation over the reals (see [Qu2, 4.6]), its geometric meaning is more apparent when the rings of abstract Nash functions are noetherian. In that case, the space $\mathcal{G}_U$ is noetherian too, and we have a convincing notion of *analytic component* as an irreducible closed subset of $\mathcal{G}_U$, because we have a decomposition into (finitely many) irreducible components. Furthermore, the notion of a closed subset of $\mathcal{G}_U$ should be local with respect to the real spectrum. This can be explained as follows. Let $X \subset \mathcal{G}_U$ be a set defined by an open covering $U = \bigcup_i U_i$ and ideals $I_i \subset \mathcal{N}_A(U_i)$ in the form $X \cap \mathcal{G}_{U_i} = \mathfrak{X}_{U_i}(I_i)$. We can assume that the U_i's are constructible, that the covering is finite (compactness of U), and that the I_i's radical, and then the Nullstellensatz (Proposition 24) implies

$$I_i\mathcal{N}_A(U_i \cap U_j) = \mathcal{J}_{U_i \cap U_j}(X \cap \mathcal{G}_{U_i \cap U_j}) = I_j\mathcal{N}_A(U_i \cap U_j).$$

Thus we have a sheaf of radical ideals $\mathcal{I} \subset \mathcal{N}_A|U$ defined over each U_i by $\mathcal{I}|_{U_i} = I_i\mathcal{N}_A|_{U_i}$. However, we cannot conclude from this that $X = \mathfrak{X}(I)$ for the ideal $I = H^0(U, \mathcal{I})$: Global Equations holds for finite sheaves, and $\mathcal{I}$ need not be finite. Now, suppose that the rings $\mathcal{N}_A(U_i)$ are noetherian. Then each I_i is finitely generated, so that $\mathcal{I}$ is in fact finite, and X is a closed set of $\mathcal{G}_U$.

After these remarks, we are interested in the conditions that guarantee that rings of abstract Nash functions are noetherian. One has of course the condition that A be noetherian. In fact, under this assumption the rings $\mathcal{N}_A(U)$ are *locally noetherian*. For, any maximal ideal $\mathfrak{m}$ of $\mathcal{N}_A(U)$ is the support of a point $\alpha \in U$, and idempotency gives a faithfully flat morphism $\mathcal{N}_A(U)_{\mathfrak{m}} \to A_\alpha$; then, A being noetherian, so are A_α and $\mathcal{N}_A(U)_{\mathfrak{m}}$. However this is not enough for $\mathcal{N}_A(U)$ to be noetherian, as one realizes when A is a field with infinitely many orderings.

To deal with this matter, we consider the ring of fractions $A' = A[S^{-1}]$ with respect to the multiplicative subset S of all $a \in A$ such that $a(\alpha) \neq 0$ for all $\alpha \in \mathrm{Spec_r}(A)$. The formal Positivstellensatz ([BCR, 4.4.1]) implies that the ring A' can also be described as the ring of fractions of A with respect to the multiplicative subset of elements of the form $1 + \sum_{i=1}^r a_i^2$. Clearly, neither the real spectrum nor the sheaf of abstract Nash functions change via $A \to A'$, and now the ind-étale morphism $A' \to \mathcal{N}_A(\mathrm{Spec_r}(A))$ is *faithfully* flat.

Proposition 26 *Suppose that for every open constructible set $U \subset \mathrm{Spec_r}(A)$ the ring of Nash functions $\mathcal{N}_A(U)$ is noetherian. Then A' is noetherian, and every constructible set of $\mathrm{Spec_r}(A)$ has finitely many connected components.*

Proof. That A' is noetherian follows readily from the fact that $\mathcal{N}_A(\mathrm{Spec_r}(A))$ is noetherian and $A' \to \mathcal{N}_A(\mathrm{Spec_r}(A))$ faithfully flat. Now let us prove the assertion concerning connected components. It is enough to prove it for a constructible set of the form $T = \{a = 0, b_1 > 0, \ldots, b_r > 0\}$. If this set has infinitely many connected components, we obtain a sequence $T_k \subset T$ of closed disjoint constructible subsets of $U = \{b_1 > 0, \ldots, b_r > 0\}$ which furthermore are open in T. Next, suppose we are given Nash functions $f_k \in \mathcal{N}_A(U)$ such that $f_k < 0$ on $S_k = T_1 \cup \cdots \cup T_k$ and $f_k > 0$ on $F_k = T \setminus S_k$. We note that $h_k = f_k - \sqrt{a^2 + f_k^2}$ is a well defined Nash function on U, which coincides with $f_k - |f_k|$ on T. Hence h_k does not vanish at any point of S_k and $h_k|_{F_k} \equiv 0$. Thus we have an infinite sequence of ideals of $\mathcal{N}_A(U)$, namely

$$(h_1) \subset (h_1, h_2) \subset \cdots \subset (h_1, h_2, \ldots, h_k) \subset \cdots,$$

and $\mathcal{N}_A(U)$ is not noetherian.

Whence it only remains to show that the f_k's do exist. But this is relative separation of disjoint closed constructible sets, which can be proved by change of base (see [Mh, 4.3]). Let $U \subset \mathrm{Spec_r}(A)$ be open constructible and $S, F \subset U$ disjoint constructible sets that are closed in U. As usual (Lemma 19), we find a finitely presented algebra A^0, and a morphism $A^0 \to A$ such that the constructible sets U, S, F come from A^0: we have $U^0 \subset \mathrm{Spec_r}(A^0)$ open constructible and $S^0, F^0 \subset U^0$ disjoint constructible sets, closed in U^0, which can be separated by a Nash function $f^0 \in \mathcal{N}_{A^0}(U^0)$ (by the relative Mostowski's separation lemma for closed semialgebraic sets over $\mathbb{R}_{\mathrm{alg}}$, [BCR, 2.7.7]). If $f^0 \mapsto f$ via this change of base, we conclude that f separates S and F. We are done. $\square$

The preceding proposition gives two necessary conditions for rings of Nash functions to be noetherian. The property concerning connected components has been used before in a different context (see [ABR, I.3.5]). To find a sufficient condition for noetherianity is a more involved matter, but an easy first result related to separation is this:

Proposition 27 *Suppose that A' is noetherian. Let $U \subset \mathrm{Spec_r}(A)$ be an open constructible set. Then $\mathcal{N}_A(U)$ is noetherian if and only if so is the space $\mathcal{G}_U$.*

Proof. (Compare with [Qu2, 1.8]) We can suppose $A = A'$. The only if part has been already mentioned. Hence, we suppose that $\mathcal{G}_U$ is noetherian. For $\mathcal{N}_A(U)$ to be noetherian, it is enough that every prime ideal $\mathfrak{p}$ of $\mathcal{N}_A(U)$ is finitely generated (Cohen's criterion). Then consider the closed set $\mathfrak{X}(\mathfrak{p})$. Since $\mathcal{G}_U$ is noetherian, there are $f_1, \dots, f_r \in \mathfrak{p}$ such that $\mathfrak{X}(\mathfrak{p}) = \mathfrak{X}(f_1, \dots, f_r)$, and from the Nullstellensatz we get

$$\mathfrak{p} = \mathcal{I}\mathfrak{X}(\mathfrak{p}) = \mathcal{I}\mathfrak{X}(f_1, \dots, f_r) = \sqrt{(f_1, \dots, f_r)B_U}$$

for some $f_1, \dots, f_r \in B_U$. Now, by the Artin-Mazur property, we find an AM-datum $\mathfrak{C} = (\mathrm{Spec_r}(A), C, \sigma)$ and elements $b_i \in C$ which represent the Nash functions f_i. But C is finitely presented over A, hence C is noetherian as A is, and $\sqrt{(b_1, \dots, b_r)C} = (c_1, \dots, c_s)C$ for some $c_j \in C$. Let $g_1, \dots, g_s \in B_U$ be the Nash functions represented by $c_1, \dots, c_s$, that is, $b_i, c_j \mapsto f_i, g_j$ via $C \to B_U$. As the morphism $C \to B_U$ is ind-étale, we get

$$\mathfrak{p} = \sqrt{(f_1, \dots, f_r)B_U} = \sqrt{(b_1, \dots, b_r)}B_U = (c_1, \dots, c_s)B,$$

and we are done. $\square$

Finally, we quote [Qu2, 3.13,4.1], which says that if the necessary conditions of Proposition 26 hold and the ring A' is *excellent*, then the spaces $\mathcal{G}_U$ are noetherian. In fact, the argument given in [Qu2] for $U = \mathrm{Spec_r}(A)$ works in general. Hence this implies immediately that $\mathcal{N}_A(U)$ is noetherian. We even obtain the following proposition (proved in [Cu] for A a finitely generated algebra over a real closed field, assuming idempotency) :

Proposition 28 *Suppose that A is excellent, and every constructible subset of $\mathrm{Spec_r}(A)$ has finitely many connected components. Then the rings $\mathcal{N}_A(U)$ of Nash functions on open constructible sets $U \subset \mathrm{Spec_r}(A)$ are excellent.*

Proof. By the preceding remark, $\mathcal{N}_A(U)$ is noetherian. On the other hand, the morphism $A \to \mathcal{N}_A(U)$ is ind-étale. This implies that $\mathcal{N}_A(U)$ is excellent ([G, 5.3]). $\square$

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