Abstract

Close insight into mathematical and conceptual structure of classical field theories shows serious inconsistencies in their common basis. In other words, we claim in this work to have come across two severe mathematical blunders in the very foundations of theoretical hydrodynamics. One of the defects concerns the traditional treatment of time derivatives in Eulerian hydrodynamic description. The other one resides in the conventional demonstration of the so-called Convection Theorem. Both approaches are thought to be necessary for cross-verification of the standard differential form of continuity equation. Any revision of these fundamental results might have important implications for all classical field theories. Rigorous reconsideration of time derivatives in Eulerian description shows that it evokes Minkowski metric for any flow field domain without any previous postulation. Mathematical approach is developed within the framework of congruences for general 4-dimensional differentiable manifold and the final result is formulated in form of a theorem. A modified version of the Convection Theorem provides a necessary cross-verification for a reconsidered differential form of continuity equation. Although the approach is developed for one-component (scalar) flow field, it can be easily generalized to any tensor field. Some possible implications for classical electrodynamics are also explored.

On Two Complementary Types of Total Time Derivative in Classical Field Theories and Maxwell's Equations

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1 Introduction

This work treats some aspects of conceptual, logical and mathematical structure of classical field theories. Put in other terms, we claim here to have stumbled upon two severe mathematical blunders in the very foundations of theoretical hydrodynamics as a corner-stone of all classical field theories. More close insight shows that there are some difficulties in the conventional approach to time derivatives. First indications of them could already be found in Euler's seminal work "*General Principles of the Motion of Fluids*" (1755)[1] whereas the other appeared in the 19th century in the demonstration of the so-called Convection Theorem. The question about whether there are some reasons these defects resisted to be seen up till now deserves special considerations elsewhere.

However, we shall make some comments in this respect in our attempt to clarify the situation.

In modern retrospective we are aware of a certain unevenness in progressive development of mathematics implying by it[2]

> *...false proofs, slips in reasoning, and inadvertent mistakes which with more care could have been avoided. Such blunders there were aplenty. The illogical development also involved inadequate understanding of concepts, a failure to recognize all the principles of logic required, and an inadequate rigor of proof; that is, intuition, physical arguments, and appeal to geometrical diagrams had taken the place of logical arguments...*

In the first place, this uneasy state of affairs concerned mainly the calculus, laid down by 16th and 17th-centuries scholars. However, the heroic extension of the subject by 18th-century mathematicians (sometimes the 18th century is called the heroic age in mathematics) to entirely new branches (ordinary and partial differential equations, the calculus of variations, differential geometry etc) did not imply a special effort in clarification of basic concepts and logical justification of operations, frequently used in the calculus. The problem of rigorization of the subject remained open. According to the Morris Kline authorized opinion[2]

> *... the 18th century ended with the logic of the calculus and of the branches of analysis built on the calculus in a totally confused state. In fact, one could say that the state of the foundations was worse in 1800 than in 1700. Giants, notably Euler and Lagrange, had given incorrect logical foundations. Because these men were authorities, many of their colleagues accepted and repeated uncritically what they proposed and even built more analysis on their foundations...*

Even fundamental concepts of calculus proper such as a continuous function and the derivative of a function were, to some extent, intuitive notions well into the 19th century. By 1800 and early in the 19th century, among mathematicians (Bolzano, Abel, Cauchy etc) certain concern was growing about what was correct, inadequacies and contradictions in existing proofs, confusions and vagueness in definitions etc. Cauchy, who started clarification of foundations in the second decade of the 19th century and exerted the most influence in rigorization of the calculus, decided to build his approach on D'Alembert's limit concept. Notably, they were Bolzano and Cauchy who gave a modern definition of the derivative[3], i.e. they defined the derivative as a limit, avoiding a purely formalistic approach adopted earlier by Euler and Lagrange. Cauchy's work was followed by many mathematicians and by Weierstrass, the most influent authority among them, who removed all dependence on intuitive notions and by 1900 completed the rigorization of the fundamentals of analysis.

At any rate, the requirement of rigor was also applicable to all branches of applied mathematics and mathematical physics. Theoretical hydrodynamics was no exception, although some aspects of specific empirical knowledge resisted to be molded into adequate mathematical abstractions and remained on a level of half-intuitive concepts for a longer period. It especially concerns the notion of *a fluid quantity* which found sound mathematical clarification only at the end of the 19th century in works on geometrical transformations (Klein, Lie etc). It is also worth reminding that the study of functions, differential equations, basic notions of differential geometry relevant to some extent in the framework of the theory of fluids and elasticity was continued well in the 20th century. So that, perhaps it would not be so groundlessly to think that some original defects might have escaped a scrutiny of experts and could still take place in the conceptual, logical or mathematical structure of classical field theories. However, a deeper question about whether the installation of rigor led to relatively major or minor corrections in mathematical foundations of hydrodynamics and other field theories is, of course, a matter of additional investigation.

In what follows, our first intention is to fix basic conceptions of mathematical hydrodynamics. We shall start with the physical background, reminding the motivation for one or another description of fluids in hydrodynamics. As a branch of mathematical physics, hydrodynamics deals with a real physical world. Any physical description implies an observer who can measure and compare measurable quantities. So that, a logically sound description in hydrodynamics is inconceivable without an idealized observer.

In general, two complementary types of observers are thought to suffice in order to provide a general description of the flow field kinematics or dynamics. The first one (let us call it *Lagrangian observer*) identifies individual elements or fluid particles and follows them along their motion. This idea associates a non-zero hydrodynamic flow with a non-zero geometrical transformation H_t on the closure Ω_0 such that the set $H_t\Omega_0$ represents the same individual bit of fluid at time t. A position-vector of Lagrangian observer $\mathbf{r} = \mathbf{r}(t)$ coincides with the position-vector of an identified fluid point-particle in a local coordinate system at rest. Therefore, the set of coupled variables $\{t, \mathbf{r}(t)\}$ can be naturally used for the mathematical description of fluid quantity f from the point of view of Lagrangian observer as a function of the type $f(t, \mathbf{r}(t))$. This description is commonly known as *Lagrangian specification* (or *representation*) and is valid only if the identification can be maintained by some kind of labeling usually denoting the initial position $\mathbf{r}_0$ at instant t_0. Further on, the set of coupled variables $\{t, \mathbf{r}(t)\}$ we shall refer sometimes as *Lagrangian variables*.

The second one (let us call it *Eulerian observer*) identifies a fixed volume element or a fixed point of space in a local coordinate system at rest. This observer conceives a description dissociated from identification of individual bits of fluid. It makes use of the flow quantity f only as a function of local position $\mathbf{r}$ and time t, i.e. as a function of the type $f(t, \mathbf{r})$. Hence, implementation of independent field (or *Eulerian*) variables $\{t, \mathbf{r}\}$ characterizes mathematical description of fluid quantities from the point of view of Eulerian observer and represents what is commonly known as *Eulerian specification*. It has a special significance for hydrodynamics and classical electrodynamics, since any practical attempt of Lagrangian identification is unattainable in both cases. Obviously, viewpoints of Lagrangian and Eulerian observers are complementary to each other. Their convenience depends entirely on a particular context.

The next cardinal matter to be taken into consideration concerns the description of time variations from the point of view of both observers. The total time derivative for Lagrangian observer is expressed traditionally in Eulerian field variables and is commonly known as Euler's directional (or substantive) derivative. However, just the very treatment of the time derivative for Eulerian observer looks like a rather delicate point in the traditional approach. As we shall discuss it further, no equal standards of rigor are applied in both cases. The rigorization of the conventional formalism may have important implications for the differential form of continuity equation as well as for some other aspects of classical field theories and, especially, classical electrodynamics.

2 Two complementary Lagrangian specifications of flow field

As the first step, we shall fix basic concepts and notations of mathematical hydrodynamics. We begin with the simplest assumptions for an ideal fluid moving in a three-dimensional Euclidean domain. Let a mapping H_t represent in Lagrangian description a geometrical transformation of the initial closure Ω_0 onto $H_t\Omega_0$ for the same individual bit of fluid at time t. Then H_t also represents the function[4]:

$$\mathbf{r} = H_t\mathbf{r}_0 = \mathbf{r}(t, \mathbf{r}_0) \tag{1}$$

where points $\mathbf{r}$ and $\mathbf{r}_0$ denote the position-vector of the fluid identifiable point-particle at time t and initial time t_0, respectively. The velocity of the particle along the trajectory is defined as:

$$\mathbf{v} = \frac{d}{dt}\mathbf{r}(t, \mathbf{r}_0) = \frac{\partial}{\partial t}\mathbf{r}(t, \mathbf{r}_0) \tag{2}$$

where the initial position-vector $\mathbf{r}_0$ is assumed to be fixed or time independent.

In the context of Eulerian description, *a priori* there is no identification and hence no explicit consideration of the function $\mathbf{r} = \mathbf{r}(t, \mathbf{r}_0)$. The primary notion is the velocity field as a function of position $\mathbf{r}$ in space and time t on a fluid domain:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}(t, \mathbf{r}) \tag{3}$$

where variables $\mathbf{r}$ and t are originally uncoupled.

Picking up some initial point $\mathbf{r}_0 = \mathbf{r}(t_0)$, one selects from a congruence (a set of integral curves of (3)) a unique solution. Thus, a formulation of the *initial Cauchy problem*

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}(t, \mathbf{r}); \qquad \mathbf{r}(t_0) = \mathbf{r}_0 \tag{4}$$

is mathematically equivalent to an act of identification, allowing any solution of (4) to be written in the form of (1). There is a general consensus that this procedure can be taken as a rule for translating from one specification to the other. However, more close insight reveals a possibility of a different way of translation, alternative to the traditional procedure. The treatment of time variations from the viewpoint of Eulerian observer will show why it acquires special significance.

Let us explore this alternative. Our purpose here is to define a new complementary Lagrangian specification by identifying a *final* closure instead of the initial one. We denote the final closure as Ω_f and assume that it has always fixed shape and fixed position in space. For further convenience, all our notation will be accompanied by *tilde* when we refer to the Lagrangian description with identification of the final closure $\tilde{\Omega}_f$. Then some geometrical transformation $\tilde{H}_t$ represents a mapping of the initial bit $\tilde{\Omega}_0$ at instant t_0 onto the final bit $\tilde{H}_t\tilde{\Omega}_0 = \tilde{\Omega}_f$ at instant t. Since the closure $\tilde{\Omega}_f$ is assumed to be fixed in a local coordinate frame, the initial closure $\tilde{\Omega}_0$ becomes a function of space and time variables: $\tilde{\Omega}_0 = \tilde{H}_t^{-1}\tilde{\Omega}_f$ (according to one of the postulates of mathematical hydrodynamics[4] for perfect fluids, the transformation H_t has the inverse H_t^{-1}, so there is no obstacle in assuming the existence of the inverse mapping $\tilde{H}_t^{-1}$). In other words, the knowledge of the inverse geometrical transformation $\tilde{H}_t^{-1}$ allows a reconstruction of initial bit $\tilde{\Omega}_0$ as well as its shape and space position at time t_0 from the knowledge of a fixed *final* closure $\tilde{\Omega}_f$ at time t.

When we deal with identifiable point-particle, $\tilde{H}_t$ represents the function:

$$\tilde{\mathbf{r}}_f = \tilde{H}_t\tilde{\mathbf{r}}_0 = \tilde{\mathbf{r}}(t, \tilde{\mathbf{r}}_0) \tag{5}$$

which maps the initial position-vector $\tilde{\mathbf{r}}_0$ at time t_0 onto $\tilde{\mathbf{r}}_f$ at time t.

The whole situation can be seen schematically as follows: the initial position of identifiable point-particle depends on a parameter t and on a fixed final position $\tilde{\mathbf{r}}_f$:

$$\tilde{\mathbf{r}}_0 = \tilde{H}_t^{-1}\tilde{\mathbf{r}}_f = \tilde{\mathbf{r}}_0(t, \tilde{\mathbf{r}}_f) \tag{6}$$

The longer is the lapse of time $t - t_0$, the larger is the distance which a particle should travel through from $\tilde{\mathbf{r}}_0$ to $\tilde{\mathbf{r}}_f$.

It is obvious that in the new framework, Lagrangian specification can not be linked to *initial Cauchy problem* for (3). Only *final Cauchy problem,* as we shall regard it, is appropriate for that purpose:

$$\frac{d\tilde{\mathbf{r}}}{dt} = \mathbf{v}(t, \tilde{\mathbf{r}}); \qquad \tilde{\mathbf{r}}(t, \tilde{\mathbf{r}}_0) = \tilde{\mathbf{r}}_f \tag{7}$$

We remind here that the trajectory $\tilde{\mathbf{r}}(t, \tilde{\mathbf{r}}_0)$ ends in the fixed point of space $\tilde{\mathbf{r}}_f$. In whole similarity with the initial Cauchy problem, the *final condition* $\tilde{\mathbf{r}}(t) = \tilde{\mathbf{r}}_f$ also selects a unique solution from a congruence (a set of integral curves of the equation (3)). The set of coupled variables $\{t, \tilde{\mathbf{r}}(t)\}$ we shall refer as *Lagrangian variables* for final Cauchy problem.

To finish this Section, we conclude that both geometrical transformations H_t and $\tilde{H}_t$ give Lagrangian description because they advance any identified fluid point-particle from its position $\mathbf{r}_0$ (or $\tilde{\mathbf{r}}_0$) at time t_0 to its position at time t. In both cases an imaginary observer *follows the motion of the fluid* in the direction of its velocity field $\mathbf{v}$. The only difference is that one starts from a fixed initial point $\mathbf{r}_0$ and the other ends in a fixed final point $\tilde{\mathbf{r}}_f$. In what follows, let us explore some possible implementations of the transformation $\tilde{H}_t$.

3 Fluid quantities and time derivatives for Lagrangian and Eulerian observers

To clarify our approach, all subsequent analysis will be based on a consideration of one-component (scalar-field) ideal fluid moving in a 3-dimensional space closure. In general terms, we denote by fluid quantity some regular function f defined on a fluid domain. Any non-zero flow is characterized by a non-zero velocity vector field $\mathbf{v}$ and *vice versa*. Thus, in Lagrangian description the function f is defined on a set of Lagrangian variables $\{t, \mathbf{r}(t)\}$, i.e. as a function of the type $f(t, \mathbf{r}(t))$. If our approach is placed within the traditional framework, then $\mathbf{r}(t)$ is an integral curve of the equation (4) linked to the *initial Cauchy problem*. In other words, the transformation H_t:

$$\mathbf{r}_0 = \mathbf{r}(t_0) \quad \rightarrow \quad \mathbf{r}(t, \mathbf{r}_0) = H_t \mathbf{r}_0 \tag{8}$$

defines the mapping of $f(t_0, \mathbf{r}_0)$ along the integral curve $\mathbf{r}(t)$ into a new function[5]:

$$f(t_0, \mathbf{r}_0) \quad \rightarrow \quad H_t f(t_0, \mathbf{r}_0) = f(H_t t_0, H_t \mathbf{r}_0) = f(t, \mathbf{r}(t, \mathbf{r}_0)) \tag{9}$$

where $H_t t_0 = t$.

When the Lagrangian description is placed within the alternative framework linked to the *final Cauchy problem* (7), the geometrical transformation $\tilde{H}_t$:

$$\tilde{\mathbf{r}}_0(t_0) \quad \rightarrow \quad \tilde{\mathbf{r}}(t, \tilde{\mathbf{r}}_0) = \tilde{H}_t \tilde{\mathbf{r}}_0 = \tilde{\mathbf{r}}_f \tag{10}$$

defines the mapping of $f(t_0, \tilde{\mathbf{r}}_0)$ along the integral curve $\tilde{\mathbf{r}}(t)$ into a new function:

$$f(t_0, \tilde{\mathbf{r}}_0) \quad \rightarrow \quad \tilde{H}_t f(t_0, \tilde{\mathbf{r}}_0) = f(\tilde{H}_t t_0, \tilde{H}_t \tilde{\mathbf{r}}_0) = f(t, \tilde{\mathbf{r}}(\tilde{\mathbf{r}}_0, t)) \tag{11}$$

where $\tilde{H}_t t_0 = t$.

Both transformations H_t and $\tilde{H}_t$ correspond to Lagrangian description, i.e. represent the viewpoint of Lagrangian observer who follows the motion of the fluid. Importantly, the geometrical interpretation based on H_t and $\tilde{H}_t$ (absent at the end of the 18th century) gives a necessary *clarification of the concept of fluid quantity.* Put in qualitative terms, one can regard a *fluid quantity* $f(t, \mathbf{r}(t))$ as a function value $f(t_0, \mathbf{r}_0)$ which is permanently submitted to a non-zero geometrical transformation H_t, i.e. as a function of the type $f(t, \mathbf{r}(t)) = H_t f(t_0, \mathbf{r}_0)$ (note again that H_t is non-zero if and only if the velocity vector field $\mathbf{v}$ of fluid flow is non-zero).

In this context, the treatment of time variation of fluid quantities for Lagrangian observer straightforwardly leads to the classical definition of the total time derivative represented in Lagrangian variables:

$$\frac{d}{dt}f(t,\mathbf{r}(t)) = \lim_{t\to t_0}\frac{H_t f(t_0,\mathbf{r}_0) - f(t_0,\mathbf{r}_0)}{t-t_0} = \lim_{t\to t_0}\frac{f(t,\mathbf{r}(t,\mathbf{r}_0)) - f(t_0,\mathbf{r}_0)}{t-t_0} \tag{12}$$

as well as:

$$\frac{d}{dt}f(t,\tilde{\mathbf{r}}(t)) = \lim_{t\to t_0}\frac{\tilde{H}_t f(t_0,\tilde{\mathbf{r}}_0) - f(t_0,\tilde{\mathbf{r}}_0)}{t-t_0} = \lim_{t\to t_0}\frac{f(t,\tilde{\mathbf{r}}(t,\tilde{\mathbf{r}}_0)) - f(t_0,\tilde{\mathbf{r}}_0)}{t-t_0} \tag{13}$$

Both definitions (12) and (13) are equivalent so that later on we shall refer only to the conventional expression (12). Important to note that in this classical definition of the total time derivative for Lagrangian observer, a function f is supposed to be submitted to non-zero geometrical transformation H_t associated with non-zero flow velocity field $\mathbf{v}$.

The next natural step to do is to consider time variations in properly Eulerian description (i.e. from the point of view of Eulerian observer). One might anticipate here that formalistic Euler's approach had struck at this point, relying on intuitive or half-intuitive concepts of the 18th century calculus. As a consequence, Euler and his close followers interpreted a time derivative of *fluid quantities* for an observer at rest as a standard partial time derivative:

$$\frac{\partial}{\partial t}f(t,\mathbf{r}) = \lim_{t\to t_0}\frac{f(t,\mathbf{r}) - f(t_0,\mathbf{r})}{t-t_0} \tag{14}$$

where $\mathbf{r}$ is a fixed point of space.

What is especially noteworthy about Euler's final result is that even after the obvious progress in mathematical rigorization of the fluid quantity concept had been achieved at the end of the 19th century, nobody seemed to have worried about the following peculiarity. In the classical definition of a partial time derivative (14), a function f *does not explicitly possess mathematical characteristics of properly fluid quantity* (recently, this fact was also critically pointed out in[6]–[9] but on different positions). Anyway, contrarily to what is explicitly assumed in (12), in the definition (14) there is no indication that f is submitted to non-zero geometrical transformation H_t associated with non-zero flow velocity field $\mathbf{v}$. Therefore, (14) is not directly applicable to treat time derivatives of fluid quantities.

To avoid traditionally formalistic approach (14), let us carefully analyze time variations of fluid quantities in properly Eulerian description. As a matter of fact, an observer should be placed at a fixed position of space in order to undertake a study of any fluid quantity coming through as a function of time. It can be easily achieved if one substitutes a permanent monitoring of a fluid point-particle by a permanent identification of a fixed point of space $\tilde{\mathbf{r}}_f$. Firstly, Eulerian observer measures a fluid quantity $f(t_0,\tilde{\mathbf{r}}_f)$ carried by a point-particle on its way through $\tilde{\mathbf{r}}_f$ at instant t_0. At time t, the previous particle has been replaced at $\tilde{\mathbf{r}}_f$ by another particle, carrying a fluid quantity

$f(t, \tilde{\mathbf{r}}(t, \tilde{\mathbf{r}}_0))$, where $\tilde{\mathbf{r}}(t, \tilde{\mathbf{r}}_0) = \tilde{\mathbf{r}}_f$. At time t_0 this particle stayed at the point $\tilde{\mathbf{r}}_0$ different from $\tilde{\mathbf{r}}_f$. Thus, we realize here the necessity of the practical implementation of the final Cauchy problem for a fluid point-particle. This reasoning compels us to introduce a definition which takes into account a non-zero geometrical transformation (associated with a fluid flow) and which, therefore, differs from (14):

$$\frac{d^*}{d^*t} f(t, \tilde{\mathbf{r}}(t)) = \lim_{t \to t_0} \frac{\tilde{H}_t f(t_0, \tilde{\mathbf{r}}_0) - f(t_0, \tilde{\mathbf{r}}_f)}{t - t_0} = \lim_{t \to t_0} \frac{f(t, \tilde{\mathbf{r}}_f) - f(t_0, \tilde{\mathbf{r}}_f)}{t - t_0} \tag{15}$$

where $\tilde{\mathbf{r}}_f = \tilde{\mathbf{r}}(t, \tilde{\mathbf{r}}_0)$.

At first glance, there is no difference in symbolic notations between the definition (14) and the right-hand side of the definition (15) so that they can be easily confused in a purely formalistic approach when geometrical transformation $\tilde{H}_t$ is not taken into account. Later on we shall prove that the mathematical object, defined by (15), virtually differs from partial time derivative $\frac{\partial}{\partial t}$. To highlight this aspect in our discussions, we shall denote the total time derivative in Eulerian description by upper *asterisk*.

As the next step, let us consider analytical expressions for (12) and (15) in Eulerian variables. The total time derivative is conceived as a linear part of the rate of change of f with respect to t. Thus, when $t - t_0$ tends to zero, only linear part of geometrical transformations H_t makes sense for further discussions:

$$\mathbf{r}(t) = \mathbf{r}(t_0) + (t - t_0)\left(\frac{d\mathbf{r}}{dt}\right)_{t=t_0} + o(t - t_0) = H_t \mathbf{r}(t_0) \tag{16}$$

where $\mathbf{r}_0 = \mathbf{r}(t_0)$ is the reference point of the corresponding Taylor series; $(\frac{d\mathbf{r}}{dt})_{t=t_0}$ is the initial velocity of a point-particle at t_0.

By analogy, the linear part of geometrical transformation $\tilde{H}_t$ is:

$$\tilde{\mathbf{r}}(t) = \tilde{\mathbf{r}}(t_0) + (t - t_0)\left(\frac{d\tilde{\mathbf{r}}}{dt}\right)_{t=t_0} + o(t - t_0) = \tilde{H}_t \tilde{\mathbf{r}}(t_0) \tag{17}$$

where $\tilde{\mathbf{r}}_0 = \tilde{\mathbf{r}}(t_0)$ is the reference point of the corresponding Taylor series; $(\frac{d\tilde{\mathbf{r}}}{dt})_{t=t_0}$ is evaluated at $\tilde{\mathbf{r}}_0$ different from $\tilde{\mathbf{r}}_f$ where Eulerian observer is placed. Importantly, for further discussions we note that the reference point $\mathbf{r}_0$ in (16) is fixed in a local coordinate system whereas the reference point $\tilde{\mathbf{r}}_0 = \tilde{H}_t^{-1} \tilde{\mathbf{r}}_f$ in (17) is a function of a parameter t.

Applying (16) to the definition (12), one gets:

$$\frac{d}{dt} f(t, \mathbf{r}(t)) = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} = \frac{Df}{Dt} \tag{18}$$

where $\mathbf{v} = (\frac{d\mathbf{r}}{dt})_{t=t_0}$. The differential operator $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}$ is usually called *substantive* or *Euler's directional derivative.*

Although this is well-known result, it needs some clarifying interpretation. The left-hand side of (18) is a symbolic expression for the total time derivative for Lagrangian observer represented in Lagrangian variables of the initial Cauchy problem. The right-hand side of (18) represents the same result in Eulerian field variables. In fact, partial derivatives $\frac{\partial f}{\partial t}$ and $\boldsymbol{\nabla} f$ make sense only when variables t and $\mathbf{r}$ are independent. Thus, Euler's derivative $\frac{D}{Dt}$ allows a calculation of the total time derivative meaningful for

Lagrangian observer but does not require any information on the particle trajectory $\mathbf{r}(t)$, indispensable in Lagrangian specification. In other words, *Euler's derivative* $\frac{D}{Dt}$ *emulates the total time derivative of properly Lagrangian description in infinitesimal vicinity of the initial point* $\mathbf{r}_0$. This interpretation is in agreement with what Euler himself thought two and a half centuries ago and what is unreservedly accepted nowadays: the directional derivative $\frac{D}{Dt}$ describes the rate of time variation of material properties *following the motion of the fluid*[10].

Let us see what form will take in Euler's variables the total time derivative (15) considered by Eulerian observer that remains at rest in a fixed point of space $\tilde{\mathbf{r}}_f$. Therefore, the set $\{t, \tilde{\mathbf{r}}_f\}$ represents Eulerian variables for (15). It means that partial derivatives $\frac{\partial f}{\partial t}$, $\boldsymbol{\nabla} f$ and the value of fluid velocity $\mathbf{v}$ have to be evaluated by Eulerian observer only locally at $\tilde{\mathbf{r}}_f$. This circumstance highlights the obvious inconvenience of the reference point $\tilde{\mathbf{r}}_0$ which appears in the Taylor series (17). In fact, any local reference system related to the reference point $\tilde{\mathbf{r}}_0 = \tilde{H}_t^{-1}\tilde{\mathbf{r}}_f$ is a function of a time parameter t (i.e. it is not a reference system at rest). On the other hand, the reference system of Eulerian observer has to be related to the fixed point of space $\tilde{\mathbf{r}}_f$. Hence, a change of the reference point is required. It can be obtained by rewriting the infinitesimal transformation (17) as follows:

$$\tilde{\mathbf{r}}_0(t) = \tilde{\mathbf{r}}_f - (t - t_0)(\frac{d\tilde{\mathbf{r}}}{dt})_{\tilde{\mathbf{r}}_0} + o(t - t_0) \tag{19}$$

where $\tilde{\mathbf{r}}(t) = \tilde{\mathbf{r}}_f$ is a fixed point of space.

Making use of the linearity of the transformation (17), one can conclude that a fluid point-particle arrives at $\tilde{\mathbf{r}}_f$ at time t with the velocity equal to the initial one $(\frac{d\tilde{\mathbf{r}}}{dt})_{\tilde{\mathbf{r}}_0} = (\frac{d\tilde{\mathbf{r}}}{dt})_{\tilde{\mathbf{r}}_f}$:

$$\tilde{\mathbf{r}}_0(t) = \tilde{\mathbf{r}}_f - (t - t_0)(\frac{d\tilde{\mathbf{r}}}{dt})_{\tilde{\mathbf{r}}_f} + o(t - t_0) \tag{20}$$

where $\tilde{\mathbf{r}}_f$ is already the reference point. Let us denote the transformation (20) by G_t. Note that G_t does not change the arrow of time: $G_t t_0 = t$.

Importantly, both transformations G_t and $\tilde{H}_t$ are equivalent only as infinitesimal transformations when higher order terms in corresponding Taylor's series are not taken into account. Moreover, G_t does not describe any mapping of function values along a congruence and is necessary only to keep Eulerian observer at a fixed point of space $\tilde{\mathbf{r}}_f$, counteracting the flow drift. To grasp schematically the underlying idea of (20), one can fancy an observer on an escalator (automatic staircase) running in a direction opposite to the direction of the fluid velocity field $\mathbf{v}$ just to be always in the same point of space $\tilde{\mathbf{r}}_f$.

Applying (20) to the definition (15), one gets the linear part of the time variation of f with respect to t from the viewpoint of Eulerian observer:

$$\frac{d^*}{d^*t} f(t, \tilde{\mathbf{r}}(t)) = \frac{\partial f}{\partial t} - \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} = \frac{D^* f}{D^* t} \tag{21}$$

where $\mathbf{v} = \lim_{t \to t_0} (\frac{d\tilde{\mathbf{r}}}{dt})_{\tilde{\mathbf{r}}_f, t} = (\frac{d\tilde{\mathbf{r}}}{dt})_{\tilde{\mathbf{r}}_f, t_0}$. In (21) we already use a common notation $\mathbf{r}$ for space variables instead of $\tilde{\mathbf{r}}_f$. This change in notation is justified by the fact that $\tilde{\mathbf{r}}_f$ coincides with the space variable $\mathbf{r}$ of a local coordinate system at rest. To make distinction between Euler's derivative $\frac{D}{Dt}$ and (21), we shall denote the latter by $\frac{D^*}{D^* t}$ and call *local directional derivative*.

The interpretation of (21) is straightforward. The left-hand side is a symbolic expression of the total time derivative for Eulerian observer represented in Lagrangian variables of the final Cauchy problem, emulating properly Eulerian description. The right-hand side represents the same result in Euler's field variables. The schematized interpretation of the transformation (20) as counteraction of the flow drift helps to grasp the meaning of the negative sign for the velocity value in (21).

Looking at (18) and (21), one can see in explicit terms the difference between total time derivatives for Lagrangian and Eulerian observers. Further we shall test the correctness of the expression (21) analyzing hydrodynamics conservation laws.

4 Fluid quantities and time derivatives in 4-dimensional notation

Before we proceed to the application of (21), it would be convenient to make a certain generalization of the previous exposition. Let us choose a 4-dimensional, metric free framework for the description of an ideal fluid in order to use the notion of *Lie's derivative* as a particularly important generalization of (12) on manifolds without metric. For this purpose, we add a trivial statement $\frac{dt}{dt} = 1$ to the differential equation (3), in order to use a 4-dimensional notation:

$$\frac{dx^i}{dt} = V^i(x) \tag{22}$$

where $x = (x^0, x^1, x^2, x^3) = (t, \mathbf{r})$; $V = (1, \mathbf{v})$. To satisfy the index conventions of modern differential geometry, upper indices are used for coordinate functions $x^i(t)$, $i = 0, 1, 2, 3$. In further discussions we shall also leave for the time variable x^0 its original denomination t.

Let us formulate the initial Cauchy problem for (22):

$$\frac{dx^i}{dt} = V^i(x) \qquad\qquad x_0 = x(t_0) \tag{23}$$

where the lower index 0 denotes the initial point on a manifold at t_0.

In full similarity with the previous Section, (23) defines a geometrical transformation H_t: $x_0 \to x(t, x_0)$ which maps the initial point x_0 along the congruence onto $x(t, x_0)$. Two points x_0 and x with parameters t_0 and t, respectively, are related by the Taylor series:

$$x^i(t) = x^i(t_0) + (t - t_0)(\frac{dx^i}{dt})_{t_0} + \frac{(t - t_0)^2}{2!}(\frac{d^2 x^i}{dt^2})_{t_0} + ... = H_t x^i(t_0) \tag{24}$$

The expression (24) gives the finite motion $H_t x_0$ along the integral curve whereas the first order (linear) operator $H_t x_0 = x^i(t_0) + (t - t_0)(\frac{dx^i}{dt})_{t=t_0}$ gives only an infinitesimal motion.

If a fluid quantity $f(x)$ is defined on the velocity vector field (22), then the transformation H_t describes the mapping of $f(x_0)$ along the congruence into a new function $f(x)$[5]:

$$H_t f(x_0) = f(H_t x_0) = f(x) \tag{25}$$

Thus, we arrive at the classical definition of the *Lie derivative* L_V along the vector field $V = (1, \mathbf{v})$ written in the traditional Lagrangian specification (i.e. linked to the initial Cauchy problem):

$$L_V\, f = [\frac{d}{dt} H_t\, f]_{t=t_0} = \lim_{t \to t_0} \frac{1}{t - t_0}[f(x) - f(x_0)] \tag{26}$$

Note that (26) gives a unique difference and therefore a unique derivative.

When $t - t_0$ is too small, all higher order terms of the kind $\frac{t^n}{n!}(\frac{d^n}{dt^n})_{t=t_0}$ $(n \geq 2)$ vanish in (24), and the mapping H_t: $x_0 \to x(t, x_0)$ is linear, holding an explicit form:

$$H_t x_0 = x_0^i + (t - t_0)V^i(x_0) + o(t - t_0) \tag{27}$$

The transformation (27) gives the linear part of time variation, providing analytic expression for Lie's derivative (26) in Eulerian field variables:

$$L_V\, f = \frac{d}{dt} f(H_t\, x_0) = V^i \frac{\partial f}{\partial x^i} \tag{28}$$

If we are in an ordinary Euclidean domain, L_V takes a familiar form of Euler's directional derivative[5]:

$$L_V\, f = V^i \frac{\partial f}{\partial x^i} = (\frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}) f = \frac{Df}{Dt} \tag{29}$$

Therefore, it is worth emphasizing here that the Lie derivative on a differentiable manifold and its Euler's equivalent on an ordinary Euclidean domain are traditionally defined entirely in the spirit of Lagrangian description, i.e. when a point on a congruence or a fluid point-particle remain permanently identified.

The infinitesimal transformation which in the previous Section we regarded as G_t, takes the following equivalent form in 4-dimensional notation:

$$G_t x_f^i(t_0) = x_f^i(t_0) + (t - t_0)V^{*i}(x_f(t_0)) \tag{30}$$

where $V^* = (1, -\mathbf{v})$ and the lower index f denotes a fixed space component $\mathbf{r}_f$. Note again that the transformation (30) does not describe any mapping along the congruence and makes sense only as infinitesimal one. However, its use is fully justified, since it effectively emulates the description of time variations perceived by Eulerian observer. In 4-dimensional notation the infinitesimal transformation G_t: $x_f(t_0) \to G_t x_f(t_0)$ implies the substitution of the function value $f(x_f(t_0))$ by $f(G_t x_f(t_0))$. In other words, it looks like some kind of mapping of $f(x_f(t_0))$ into $f(G_t x_f(t_0))$:

$$G_t f(x_f(t_0)) = f(G_t x_f(t_0)) \tag{31}$$

This formulation allows us to adjust the framework of the conventional definition (28) for our attempt to express analytically Eulerian total time derivative on a 4-dimensional manifold as some kind of Lie derivative on *effective* local vector field V^*:

$$L_{V^*} f = [\frac{d}{dt} f(G_t \, x_f(t_0))]_{t=t_0} = V^{*i} \frac{\partial f}{\partial x^i} \tag{32}$$

Here we shall call $V^* = (1, -\mathbf{v})$ as *effective velocity vector field*, since it is meaningful for Eulerian observer in order to counteract a flow drift.

In an ordinary Euclidean domain (32) takes the form of *local directional derivative* established in the previous Section:

$$L_{V^*} f = V^{*i} \frac{\partial f}{\partial x^i} = (\frac{\partial}{\partial t} - \mathbf{v} \cdot \boldsymbol{\nabla}) f = \frac{D^* f}{D^* t} \tag{33}$$

Let us remind that this result manifests the viewpoint of Eulerian observer, i.e. it does not imply any sort of identification of points on a congruence or fluid elements. Therefore, the directional derivative (32) defined on a general differentiable manifold and on an effective velocity field V^* should be regarded as a complementary counterpart of the standard Lie derivative. The character of complementarity between (28) and (32) is of the same nature as for the relationship between their Euclidean analogies (29) and (33) given in an ordinary Euclidean domain. Having in mind this similarity with Lie's directional derivative, we can regard total time derivatives $\frac{Df}{Dt}$ and $\frac{D^* f}{D^* t}$ for Lagrangian and Eulerian observers, respectively, as *complementary 4-dimensional directional derivatives* defined on 4-dimensional space-time manifold.

Both types of directional derivatives $\frac{Df}{Dt}$ and $\frac{D^* f}{D^* t}$ can be analyzed in terms of 1-forms or real-valued functions of vectors in 4-dimensional manifolds:

$$\omega = (\omega_i) = (\frac{\partial f}{\partial x^i}) \tag{34}$$

where $i = 0, 1, 2, 3$ and $(\frac{\partial f}{\partial x^i}) = (\frac{\partial f}{\partial t}, \boldsymbol{\nabla} f)$ in an ordinary Euclidean domain.

Now we point out that in tensor algebra the set $\{\omega_i V^j\}$ are components of a linear operator or $\binom{1}{1}$ tensor. The formation of a scalar $\omega(V)$ is called the contraction of the 1-form ω with the vector V and it is an alternative representation of directional derivatives:

$$\frac{Df}{Dt} = \omega_i V^i; \qquad \frac{D^* f}{D^* t} = \omega_i V^{*i} \tag{35}$$

The contraction of diagonal components of the tensor $\omega_i V^j$ is independent of the basis. Importantly, this law shows that both types of directional derivatives $\frac{Df}{Dt}$ and $\frac{D^* f}{D^* t}$ are invariant and do not depend on a particular choice of a local coordinate system.

If there is a metric tensor defined on a manifold, then it maps 1-forms into vectors in a 1-1 manner. This pairing is usually written as:

$$\omega_i = g_{ij}\omega^j; \qquad V^i = g^{ij}V_j \tag{36}$$

Therefore, from the point of view of tensor algebra, (35) can be represented as a scalar product in a 4-dimensional manifold with metric:

$$\frac{Df}{Dt} = g_{ii}\omega^i V^i; \qquad \frac{D^*f}{D^*t} = g_{ii}\omega^i V^{*i} \tag{37}$$

where $g_{ij} = \delta_{ij}$ is the Euclidean metric tensor. Important to note that Euclidean metric appears here as an *effective metric* defined in infinitesimal 4-dimensional vicinity of $(t_0, \mathbf{r}_0)$ where $\mathbf{r}_0$ is the initial point from which Lagrangian observer starts moving with the fluid.

A Minkowski metric is also consistently singled out for the local directional derivative $\frac{D^*f}{D^*t}$:

$$\frac{D^*f}{D^*t} = g_{ii}\omega^i V^{*i} = g^*_{ii}\omega^i V^i \tag{38}$$

where $V^* = (1, -\mathbf{v})$; $g^*_{ij} = diag(1, -1, -1, -1)$ is indefinite or Minkowski metric tensor. Similarly, Minkowski metric should be understood here as an *effective metric* defined in infinitesimal 4-dimensional vicinity of $(t_0, \mathbf{r}_f)$ where $\mathbf{r}_f$ is the fixed point of space in which Eulerian observer is placed.

One of the advantages of the scalar product form is that it gives orthonormal bases for space-time manifolds. For Lagrangian description, a basis is Cartesian and a transformation matrix Λ_c from one such basis to another is orthogonal matrix:

$$\Lambda_c^T = \Lambda_c^{-1}; \qquad {}'g_{ij} = \Lambda_c^{-1} g_{ij} \Lambda_c \tag{39}$$

These matrices Λ_c form the symmetry group $O(4)$.

Likewise, for Eulerian description a Minkowski metric picks out a preferred set of bases known as pseudo-Euclidean or Lorentz bases. A transformation matrix Λ_L from one Lorentz basis to another satisfies:

$$\Lambda_L^T = \Lambda_L^{-1}; \qquad {}'g^*_{ij} = \Lambda_c^{-1} g^*_{ij} \Lambda_c \tag{40}$$

Λ_L is called a Lorentz transformation and belongs to the Lorentz group $L(4)$ or $O(3,1)$.

The point that needs to be emphasized here is the remarkable circumstance of properly Eulerian description in evoking of the Minkowski metric *without any previous postulation*. In other words, consistent mathematical description of fluids is also perfectly compatible with the Lorentz symmetry group. From the complementary standpoints of Lagrangian and Eulerian observers it is clear that both kinds of total time (or 4-dimensional directional) derivatives are valid only in their complementary contexts. Hence, it also concerns the complementary relationship between Euclidean and Minkowski metrics. In what follows we shall confine our attention on some practical implications of (33) in the classical field theory.

5 Fluid contents in Lagrangian and Eulerian descriptions

Let us now consider a fluid f-content in a 3-dimensional space domain V, i.e a volume integral of the type $\int f dV$. Here we assume the function f fulfils all standard conditions on integrability that allows us to consider not only smooth or continuous integrand functions but a more general class of fluid quantities with spatial discontinuities which may take place in many practical examples (for instance, some extra particles of dust moving in water introduce finite discontinuity into fluid density). Nevertheless, regarding discontinuities, we shall restrict our approach only by the simplest class or elementary discontinuities of finite size. Infinite or delta-function types of discontinuities need additional suppositions on integrability and, therefore, can be taken into consideration elsewhere on a more rigorous basis.

In properly Lagrangian description V is an identified macroscopic volume domain moving with a fluid. If the bounding surface of a closure always consists of the same fluid particles regardless any change of shape of the volume V then, as a result, no fluid flows through the volume surface. Obviously, no time variation of a fluid content takes place and a mathematical description can be used to express a conservation of f-content. If the above condition is not fulfilled and Lagrangian observer identifies all fluid particles coming in and out the closure, there is a general expression of the time variation, commonly known under the name of the *Convection Theorem*[4],[10]:

$$\frac{d}{dt}\int\limits_{V(t)} f(t,\mathbf{r}(t))dV = \int\limits_{V(t)} (\frac{Df}{Dt} + f\boldsymbol{\nabla}\cdot\mathbf{v})dV \tag{41}$$

In the left-hand side of (41) both volume $V(t) = H_t V(t_0)$ and a fluid quantity $f(t,\mathbf{r}(t)) = H_t f(t_0,\mathbf{r}_0)$ are permanently submitted to a non-zero geometrical transformation H_t, i.e. they are given in Lagrangian specification linked to the initial Cauchy problem whereas the right-hand side represents the same result in Eulerian independent variables of the local reference system at rest. If there is no flux through the surface, the Convection Theorem gives the conservation of the fluid f-content. The restriction on conservation is usually written in the standard differential form of continuity equation:

$$\frac{Df}{Dt} + f\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{\partial f}{\partial t} + \boldsymbol{\nabla}\cdot f\mathbf{v} = 0 \tag{42}$$

The point of view of Eulerian observer is complementary. He picks out a fixed 3-dimensional volume V_0 and studies a fluid f-content as a function of time. Since the volume element is now fixed, the traditional formalistic approach (see (14) in the Section 3) takes the time derivative of Eulerian observer for the partial time derivative:

$$\frac{d}{dt}\int\limits_{V_0} f dV = \lim_{t\to t_0}\int\limits_{V_0}\left[\frac{f(t,\mathbf{r}) - f(t_0,\mathbf{r})}{t-t_0}\right]dV = \int\limits_{V_0}\frac{\partial f}{\partial t}dV \tag{43}$$

In order to express the restriction on conservation when the time variation of f-content is not zero, (43) should be equaled to the fluid inflow or outflow through a bounding surface ∂V_0 of the volume V_0. It immediately leads to the integro-differential form of continuity equation:

$$\int\limits_{V_0} \frac{\partial f}{\partial t} dV = -\int\limits_{\partial V_0} f\mathbf{v}\cdot d\mathbf{S} = -\int\limits_{V_0} \boldsymbol{\nabla}\cdot f\mathbf{v} dV \tag{44}$$

The remarkable circumstance that the differential form of (44) coincides with (42) derived for the volume $V(t)$ in motion, is traditionally associated with the cross-verification of the standard differential form of continuity equation (42). However, from the Section 3 we know that, generally speaking, partial time derivatives do not represent time derivatives of fluid quantities perceived by Eulerian observer. What attitude should we take then on the fact that the Convection Theorem leads to the result obtained in the formalistic approach (43) which in no way refers to fluid quantities? In fact, we claim here to have come across another mathematical blunder in the very demonstration of the Convection Theorem. Elimination of defects implies mathematical modifications in the conventional form of (41) (see Appendix A):

$$\frac{d}{dt} \int\limits_{V(t)} f(t, \mathbf{r}(t)) dV = \int\limits_{V(t)} (\frac{\partial f}{\partial t} + f \boldsymbol{\nabla} \cdot \mathbf{v}) dV \tag{45}$$

where the set $\{t, \mathbf{r}\}$ represents Eulerian field variables associated with the local reference system at rest.

Before going further with implementation of the *modified version of the Convection Theorem* (45), let us substitute (43) by a rigorized description of the time derivative for Eulerian observer (see (15) in the Section 3). As a matter of fact, the terminology of the Lagrangian description linked to the final Cauchy problem provides us with the appropriate framework. Put in quantitative terms, Eulerian observer measures a fluid quantity $f(t_0, \tilde{\mathbf{r}}_f)$ carried by a point-particle on its way through $\tilde{\mathbf{r}}_f$ at instant t_0. At time t, the previous particle has been replaced at $\tilde{\mathbf{r}}_f$ by another particle, carrying a fluid quantity $f(t, \tilde{\mathbf{r}}(t, \tilde{\mathbf{r}}_0))$. If we apply this procedure to all fixed points of space which form part of the fixed volume V_f, then we arrive at the integral formulation:

$$\frac{d}{dt} \int\limits_{V_f} f(t, \tilde{\mathbf{r}}(t)) dV = \lim_{t \to t_0} \int\limits_{V_f} \left[\frac{f(t, \tilde{\mathbf{r}}(t, \tilde{\mathbf{r}}_0)) - f(t_0, \tilde{\mathbf{r}}_f)}{t - t_0} \right] dV_f \tag{46}$$

where $\tilde{\mathbf{r}}(t, \tilde{\mathbf{r}}_0) = \tilde{\mathbf{r}}_f$ so that both values of f represent the fluid property at the same point of space $\tilde{\mathbf{r}}_f$.

The right-hand side of (46) makes use of Lagrangian variables of the final Cauchy problem. If Eulerian variable are implemented, then according to (21) (or (33)), the time variation of f-content for Eulerian observer takes the following form:

$$\frac{d}{dt} [\int\limits_{V_f} f(t, \tilde{\mathbf{r}}(t)) dV]_{t=t_0} = \int\limits_{V_f} (\frac{\partial f}{\partial t} - \mathbf{v} \cdot \boldsymbol{\nabla} f) dV \tag{47}$$

In our derivation of this general result we assumed that the gradient $\boldsymbol{\nabla} f$ is defined as a continuous function in all point of a fluid domain.

To mark out the fact of complementarity between this Eulerian description (for fixed volume domains) and the Lagrangian description (for volumes in motion) regarded as the *Convection Theorem* (45), from now on we shall call the result (47) as *local Convection Theorem* providing the following formal formulation:

Theorem 1 *(Local Convection Theorem): Let* $\mathbf{v}$ *be a vector field generating a fluid flow through a fixed 3-dimensional domain* V *and if* $f(\mathbf{r},t) \in C^1(\bar{V})$*, then*

$$\frac{d}{dt}\int\limits_V f dV = \int\limits_V (\frac{\partial f}{\partial t} - \mathbf{v}\cdot\boldsymbol{\nabla} f) dV \tag{48}$$

where dV *denotes the fixed volume element.*

Note that (48) is also applicable to arbitrary 1- and 2-dimensional closures of flow domains. Mathematical soundness of this theorem can be easily seen on a simple 1-dimensional example considered in Appendix B.

Since the volume domain V_f is fixed in a local reference system at rest and the flow vector field $\mathbf{v}$ is supposed to be non-zero, then a time variation of f-content is unambiguously related to a flux of fluid through the bounding surface ∂V_f. Thus, if (47) equals the right-hand side of the equation (44), we obtain a modified integro-differential version of f-content conservation law:

$$\int\limits_{V_f} (\frac{\partial f}{\partial t} - \mathbf{v}\cdot\boldsymbol{\nabla} f) dV = -\int\limits_{\partial V_f} f\mathbf{v}\cdot d\mathbf{S} = -\int\limits_{V_f} (\boldsymbol{\nabla}\cdot f\mathbf{v}) dV \tag{49}$$

As integral form of the general conservation law, (49) should make sense for continuous as well as for spatially discontinuous flows. In points of spatial discontinuity of the function f, gradients $\boldsymbol{\nabla} f$ have singularities, hence invalidating the integrability properties of integrands in both sides of the equation (49). This difficulty is surmounted by itself when we realize that gradients $\boldsymbol{\nabla} f$, which appear in both sides of (49), mutually cancel each other. However, the problem remains open for the standard integro-differential form of the conservation law. In fact, in the case of discontinuity of f, the right-hand side of (44) contains a singularity which can not be canceled.

If the volume V_f in (49) tends to zero, we obtain *a modified differential form of continuity equation*:

$$\frac{\partial f}{\partial t} + f\boldsymbol{\nabla}\cdot\mathbf{v} = 0 \tag{50}$$

Therefore, it is worthy to note that in the case of finite spatial discontinuities of the function f, the partial differential equation (50) does not possess untractable infinities related to the gradient $\boldsymbol{\nabla} f$. However, these singularities still take place in the standard differential form of continuity equation (42).

The usefulness of the concept of conservation of fluid f-content comes from its generality and also from its capability of cross-verifications of the results obtained in complementary Lagrangian and Eulerian descriptions. Thus, turning back to (45), if there is no time variation, the right-hand side equals to zero and in the left-hand side we arrive again at the same conclusion (50) by means of the modified Convection Theorem. The question about whether the previous cross-verification based on ill-founded considerations (41) and (43) was just that stumbling block which hindered the process of rigorization of foundations of classical field theory, deserves some special clarifications elsewhere. Let turn our attention on possible implications of the above-stated results for classical electrodynamics.

6 Local Convection Theorem and Maxwell's equations

Another interesting task would be an application of the *local Convection Theorem* (48) to the integral form of Maxwell's equations. To implement Eulerian description in hydrodynamics, one needs the knowledge of fluid quantities and the velocity flow field as functions of Eulerian variables in a local reference system at rest. The situation is somewhat different in the case of classical electrodynamics. Evidently, *a priori* unknown nature of the velocity vector field for electromagnetic field components cancels the validity of hydrodynamics common sense. However, virtual applicability of the local Convection Theorem looks like viable if, for instance, we restrict our approach to the consideration of finite size charged particles moving with a constant velocity (if in the limited case, the charge is of a delta-function type, then we suppose that some additional conditions are added to the formulation of the local Convection Theorem (48) in order to be applicable to this type of charge density discontinuity).

In fact, Einstein's special relativity theory firmly established the equivalence of inertial frames of reference in classical electrodynamics. If a single electric charge is at rest in a local frame then its electromagnetic field components do not explicitly depend on time from the point of view of an observer in uniform motion. In other words, if we are in the observer's inertial frame, electric field components will keep up appearances with straight lines coming out of the charge source. Therefore, if the charge velocity $\mathbf{v}_q$ is known, the velocity vector field $\mathbf{v} = \mathbf{v}_q$ for components of electric and magnetic field is also defined in the whole closure V_0.

Let us formulate Maxwell's equations in this particular case of one charge system (microscopic version), using general notations. For Eulerian observer all kind of space closures (volumes, bounding surfaces or curves) are fixed in a local frame of reference at rest. The first pair of Maxwell's equations does not refer to any time variation and describes the source of electric $\mathbf{E}$ and magnetic $\mathbf{H}$ field, respectively:

$$\int_{S_0} \mathbf{E}\cdot d\mathbf{S} = \int_{V_0} \boldsymbol{\nabla} \cdot \mathbf{E} dV = 4\pi Q; \qquad \int_{S_0} \mathbf{H}\cdot d\mathbf{S} = \int_{V_0} \boldsymbol{\nabla} \cdot \mathbf{H} dV = 0 \tag{51}$$

where V_0 is a fixed volume; S_0 is a closed surface bounding V_0 and Q is a whole electric charge inside V_0.

The differential formulation of (51) is

$$\boldsymbol{\nabla}\cdot\mathbf{E}=4\pi\rho;\qquad \boldsymbol{\nabla}\cdot\mathbf{H}=0 \tag{52}$$

where ρ is the charge density.

The second pair of Maxwell's integral equations refers to the time variation of electric $\mathbf{E}$ and magnetic $\mathbf{H}$ field fluxes through a fixed open surface S_0 bounded by a closed curve C_0:

$$\int\limits_{C_0}\mathbf{H}\cdot d\mathbf{l}=\frac{4\pi}{c}\int\limits_{S_0}\mathbf{j}\cdot d\mathbf{S}+\frac{1}{c}\frac{d}{dt}\int\limits_{S_0}\mathbf{E}\cdot d\mathbf{S} \tag{53}$$

$$\int\limits_{C_0}\mathbf{E}\cdot d\mathbf{l}=-\frac{1}{c}\frac{d}{dt}\int\limits_{S_0}\mathbf{H}\cdot d\mathbf{S} \tag{54}$$

where $\mathbf{j}=\rho\mathbf{v}$ is the charge current density.

Let us introduce a fluid quantity f as a scalar product of two vectors $\mathbf{A}$ and $\mathbf{n}$, defined on the surface, i.e. $f=\mathbf{A}\cdot\mathbf{n}$, where $\mathbf{A}$ is some general vector field and $\mathbf{n}$ is a unit vector normal to the surface. Since the local Convection Theorem (48) is also meaningful for 1- and 2- dimensional flow domains, it can be easily checked out that:

$$\frac{d}{dt}\int\limits_{S_0}(\mathbf{A}\cdot\mathbf{n})dS=\int\limits_{S_0}\left[\frac{\partial\mathbf{A}}{\partial t}-(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}\right]\cdot\mathbf{n}dS=\int\limits_{S_0}\frac{D^*\mathbf{A}}{D^*t}\cdot d\mathbf{S} \tag{55}$$

Despite of the limited validity of hydrodynamics approach to electromagnetic field description, we make here an attempt to use (55) in the Eulerian representation of the second pair Maxwell's integral equations (53) and (54) as follows:

$$\int\limits_{C_0}\mathbf{H}\cdot d\mathbf{l}=\int\limits_{S_0}[\boldsymbol{\nabla},\mathbf{H}]\cdot d\mathbf{S}=\frac{4\pi}{c}\int\limits_{S_0}\mathbf{j}\cdot d\mathbf{S}+\frac{1}{c}\int\limits_{S_0}\frac{D^*\mathbf{E}}{D^*t}\cdot d\mathbf{S} \tag{56}$$

$$\int\limits_{C_0}\mathbf{E}\cdot d\mathbf{l}=\int\limits_{S_0}[\boldsymbol{\nabla},\mathbf{E}]\cdot d\mathbf{S}=-\frac{1}{c}\int\limits_{S_0}\frac{D^*\mathbf{H}}{D^*t}\cdot d\mathbf{S} \tag{57}$$

or in a compact differential form:

$$[\boldsymbol{\nabla},\mathbf{H}]=\frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{D^*\mathbf{E}}{D^*t};\qquad [\boldsymbol{\nabla},\mathbf{E}]=-\frac{1}{c}\frac{D^*\mathbf{H}}{D^*t} \tag{58}$$

If in this hydrodynamic formulation the velocity flow fields $\mathbf{v}$ for electromagnetic components $\mathbf{E}$ and $\mathbf{H}$ are known, then the time derivative from the viewpoint of Eulerian observer is also defined in explicit terms as $\frac{D^*}{D^*t}=\frac{\partial}{\partial t}-\mathbf{v}\cdot\boldsymbol{\nabla}$.

On the other hand, Maxwell's equations are compatible with the charge conservation law. Since velocities of charge particles are measurable values, their knowledge makes meaningful the direct application of the local Convection Theorem (48) and, as a consequence, the law of the conservation of f-content (49):

$$\int\limits_{V_f}(\frac{\partial\rho}{\partial t}-\mathbf{v}\cdot\boldsymbol{\nabla}\rho)dV=-\int\limits_{\partial V_f}\rho\mathbf{v}\cdot d\mathbf{S}=-\int\limits_{V_f}(\boldsymbol{\nabla}\cdot\rho\mathbf{v})dV \tag{59}$$

or in modified differential form of continuity equation:

$$\frac{\partial\rho}{\partial t}+\rho\cdot\boldsymbol{\nabla}\mathbf{v}=0 \tag{60}$$

Let us write the differential form (58) explicitly in terms of partial derivatives and velocity vector field if they are presumably known:

$$[\boldsymbol{\nabla},\mathbf{H}]=\frac{4\pi}{c}\mathbf{j}+\frac{1}{c}(\frac{\partial\mathbf{E}}{\partial t}-(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{E});\qquad[\boldsymbol{\nabla},\mathbf{E}]=-\frac{1}{c}(\frac{\partial\mathbf{H}}{\partial t}-(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{H}) \tag{61}$$

Applying a general expression valid for any vector field $\mathbf{E}$ (or $\mathbf{H}$):

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{E}=\mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{E})-[\boldsymbol{\nabla},[\mathbf{v},\mathbf{E}]] \tag{62}$$

and having in mind the first pair of Maxwell's source equations (52) we arrive at:

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{E}=4\pi\mathbf{j}-[\boldsymbol{\nabla},[\mathbf{v},\mathbf{E}]];\qquad(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{H}=-[\boldsymbol{\nabla},[\mathbf{v},\mathbf{H}]] \tag{63}$$

where $\mathbf{j}=\rho\mathbf{v}$.

It is worth reminding here that our approach to integral formulation of Maxwell's equation has been originally submitted to a consideration of uniformly moving charged particle. In this case the time dependence of fields is implicit and all partial time derivatives vanish from (61). Substituting (63) into (61), we arrive at the well-established relationship between quasistatic magnetic and electric field strengths of uniformly moving charge or magnetic source, respectively[11]:

$$\mathbf{H}=\frac{1}{c}[\mathbf{v},\mathbf{E}];\qquad\mathbf{E}=-\frac{1}{c}[\mathbf{v},\mathbf{H}] \tag{64}$$

where $\mathbf{v}$ is the velocity of a source.

Note that (64) makes sense only from the viewpoint of Eulerian observer placed in the local inertial reference system at rest so that no use of the special relativity relationships for field transformations has been necessary.

It is very important to stress here that in our attempt to reconsider basic differential equations, we leave without any modification the original integral form of continuity as well as Maxwell's equations. Note that this is the only form (not differential one) which had been verified by experiments. So that no additional experimental confirmation of these fundamental laws is implied in this approach. In fact, in this work we call into question whether the conventional mathematical procedure of the transition from integral equations with total time derivative for *fluid quantities* to their differential form is correct. On the other hand, the differential form of continuity as well as Maxwell's equations constitutes the standard basis for providing mathematical solutions for the classical electromagnetic theory. Hence what kind of differential equations of classical electrodynamics have been under a scrutiny of experts for more than a century?

The major fact that emerges from the above considerations is that there appear two conflicting approaches to the differential form of Maxwell's equations from the viewpoint of Eulerian observer. The traditional one is indifferent to any specific manifestations of fluid quantities, treating the time derivative for an observer at rest as a simple partial time derivative $\frac{\partial}{\partial t}$. The other starts out the same integral formulation of Maxwell's equations but the time derivative for Eulerian observer no longer coincides with the partial one. There appears an extra term $\mathbf{v} \cdot \boldsymbol{\nabla}$ which is one of the outcomes of the non-zero flow velocity field, in what case it can be regarded as a convective term.

Therefore, the denial of the exclusive use of partial time derivatives to describe time variation for an observer at rest and a recognition of a deeper underlying meaning of the total time derivative in Eulerian description imply inevitable changes in the structure of mathematical solutions to Maxwell's equations (as regards this aspect, some alternative frameworks for classical electrodynamics were recently discussed in [6],[12]–[15]). However, a wider analysis of the integral form of Maxwell's equations on basis of the local Convection Theorem does not look tractable at present stage. Perhaps, it is possible to approach a description of electromagnetic field of a classical spinning charged particle using above-considered approximations. Nevertheless, the question about whether some postulates are indispensable in this and general cases should be studied carefully elsewhere.

7 Conclusions

In this work we attempted to get a more detailed insight towards some traditional aspects of mathematical and conceptual structure of theoretical hydrodynamics. We found that no equal standards of rigor take place in Lagrangian and Eulerian descriptions. The reconsidered account provides a rigorous analytical approach to the treatment of time derivatives in properly Eulerian description. To avoid traditional formalistic approach in which the total time derivative of Eulerian observer is taken for the partial time derivative, we realized the necessity of the practical implementation of the final Cauchy problem for velocity field differential equation. It justified a new definition for the total time derivative of Eulerian observer which was regarded in this work also as the *local directional derivative.* The point that needs to be emphasized here is the complementary character of the above introduced concept. It can be considered as a complementary counter-part of the well-known Euler's derivative.

By no means, the local directional derivative substitutes the Euler mathematical construction. By contrary, it is shown that both types of total time derivatives for Lagrangian and Eulerian observers (which could be interpreted also as *two complementary 4-dimensional directional derivatives*) are equally valid but should be used in different contexts.

One of the interesting conclusions of the analytic expression for both 4-dimensional directional derivatives is that the choice between Lagrangian and Eulerian types of flow field specification is equivalent to the choice between space-time manifolds with Euclidean and Minkowski metric respectively. Therefore, a consistent mathematical description of fluid kinematics can be also compatible with the Lorentz group symmetry. Although our approach was restricted by the consideration of one-component (scalar) ideal flow field, the notion of the local directional derivative can be easily generalized on Lie's derivatives for any general tensor field on differentiable manifolds. Both types of Lie's derivative will correspond to both complementary types of descriptions.

The concept of local directional derivative has been also applied to analyze time variation of fluid contents in Eulerian description. The result has been formulated in form of a theorem called here as the local Convection Theorem meaningful for fixed space domains. Therefore, it should be refer as complementary to the *Convection Theorem* established for Lagrangian description, i.e. for space domains moving with a fluid. Another unexpected outcome of the approach developed in this work consists in modification of the standard differential form of continuity equations. It also implies a reconsideration of the differential form of Maxwell's equations since they are compatible with the law of charge conservation, i.e continuity equation for the charge density.

In place of concluding remark let us remind asserting and encouraging attitude of a great mathematician. Gauss once wrote in his letter to Bessel (quoted from[2])

> *...One should never forget that the function [of complex variable], like all mathematical constructions, are only our own creations, and that when the definition with which one begins ceases to make sense, one should not ask, what is, but what is convenient to assume in order that it remain significant...*

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APPENDIX A. Modified version of the Convection Theorem

Let us consider a fluid f-content (which we shall denote by $F = \int f dV$) when a macroscopic volume domain $V(t)$ is identified and is moving with the fluid:

$$F(t) = \int_{V(t)} f(t, \mathbf{r}(t)) dV \tag{65}$$

In the framework of traditional Lagrangian description linked to the initial Cauchy problem, it implies the existence of non-zero geometrical transformation H_t (associated with non-zero velocity flow field $\mathbf{v}$):

$$V(t) = H_t V(t_0); \qquad f(t, \mathbf{r}(t)) = H_t f(t_0, \mathbf{r}_0) \tag{66}$$

Since the time derivative of $F(t)$ is going to be treated here, we have to consider only linear part of geometrical transformations H_t when $t - t_0$ tends to zero:

$$\mathbf{r}(t) = H_t\mathbf{r}_0 = \mathbf{r}_0 + (t - t_0)\mathbf{v} + o(t - t_0) \tag{67}$$

where $\mathbf{r}_0 = \mathbf{r}(t_0)$ belongs to the initial volume $V(t_0) = V_0$ and $\mathbf{v} =(\frac{d\mathbf{r}}{dt})_{t=t_0}$ is a local value of flow field velocity.

The geometrical transformation H_t is algebraically represented by the Jacobian determinant $\det\left|\frac{\partial H_t\mathbf{r}_0}{\partial \mathbf{r}_0}\right|$ which for the infinitesimal transformation (67) takes the following form[16]:

$$\det\left|\frac{\partial H_t\mathbf{r}_0}{\partial \mathbf{r}_0}\right| = 1 + (t - t_0)\boldsymbol{\nabla}\cdot\mathbf{v} + o(t - t_0) \tag{68}$$

Thus, the evolution of a fluid content $F(t)$ can be written in original variables $\mathbf{r}_0$ if the Jacobian has been specified for each value of the parameter t:

$$F(t) = \int\limits_{V_0} f(t, \mathbf{r}_0)\det\left|\frac{\partial H_t\mathbf{r}_0}{\partial \mathbf{r}_0}\right| dV_0 \tag{69}$$

Let us now analyze the time derivative of a fluid f-content:

$$\frac{d}{dt}F(t) = \frac{d}{dt}\int\limits_{V_0} f(t, \mathbf{r}_0)\det\left|\frac{\partial H_t\mathbf{r}_0}{\partial \mathbf{r}_0}\right| dV_0 \tag{70}$$

or in the case of infinitesimal transformation:

$$\frac{d}{dt}F(t) = \frac{d}{dt}\int\limits_{V_0} f(t, \mathbf{r}_0)\left[1 + (t - t_0)\boldsymbol{\nabla}\cdot\mathbf{v} + o(t - t_0)\right] dV_0 \tag{71}$$

Importantly, in the representation (69) of the integrand function $f(t, \mathbf{r}_0)$ space variable $\mathbf{r}_0$ does not already depend on time parameter t. Put in other terms, the expression (69) implements the Eulerian independent variable $\mathbf{r}_0$ for the integrand function $f(t, \mathbf{r}_0)$ and the integration volume V_0 instead of the Lagrangian flow variable $\mathbf{r}(t)$ used in (65). The original time dependence of the function $f(t, \mathbf{r}(t))$ through Lagrangian variable $\mathbf{r}(t)$ is now replaced by the time dependence of the Jacobian determinant. Therefore, since $\mathbf{r}_0$ is assumed to be time independent in (71), the time derivative of $f(t, \mathbf{r}_0)$ coincides with the partial time derivative according to the classical definition:

$$\frac{d}{dt}f(t, \mathbf{r}_0) = \frac{\partial}{\partial t}f(t, \mathbf{r}_0) \tag{72}$$

Thus we arrive at the *modified version of the Convection Theorem*:

$$\frac{d}{dt}[F(t)]_{t=t_0} = \int\limits_{V(t)} (\frac{\partial f}{\partial t} + f\boldsymbol{\nabla}\cdot\mathbf{v})dV \tag{73}$$

under the condition $t \to t_0$, i.e. $V(t) \to V_0$.

From these considerations it is now clear that the conventional approach took the partial time derivative (72) for Euler's directional derivative $\frac{Df}{Dt}$.

APPENDIX B. Example of applicability of the Local Convection Theorem

To highlight mathematical soundness of the *local Convection Theorem* (47) let us consider a simple example of 1-dimensional ideal flow defined on a fixed 1-dimensional interval $[a, b]$ on x-axis. Let the set $\{t, x\}$ be Eulerian variables and f a regular function, for instance:

$$f(t, x) = t + x^3 \tag{74}$$

Additionally, the fluid flow is defined by the velocity vector field $\mathbf{v}$:

$$\frac{dx}{dt} = \mathbf{v}(t, x) = \frac{t}{x^2} \tag{75}$$

Let us choose some fixed point $\tilde{x}_f$ from the interval $[a, b]$ and formulate the final Cauchy problem for the equation (75). It is equivalent to the definition of Lagrangian variables $\{t, \tilde{x}(t)\}$ for some identified point-particle:

$$\mathbf{v}(t, \tilde{x}) = \frac{d\tilde{x}}{dt} = \frac{t}{\tilde{x}^2}; \quad \to \quad \int_{\tilde{x}_0}^{\tilde{x}_f} \tilde{x}^2 d\tilde{x} = \int_{t_0}^{t} t dt \tag{76}$$

Note again that in any solution of the final Cauchy problem, the initial starting point $\tilde{x}_0$ is a function of t_0, t and $\tilde{x}_f$:

$$\tilde{x}^3(t_0, t, \tilde{x}_f) = \tilde{x}_f^3 + \frac{3}{2}t_0^2 - \frac{3}{2}t^2 \tag{77}$$

Let us now calculate separately the time variation of the fluid f-content considered in both parts of the *local Convection Theorem* (47). We first evaluate the left-hand side integral:

$$\int_a^b f(t, \tilde{x}(t)) d\tilde{x}_f = \int_a^b [\tilde{x}_f^3 + \frac{3}{2}t_0^2 - \frac{3}{2}t^2 + t] d\tilde{x}_f \tag{78}$$

then we have:

$$\int_a^b f(t, \tilde{x}(t)) d\tilde{x}_f = \frac{1}{4}(b^4 - a^4) + (\frac{3}{2}t_0^2 - \frac{3}{2}t^2 + t)(b - a) \tag{79}$$

as well as its time derivative:

$$\frac{d}{dt}\int\limits_a^b f(t,\tilde{x}(t))d\tilde{x}_f = (1-3t_0)(b-a) \tag{80}$$

taken under the condition $t \to t_0$.

The right hand-side of (47) considers f-quantity in Eulerian independent variables $\{t, \tilde{x}_f\}$. Since x_f coincides with space variable of the local x-axes at rest, we shall use for it the common notation x (i.e. without *tilde)*. Partial derivatives are calculated at t_0 in all fixed points x from $[a, b]$:

$$\frac{\partial f}{\partial t} = 1; \qquad \frac{\partial f}{\partial x} = 3x^2; \qquad \mathbf{v} = \frac{dx}{dt} = \frac{t_0}{x^2} \tag{81}$$

and we proceed to the evaluation of the following integral:

$$\int\limits_a^b (\frac{\partial f}{\partial t} - \mathbf{v}\frac{\partial f}{\partial x})dx = \int\limits_a^b (1-3t_0)dx = (1-3t_0)(b-a) \tag{82}$$

A simple comparison of (80) and (82) validates the applicability of the local Convection Theorem to the ideal fluid flow defined by (74) and (75):

$$\frac{d}{dt}[\int\limits_a^b f(t,x(t))dx]_{t=t_0} = \int\limits_a^b (\frac{\partial f}{\partial t} - \mathbf{v}\frac{\partial f}{\partial x})dx \tag{83}$$

where $\frac{dx}{dt} = \mathbf{v}(t_0, x)$.

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