Generalized complementing maps

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Dedicated to professor Enrique Outerelo on the occasion of his retirement.

ABSTRACT

In this paper we apply the generalized degree introduced by Geba, Massabo and Vignoli, in [3], to extend the notion of complementing maps defined by Fitzpatrick, Massabo and Pejsachowicz, in [1] and [2]. On the other hand, we obtain, in low dimension, a bifurcation result in terms of the linking number of some 1-dimensional manifolds. We also present a global theorem that improves a Rabinowitz's type result contained in [3] concerning the generalized degree.

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Introduction

P.M. Fitzpatrick, I. Massabo and J. Pejsachowicz, in [1] and [2], proved some theorems whose only assumption is that the Leray–Schauder degree of a given compact vector field is nonzero and whose conclusions allow to obtain information about the structure and dimension of the set of solutions of some nonlinear equations with several parameters.

The aim of this paper is to generalize the idea of complementing maps (introduced in [1] and [2]) via the generalized degree presented by K. Geba, I. Massabo and A. Vignoli in [3]. In [3] it is defined a degree theory for continuous maps $f : (\overline{U}, \partial U) \to (\mathbb{R}^n, \mathbb{R}^n \backslash \{0\})$ where U is an open and bounded subset of $\mathbb{R}^{n+k}$, $k \in \mathbb{N} \cup \{0\}$. The authors used algebraic topology methods (homotopy groups of spheres). In order to solve essentially the additivity property problem, we gave, see [11] and [14], a

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differential version of this degree based on cobordism theory (see [10]). We extract here some of these notions that we will use consistently:

Let $M^k \subset \mathbb{R}^{n+k}$ be a smooth (C^∞) closed manifold of dimension k and let $F = \{u_1, u_2, \cdots, u_n\}$ be a family of smooth sections of the normal vector bundle, $\nu(M^k)$, of $M^k \subset \mathbb{R}^{n+k}$.

F is a *frame* for M^k if $\{u_1(x), u_2(x), \cdots, u_n(x)\}$ is a linearly independent system for every $x \in M^k$. A *k-framed manifold* of $\mathbb{R}^{n+k}$ $(k - FM)$ $(k \in \mathbb{N} \cup \{0\})$ is a pair (M^k, F), where M^k is a closed manifold of dimension k and F is a frame for M^k. Two $k - FM$ (M_1^k, F_1) and (M_2^k, F_2) are said to be *homologous*, written $(M_1^k, F_1) \approx (M_2^k, F_2)$, if there exists a compact $(k+1)$-dimensional smooth submanifold, M^{k+1}, of $\mathbb{R}^{n+k} \times I$ and there exists a frame F for M^{k+1} such that:

a) $\partial M^{k+1} = M^{k+1} \cap (\mathbb{R}^{n+k} \times \{0\}) \cup M^{k+1} \cap (\mathbb{R}^{n+k} \times \{1\})$.

b) It exists $1/2 > \delta > 0$ such that $M^{k+1} \cap (\mathbb{R}^{n+k} \times \{t\}) = M_1^k \times \{t\}$ for every $t \in [0, \delta)$ and $M^{k+1} \cap (\mathbb{R}^{n+k} \times \{t\}) = M_2^k \times \{t\}$ for every $t \in (1-\delta, 1]$.

c) $F|_{M_1^k \times \{0\}} = F_1$ and $F|_{M_2^k \times \{1\}} = F_2$.

Now we consider $F^k(\mathbb{R}^{n+k})$ the set of homology classes of $k - FM$ of $\mathbb{R}^{n+k}$. Take a C^∞-map $f : S^{n+k} \to S^n$. Let q' and p' be Let q' (q) and p' (p) be respectively the north and south poles of S^{n+k} (S^n).

Let $\psi_{n+k} : \mathbb{R}^{n+k} \to S^{n+k} \backslash \{q'\}$ $(\psi_n : \mathbb{R}^n \to S^n \backslash \{q\})$ be the inverse of the projection from q' (q). Take any regular value of f $((r.v)(f))$ $s \in S^n$ such that $s \neq f(q')$, and $V = \{v_1, v_2, \cdots, v_n\}$ any positive basis of $T_s S^n$ (we assume that S^n has the usual orientation). Then $f \circ \psi_{n+k}$ is a C^∞-map and $s \in (r.v.)(f \circ \psi_{n+k})$. From f, s and V, we can define the $k - FM$ (M_f^k, F_f), where $M_f^k = (f \circ \psi_{n+k})^{-1}(s)$ and $F_f^V = \{u_1, u_2, \cdots, u_n\}$ is the frame such that $T_x(f \circ \psi_{n+k})(u_j(x)) = v_j$, $j \in \{1, \cdots, n\}$. $[(M_f^k, F_f^V)]$ only depends on the homotopy class of f and the map $\Pi_n^k : \Pi_{n+k}(S^{n+k}) \to F^k(\mathbb{R}^{n+k})$, defined by $\Pi_n^k([f]) = [(M_f^k, F_f)]$ is an isomorphism (see [10]).

Let $f : (\overline{U}, \partial U) \to (R^n, R^n \backslash \{0\})$ be a C^∞-map (i.e. f has an extension to an open set V, containing $\overline{U}$, which is C^∞) and assume that $0 \in (r.v.)(f|_U)$. In this situation, we assign to f the element $(M_f^k, F_f) \in F^k(\mathbb{R}^{n+k})$, where $M_f^k = f^{-1}(0) \subset U$ and $F = \{u_1, u_2, \cdots, u_n\}$ is a frame for M_f^k such that $Df(x)(u_j(x)) = e_j$ for any $j \in \{1, \cdots, n\}$ and $x \in M_f^k$ ($\{e_1, \cdots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$). We define the *generalized degree* of f on U by $d(f, U) = (\Pi_n^k)^{-1}([(M_f^k, F_f)])$. If f is a continuous map we define $d(f, U)$ by considering an adequate smooth approximation of f ([11]).

The structure of the present work is the following: after this introduction, Section 1 is devoted to introduce G-complementing maps and to check the relations between complemented and G-complemented maps (Remark 2 and Example 3). Later we essentially prove Proposition 1.7 which contains Theorem 2.1 of [1] as a particular case.

We also show how to apply G-complementing maps in order to obtain similar results to Theorem 1.1 of [1], when U is bounded, using 0-epi maps ([8]). Finally, Proposition 2.6 in Section 2 presents an application to bifurcation problems in the spirit of [3] that improves a global Rabinowitz's type theorem where we make no dimensional assumptions (see [4]). The results of [11] where we give sufficient conditions for the degree's additivity property to hold are useful in the proof of the mentioned Proposition 2.6.

On the other hand, we obtain, in low dimension, a bifurcation result in terms of the linking number of some 1-dimensional manifolds (Corollary 2.5).

1. G-complementing mappings

Definition 1.1 *Let E be a normed vectorial space, let $U \subset \mathbb{R}^m \times E$ be an open subset such that the first projection $p_1(U) \subset \mathbb{R}^m$ is bounded. Consider $f : \overline{U} \to E$ to be a continuous map of the form $f(\lambda, x) = x - F(\lambda, x)$ where $F : \overline{U} \to E$ is a compact map in the sense that F is continuous and $F(\overline{U})$ has compact closure. We will say that a compact map $g : \overline{U} \to \mathbb{R}^k$, $k \leq m$, is a* generalized complement *(simplified G-complement) for $f : \overline{U} \to E$ provided the map $(g, f) : \overline{U} \subset \mathbb{R}^m \times E \to \mathbb{R}^k \times E$ does not vanish on ∂U and has nonzero generalized degree.*

Obviously if f can be complemented then f also can be G-complemented.

Notice that the generalized degree theory can be extended to compact perturbations of the p_2 projection. It is not necessary E to be a Banach space neither U to be bounded. It is sufficient that the above assumptions hold (see [7]).

Remark 1.2 *There exist maps $f : \overline{U} \to E$ that can be complemented but not G-complemented by any $g : \overline{U} \to \mathbb{R}^k$ where $k < m$. Indeed, if $h : (\overline{B}^{m+1}(0), S^m) \to (\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \backslash \{0\})$ is a continuous map and one defines the map $h_0 : S^m \to S^n$ by $h_0(x) = h(x)/\|h(x)\|$, using the suspension property, [3] page 65, one obtains that $d(h, B^{m+1}(0)) = \Sigma([h_0])$ ($\Sigma : \Pi_m(S^n) \to \Pi_{m+1}(S^{n+1})$ denotes the suspension homomorphism). Therefore if we take $f : \overline{B}^3(0) \to \mathbb{R}$, defined by $f(x_1, x_2, x_3) = x_3$, it is clear that $g : \overline{B}^3(0) \to \mathbb{R}^2$ defined by $g(x_1, x_2, x_3) = (x_1, x_2)$, is a complement of f. However there is no $h : \overline{B}^3(0) \to \mathbb{R}$ such that $(h, f)(S^2) \subset \mathbb{R}^2 \backslash \{0\}$ and $d((h, f), B^3(0)) \neq 0$ because $\Pi_2(S^1) = 0$ and then $d((h, f), B^3(0)) \in Im\Sigma = \{0\}$.*

G-complementing maps are of interest because there exist open subsets U and maps $f : \overline{U} \to E$ that admit no complement using the classical degree but they can be G-complemented. Now we will construct such an example.

Example 1.3 *We will find an open and bounded subset U of $\mathbb{R}^5$ and a continuous map $f : \overline{U} \to \mathbb{R}^3$ that can be G-complemented but no complemented. It suffices to construct an open and bounded set $U \subset \mathbb{R}^5$ and $f : \overline{U} \to \mathbb{R}^3$ G-complemented such* *that* $f^{-1}(0) \cap \partial U = \cup_{j=1}^{r} S_j^4$ *where* $S_i^4 \cap S_j^4 = \emptyset$ *if* $i \neq j$ *and* S_j^4 *is homeomorphic to* S^4 *for every* $j \in \{1, 2, \cdots, r\}$. *In fact, if* $f : \overline{U} \to \mathbb{R}^3$, *satisfying the previous assumptions, could be complemented, there would exist a continuous map* $g : \overline{U} \to \mathbb{R}^2$, *such that* $d((g, f), U) \neq 0$. *Using Prop. 3.1 of [1] (page 784) one obtains that* $g : (f^{-1}(0), \cup_{j=1}^{r} S_j^4) \to (\mathbb{R}^2, \mathbb{R}^2 \backslash \{0\})$ *would induce nontrivial homomorphisms in Čech cohomology groups*

$$g^* : H^*(\mathbb{R}^2, \mathbb{R}^2 \backslash \{0\}) \to H^*(f^{-1}(0), \cup_{j=1}^{r} S_j^4).$$

On the other hand, since $\Pi_4(S^1) = 0$ *we deduce that the restriction of* $g : \cup_{j=1}^{r} S_j^4 \to \mathbb{R}^2 \backslash \{0\}$ *is homotopic to a constant map and therefore* g^* *is trivial.*

We will make the construction as follows:

Take a smooth map $h : (\overline{B}^5(0), S^4) \to (\mathbb{R}^4, \mathbb{R}^4 \backslash \{0\})$ such that $d(h, B^5(0)) \neq 0$. By the homotopy invariance property there is no loss of generality on assuming that $h(x) = (g_3(x), h_4(x)) \in \mathbb{R}^3 \times \mathbb{R}$ where g_3 is a C^∞-map and $0 \in (r.v.)(g_3|_{S^4})$. It is clear that g_3 is G-complemented by h_4 and $g_3^{-1}(0) \cap S^4 = \cup_{j=1}^{r} M_j$, where M_j is a smooth manifold of S^4 diffeomorphic to S^1 for every $j \in \{1, \cdots, r\}$.

Now we consider local coordinates $\{c_j : j \in \{1, \cdots, r\}\}$ of $\overline{B}^5(0)$, as a submanifold of $\mathbb{R}^5$, $c_j = (V_j, \phi_j)$, such that for any j:

a) $V_i \cap V_j = \emptyset$ if $i \neq j$.

b) $0 \in \phi_j(V_j)$ and $\phi(V_j)$ is an open, bounded and convex subset of $\mathbb{R}^5$.

c) $\phi_j(V_j \cap \overline{B}^5(0)) = \phi_j(V_j) \cap \mathbb{R}^5_+$.

For every $j \in \{1, \cdots, r\}$, we define $S_j^1 = \{x = (x_1, x_2, 0, 0, 0) \in \mathbb{R}^5 : x_1^2 + x_2^2 = \delta_j\} \subset \phi_j(V_j)$ (we choose δ_j to be small enough such that $S_j^1 \subset \phi_j(V_j)$). Let S_j^2 be a two-dimensional sphere contained in $[(\{0, 0\} \times \mathbb{R}^3_-) \cap \phi_j(V_j))] \backslash \mathbb{R}^5_+$.

Consider $S_j^1 * S_j^2 = \{t\mathbf{x} + (1-t)\mathbf{y} : t \in I, \mathbf{x} \in S_j^1, \mathbf{y} \in S_j^2\}$. Then $S_j^1 * S_j^2$ is homeomorphic to S^4 and $(S_j^1 * S_j^2) \cap \mathbb{R}^5_+ = S_j^1$.

As a consequence of the Jordan Separation Theorem, there exists an open and bounded subset U_j of $\mathbb{R}^5$ such that $\partial U_j = S_j^1 * S_j^2$, $U_j \subset \phi_j(V_j) \cap \mathbb{R}^5_-$ (U_j is the bounded component of $\mathbb{R}^5 \backslash (S_j^1 * S_j^2)$). Thus, $\phi_j^{-1}(S_j^1 * S_j^2) = \partial \phi_j^{-1}(U_j)$ (homeomorphic to S^4).

Let $D_j = \phi_j^{-1}(U_j)$, using that $\phi_j(V_j \cap \overline{B}^5(0)) = \phi_j(V_j) \cap \mathbb{R}^5_+$ and $\overline{U}_j \cap \mathbb{R}^5_+ = S_j^1$ it follows that $D_j \cap \overline{B}^5(0) = \phi_j^{-1}(S_j^1)$.

There exists a C^∞-isotopy $H : S^4 \times I \to S^4$ such that $H_0 = Id$ and $H_1(\phi_j^{-1}(S_j^1)) = M_j, j \in \{1, \cdots, r\}$. Now we extend the map $\overline{H} : ((\overline{B}^5(0) \times \{0\}) \cup (S^4 \times I)) \to \overline{B}^5(0)$ defined by $\overline{H}(x, 0) = x$ if $x \in \overline{B}^5(0)$ and $\overline{H}(x, t) = H(x, t)$ if $(x, t) \in S^4 \times I$, to a continuous map $G : (\overline{B}^5(0) \times I, S^4 \times I) \to (\overline{B}^5(0), S^4)$.

Let $h \circ G : (\overline{B}^5(0) \times I, S^4 \times I) \to (\mathbb{R}^4, \mathbb{R}^4 \backslash \{0\})$.

Then,

$$d(h \circ G_1, B^5(0)) = d(h \circ G_0, B^5(0)) = d(h, B^5(0)) \neq 0.$$

Therefore, writing $(h\circ G_1)(x) = ((g_3\circ G_1)(x), (h_4\circ G_1)(x)) \in \mathbb{R}^3\times\mathbb{R}$, we deduce that $g_3\circ G_1$ can be G-complemented. Moreover $((g_3\circ G_1)|_{S^4})^{-1}(0) = \cup_{j=1}^r \phi_j^{-1}(S_j^1)$.

We define $f : \overline{B}^5(0)\cup(\cup_{j=1}^r D_j) \to \mathbb{R}^3$ by

$$f(x) = 0 \quad \text{if} \quad x\in \cup_{j=1}^r D_j \quad \text{and} \quad f(x) = (g_3\circ G_1)(x) \quad \text{if} \quad x\in \overline{B}^5(0).$$

Since $\overline{B}^5(0)\cap D_j = \phi_j^{-1}(S_j^1)$ for every $j\in\{1,\cdots,r\}$, it follows that f is a continuous map.

On the other hand, $(h_4\circ G_1)|_{\cup_{j=1}^r \phi_j^{-1}(S_j^1)}$ does not vanish.

Thus, $(h_4\circ G_1)|_{\phi_j^{-1}(S_j^1)}$ has constant sign. Then, we can obtain a non vanishing extension L_j of $(h_4\circ G_1)|_{\phi_j^{-1}(S_j^1)}$ to D_j.

Let $L : \overline{B}^5(0)\cup(\cup_{j=1}^r D_j)\to\mathbb{R}$ be the continuous map defined by $L(x) = L_j(x)$ if $x\in D_j$ and $L(x) = (h_4\circ G_1)(x)$ if $x\in\overline{B}^5(0)$.

Take $(f,L) : \overline{B}^5(0)\cup(\cup_{j=1}^r D_j)\to\mathbb{R}^4$, then $(f,L)^{-1}(0)\subset B^5(0)$. It follows that $d((f,L),\overline{B}^5(0)\cup(\cup_{j=1}^r D_j)) = d((f,L),\overline{B}^5(0)) \neq 0$. Therefore, f can be G-complemented.

However $f^{-1}(0)\cap\partial(\overline{B}^5(0)\cup(\cup_{j=1}^r D_j)) = \cup_{j=1}^r\phi_j^{-1}(S_j^1 * S_j^2)$ and each $\phi_j^{-1}(S_j^1 * S_j^2)$ is homeomorphic to S^4. □

Notice that the above construction works by starting with a nontrivial class $[\psi]\in\Pi_n(S^{n-1})$ $(n>3)$. Thus we obtain that for every $n>3$, there exist an open and bounded $U\subset\mathbb{R}^{n+1}$ and a continuous map $f:\overline{U}\to\mathbb{R}^{n-1}$ that can be G-complemented but not complemented.

Next we will obtain some results for G-complementing maps extending Theorems 2.1 and 1.1 of [1], when U is bounded.

In order to carry out our first purpose we need a generalized degree theory for maps $g : M^{n+k}\to M^n$ where M^{n+k} and M^n are compact $n+k$ and n-manifolds respectively. We are going to recall briefly here the most important concepts and propositions contained in [12] that we will use later. The reader is referred to the text of [9], [10] and [11] for more information about this point of view of considering the generalized degree, and the necessary differential topology machinery.

A k-framed manifold of a oriented riemannian manifold M^{n+k} is a pair (M^k,F) where M^k is a compact k-submanifold of M^{n+k} without boundary, contained in $Int(M^{n+k}) = M^{n+k}\backslash\partial M^{n+k}$ and $F=\{u_1,\cdots,u_n\}$ is a frame for M^k in M^{n+k} (see the introduction).

Observe that if (M^k,F) is a framed manifold then it is clear that the normal vector bundle of $M^k\subset M^{n+k}$, denoted by $\nu(M^k)$, is trivial. Let $k-FM(M^{n+k}) = \{(M^k,F) : (M^k,F)$ is a k-framed manifold of $M^{n+k}\}$. By following the same ideas exposed at the beginning of the paper one can define a homology relation between two framed manifolds of M^{n+k}, of course one must require the $(k+1)$-manifold that achieves the homology to be contained in $Int(M^{n+k}) \times I$. We will denote by $F^k(M^{n+k}, \partial M^{n+k})$ the corresponding quotient set of homology classes.

Denote the set of homotopy classes of the maps $f : (M^{n+k}, \partial M^{n+k}) \to (S^n, q)$ by $[M^{n+k}, \partial M^{n+k}; S^n, q]$. Now we are in a position to introduce the following generalization of Pontryagin's theorem (see the introduction).

Let M^{n+k} be a compact $(n+k)$-manifold satisfying the above assumptions.

Define $P_n^k : [M^{n+k}, \partial M^{n+k}; S^n, q] \to F^k(M^{n+k}, \partial M^{n+k})$ by $P_n^k([h]) = [(M_f^k, F_f)]$ where $f : (M^{n+k}, \partial M^{n+k}) \to (S^n, q)$ is a smooth map such that $[h] = [f]$, $p \in (r.v.)(f)$, $M_f^k = f^{-1}(p)$ and $F_f = \{u_1, \cdots, u_n\}$ is the frame for M^k in M^{n+k} such that $T_x f(u_j(x)) = \Theta_{c'}^p(e_j)$ for every $j = 1, \cdots, n$ and $x \in M_f^k$. ($\Theta_{c'}^p : \mathbb{R}^n \to T_p S^n$ denotes the isomorphism induced by the local coordinates $c' = (U, \phi_n^{-1}), p \in U$).

Theorem 1.4 *The map $P_n^k : [M^{n+k}, \partial M^{n+k}; S^n, q] \to F^k(M^{n+k}, \partial M^{n+k})$ is an isomorphism.*

Next proposition give us a description in terms of framed manifolds of the coboundary map on cohomotopy groups that we shall use later. Observe that boundary map has an important role when Theorem 2.1 is proved in [1].

Proposition 1.5 *Consider M^{n+k} as in Theorem 1.4. Let*

$$\delta : \Pi^{n-1}(\partial M^{n+k}) \to \Pi^n(M^{n+k}, \partial M^{n+k})$$

be the coboundary mapping. Let (U, H) be a collared neighborhood of ∂M^{n+k}, $H : U \to \partial M^{n+k} \times [0,1)$.

Consider

$$[(M^k, F)] \in F^k(\partial M^{n+k})$$

and

$$[(M^k \times \{1/2\}, \delta F)] \in F^k(\partial M^{n+k} \times [0,1), \partial M^{n+k} \times \{0\}),$$

where $\delta F = \{F, \Theta_{c_1(1)}^{1/2}\} = \{u_1, \cdots, u_{n-1}, u_n\}$ ($c_1 = [0,1), 1_{[0,1)}$).

Define

$$\overline{\delta}([(M^k, F)]) = [(H^{-1}(M^k \times \{1/2\}, H^*(\delta F)] \in \Pi^n(M^{n+k}, \partial M^{n+k})$$

where $H^*(\delta F) = \{v_1, \cdots, v_n\}$ is such that $T_x H(v_i(x)) = u_i(H(x))$ for any $i = 1, ..., n$. Then the following diagram

$$\begin{array}{ccc} \Pi^{n-1}(\partial M^{n+k}) & \xleftarrow{\delta} & \Pi^n(M^{n+k}, \partial M^{n+k}) \\ {\scriptstyle \Pi_{n-1}^k}\big\downarrow & & {\scriptstyle \Pi_n^k}\big\downarrow \\ F^k(\partial M^{n+k}) & \xleftarrow{\overline{\delta}} & F^k(M^{n+k}, \partial M^{n+k}) \end{array}$$

commutes.

We will work hereafter with π-*manifolds.* We will say that a manifold M is a π-manifold if there is an embedding $f : M \to \mathbb{R}^m$ such that the normal bundle $\nu(f(M))$ is trivial.

Let M^{n+k} be a compact π-manifold, $n \geq k+2$. Let $f : M^{n+k} \to \mathbb{R}^{n+k+s}$ be any embedding such that $\nu(f(M^{n+k})$ is trivial. Take any frame for $f(M^{n+k})$, $U = \{u_1, \cdots, u_s\}$.

We define a homomorphism $U_f^* : F^k(M^{n+k}, \partial M^{n+k}) \to F^k(\mathbb{R}^{n+k+s}) \equiv F^k(S^{n+k+s})$ (see [10] and [9]) by $U_f^*([(M^k, F)]) = [(M^k, \{F, U\})]$, where $\{F, U\}$ denotes the sections of frame F followed by the sections of U.

A result about the behavior of the maps U_f^* that we will need later is the following proposition ([12]).

Proposition 1.6 *Let M^{n+k} be a compact π-manifold, $f : M^{n+k} \to \mathbb{R}^{n+k+s}$ be an embedding such that $\nu(f(M^{n+k}))$ is trivial and U a frame for $f(M^{n+k})$. If $n \geq k+2$ the homomorphism U_f^* is onto.*

Now we are ready for our announced generalization of Theorem 2.1 of [1].

Proposition 1.7 *Let $U \subset \mathbb{R}^{n+k+m}$ be an open bounded such that ∂U is a $(n+m+k-1)$-submanifold of $\mathbb{R}^{n+k+m}$. Let $f : \overline{U} \to \mathbb{R}^m$ be a smooth map such that $0 \in (r.v.)(f) \cap (r.v.)(f|_{\partial U})$ and $f^{-1}(0) \cap \partial U \neq \emptyset$. If $\Pi_{n+k+m}(S^{n+m}) \neq 0$, $\delta : \Pi^{n-1}(\partial f^{-1}(0)) \to \Pi^n(f^{-1}(0), \partial f^{-1}(0))$ is onto and $n \geq k+2$ then f is G-complemented by a map $g : \overline{U} \to \mathbb{R}^n$.*

Before proving this proposition it is useful to state the following lemma.

Lemma 1.8 *In the previous hypothesis, $d((g, f), U) = d((g \circ r, f), W)$ where W is a tubular neighborhood of $f^{-1}(0)$ in $\overline{U}$ and $r : W \to f^{-1}(0)$ is the usual retraction.*

Proof. By the excision property $d((g, f), U) = d((g, f), W)$.

Let $\nu(f^{-1}(0))$ be the normal vector bundle of $f^{-1}(0) \subset \overline{U}$, D be an open neighborhood of the zero section and $exp : D \to W$ be the exponential diffeomorphism.

Define $H : D \times I \to \mathbb{R}^{n+m}$ by

$$H(x, v_x, t) = ((g \circ exp)(x, tv_x), (f \circ exp)(x, v_x)).$$

It follows that

$$d(H_0 \circ exp^{-1}, W) = d(H_1 \circ exp^{-1}, W) = d((g, f), W).$$

Since $H_0 \circ exp^{-1} = (g \circ r, f)$ the proof is complete. □

Proof of Proposition 1.7. Since $0 \in (r.v.)(f) \cap (r.v.)(f|_{\partial U})$, $f^{-1}(0)$ is a $(n+k)$-submanifold of $\overline{U}$. We will denote $f^{-1}(0)$ by M^{n+k} and $\partial M^{n+k} = f^{-1}(0) \cap \partial U \neq \emptyset$.

Let $V = \{v_1, \cdots, v_m\}$ be a frame for M^{n+k} (in $\mathbb{R}^{n+m+k}$) such that $Df(x)(v_j(x)) = e_j$ for every $j = 1, \cdots, m$ and $x \in M^{n+k}$.

Consider $g : (M^{n+k}, \partial M^{n+k}) \to (\mathbb{R}^n, S^{n-1})$ to be a smooth map, $0 \in (r.v.)(g)$.

Associated to g, one has $M^k = g^{-1}(0)$, a submanifold without boundary of $M^{n+k} \subset Int(M^{n+k}) \subset U$, $F_g = \{w_1, \cdots, w_n\}$ a frame for M^k (in M^{n+k}) such that $[(M^k, F_g)] \in F^k(M^{n+k}, \partial M^{n+k})$.

Let $\overline{g} : \overline{U} \to \mathbb{R}^n$ be a continuous extension of g. First we will prove that $d((g, f), U) = ((\Pi^k_{n+m})^{-1} \circ V^*)([(M^k, F_g)]) \in \Pi_{n+m+k}(S^{n+m})$.

It suffices to see that

$$d((\overline{g} \circ r, f), W) = ((\Pi^k_{n+m})^{-1} \circ V^*)([(M^k, F_g)]).$$

Indeed, since $\overline{g} \circ r = g \circ r$ it is clear that $\overline{g} \circ r$ is a smooth map and $T_x(\overline{g} \circ r) = T(g \circ r) : TW \to T\mathbb{R}^n$. We know that $T_x f(v_i(x) = \Theta^x_{c_m}(e_i)$ for every $i \in \{1, \cdots, m\}$ and $x \in M^{n+k}$ ($c_m = (\mathbb{R}^m, 1_{\mathbb{R}^m})$) and that $T_x g(w_j(x) = \Theta^x_{c_n}(e_j)$ for every $j \in \{1, \cdots, n\}$ and $x \in M^k$ ($c_n = (\mathbb{R}^n, 1_{\mathbb{R}^n})$).

Then it is easy to see that the normal frame $F_{(\overline{g} \circ r, f)}$ for M^k is $\{F_g, V\}$, therefore $d((\overline{g} \circ r, f), W) = ((\Pi^k_{n+m})^{-1} \circ V^*)([(M^k, F_g)])$.

There exists $[(M^k_*, F)] \in F^k(M^{n+k}, \partial M^{n+k})$, such that $V^*([(M^k_*, F)]) \neq 0$ (see Proposition 1.6). As a consequence, if we find a smooth map $g : (M^{n+k}, \partial M^{n+k}) \to (\mathbb{R}^n, S^{n-1})$ such that $0 \in (r.v.)(g)$ and $[(g^{-1}(0), F_g)] = [(M^k_*, F)]$, by extending g to a continuous map $\overline{g} : \overline{U} \to \mathbb{R}^n$ we will have that $\overline{g}$ is a G-complement for f and the proof will be completed.

Since $\delta : \Pi^{n-1}(\partial M^{n+k}) \to \Pi^n(M^{n+k}, \partial M^{n+k})$ is onto, there is $[(M^k_1, F_1)]$ such that $\overline{\delta}([(M^k_1, F_1)]) = [(M^k_*, F)]$ (Proposition 1). Then, there exists a smooth map $g : (M^{n+k} \partial M^{n+k}) \to (\mathbb{R}^n, S^{n-1})$ such that $0 \in (r.v.)(g)$ and $[(g^{-1}(0), F_g)] = [(M^k_*, F)]$. □

Remark 1.9 *Obviously the hypothesis* $\Pi_{n+m+k}(S^{n+m}) \neq 0$ *in the previous proposition is essential, the assumption that* $\delta : \Pi^{n-1}(\partial M^{n+k}) \to \Pi^n(M^{n+k}, \partial M^{n+k})$ *) to be an epimorphism, not imposed in the case* $k = 0$ *in [1], automatically holds if* $k = 0$ *because of the fact that for any* $m \in \mathbb{Z}$ *there exists a map* $g : (M^n, \partial M) \to (\overline{B}^n(0), S^n)$ *such that* $d(g) = m$. *Thus the additional hypothesis that we request are quite natural.*

Notice also that we actually used the fact Imd *is not completely contained in* $KerV^*$ *. Then this assumption may substitute the hypothesis that* δ *to be onto.*

G-complementing maps, besides of showing some aspects about the generalized degree, can be used as a tool to find out if a given map is 0-epi on a given subset (see [8]) and consequently to obtain information about the structure and dimension of the set of solutions of some nonlinear equations.

As a example, from 0-epi maps theory one has a direct generalization of Theorem 1.1 of [1] (in the bounded case) once one checks the following proposition.

Proposition 1.10 *Let U be an open subset of $\mathbb{R}^m \times E$ such that $p_1(U)$ is bounded in $\mathbb{R}^m$. Let $h : \overline{U} \to E$ be a continuous map of the form $h(\lambda, x) = x - h_1(\lambda, x)$ where $h_1 : \overline{U} \to E$ is a compact map and $h(\partial U) \subset E \backslash \{0\}$. If $d(h, U) \neq 0$ then h is 0-epi on U and $\overline{U}$ (consequently is 0-essential on U and $\overline{U}$).*

Corollary 1.11 *Let U be an open subset of $\mathbb{R}^m \times E$ such that $p_1(U)$ is bounded in $\mathbb{R}^m$. Let $f : \overline{U} \to E$ be a continuous map of the form $f(\lambda, x) = x - F(\lambda, x)$ where $F : \overline{U} \to E$ is a compact map and let $g : \overline{U} \to \mathbb{R}^s$, $s \leq m$, be a G-complement for f. Then g is 0-epi on $f^{-1}(0) \cap U$ and $f^{-1}(0) \cap \overline{U}$ (and f is 0-epi on $g^{-1}(0) \cap U$ and $g^{-1}(0) \cap \overline{U}$). Consequently, we have:*

a) $dim(f^{-1}(0) \cap \overline{U}) \geq s$ and $dim(f^{-1}(0) \cap \partial U) \geq s - 1$.

b) $dim(g^{-1}(0) \cap \overline{U}) = dim(g^{-1}(0) \cap \partial U) = \infty$ (if $dim E = \infty$).

Moreover, if U is bounded there exists a closed and connected subset $\Sigma f^{-1}(0)$ such that g is 0-epi on $\Sigma \cap \overline{U}$ thus $dim(\Sigma \cap \overline{U}) \geq s$, $dim(\Sigma \cap \partial U) \geq s - 1$ and g is 0-essential on $\Sigma \cap \overline{U}$.

2. Some results on bifurcation

In what follows E is a Banach space, U is an open subset, not necessarily bounded, of $\mathbb{R}^k \times E$ such that $U \cap (\mathbb{R}^k \times \{0\}) \neq \emptyset$.

Let $f : U \to E$ be a continuous map and assume that $f(\lambda, 0) = 0$ for any $\lambda \in \mathbb{R}^k$ such that $(\lambda, 0) \in U$.

Consider the equation $f(\lambda, x) = 0$. The set $\{(\lambda, 0) \in U\}$ is called *set of trivial solutions* of the equation. The remaining solutions are called *nontrivial*.

Now let $f : S^{k-1} \times S^{n-1} \to \mathbb{R}^n \backslash \{0\}$ be a continuous map. Consider $\phi : \overline{B}^k(0) \times \overline{B}^n(0) \to \mathbb{R}$ to be any continuous map such that

a) $\phi(x, y) < 0$ if $(x, y \in B^k(0) \times S^{n-1}$.

b) $\phi(x, y) > 0$ if $(x, y) \in S^{k-1} \times B^n(0)$.

(thus $\phi(x, y) = 0$ if $(x, y) \in S^{k-1} \times S^{n-1}$).

Extend f to a map $\overline{f} : \overline{B}^k(0) \times \overline{B}^n(0) \to \mathbb{R}^n$ and define

$$F : (\overline{B}^k(0) \times \overline{B}^n(0), \partial(\overline{B}^k(0) \times \overline{B}^n(0))) \to (\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \backslash \{0\})$$

by $F(x, y) = (\overline{f}(x, y), \phi(x, y))$.

The homotopy class of F does not depend on the extension $\overline{f}$ of f neither on the choice of ϕ (whenever ϕ satisfies a) and b)).

The equality $\widehat{\chi}([f]) = d(F, B^k(0) \times B^n(0))$ defines a map
$\widehat{\chi} : [S^{k-1} \times S^{n-1}; \mathbb{R}^n \backslash \{0\}] \to \Pi_{n+k}(S^{n+1})$ (see [3] page 66).

Definition 2.1 *Let f be a map as above. Let $\lambda_0 \in \mathbb{R}^k$ such that $(\lambda_0, 0) \in U$. If every neighborhood of $(\lambda_0, 0)$ contains nontrivial solutions of $f(\lambda, x) = 0$, λ_0 is called a bifurcation point of the equation.*

Next proposition is a remark motivated by the proof of Proposition 4.2 of [3] where one can observe that the assumption $\widehat{\chi}([f]) \neq 0$ implies the existence of a sequence $\{\epsilon_m\} \to 0$ and G-complements $g_m : \overline{B}^k(0) \times \overline{B}^n_{\epsilon_m}(0) \to \mathbb{R}$ for f. It shows that one can obtain the same conclusion even though g has image in $\mathbb{R}^2$.

Proposition 2.2 *Let $f : U \to E$ be a continuous map of the form $f(\lambda, x) = x - F(\lambda, x)$, where $F : U \to E$ is a compact map. Assume that $f(\lambda, 0) = 0$ for any $\lambda \in \mathbb{R}^k$, $(\lambda, 0) \in U$, and that there exist $(\lambda_0, 0) \in U$ and $\epsilon_0 > 0$ such that $f(\lambda, x) \neq 0$ if $(\lambda, x) \in S^{k-1}_{\epsilon_0}(\lambda_0) \times (\overline{B}^E_{\epsilon_0}(0)\backslash\{0\}) \subset U$ ($S^{k-1}_{\epsilon_0}(\lambda_0) = \{\lambda \in \mathbb{R}^k : \|\lambda - \lambda_0\| = \epsilon_0\}$) is the boundary of $\overline{B}^k_{\epsilon_0}(\lambda_0) = \{\lambda \in \mathbb{R}^k : \|\lambda - \lambda_0\| \leq \epsilon_0\}$) and $\overline{B}^E_{\epsilon_0}(0) = \{x \in E : \|x\| \leq \epsilon_0\}$). Consider the restriction of f to $\overline{B}^k_{\epsilon_0}(\lambda_0) \times \overline{B}^E_{\epsilon_0}(0)$. If there is a sequence $\{\epsilon_m\} \to 0$ where $\epsilon_m \leq \epsilon_0$ for every $m \in \mathbb{N}$, such that for every $m \in \mathbb{N}$ f has a G-complement g_m onto either $\mathbb{R}$ if $k = 2$ or $\mathbb{R}^2$ if $k \geq 3$, then there exists $\lambda^* \in \overline{B}^k_{\epsilon_0}(\lambda_0)$ a bifurcation point of the equation $f(\lambda, 0) = 0$.*

We are interested on finding conditions that allow us to assert that $\widehat{\chi}([f]) \neq 0$, without using the auxiliary map ϕ.

In order to do that let us consider a map $f : S^{k-1} \times S^{n-1} \to \mathbb{R}^s\backslash\{0\}$ and take any continuous extension $\overline{f} : S^{k-1} \times \overline{B}^n(0) \to \mathbb{R}^s$ of f.

Let $H : S^{k-1} \times \overline{B}^n(0) \to S^{n+k-1}\backslash\{q\}$ be defined by $H(x, y) = (x, y)/\|(x, y)\|$. Note that H is a homeomorphism onto its image and $H(S^{k-1} \times B^n(0))$ is an open set of S^{n+k-1} with boundary $H(S^{k-1} \times S^{n-1})$.

Let $U = (\psi^{-1}_{n+k-1} \circ H)(S^{k-1} \times B^n(0))$. Thus U is an open set of $\mathbb{R}^{n+k-1}$. Consider the map $\overline{f} \circ H^{-1} \circ \psi_{n+k-1} : (\overline{U}, \partial U) \to (\mathbb{R}^s, \mathbb{R}^s\backslash\{0\})$.

We define $\delta(f) = d(\overline{f} \circ H^{-1} \circ \psi_{n+k-1}, U) \in \Pi_{n+k-1}(S^s)$.

The reader can show easily that $\delta(f)$ just depends on the homotopy class of f. Consequently, given a map $f : S^{k-1} \times S^{n-1} \to \mathbb{R}^s\backslash\{0\}$, in order to compute $\delta(f)$ we can assume that $f(S^{k-1} \times S^{n-1}) \subset S^{s-1}$ because $\delta(f) = \delta(f/\|f\|)$ (analogously $\widehat{\chi}([f]) = \widehat{\chi}([f/\|f\|])$ if $s = n$).

Proposition 2.3 *For any $f : S^{k-1} \times S^{n-1} \to \mathbb{R}^n\backslash\{0\}$ it follows that $\widehat{\chi}([f]) = \Sigma(\delta(f))$.*

Proof. There is no loss of generality on assuming that $Imf \subset S^{n-1}$. Observe that $H : S^{k-1} \times \overline{B}^n(0) \to S^{n+k-1}$ is restriction to $S^{k-1} \times \overline{B}^n(0)$ of the homeomorphism, that we will continue denoting by H, $H : \overline{B}^k(0) \times S^{n-1} \cup S^{k-1} \times \overline{B}^n(0) \to S^{n+k-1}$ defined again by $H(x, y) = (x, y)/\|(x, y)\|$.

Write $E^n_+ = \{(x_1, x_2, \cdots, x_{n+1}) \in S^n : x_{n+1} > 0\}$ and E^n_- is defined on the obvious way.

Extend f to a map $\overline{f} : \overline{B}^k(0) \times S^{n-1} \cup S^{k-1} \times \overline{B}^n(0) \to S^n$ such that $\overline{f}(B^k(0) \times S^{n-1}) \subset E^n_+\backslash S^{n-1}$ and $\overline{f}(S^{k-1} \times B^n(0) \subset E^n_-\backslash S^{n-1}$.

Therefore $\delta(f) = [\overline{f} \circ H^{-1}]$.

We consider the obvious extension $W : (\overline{B}^k(0) \times \overline{B}^n(0)) \backslash \{0\} \to S^{n+k-1}$ ($W(x,y) = (x,y)/\|(x,y)\|$) of H.

Take $\overline{f} \circ H^{-1} \circ W : (\overline{B}^k(0) \times \overline{B}^n(0)) \backslash B^{n+k}(0) \to S^n$.

Let $F : \overline{B}^k(0) \times \overline{B}^n(0) \to \mathbb{R}^{n+1}$ be a continuous extension of $\overline{f} \circ H^{-1} \circ W$. If we write $F(x,y) = (g(x,y), -\phi(x,y)) \in \mathbb{R}^n \times \mathbb{R}$, it follows that $F|_{S^{k-1} \times B^n(0)} = \overline{f}$, thus $\phi(x,y) > 0$ for every $(x,y) \in S^{k-1} \times B^n(0)$. On the other hand, $F|_{B^k(0) \times S^{n-1}} = \overline{f}$, thus $\phi(x,y) < 0$ for every $(x,y) \in B^k(0) \times S^{n-1}$. Consequently, $F|_{S^{k-1} \times S^{n-1}} = (f, 0)$ and g is an extension of f.

Therefore $\widehat{\chi}([f]) = d(F, B^k(0)) \times B^n(0))$. Since $F(x,y) \neq 0$ for all $(x,y) \in (\overline{B}^k(0) \times \overline{B}^n(0)) \backslash B^{n+k}(0)$ we have that $\widehat{\chi}([f]) = d(F, B^{n+k}(0))$.

Finally,

$$\widehat{\chi}([f]) = \Sigma([F|_{S^{n+k-1}}]) = \Sigma([\overline{f} \circ H^{-1} \circ W|_{S^{n+k-1}}]) = \Sigma([\overline{f} \circ H^{-1}]) = \Sigma(\delta(f)).$$

□

Corollary 2.4 *Let $f : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ be a map such that $f(\lambda, 0) = 0$ for every $\lambda \in \mathbb{R}^k$ and $f(S^{k-1} \times \overline{B}^n(0) \backslash \{0\})) \subset \mathbb{R}^n \backslash \{0\}$. If $\delta(f) \neq 0$ then there exists a bifurcation point of the equation $f(\lambda, x) = 0$, provided $k = 3$ and $n \geq 2$ or $n \geq k+2$.*

Computation of $\delta(f)$ instead of $\widehat{\chi}([f])$ presents the advantages of working in a lower dimension without the auxiliary map ϕ defined above. Therefore it is in low dimension when this advantage is better appreciated. As a consequence of Proposition 2.3 we obtain next nice corollary.

Corollary 2.5 *Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be a map such that $f(\lambda, 0) = 0$ and $f(S^1 \times \overline{B}^2(0) \backslash \{0\})) \subset \mathbb{R}^2 \backslash \{0\}$. Let $H : S^1 \times \overline{B}^2(0) \to S^3$ be as above, $U = \psi_3^{-1} \circ H)((S^1 \times B^2(0) \subset \mathbb{R}^3$ and $\epsilon = \frac{1}{2} dist(f \circ H^{-1} \circ \psi_3)(\partial U), 0)$. Let $g : (U, \partial U) \to (\mathbb{R}^2, \mathbb{R}^2 \backslash \{0\})$ be a smooth map ϵ-near to $f \circ H^{-1} \circ \psi_3$ and $a_0, a_1 \in B^2_\epsilon(0)$ be two regular values of g. If the linking number $L(g^{-1}(a_0), g^{-1}(a_1))$ is odd, then there exists $\lambda_0 \in B^2(0)$ bifurcation point of the equation $f(\lambda, x) = 0$.*

Proof. From Proposition 2.3, we have

$$\widehat{\chi}([f]) = \Sigma(\delta(f)) = \Sigma(d(f \circ H^{-1} \circ \psi_3, U)) = \Sigma(d(g, U)) \in \Pi_4(S^3).$$

Since a map $h : S^3 \to S^2$ satisfies that $\Sigma([h]) \neq 0$ if and only if the Hopf invariant of h, $\gamma(h)$, is odd ([10]), we have that $L(g^{-1}(a_0), g^{-1}(a_1)) = \gamma(d(g, U))$ is odd if and only if $\widehat{\chi}([f]) = \Sigma(d(g, U)) \neq 0$. □

Now let us consider $f : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ to be a smooth map such that

a) $f(\lambda, 0) = 0$ for any $\lambda \in \mathbb{R}^k$,

b) $\Delta = \{\lambda \in \mathbb{R}^k : D_x f(\lambda, 0) \notin GL(\mathbb{R}^n)\}$ is discrete.

Let S be the closure of the nontrivial solutions subset of the equation $f(\lambda, x) = 0$.

Let $\lambda_1 \in \Delta$ and $C(\lambda_1)$ be the connected component of S containing $(\lambda_1, 0)$. We assume $C(\lambda_1)$ to be bounded (compact). Hence $C(\lambda_1) \cap (\mathbb{R}^k \times \{0\}) = \{\lambda_1, \cdots, \lambda_m\} \times \{0\}$, where $\lambda_i \neq \lambda_j$ for $i \neq j$.

Proposition 4.2 of [3] can be improved. By applying Theorem 4.2 of [8] one obtains as a consequence that there is $\lambda_2 \in \Delta \backslash \{\lambda_1\}$ such that $(\lambda_2, 0) \in C(\lambda_1)$. Nevertheless this result is a direct corollary of next global proposition. Note that we won't require any dimensional condition, compare to [4], In the proof we can suppress the assumption $n \geq k + 4$ by using the sufficient conditions for the degree's additivity property to hold given in [11].

Proposition 2.6 *Let $f : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ to be a smooth map satisfying a) and b) as above. If $\lambda_1 \in \Delta$ and $C(\lambda_1)$ is bounded it follows that $\sum_{\alpha=1}^m \widehat{\chi}_{\lambda_\alpha}([f]) = 0$ ($\widehat{\chi}_{\lambda_\alpha}([f])$ denotes $\widehat{\chi}([f|_{S^k(\lambda_\alpha) \times S^n}]) \in \Pi_{n+k}(S^{n+1})$).*

Proof. Let U be an open and bounded subset of $\mathbb{R}^k \times \mathbb{R}^n$ $r > 0$ and $\rho > 0$ such that $C(\lambda_1) \subset U$, $\partial U \cap S = \emptyset$, $U \cap (\mathbb{R}^k \times B_r^n(0)) = \cup_{i=1}^m B_\rho^k(\lambda_i) \times B_r^n(0)$, $\overline{B}_\rho^k(\lambda_i) \cap \overline{B}_\rho^k(\lambda_j) = \emptyset$ for $i \neq j$ and $f(\cup_{i=1}^m S_\rho^{k-1}(\lambda_i) \times \overline{B}_r^n(0) \backslash \{0\})) \subset \mathbb{R}^n \backslash \{0\}$.

Consider $\phi : \cup_{i=1}^m \overline{B}_\rho^k(\lambda_i) \times \overline{B}_r^n(0) \to \mathbb{R}$ to be a continuous map such that

$\phi(\cup_{i=1}^m \overline{B}_\rho^k(\lambda_i) \times S_r^{n-1}(0)) \subset (-\infty, 0)$ and $\phi(\cup_{i=1}^m S_\rho^{k-1}(\lambda_i) \times B_r^n(0)) \subset (0, \infty)$.

Let H be the compact subset of $\overline{U}$ defined by

$H = C(\lambda_1) \cap (\cup_{i=1}^m \overline{B}_\rho^k(\lambda_i) \times S^{n-1}(0)) = C(\lambda_1) \cap (\cup_{i=1}^m B_\rho^k(\lambda_i) \times S_r^{n-1}(0))$.

$\phi(H) \subset (-\infty, m]$ for some $m < 0$. Thus ϕ admits a continuous extension $\overline{\phi}$: $C(\lambda_1) \cup (\cup_{i=1}^m \overline{B}_\rho^k(\lambda_i) \times \overline{B}_r^n(0)) \to \mathbb{R}$ such that $\overline{\phi}(C(\lambda_1) \backslash (\cup_{i=1}^m B_\rho^k(\lambda_i) \times B_r^n(0))) \subset (-\infty, m]$.

Finally take $\phi_1 : \overline{U} \to \mathbb{R}$ any continuous extension of $\overline{\phi}$.

Then the degree $d((f, \phi_1), U) \in \Pi_{n+k}(S^{n+1})$ of

$$(f, \phi_1) : (\overline{U}, \partial U) \to (\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \backslash \{0\})$$

is well defined.

Moreover $(f, \phi_1)^{-1}(0) \subset \cup_{i=1}^m B_\rho^k(\lambda_i) \times B_r^n(0)$, then

$d((f, \phi_1), U) = d((f, \phi_1)|_{\cup_{i=1}^m \overline{B}_\rho^k(\lambda_i) \times \overline{B}_r^n(0)}, \cup_{i=1}^m B_\rho^k(\lambda_i) \times B_r^n(0))$.

Since the sets $B_\rho^k(\lambda_i) \times B_r^n(0))$ are separated in the sense of [11], the additivity property holds and

$$\begin{aligned} d((f, \phi_1), U) &= d((f, \phi_1)|_{\cup_{i=1}^m \overline{B}_\rho^k(\lambda_i) \times \overline{B}_r^n(0)}, \cup_{i=1}^m B_\rho^k(\lambda_i) \times B_r^n(0)) = \\ &= \sum_{i=1}^m d((f, \phi_1)|_{\overline{B}_\rho^k(\lambda_i) \times \overline{B}_r^n(0)}, B_\rho^k(\lambda_i) \times B_r^n(0)) = \sum_{i=1}^m \widehat{\chi}_{\lambda_i}([f]). \end{aligned}$$

It only remains to prove that $d((f, \phi_1), U) = 0$.

If $d((f,\phi_1),U) \neq 0$ $\phi : (f^{-1}(0) \cap \overline{U}, f^{-1}(0) \cap \partial U) \to (\mathbb{R}, \mathbb{R}\backslash\{0\})$ is 0-epi on $f^{-1}(0) \cap \overline{U}$ which is contradictory because $f^{-1}(0) \cap \partial U = \cup_{i=1}^{m} S_\rho^{k-1}(\lambda_i) \times \{0\}$ and there the sign of ϕ_1 is constant. □

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