

ε -CONNECTEDNESS, FINITE APPROXIMATIONS, SHAPE THEORY AND COARSE GRAINING IN HYPERSPACES

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ABSTRACT. We use upper semifinite hyperspaces of compacta to describe ε -connectedness and to compute homology from finite approximations. We find another connection between ε -connectedness and the so called *Shape Theory*. We construct a geodesically complete \mathbb{R} -tree, by means of ε -components at different resolutions, whose behavior at infinite captures the topological structure of the *space of components* of a given compact metric space. We also construct inverse sequences of finite spaces using internal finite approximations of compact metric spaces. These sequences can be converted into inverse sequences of polyhedra and simplicial maps by means of what we call the Alexandroff-McCord correspondence. This correspondence allows us to relate upper semifinite hyperspaces of finite approximation with the Vietoris-Rips complexes of such approximations at different resolutions. Two motivating examples are included in the introduction. We propose this procedure as a different mathematical foundation for problems on data analysis. This process is intrinsically related to the methodology of shape theory. Finally this paper reinforces Robins's idea of using methods from shape theory to compute homology from finite approximations.

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1. INTRODUCTION

The aim of this paper is to show how hyperspaces, concretely upper semifinite hyperspaces, can be used to study topological data analysis. In particular to treat ε -connectedness and the computing of homology from finite approximations.

The germ of the concept of ε -connectedness is contained in Cantor's work, see [1], and it has been recently used in computational topology mainly by Vanessa Robins and her collaborators [2], [3].

The task of computing homology from finite approximations have been accomplished by Vanessa Robins in her thesis [4] and in [5] on one hand, and, on the other hand, by Carlsson, de Silva, Edelsbrunner, Harer and Zomorodian among others. See the recent Ghrist's paper [6] and the references therein.

We have also to say that a different and independent approach to finite approximation to homology have been given by A. Giraldo, M. A. Morón, F.R. Ruiz del Portal and J.M.R. Sanjurjo in [7].

Our contribution on ε -connectedness is mainly focused on the foundation of the theory. In this sense, this paper is related to [8]. Our point of view is, in some sense, different. In fact we think that we have found an adequate theoretical framework to give a more complete picture than that drawn in [8]. Moreover all results in [8] can be reproved using our techniques.

Let us recall that an ε -chain in a metric space (X, d) joining two points $x, y \in X$ is a finite subset $\{x_0, \dots, x_n\} \subset X$ such that $x_0 = x$ and $x_n = y$ and $d(x_i, x_{i+1}) < \varepsilon$ for any $i \in \{0, \dots, n-1\}$. The metric space (X, d) is said to be ε -connected if any two points in X can be joined by an ε -chain.

Concretely our approach to ε -connectedness is as follows:

We have an ideal figure (that is to say a compact metric space (X, d)) and we consider a macro-structure intrinsically related to (X, d) , called the upper semifinite hyperspace of (X, d) , where the figure is canonically embedded. Finally we take a coarse look, i.e. we use a coarse graining process, at this macro-structure near the canonical copy of the figure.

To give a quick description for the reader of what is the upper semifinite hyperspace of (X, d) we have included some topological background in Section 2. We denote this hyperspace by 2_U^X . The points in the space 2_U^X are just the closed non-empty subsets of (X, d) . The main tool we are going to use is the following:

Let (X, d) be a compact metric space. Then the family $U = \{U_\varepsilon\}_{\varepsilon>0}$ is a base of open neighborhoods of the canonical copy X inside 2_U^X , where $U_\varepsilon = \{C \in 2^X \mid \text{diam}(C) < \varepsilon\}$

The above result can be found in [9], see also [10] for more information on upper semifinite hyperspaces.

The canonical copy of X is formed by those closed, non-empty, subsets of X with diameter equal zero, i.e. the points of X .

The important fact about the topology of 2_U^X is that if we have $C, D \in U_\varepsilon$ with $C \subset D$ then they are undistinguishable under an ε -resolution. So if one can see any of them it is just the greater one. This can be explained studying the closure of a point in the space 2_U^X . So we think that it is an example of spaces with very bad separation properties that are suitable for getting geometric and computational results.

This point of view allows us to detect, in particular, the ε -components of (X, d) . The concept of ε -component in the theory of ε -connectedness corresponds to the concept of connected component in the usual theory of topological connectedness.

One of the consequences of this approach is that the abstract topology of the macro-structure can help in order to study computational aspects of the ideal figure. In particular we include herein an specially aesthetic result, $C(\varepsilon) = Sh(U_\varepsilon)$, where $C(\varepsilon)$ is the number of ε -components, [2], [3], and Sh is the *topological shape* as described by Mardešić in [11], see also [12] and [13]. The above equality means that the spaces U_ε has the same shape as a discrete space with $C(\varepsilon)$ points. An striking difference between shape and homotopy theory is the fact, see [9], that U_ε has not in general the same homotopy type of any space with good separation properties.

We will also construct a geodesically complete \mathbb{R} -tree, see [14] for definition, using the ε -components for a countable cofinal family of numbers $\varepsilon > 0$ in such a way that the end space of this tree describes not only the set of real connected-components of the figure, (X, d) , but also the natural quotient topology induced on it.

One of the simplest and effective ideas in this paper is to characterize the existence of ε -chains, in the sense of Cantor, in terms of the existence of real topological paths *near* the figure joining the same points as the ε -chains. In this sense we *prove* that the *intuition* of Snyder calling *paths* to ε -chains, see [8], was accurate.

Vanessa Robins in her thesis [4] and in [5] considered the problem of extrapolating the homology of a compact metric space, X , from a finite point set approximation, S . She first considered both of them as subsets of a metric space (M, d) . Her main motivation in [5] was to identify

the topological and geometric structure of attractors and other invariant sets in dynamical systems. Her approach is based on inverse systems of ε -neighborhoods and inclusions maps and then related in nature to shape theory [12]. A different way to use shape theory to get global properties of attractors and invariant sets in dynamical systems has been developed along the past few years, and it is still being. This is the reason why we have included in Section 2 a short history about some applications of shape theory to dynamical systems.

To give the compact space, X , and its finite approximation, S , comparable topological structure, Robins used their closed ε -neighborhoods:

$$X_\varepsilon = \{x \in M \mid d(x, X) \leq \varepsilon\} \quad , \quad S_\varepsilon = \{x \in M \mid d(x, S) \leq \varepsilon\}$$

Her point of view, as explained in [5], is that if X and S are within a distance ρ , their ε -neighborhoods should have similar properties for $\varepsilon > \rho$. Later she explains her way to replace the ε -neighborhoods by finite simplicial complexes to apply the machinery of algebraic topology to treat, in particular, the important point of view of persistency.

In the last part of the paper, Section 6, we propose a different way to assign finite simplicial complexes related to finite approximations and using upper semifinite hyperspaces. We have also the advantage of avoiding any embedding of the compact metric space X in any prefixed ambient space M . On the contrary we always consider that the finite approximation of X is formed by points inside X .

We mainly use some constructions due to Alexandroff [15], and, overall, a related beautiful paper due to M. McCord [16], see also Stong [17]. Our intention is to show that upper semifinite hyperspaces of finite spaces can be used to infer high-dimensional structure of a compactum from its finite approximations. This is our way to attack the two following fundamental tasks, as mentioned in [6] page 61:

- (1) How one infers high-dimensional structure from low-dimensional representations
and
- (2) How one assembles discrete points into global structure.

Our main tool here is the following result extracted from McCord [16]:

For each finite topological space X there is a finite simplicial complex $\mathcal{K}(X)$ and a weak homotopy equivalence $f_X : |\mathcal{K}(X)| \longrightarrow X$. Conversely, for each finite simplicial complex K there

exist a finite topological space $\mathcal{X}(K)$ and a weak homotopy equivalence $f_K : |K| \longrightarrow \mathcal{X}(K)$. Moreover each map $\varphi : X \longrightarrow Y$ between finite T_0 spaces is also a simplicial map $\mathcal{K}(X) \longrightarrow \mathcal{K}(Y)$ and $\varphi f_X = f_Y | \varphi |$.

Where $|W|$ represents the geometric realization of the complex W . We are going to treat only the case when the finite space is a T_0 spaces because it is the case of any topological subspace of an upper semifinite hyperspace. Recall that a topological space X is a T_0 space, or satisfies the T_0 separation axiom, if for any pair of different points there is always an open subset of X containing one of them and not to the other one.

Recall that a map $f : X \longrightarrow Y$ is said to be a *weak homotopy equivalence* if the induced maps $f_* : \pi_i(X, x) \longrightarrow \pi_i(Y, f(x))$, on the homotopy groups, are isomorphisms for all $x \in X$ and $i \geq 0$. As mentioned by McCord in [16], it is a well-known theorem of J.H.C. Whitehead that every weak homotopy equivalence induces isomorphisms on singular homology groups. One can not expect that any of the maps in the above result are homotopy equivalences because of the images of both of them are finite spaces. The point here is that if the topological space X is the upper semifinite hyperspace of a finite metric space (A, d) with $n + 1$ points, then the finite simplicial complex $\mathcal{K}(X)$ is the first barycentric subdivision of an n -simplex. Moreover, if we denote by $U(A) = \{U_\varepsilon(A)\}_{\varepsilon > 0}$ the corresponding family given by $U_\varepsilon(A) = \{C \in 2^A \mid \text{diam}(C) < \varepsilon\}$ then $\mathcal{K}(U_\varepsilon(A))$ is the first barycentric subdivision of the corresponding Vietoris-Rips complex (see [6] and [19] or section 6 herein for the definition) $\mathcal{R}_\varepsilon(A)$ for the metric space (A, d) . Note that, in the notation, we forget the metric d if it not causes confusion.

Let us say only few words about what we call the Alexandroff-McCord correspondence, described above, between finite topological spaces and finite simplicial complexes. We strongly recommend to read the originals [16] and [17]. All of these results are extracted from [16] and from some notes due to J.P. May [18].

For any finite topological space X and any point $x \in X$ we denote by V_x the minimal open set containing the point x . Obviously V_x is the intersection of all open subsets of X containing x . It can be also defined a relation \leq on X by saying $x \leq y$ if $x \in V_y$ (equivalently, $V_x \subset V_y$). It is written $x < y$ if $x \leq y$ but $x \neq y$. The relation \leq is transitive and reflexive. Moreover \leq is a partial order if and only if X is a T_0 space. The continuity of a function $f : X \longrightarrow Y$ between

finite topological spaces is equivalent to the fact that f is order preserving for the corresponding relations.

Given a finite T_0 space X , consider the partial order \leq associated to X as above. The finite complex $\mathcal{K}(X)$ is defined by the following way:

The vertices of $\mathcal{K}(X)$ are the points of X . The simplices of $\mathcal{K}(X)$ are the totally ordered subsets of X . As showed in [16], if $Y \subset X$ then $\mathcal{K}(Y)$ is a full subcomplex of $\mathcal{K}(X)$.

The map $f_X : |\mathcal{K}(X)| \rightarrow X$ is defined as follows:

given $u \in |\mathcal{K}(X)|$, then u is contained in a unique open simplex (x_0, x_1, \dots, x_r) where $x_0 < x_1 < \dots < x_r$ in X . Let $f_X(u) = x_0$.

Let us analyze two simple and *ideal* examples of our approach to compute homology from finite approximations. Later, in Section 6, we will describe a general procedure related to these examples. In our main construction in Section 6 we have to be a little bit more careful

1.1. Two Examples.

Example 1. Consider the unit circle S^1 in the complex plane with the geodesic distance. That is, the distance between two points is the minimum of the lengths of the two arcs joining them. We denote this metric by d .

As in [19], for any natural number n consider:

$$A_n = \{e^{\frac{2\pi ik}{n}} \mid k = 0, 1, \dots, n-1\}$$

the set of n^{th} roots of 1 with the induced metric. Take $\rho_n = \frac{2\pi}{n}$. Then A_n is a finite ρ_n -approximation to S^1 .

What about $\mathcal{K}(U_\varepsilon(A_n))$?:

If $\varepsilon \leq \rho_n$:

Then $\mathcal{K}(U_\varepsilon(A_n))$ is a complex having only vertices as simplices. Consequently $|\mathcal{K}(U_\varepsilon(A_n))|$ is a discrete space with n points because, in fact, $U_\varepsilon(A_n)$ is a discrete space with n points.

If $\varepsilon > \rho_n$:

Then $U_\varepsilon(A_n) = 2A_n$. $\mathcal{K}(U_\varepsilon(A_n))$ is the first barycentric subdivision of the $(n-1)$ -simplex and consequently $|\mathcal{K}(U_\varepsilon(A_n))|$ is a topological closed cell of dimension $n-1$.

If $\rho_n < \varepsilon \leq 2\rho_n$ and $n \geq 4$:

Then $\mathcal{K}(U_\varepsilon(A_n))$ is the first barycentric subdivision of a regular polygon with n sides. Consequently $|\mathcal{K}(U_\varepsilon(A_n))|$ is, topologically, S^1 .

Consider now the sequence $\varepsilon_n = \rho_{2^{3(n-1)}}$, $n \geq 1$. So, $\varepsilon_{n+1} = \frac{\varepsilon_n}{8}$. For every $k \geq 0$ we have that $A_{2^{3k}}$ is an ε_{k+1} -approximation of S^1 .

Consider for $n \geq 1$

$$A_{2^{3(n-1)}} = \{\theta_k = e^{\frac{2\pi ik}{2^{3(n-1)}}} \mid k = 0, 1, \dots, 2^{3(n-1)} - 1\}$$

and

$$A_{2^{3n}} = \{\eta_j = e^{\frac{2\pi ij}{2^{3n}}} \mid j = 0, 1, \dots, 2^{3n} - 1\}$$

For every $\eta_j \in A_{2^{3n}}$, define

$$A_{2^{3(n-1)}}(\eta_j) = \{\theta \in A_{2^{3(n-1)}} \mid d(\eta_j, \theta) = d(\eta_j, A_{2^{3(n-1)}})\}$$

Finally we can define a continuous function $p_{n+1,n} : U_{2\varepsilon_{n+1}}(A_{2^{3n}}) \longrightarrow U_{2\varepsilon_n}(A_{2^{3(n-1)}})$ by $p_{n+1,n}(C) = \bigcup_{c \in C} A_{2^{3(n-1)}}(c)$ for any $C \in U_{2\varepsilon_{n+1}}(A_{2^{3n}})$.

In fact we have only to prove that $p_{n+1,n}(C) \in U_{2\varepsilon_n}(A_{2^{3(n-1)}})$ for every $C \in U_{2\varepsilon_{n+1}}(A_{2^{3n}})$. For $n = 1$ it is obvious. Suppose then that $n \geq 2$. Consequently $2^{3(n-1)} > 4$. If $C \in U_{2\varepsilon_{n+1}}(A_{2^{3n}})$ then $\text{diam}(C) < \frac{2\pi}{2^{3n-1}}$. So, C contains at most two points. Moreover if C contains just two points then $C = \{\eta_j, \eta_{j+1}\}$ understanding that $j+1 = 0$ if $j = 2^{3n} - 1$. Fix $j \in \{0, 1, \dots, 2^{3n} - 1\}$ and suppose that $\theta_l, \theta_m \in A_{2^{3(n-1)}}(\eta_j)$. Consequently $d(\theta_l, \theta_m) < 2\varepsilon_n$ by the triangle inequality. So, $\text{diam}(p_{n+1,n}(\eta_j)) < 2\varepsilon_n$ because $p_{n+1,n}(\eta_j) = A_{2^{3(n-1)}}(\eta_j)$.

If $C \in U_{2\varepsilon_{n+1}}(A_{2^{3n}})$ contains just two points then $C = \{\eta_j, \eta_{j+1}\}$. Consequently $p_{n+1,n}(C) = A_{2^{3(n-1)}}(\eta_j) \cup A_{2^{3(n-1)}}(\eta_{j+1})$. Since $2^{3(n-1)}$ divides 2^{3n} , it follows that $A_{2^{3(n-1)}} \subset A_{2^{3n}}$. This implies that there is a unique $k \in \{0, 1, \dots, 2^{3(n-1)} - 1\}$ such that $C = \{\eta_j, \eta_{j+1}\}$ is contained in the minimum arc $[\theta_k, \theta_{k+1}]$ joining them in S^1 . Here again $k+1 = 0$ if $k = 2^{3(n-1)} - 1$. Hence $p_{n+1,n}(C) \subset \{\theta_k, \theta_{k+1}\}$. Therefore $\text{diam}(p_{n+1,n}(C)) \leq \varepsilon_n < 2\varepsilon_n$. The continuity of the maps $p_{n+1,n}$ can be easily proved using the upper semifinite topology. In particular using Lemma 3 in [9]. So we have constructed an inverse sequence of finite topological spaces:

$$\dots \longrightarrow U_{2\varepsilon_{n+1}}(A_{2^{3n}}) \longrightarrow U_{2\varepsilon_n}(A_{2^{3(n-1)}}) \dots \longrightarrow U_{2\varepsilon_2}(A_{2^3}) \longrightarrow U_{2\varepsilon_1}(A_1)$$

with the corresponding $p_{n+1,n}$ as bonding maps. Using now Theorem 2 in page 466 in [16] we have the existence of a simplicial map $|p_{n+1,n}|$ making the following diagram commutative:

$$\begin{array}{ccc} | \mathcal{K}(U_{2\varepsilon_{n+1}}(A_{2^{3n}})) | & \xrightarrow{|p_{n+1,n}|} & | \mathcal{K}(U_{2\varepsilon_n}(A_{2^{3(n-1)}})) | \\ f_{n+1} \downarrow & & \downarrow f_n \\ U_{2\varepsilon_{n+1}}(A_{2^{3n}}) & \xrightarrow{p_{n+1,n}} & U_{2\varepsilon_n}(A_{2^{3(n-1)}}) \end{array}$$

where f_{n+1} and f_n are the corresponding McCord's weak homotopy equivalences quoted above.

So we have two inverse sequences and a level map of inverse sequences where, at any level, the connecting map is a weak homotopy equivalence as described in the following diagram:

$$\begin{array}{ccccccc} \longrightarrow & | \mathcal{K}(U_{2\varepsilon_{n+1}}(A_{2^{3n}})) | & \xrightarrow{|p_{n+1,n}|} & | \mathcal{K}(U_{2\varepsilon_n}(A_{2^{3(n-1)}})) | & \longrightarrow \cdots \longrightarrow & | \mathcal{K}(U_{2\varepsilon_2}(A_{2^3})) | & \xrightarrow{|p_{2,1}|} & | \mathcal{K}(U_{2\varepsilon_1}(A_1)) | \\ & f_{n+1} \downarrow & & \downarrow f_n & & f_2 \downarrow & & \downarrow f_1 \\ \longrightarrow & U_{2\varepsilon_{n+1}}(A_{2^{3n}}) & \xrightarrow{p_{n+1,n}} & U_{2\varepsilon_n}(A_{2^{3(n-1)}}) & \longrightarrow \cdots \longrightarrow & U_{2\varepsilon_2}(A_{2^3}) & \xrightarrow{p_{2,1}} & U_{2\varepsilon_1}(A_1) \end{array}$$

Passing at the corresponding inverse sequences of singular homology groups we have, at any square, the commutativity of:

$$\begin{array}{ccc} H_k(| \mathcal{K}(U_{2\varepsilon_{n+1}}(A_{2^{3n}})) |) & \xrightarrow{H_k(|p_{n+1,n}|)} & H_k(| \mathcal{K}(U_{2\varepsilon_n}(A_{2^{3(n-1)}})) |) \\ H_k(f_{n+1}) \downarrow & & \downarrow H_k(f_n) \\ H_k(U_{2\varepsilon_{n+1}}(A_{2^{3n}})) & \xrightarrow{H_k(p_{n+1,n})} & H_k(U_{2\varepsilon_n}(A_{2^{3(n-1)}})) \end{array}$$

$H_k(f_n)$ and $H_k(f_{n+1})$ are isomorphisms because f_{n+1} and f_n are weak homotopy equivalences. Consequently the inverse sequences $(H_k(| \mathcal{K}(U_{2\varepsilon_{n+1}}(A_{2^{3n}})) |), H_k(| p_{n+1,n} |))$ and $(H_k(U_{2\varepsilon_{n+1}}(A_{2^{3n}})), H_k(p_{n+1,n}))$ are isomorphic in pro-Group for any $k \geq 0$. In particular the inverse limit groups of both of them are isomorphic as groups.

In order to end let us say that, in this particular case, the maps $|p_{n+1,n}|$ are homotopy equivalences for $n \geq 2$. So, the inverse limit of $(H_k(| \mathcal{K}(U_{2\varepsilon_{n+1}}(A_{2^{3n}})) |), H_k(| p_{n+1,n} |))$ is just the k -th Čech homology group of S^1 (remember that $| \mathcal{K}(U_{2\varepsilon_{n+1}}(A_{2^{3n}})) |$ is homeomorphic to

S^1 for every $n \geq 1$). The same thing can be said for Čech cohomology groups and the shape groups. Consequently the inverse sequence of finite spaces:

$$\dots \longrightarrow U_{2\varepsilon_{n+1}}(A_{2^{3n}}) \longrightarrow U_{2\varepsilon_n}(A_{2^{3(n-1)}}) \dots \longrightarrow U_{2\varepsilon_2}(A_{2^3}) \longrightarrow U_{2\varepsilon_1}(A_1)$$

which depends only on finite approximations of S^1 can be used to extrapolate properties of the space S^1 .

A more careful analysis of the maps $|p_{n+1,n}|$ in this example, for $n \geq 2$, shows that they are near-homeomorphisms (because they are cell-like maps) and then the inverse limit of the inverse sequence $(|K(U_{2\varepsilon_{n+1}}(A_{2^{3n}}))|, |p_{n+1,n}|)$ is even homeomorphic to S^1

Example 2. The second example that we consider is the usual middle third Cantor set, we denote it here by \mathcal{C} , with the metric induced by the absolute value in the real line \mathbb{R} . We are going to use in this example the same notation as in the previous one.

To construct the usual middle third Cantor set, start with the interval $[0, 1]$ and throw away the middle third $(\frac{1}{3}, \frac{2}{3})$. We now have two pieces, $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. From each of these, throw away the middle third, leaving four pieces. From each of those, throw away the middle third. Etc...

What is left?. Just the middle third Cantor set. Each step removes $1/3$ of the points one has. After n steps, the total length of the intervals in the set is $(\frac{2}{3})^n$.

Take $\varepsilon_1 = 2$ and consider the sequence $\varepsilon_{n+1} = \frac{1}{3^n}$ for $n \geq 1$. Let us consider the following sequence of finite subsets of \mathcal{C} :

$A_1 = \{0\}$. Let $A_2 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ the set of end points of every of the two pieces after the first operation of throwing away. In general if $n \geq 1$ then A_{n+1} is the set of end points of every of the 2^n interval obtained after n steps of the throwing away operation in the construction of \mathcal{C} .

It is clear that A_k is a finite ε_k -approximation of \mathcal{C} . If $k \geq 2$, then A_k contains exactly 2^k points. For example $A_3 = \{0, \frac{1}{3^2}, \frac{2}{3^2}, \frac{1}{3}, \frac{2}{3}, \frac{2 \cdot 3 + 1}{3^2}, \frac{2 \cdot 3 + 2}{3^2}, 1\}$. In fact if $n \geq 1$, then $A_{n+1} = \bigcup_{k=1, \dots, 2^n} \{a_k^n, b_k^n\}$ where:

- (a) For every $k \in \{1, \dots, 2^n\}$, $b_k^n - a_k^n = \frac{1}{3^n} = \varepsilon_{n+1}$.
- (b) If $k, k' \in \{1, \dots, 2^n\}$, $k \neq k'$, $t \in [a_k^n, b_k^n]$ and $t' \in [a_{k'}^n, b_{k'}^n]$. Then $|t - t'| \geq \frac{1}{3^n} = \varepsilon_{n+1}$.

(c) $a_1^n = 0 < a_2^n < \dots < a_{2^n}^n = \sum_{i=1}^n \frac{2}{3^i}$ and

$$\{a_k^n \mid k \in \{1, \dots, 2^n\}\} = \{a_k^{n-1} \mid k \in \{1, \dots, 2^{n-1}\}\} \cup \{a_k^{n-1} + \frac{2}{3^n} \mid k \in \{1, \dots, 2^{n-1}\}\}.$$

Consequently $\{a_k^n \mid k \in \{1, \dots, 2^n\}\} = \{\sum_{i=1}^n \frac{x_i}{3^i} \mid x_i = 0 \text{ or } x_i = 2 \ i \in \{1, \dots, n\}\}$.

What about $U_{2\varepsilon_{n+1}}(A_{n+1})$?

If $n = 1$. Then $U_{\frac{2}{3}}(A_2) = \{\{0\}, \{0, \frac{1}{3}\}, \{\frac{1}{3}\}, \{\frac{1}{3}, \frac{2}{3}\}, \{\frac{2}{3}\}, \{\frac{2}{3}, 1\}, \{1\}\}$. Consequently $\mathcal{K}(U_{\frac{2}{3}}(A_2))$ has as vertices the points in $U_{\frac{2}{3}}(A_2)$ and contains no simplices of dimension higher than one. The one-dimensional simplices in $\mathcal{K}(U_{\frac{2}{3}}(A_2))$ are just $\langle \{0\}, \{0, \frac{1}{3}\} \rangle$, $\langle \{0, \frac{1}{3}\}, \{\frac{1}{3}\} \rangle$, $\langle \{\frac{1}{3}\}, \{\frac{1}{3}, \frac{2}{3}\} \rangle$, $\langle \{\frac{1}{3}, \frac{2}{3}\}, \{\frac{2}{3}\} \rangle$, $\langle \{\frac{2}{3}\}, \{\frac{2}{3}, 1\} \rangle$ and $\langle \{\frac{2}{3}, 1\}, \{1\} \rangle$

In fact $|\mathcal{K}(U_{\frac{2}{3}}(A_2))|$ is homeomorphic to the unit interval $[0, 1]$. The maps $p_{2,1} : U_{\frac{2}{3}}(A_2) \longrightarrow U_4(A_1)$ is, obviously, the constant one.

A more interesting analysis, that reveals the general behavior, appears when we compare $U_{\frac{2}{3^2}}(A_3)$ with $U_{\frac{2}{3}}(A_2)$ by means of the map $p_{3,2}$. First of all note that if $D \in U_{\frac{2}{3^2}}(A_3)$ then D contains at most two points. If D contains just two points, then they have to be consecutive in the natural order given in $A_3 = \{0, \frac{1}{3^2}, \frac{2}{3^2}, \frac{1}{3}, \frac{2}{3}, \frac{2 \cdot 3 + 1}{3^2}, \frac{2 \cdot 3 + 2}{3^2}, 1\}$. But not every set formed by couple of consecutive points above belongs to $U_{\frac{2}{3^2}}(A_3)$ and it is because $\frac{1}{3} > \frac{2}{3^2}$ (in general $\frac{1}{3^n} > \frac{2}{3^{n+1}}$). So in this case $\{\frac{1}{3}, \frac{2}{3}\} \notin U_{\frac{2}{3^2}}(A_3)$. This implies that $\mathcal{K}(U_{\frac{2}{3^2}}(A_3))$ has two connected components. Consequently $|\mathcal{K}(U_{\frac{2}{3^2}}(A_3))|$ has exactly two connected components each of them homeomorphic to $[0, 1]$ as before. So, homotopically, $|p_{3,2} : |\mathcal{K}(U_{\frac{2}{3^2}}(A_3))| \longrightarrow |\mathcal{K}(U_{\frac{2}{3}}(A_2))|$ is equivalent to the unique map from a discrete space with two points onto a space with one point.

In fact what happens is that $U_{\frac{2}{3^n}}(A_{n+1})$ has exactly 2^n connected components. The same happens with $|\mathcal{K}(U_{\frac{2}{3^2}}(A_3))|$. When we pass from $U_{\frac{2}{3^n}}(A_{n+1})$ to the next step $U_{\frac{2}{3^{n+1}}}(A_{n+2})$, then any component Λ in $U_{\frac{2}{3^n}}(A_{n+1})$ splits into two components in $U_{\frac{2}{3^{n+1}}}(A_{n+2})$ which are mapped by $p_{n+2,n+1}$ into Λ . The same happens for $|\mathcal{K}(U_{\frac{2}{3^n}}(A_{n+1}))|$ and $|\mathcal{K}(U_{\frac{2}{3^{n+1}}}(A_{n+2}))|$ and the maps $|p_{n+2,n+1}|$. In this case the components of $|\mathcal{K}(U_{\frac{2}{3^n}}(A_{n+1}))|$ and $|\mathcal{K}(U_{\frac{2}{3^{n+1}}}(A_{n+2}))|$ are all homeomorphic to $[0, 1]$. This means that in this case, homotopically, the diagram

$$|p_{n+2,n+1} : |\mathcal{K}(U_{\frac{2}{3^{n+1}}}(A_{n+2}))| \longrightarrow |\mathcal{K}(U_{\frac{2}{3^n}}(A_{n+1}))|$$

is equivalent to that of a map from a discrete space with 2^{n+1} points onto a discrete space with 2^n such that each fiber has exactly two points. Consequently the inverse sequences $(H_k(|$

$\mathcal{K}(U_{2\varepsilon_{n+1}}(A_{n+1}))$), $H_k(| p_{n+1,n} |)$ and $(H_k(U_{2\varepsilon_{n+1}}(A_{n+1})), H_k(p_{n+1,n}))$, which are isomorphic in pro-Group for any $k \geq 0$, can be used to get all Čech homology groups of \mathcal{C} . Obviously all of them are trivial if $k \geq 1$. Moreover for $k = 0$ it is the inverse limit of the corresponding sequence of free abelian groups.

2. SOME TOPOLOGICAL BACKGROUND

2.1. Hyperspaces. The use of hyperspace's point of view and techniques is spread along all branches in Mathematics both pure and applied. In Topology it is a basic tool to construct new spaces from old. Usually, given a topological spaces Z we consider the set of all non-empty closed subsets of Z (denote it by 2^Z) and construct some natural topologies on it. When the topology of Z is sufficiently good, i. e. if the points of Z are closed sets, then we have a canonical topological embedding of Z into its hyperspace. Note that this ambient space depends only on Z but it could happen, depending on the endowed topology, that the corresponding hyperspaces of two non-homeomorphic spaces could be homeomorphic. Some global topological properties of Z can be studied looking at the relative situation of the canonical copy of Z inside 2^Z . This is, for example, the point of view in the papers [20], [10] and [9], and this is our point of view in this paper too.

For general information on hyperspaces we recommend [21], where the definition of upper semifinite topology appears, and the books [22], [23].

Herein, and for completeness, we are going to recall some definitions and basic results about the upper semifinite topology in hyperspaces that we need to treat ε -connectedness and finite approximations . They are contained in [10] and [9].

Given a topological space (X, T) and an open set $U \subset X$, i. e. $U \in T$, we define $B_U = \{C \in 2^X, C \subset U\}$. The family $B = \{B_U\}_{U \in T}$ is a base for the upper semifinite topology on 2^X . As a general rule in the sequel we will use the same notation C for a closed subset of X and for an element of 2^X (instead of $\{C\}$). From now on we denote by 2_U^X the corresponding topological spaces.

One of the main facts, with trivial proof , that we need is the description of the closure of a point in 2_U^X , see [10].

Proposition 3. *Let (X, T) be a T_1 topological space and $C \in 2^X$. Then $\overline{\{C\}} = \{D \in 2^X \mid C \subset D\}$.*

Recall that a topological space is a T_1 space if any point is closed.

Suppose that (X, T) is a T_1 topological space. Then the map $\Phi : (X, T) \rightarrow 2^X_U$ given by $\Phi(x) = \{x\}$ is a topological embedding. This means that (X, T) and $\Phi(X)$, with the relative topology, are homeomorphic. From now on we will refer to $\Phi(X)$ as the canonical copy of X and we will denote it (if there is no possibility of confusion) by X again.

The main tool that we are going to use is the following result, see [9]. The proof relies on the existence of a Lebesgue number associated to any cover in a compact metric space.

Proposition 4. *Let (X, d) be a compact metric space. Then the family $U = \{U_\varepsilon\}_{\varepsilon>0}$ is a base of open neighborhoods of the canonical copy X inside 2^X_U , where $U_\varepsilon = \{C \in 2^X \mid \text{diam}(C) < \varepsilon\}$. Consequently for any positive decreasing sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ with $\varepsilon_n \rightarrow 0$, we have that $\{U_{\varepsilon_n}\}_{n \in \mathbb{N}}$ is a nested countable base.*

Note that the set U_ε seems to be adequate to treat approximations up to ε resolutions. Not only because any set with diameter less than ε becomes a point in U_ε but also because if $A \subset B$ are points in U_ε , then A and B can not be distinguishable in U_ε . It is modelled by the fact that, in this topological space, $B \in \overline{\{A\}}$. That is, in some sense, they are glue together.

2.2. Shape Theory. With this title we are refereing to the topological theory introduced by Karol Borsuk in [24], see also [25], [12] and [13]. We will use this theory to give a theoretical approach to the calculus of the number of ε -components. We will also use ideas from shape theory, as V.Robins in [5], to propose a different approach to compute homology from finite approximations.

Not only in pure mathematics but also in the mathematical modelling of natural phenomena appear some strange objects. For example solenoids or strange attractors. These geometrical objects have not good local properties in general. It avoids the classical homotopy or homology theory to be an adequate tool to treat them. Shape theory fills, in some sense, this gap. Moreover in the good cases, as manifolds or polyhedra, this theory does not modify the classical homotopy theory. Shape theory takes into account only global properties of spaces. The techniques used to

do that are either considering the system of neighborhoods of a copy of a space embedded into an ambient with very good local and global properties or approximating, by means of inverse systems, bad spaces by polyhedra. So, shape theory is an approximation theory in nature.

The shape category classifies the family of topological spaces up the so called *shape types*. So two objects are isomorphic if and only if they have the same shape. To give an idea of the geometrical flavor of shape theory let us mention that Borsuk, see [25] for example, proved that two compact connected subsets of the plane have the same shape if and only if they decompose the plane into the same *number*, finite or countable, of connected components.

The following notation will be used in this paper: Given a topological space X and a natural number n , by the equality $Sh(X) = n$ (read it as the shape of X is equal to n) we understand that the topological space X has the same shape as a discrete space with just n points. So $Sh(X) = 1$ means that X has trivial shape or the shape of a point, $Sh(X) = 2$ means that X is, in the shape category, the same thing as the space $\{1, 2\}$, with the discrete topology, and so on.

Not all shape morphisms are represented by continuous functions, as opposed to the homotopy category, but any continuous function induces a shape morphism. One of the results from shape theory that we need is a characterization of when a continuous map $f : X \rightarrow Y$ between topological spaces induces an equivalence in the shape category. Following Mardešić [11], see also [12], we have:

Proposition 5. *Let X and Y be two topological spaces and suppose that $f : X \rightarrow Y$ is a continuous function. Then f is a shape equivalence (that is the shape morphism induced by f is an isomorphism in this category) if and only if for any CW-complex P the function*

$$\begin{aligned} f_* : [Y, P] &\longrightarrow [X, P] \\ [h] &\longmapsto [h \circ f] \end{aligned}$$

is a bijection

$[Z, R]$ represents the set of all homotopy classes of continuous functions between Z and R . Given a continuous function $h : Z \rightarrow R$, we denote by $[h]$ the homotopy class of h . Recall that two maps $h, h' : Z \rightarrow R$ are said to be homotopic if they can be continuously deformed one into the other. Concretely h and h' are homotopic if there is a continuous function, called a

homotopy, $H : X \times I \rightarrow Y$ such that $H(x, 0) = h(x)$ and $H(x, 1) = h'(x)$ for any $x \in X$, where $I = [0, 1]$ is the unit interval endowed with its usual topology inherited from the real line \mathbb{R} .

CW -complexes are topological generalizations of polyhedra constructed gluing, coherently, pieces of topological cells. See for example the appendix in [12].

2.2.1. *Shape Theory and dynamical systems.* Even in phenomena modelled by plane dynamics one has situations of spaces having the same shape but not the same homotopy type. For example consider an smooth flow in the plane with a limit cycle. One of the subsets that we consider is the limit cycle itself, a *good* space, and the other is the union of the cycle with the positive semiorbit of a point which is not in the limit cycle and such that the limit cycle is the ω -limit of this point. It is a *bad* space because it is not , in particular, locally connected. Using the above *visual* Borsuk's criterium both of them are compact connected subsets decomposing the plane into two components and then they have the same shape. They have not the same homotopy type because the limit cycle has only one path-component and the other figure has two different path components. In this case it can be even viewed by the uniqueness of solutions: the semiorbit can not meet the limit cycle.

The use of shape theory in the study of dynamical systems was initiated by Hastings [26], [27]. Other authors have shown how to apply shape theory to obtain global properties of attractors and other type of invariant sets in the papers [28], [29], [30] [31], [32], [33], [34], [35], [36], [37], [38], [39], [40] and [41] . Ideas from shape theory were also important in the investigation of finite-dimensional dynamics on global attractors in [42]. Shape theory was related with differential equations in [43] and [42] and it is the main tool used in [44] and [45] to define a Conley index for discrete dynamical systems.

To give some ideas of how shape theory can be useful in the study of dynamics, let us mention that Hastings [26, 27] developed an analogue of the Poincaré-Bendixon theorem in the Euclidean n -space using shape. Motivated by Hastings's result several authors established at different levels of generality the following:

Suppose that a flow on a manifold has the compact subset K as an asymptotically stable attractor. Then K has polyhedral shape.

The above result was first proved by Bogatyı and Gutsu [28] for differentiable flows and later by Günter and Segal [31] for continuous flows in manifolds and by Sanjurjo [32, 33] for (non-necessarily finite-dimensional) Absolute Neighborhood Retracts. As a consequence of this result attractors of flows in manifolds have finitely generated Čech homology and cohomology which vanishes in higher dimensions. This means that Algebraic Topology can be used as an efficient tool for the study of attractors.

Kapitanski and Rodnianski in [35] proved the following:

Assume that a continuous semi-dynamical system on a complete metric space M possesses a compact global attractor K . Assume that the system has an equilibrium $z \in K$. Then the inclusion $i : (K, z) \longrightarrow (M, z)$ induces a shape equivalence of pointed spaces.

A more general related result was given by Giraldo, Morón, Ruiz del Portal and Sanjurjo in [38]. Another result, eliminating the requirement of existence of an equilibrium of the system, is presented in [41]. In the same paper [41] some results are presented which study the properties of the connected components of attractors and their relations with the components of the phase space.

3. ε -CHAINS AND TOPOLOGICAL PATHS IN UPPER SEMIFINITE HYPERSPACES

We recommend [46] for definitions of topological concepts related to connectedness that we use here. In [8] page 131, Snyder decided to refer to ε -chains as ε -paths. Our first result gives a full meaning to this decision. Recall that by a path, or a topological path, joining to points x, y in a topological space X we understand a continuous function $p : [0, 1] \rightarrow X$ such that $p(0) = x$ and $p(1) = y$. Here we consider again the usual topology in the interval $[0, 1]$.

Theorem 6. *Let (X, d) be a compact metric space and consider the upper semifinite hyperspace 2_U^X . Suppose that $x, y \in X$ and that $\varepsilon > 0$. Then there is an ε -chain in (X, d) joining x to y if and only if there is a path $p : [0, 1] \rightarrow 2_U^X$ with $p(0) = x$, $p(1) = y$ and $p([0, 1]) \subset U_\varepsilon$*

Proof. Suppose first that we have an ε -chain $\{x_0, \dots, x_n\} \subset X$ joining x to y . Then $x_0 = x$, $x_n = y$ and $d(x_i, x_{i+1}) < \varepsilon$ for any $i \in \{0, \dots, n-1\}$. Choose $t_0 = 0 < t_1 < t_2 < \dots < t_n < 1$

$t_{n+1} = 1$ and define $p : [0, 1] \rightarrow 2^X$ by the following formula:

$$p(t) = \begin{cases} \{x_0\}, & \text{if } t \in [0, t_1); \\ \{x_0, x_1\}, & \text{if } t = t_1; \\ \{x_1\}, & \text{if } t \in (t_1, t_2); \\ \{x_1, x_2\}, & \text{if } t = t_2; \\ \vdots & \\ \{x_{n-1}, x_n\}, & \text{if } t = t_n; \\ \{x_n\}, & \text{if } t \in (t_n, t_{n+1}]. \end{cases}$$

Of course the one point and the two points subsets of X are closed in X . Moreover $\text{diam}(p(t)) < \varepsilon$ because of the definition of ε -chain. So $p([0, 1]) \subset U_\varepsilon$. We only have to prove the continuity of p at the points t_1, \dots, t_n . Here is where the upper semifinite topology plays its role. Let $k \in \{1, \dots, n\}$ and suppose that B_V is a basic neighborhood of $p(t_k)$ in 2_U^X . So $V \subset X$ is an open set with $\{x_{k-1}, x_k\} \subset V$. Recall that $B_V = \{C \subset X \mid C \text{ is closed and non-empty with } C \subset V\}$. In what follows we denote by $\mathcal{B}(a, r)$ the ball of center a and radius r . So if $\delta_k < \min\{d(t_k, t_{k+1}), d(t_k, t_{k-1})\}$ then $p(\mathcal{B}(t_k, \delta_k)) \subset B_V$ and the proof is finished.

On the other hand, if $p : [0, 1] \rightarrow 2_U^X$ is a path with $p(0) = x$, $p(1) = y$ and $p([0, 1]) \subset U_\varepsilon$, then for any $t \in [0, 1]$ we have $\text{diam}(p(t)) < \varepsilon$. Consider a number $\gamma(t) > 0$ such that $\text{diam}(p(t)) + 2\gamma(t) < \varepsilon$. Consequently $p(t) \in B_{\mathcal{B}(p(t), \gamma(t))} \subset U_\varepsilon$ for any $t \in [0, 1]$. The family $\{B_{\mathcal{B}(p(\alpha), \gamma(\alpha))}\}_{\alpha \in [0, 1]}$ is a cover of $p([0, 1])$ by open subsets of 2_U^X . Consequently

$$\{W_\alpha = p^{-1}(B_{\mathcal{B}(p(\alpha), \gamma(\alpha))} \cap p([0, 1]))\}_{\alpha \in [0, 1]}$$

is an open cover of the unit interval. So there is a finite subcover $\{W_1, \dots, W_m\}$. Using the properties of the unit interval $[0, 1]$ we can suppose, taking refinements if necessary, that $0 \in W_1$, $1 \in W_m$ and $\supremum(W_i) \in W_{i+1}$. Consider $t_1, \dots, t_{m-1} \in [0, 1]$ with the property that $t_i \in W_i \cap W_{i+1}$ and fix $x_0 = x$, $x_m = y$. For any $i \in \{1, \dots, m-1\}$ choose a point $x_i \in p(t_i)$. By the choice of $\gamma(t)$ it is clear that $\{x_0, x_1, \dots, x_m\}$ is an ε -chain in X joining x to y . \square

In general the connected components and the path components of a topological space do not coincide. Any path component is contained in a connected-component. Moreover the connected components induce a partition of the space into closed, disjoint, sets.

The next result, that will be useful for our approach to ε -connectedness, clarifies the situation for the open set U_ε of 2_U^X .

Proposition 7. *Let (X, d) be a compact metric space and $\varepsilon > 0$. Consider the corresponding $U_\varepsilon = \{C \in 2^X / \text{diam}(C) < \varepsilon\}$, then the path components of U_ε are open and closed in U_ε , with the relative topology inherited from 2_U^X .*

Proof. Let $\Lambda \subset U_\varepsilon$ be a path-component and suppose that $D \in \bar{\Lambda} \subset U_\varepsilon$ where the closure operator $-$ is taken in the relative topology of U_ε .

Let $0 < \delta < \frac{\varepsilon - \text{diam}(D)}{2}$ and take $V = \mathcal{B}(D, \delta) = \{x \in X / d(x, D) < \delta\}$. Since $D \in \bar{\Lambda}$ it follows that $B_V \cap \Lambda \neq \emptyset$. Then there is $C \in B_V \cap \Lambda$ and consequently $C \subset V$. Since C is non-empty it follows that there is a point $\alpha \in C$. Fix now a point $\beta \in D$ (recall that D is also non-empty) with $d(\alpha, \beta) < \delta$. In fact $d(x, y) < \varepsilon$ for any $x \in D$ and any $y \in C$. Define now $p_{\beta, \alpha} : [0, 1] \rightarrow U_\varepsilon$ by

$$p_{\beta, \alpha}(t) = \begin{cases} \{\beta\}, & \text{if } t \in [0, \frac{1}{2}); \\ \{\beta, \alpha\}, & \text{if } t = \frac{1}{2}; \\ \{\alpha\}, & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

It is obvious, as before, that p is continuous and then a path joining β to α . By the same arguments the functions

$$p_{\beta, D}(t) = \begin{cases} \{\beta\}, & \text{if } t \in [0, 1); \\ D, & \text{if } t = 1. \end{cases}$$

and

$$p_{\alpha, C}(t) = \begin{cases} \{\alpha\}, & \text{if } t \in [0, 1); \\ C, & \text{if } t = 1. \end{cases}$$

are paths in U_ε . By concatenation of paths we have finally a path $\omega : [0, 1] \rightarrow U_\varepsilon$ with $\omega(0) = D$ and $\omega(1) = C$. Consequently $D \in \Lambda$ and then Λ is closed in U_ε .

We are going to prove that Λ is also open in U_ε (in fact it is open in 2_U^X). Let $C \in \Lambda$. Consider as before $0 < \delta < \frac{\varepsilon - \text{diam}(C)}{2}$. Then $B_{\mathcal{B}(C, \delta)}$, which is a basic open set in 2_U^X , is contained in U_ε . In fact we are going to prove that $B_{\mathcal{B}(C, \delta)} \subset \Lambda$:

So let $D \in B_{\mathcal{B}(C, \delta)}$, then $D \subset \mathcal{B}(C, \delta)$. For any point $\beta \in D$ there is a point $\alpha \in C$ with $d(\beta, \alpha) < \delta < \varepsilon$. As before we can construct topological paths from $\{\alpha\}$ to $\{\beta\}$, from $\{\alpha\}$ to C and from $\{\beta\}$ to D , all of them inside U_ε . Consequently, by concatenation, D can be joined

to C by a path in U_ε . Since Λ is the, unique, path component of U_ε containing C we infer that $B_{\mathcal{B}(C,\delta)} \subset \Lambda$ and then Λ is open. \square

Corollary 8. *For any compact metric space (X, d) and any $\varepsilon > 0$, the connected components and the path components of U_ε coincide.*

Proof. First of all any path component in any topological space is contained in a unique connected component. Path components are always connected and, in the case of U_ε , they are open and closed in U_ε . Obviously this means that it is a maximal, by inclusions, connected subset of U_ε and then a connected components. \square

Using the above results we have:

Corollary 9. *Given a compact metric space (X, d) and two points $x, y \in X$. Then x and y can be joined by an ε -chain in X if and only if they are in the same connected component of the corresponding U_ε .*

4. APPLICATIONS TO ε -CONNECTEDNESS

We recommend [2],[3] and mainly Section 2 in [8] for the basic definitions about ε -connectedness that we are going to use. The first thing we are going to do in this section is to give our own proof of Cantor's Theorem.

Corollary 10. *(CANTOR THEOREM) Let (X, d) be a compact metric space. Then X is connected if and only if for any $\varepsilon > 0$ and any pair of points $x, y \in X$ there is an ε -chain $\{x_0, \dots, x_n\}$ joining x to y .*

Proof. Suppose first that X is connected. It is obvious that the canonical copy X is dense in 2_U^X , see [10], and then it is dense in any U_ε . Consequently U_ε is connected for any ε . This means, as we proved before, that U_ε is path-connected. Using again the first result proved in the last section we infer that for any pair of points $x, y \in X$ and any $\varepsilon > 0$ there is an ε -chain joining x to y .

Suppose now that for any $\varepsilon > 0$ and any pair of points $x, y \in X$ there is an ε -chain joining x to y . This means that X is contained in an unique component of U_ε . Since X is dense in 2_U^X and any component of U_ε is open in 2_U^X it follows that any component of U_ε meet X and then U_ε is

connected for any $\varepsilon > 0$. Recall that $\{U_\varepsilon\}_{\varepsilon>0}$ is a base of open neighborhoods of the canonical copy X in 2_U^X .

Suppose that X is not connected. Then there are two open non-empty subsets X_1, X_2 such that $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$. So $B_{X_1} \cup B_{X_2}$ is open in 2_U^X and contains the canonical copy X . Since $\{U_\varepsilon\}_{\varepsilon>0}$ is a base of open neighborhoods for this canonical copy it follows that there is an $\alpha > 0$ such that $X \subset U_\alpha \subset B_{X_1} \cup B_{X_2}$. Obviously $B_{X_1} \cap B_{X_2} = \emptyset$. Then $U_\alpha = (U_\alpha \cap B_{X_1}) \cup (U_\alpha \cap B_{X_2})$ being both of them non-empty which is not possible because U_α is connected. Consequently X is connected. \square

Recall, see for example [3] page 278, that X is ε -disconnected if there are two closed subsets, C and F , with $X = C \cup F$ and $d(C, F) = \inf_{x \in C, y \in F} \{d(x, y)\} \geq \varepsilon$. Otherwise, X is ε -connected.

Now we have the following

Corollary 11. *Let (X, d) be a compact metric space and $\varepsilon > 0$. Then the following are equivalent:*

- a) for any $x, y \in X$ there is an ε -chain joining x to y .
- b) X is ε -connected.
- c) U_ε is connected.

Proof. a) \Rightarrow b) Suppose that X is not ε -connected. Then there are closed non-empty subsets C, F , with $X = C \cup F$ and $d(C, F) \geq \varepsilon$. This obviously implies that there is not an ε -chain joining a point of C with a point of F .

b) \Rightarrow c) Suppose that X is ε -connected and that U_ε is not connected. Then there are open non-empty subsets V and W such that $U_\varepsilon = V \cup W$ and $V \cap W = \emptyset$. Obviously $V \cap X \neq \emptyset$ and $X \cap W \neq \emptyset$. Suppose that $x \in V \cap X$ and $y \in X \cap W$. Then $d(x, y) \geq \varepsilon$ because if not, the map $p : [0, 1] \rightarrow U_\varepsilon$ given by

$$p(t) = \begin{cases} \{x\}, & \text{if } t \in [0, \frac{1}{2}); \\ \{x, y\}, & \text{if } t = \frac{1}{2}; \\ \{y\}, & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

is a topological path joining x to y in U_ε and it is not possible because $p(0) \in V$ and $p(1) \in W$. Consequently $p([0, 1])$ is not connected what is a contradiction with the continuity of p . So U_ε is connected.

c) \Rightarrow a) Since U_ε is connected, by the previous section, it is path connected. So any pair of point of X can be joined by a topological path in U_ε and this implies, using again the previous section, that any pair of points can be joined by means of an ε -chain. \square

More can be said, as it is contained in the following:

Corollary 12. *Let (X, d) be a compact metric space and $\varepsilon > 0$. Consider the corresponding $U_\varepsilon = \{C \in 2^X \mid \text{diam}(C) < \varepsilon\}$. Then U_ε has a finite number of components, say $\{\Lambda_{1,\varepsilon}, \dots, \Lambda_{n(\varepsilon),\varepsilon}\}$. Moreover $P(\varepsilon) = \{\Lambda_{1,\varepsilon} \cap X, \dots, \Lambda_{n(\varepsilon),\varepsilon} \cap X\}$ is a partition of X into open and closed sets whose elements are just the ε -components in the sense of [2], [3] or [8]. Consequently $n(\varepsilon) = C(\varepsilon)$ as denoted in [2].*

Proof. Let $\varepsilon > 0$ and suppose that $\{\Lambda_{\omega,\varepsilon}\}_{\omega \in \Omega}$ is the set of connected components of U_ε . As proved in the previous section $\Lambda_{\omega,\varepsilon}$ is open and closed in U_ε and then they are open in 2_U^X . Consequently $\Lambda_{\omega,\varepsilon} \cap X \neq \emptyset$, because $\overline{X} = 2_U^X$, and $X \subset \cup_{\omega \in \Omega} \Lambda_{\omega,\varepsilon}$. From compactness of X we obtain that $\text{Card}(\Omega) = n(\varepsilon)$ is a finite number. $P(\varepsilon) = \{\Lambda_{1,\varepsilon} \cap X, \dots, \Lambda_{n(\varepsilon),\varepsilon} \cap X\}$ is a partition of X into open and closed sets because X , the canonical copy, is topologically embedded into any U_ε . It only remains to prove that $\Lambda_{k,\varepsilon} \cap X$, $k = 1, \dots, n(\varepsilon)$, are just the ε -components of X . So, consider the compact subsets $X_k = \Lambda_{k,\varepsilon} \cap X$ with the restriction of the metric d . Since any two points in X_k are in the same component of U_ε we have an ε -chain joining them and applying the previous corollary we obtain that X_k is ε -connected for any $k \in \{1, \dots, n(\varepsilon)\}$. Moreover if $k \neq k'$ and $x \in X_k$ $y \in X_{k'}$ then, following the same arguments as in the above corollary, $d(x, y) \geq \varepsilon$ because they are in different connected components of U_ε . Consequently $X_1, \dots, X_{n(\varepsilon)}$ are just the ε -components of X in the sense of [8]. The number $n(\varepsilon)$ is denoted by $C(\varepsilon)$ in [2] and [3]. \square

Related to the number $C(\varepsilon)$ of ε -components of a compact metric space X , we proved before that it is just the number of connected components of U_ε . The *space* of ε -components of X is then the quotient, obtained from U_ε by means of the equivalence relation given $C, D \in U_\varepsilon$ then $CR_\varepsilon D$ if and only if C and D belong to the same connected component of U_ε . In another words the space of ε -components of X coincide, topologically, with the so called the spaces of components of U_ε .

We are going to prove more. In fact we will prove a stronger relation involving shape theory. That is to say, from the point of view of shape theory, which is a *coarse graining* of the category of topology, U_ε and the space of ε -components of X are the same things. To prove this we are going to use the characterization of continuous maps inducing shape equivalences sated in the topological background section, but first of all we need the following

Proposition 13. *let (X, d) be a compact metric space and consider $U_\varepsilon = \{C \in 2^X \mid \text{diam}(C) < \varepsilon\}$. Suppose that Λ is a connected component of U_ε . Given any topological space (Y, T) with the T_1 separation property we have that any continuous function $f : \Lambda \rightarrow Y$ is constant. Consequently $Sh(\Lambda) = 1$ (or Λ has trivial shape).*

Proof. Let $f : \Lambda \rightarrow Y$ be a continuous function, of course we are considering on Λ the relative topology induced by that of U_ε or 2^X . First of all note that if $D \in \Lambda$ then there is a point $x_D \in \Lambda \cap X$ such that $f(D) = f(x_D)$. In fact if $D \in \Lambda$ then $D \subset X$ is a non-empty closed subset. Choose a point $x_D \in D$. We have a path $p : [0, 1] \rightarrow \Lambda$ $p(t) = x_D$ if $t \in [0, 1)$ and $p(1) = D$. Then $f \circ p : [0, 1] \rightarrow Y$ is a continuous function and $1 \in \overline{[0, 1]}$. Consequently $f \circ p(\overline{[0, 1]}) \subset \overline{f \circ p([0, 1])}$ by continuity. Then $f \circ p(1) \in \overline{f \circ p([0, 1])}$ but $f \circ p([0, 1)) = f(x_D)$ and the point $f(x_D)$ is closed in Y by the T_1 property. Consequently $f \circ p(1) = f(D) = f(x_D)$. So the possible variation of the function f should be reflected on the piece of the canonical copy X inside Λ .

So consider now $x, y \in \Lambda \cap X$. Then there is a path in Λ joining x to y (because Λ is a component in U_ε). Consequently, using our theorem in Section 3, we have an ε -chain $x_0 = x, x_1, \dots, x_n = y$ in $\Lambda \cap X$, with the relative metric. Applying again the theorem we have a topological path

$$p(t) = \begin{cases} \{x\}, & \text{if } t \in [0, t_1); \\ \{x, x_1\}, & \text{if } t = t_1; \\ \{x_1\}, & \text{if } t \in (t_1, t_2); \\ \{x_1, x_2\}, & \text{if } t = t_2; \\ \vdots, & \\ \{x_{n-1}, x_n\}, & \text{if } t = t_n; \\ \{y\}, & \text{if } t \in (t_n, 1]. \end{cases}$$

where $t_0 = 0 < t_1 < \dots < t_n < t_{n+1} = 1$.

Using the same arguments as before we have that the restrictions $f \circ p|_{[t_k, t_{k+1}]}$ are constant maps. So $f(x) = f(\{x, x_1\}) = f(x_1) = \dots = f(\{x_{n-1}, x_n\}) = f(y)$. This means that, finally, $f : \Lambda \rightarrow Y$ is constant. Consider now a one point space, with the unique possible topology on it, $\{\alpha\}$ and take the constant map $f : \Lambda \rightarrow \alpha$. If one consider any CW -complex P then $f_*[\alpha, P] \rightarrow [\Lambda, P]$ induces a bijection because P is always a T_1 space. In this case $[\alpha, P]$ and $[\Lambda, P]$ are naturally bijective with the set of path-components of P . So we have proved that $Sh(\Lambda) = 1$, or Λ has trivial shape. \square

Corollary 14. $C(\varepsilon) = Sh(U_\varepsilon), \forall \varepsilon > 0$.

Proof. Consider $\{\Lambda_1, \dots, \Lambda_{C(\varepsilon)}\}$ the different connected components of U_ε . Since they are open and closed in U_ε , we have $U_\varepsilon = \Lambda_1 \oplus \dots \oplus \Lambda_{C(\varepsilon)}$ where \oplus represents here the topological sum, or the co-product in the topological category. Since $Sh(\Lambda_k)$ is trivial for any $k \in \{1, \dots, C(\varepsilon)\}$ then it follows, it is known and trivial to prove, that $Sh(\Lambda_1 \oplus \dots \oplus \Lambda_{C(\varepsilon)}) = Sh(\{1, \dots, C(\varepsilon)\}, \text{discrete topology}) = C(\varepsilon)$ as denoted before. \square

5. TREES OF ε -COMPONENTS: A TOPOLOGICAL APPROACH TO THE SPACE OF COMPONENTS

In [2] and [3] are some discussions about the possibility to obtain topological properties related to connectedness, of the ideal figure X in terms of the analysis of ε -connectedness for different ε -resolutions. In particular the possibility to get information on the *number*, finite or infinite, of connected components of X by the behavior of the natural number $C(\varepsilon)$ at different $\varepsilon > 0$.

Our construction in this section will say that, theoretically, if we know certain relations between the numbers $C(\varepsilon_n)$ for a decreasing sequence, with $\varepsilon_n \rightarrow 0$, then we have the possibility to reconstruct, topologically, and not only *numerically*, the structure of the set of components.

First of all it is usual, in topology, to consider the *space of components* denoted by $\square X$ of a topological space X . This space reflects not only the number of connected components but also the relative topological situation of those components. Concretely given a topological space (X, T) , we can define the equivalence relation on X by xRy if and only if they belong to the same connected component of X . Let $p : X \rightarrow \square X$ be the natural projection then we consider

the quotient topology on $\square X$ by means of p . This means that $A \subset \square X$ is open in this topology if and only if $p^{-1}(A)$ is open in the topological space X .

This natural topology on $\square X$ can be quite unpleased but in the compact metric case this topology is metrizable compact and 0-dimensional.

Here, 0-dimensional means that any point has a base of neighborhoods which are open and closed. This space has been used, in particular, by Borsuk in [25] to get invariants in shape theory for compact metric spaces.

Our construction in this section is as follows:

Let (X, d) be a compact metric space. Suppose that $\text{diam}(X) = \sup_{x,y \in X} \{d(x, y)\} = M$. Fix a number $N > M$ and consider the sequence $\varepsilon_n = \frac{N}{n}$. Take the inverse sequence $\{U_{\varepsilon_n}, i_{n,n+1}\}_{n \in \mathbb{N}}$ where the bonding maps $i_{n,n+1}$ are just inclusions $i_{n,n+1} : U_{\varepsilon_{n+1}} \rightarrow U_{\varepsilon_n}$ which are obviously continuous functions. Then $i_{n,n+1}$ transforms connected components of $U_{\varepsilon_{n+1}}$ into connected components U_{ε_n} because the continuous image of a connected set is connected. Let us denote as before by $\{\Lambda_{1,\varepsilon_n}, \dots, \Lambda_{C(\varepsilon_n),\varepsilon_n}\}$ the topological space, with the discrete topology, of components of U_{ε_n} which is the *same* thing, intersecting with X , as the ε_n -components of X . Note that $C(\varepsilon_1) = 1$ because $\varepsilon_1 > \text{diam}(X)$. The map $i_{n,n+1} : U_{\varepsilon_{n+1}} \rightarrow U_{\varepsilon_n}$ induces a map $p_{n,n+1} : \{\Lambda_{1,\varepsilon_{n+1}}, \dots, \Lambda_{C(\varepsilon_{n+1}),\varepsilon_{n+1}}\} \rightarrow \{\Lambda_{1,\varepsilon_n}, \dots, \Lambda_{C(\varepsilon_n),\varepsilon_n}\}$ where $p_{n,n+1}(\Lambda_{k,\varepsilon_{n+1}}) = \Lambda_{k',\varepsilon_n}$ if and only if $\Lambda_{k',\varepsilon_n}$ is the unique connected component, or equivalently ε -component of X , of U_{ε_n} containing $\Lambda_{k,\varepsilon_{n+1}}$. We construct the following simplicial tree, denoted by $T_{(X,\{\varepsilon_n\})}$, whose set of vertices is just $\{\Lambda_{k,\varepsilon_n} / n \in \mathbb{N}, 1 \leq k \leq C(\varepsilon_n)\}$ or equivalently the set of the ε_n -components for any $n \in \mathbb{N}$. We have an edge joining $\Lambda_{k,\varepsilon_n}$ with $\Lambda_{j,\varepsilon_m}$ if and only if $|m - n| = 1$ and $\Lambda_{j,\varepsilon_m} \subset \Lambda_{k,\varepsilon_n}$ or $\Lambda_{k,\varepsilon_n} \subset \Lambda_{j,\varepsilon_m}$.

We fix $\Lambda_{1,\varepsilon_1}$ as the root. To introduce a metric in $T_{(X,\{\varepsilon_n\})}$ to convert it in a so called \mathbb{R} -tree, or real tree [14], we impose that any edge is isometric to the unit Euclidean real interval $[0, 1]$ and the distance $\rho_{(X,\{\varepsilon_n\})}$ between two points $\alpha, \beta \in T_{(X,\{\varepsilon_n\})}$ is just the length of the unique arc in $T_{(X,\{\varepsilon_n\})}$ joining α to β , which is isometric to $[0, \rho_{(X,\{\varepsilon_n\})}(\alpha, \beta)]$.

In order to give our technical result in this section we are going to recall some definitions in [14] page 153.

First of all a *rooted \mathbb{R} -tree* consists of an \mathbb{R} -tree (T, ρ) and a point $v \in T$ called the *root*.

A rooted \mathbb{R} -tree (T, v) is *geodesically complete* if every isometric embedding $f : [0, t] \rightarrow T$, $t > 0$ with $f(0) = v$, extends to an isometric embedding $\tilde{f} : [0, \infty) \rightarrow T$. In this case, we say $[v, f(t)]$ can be *extended to a geodesic ray*.

Suppose we have a rooted \mathbb{R} -tree (T, v) with a metric ρ . The *end space* of (T, v) is given by $end(T, v) = \{f : [0, \infty) \rightarrow T / f(0) = v \text{ and } f \text{ is an isometric embedding}\}$.

for $f, g \in end(T, v)$, define

$$\rho_e(f, g) = \begin{cases} 0, & \text{if } f = g \\ \frac{1}{e^{t_0}}, & \text{if } f \neq g \text{ and } t_0 = \sup\{t \geq 0 / f(t) = g(t)\}. \end{cases}$$

As proved in [14] we have that $(end(T, v), \rho_e)$ is a complete *ultrametric* space of diameter ≤ 1 .

We have the following .

Theorem 15. *Let (X, d) be a compact metric space and $\varepsilon_n \rightarrow 0$ when $n \rightarrow \infty$ a decreasing sequence of positive numbers such that $\varepsilon_1 > diam(X)$. Consider the rooted \mathbb{R} -tree $((T_{(X, \varepsilon_n)}, \Lambda_{1, \varepsilon_1}), \rho_{(X, \{\varepsilon_n\})})$ constructed before where $\Lambda_{1, \varepsilon_1}$ is the unique connected component of the corresponding U_{ε_1} . Then, the topological spaces $\square X$ and $end(T_{(X, \varepsilon_n)}, \Lambda_{1, \varepsilon_1})$ are homeomorphic. Where the topology considered on $\square X$ is the quotient topology by means of the canonical projection and the topology considered on $end(T_{(X, \varepsilon_n)}, \Lambda_{1, \varepsilon_1})$ is that induced by the corresponding ultrametric ρ_e induced on the end space by the metric $\rho_{(X, \{\varepsilon_n\})}$.*

Proof. Let $f \in end(T_{(X, \varepsilon_n)}, \Lambda_{1, \varepsilon_1})$ then $f : [0, \infty) \rightarrow T_{(X, \varepsilon_n)}$ is an isometric embedding with $f(0) = \Lambda_{1, \varepsilon_1} \supset X$. By the definition of the metric $\rho_{(X, \varepsilon_n)}$ we have that $f(n)$ is a connected component of $U_{\varepsilon_{n+1}}$. By the construction of the tree we have $f(n) \supset f(n+1)$ for every $n = 0, 1, 2, \dots$,

Define $X_{n,f} = f(n) \cap X$. We have that $X_{n,f}$ is in fact an ε_{n+1} -component in X . Using the above results we obtain that $X_{n,f}$ are open and closed in X in particular the family $\{X_{n,f}\}_{n \in \mathbb{N}}$ is a nested, decreasing by inclusions, sequence of compact subsets and then $X_f = \bigcap_{n \in \mathbb{N}} X_{n,f}$ is a non-empty compact subset of X . Moreover as X_f is the intersection of open and closed sets we obtain that if $x \in X_f$. Then the whole connected component of x is contained in X_f . That is, X_f is union of connected components. In fact we are going to prove that X_f is a connected component of X . Suppose that X_f contains two different connected components Ω_1 and Ω_2 of X . Since $p : X \rightarrow \square X$ is a closed map and $\square X$ is metrizable and 0-dimensional, it follows that

there is an open and closed set $F \subset X$ such that $\Omega_1 \subset F$ and $\Omega_2 \subset X \setminus F$. Take the open set $B_F \cup B_{X \setminus F}$ in the hyperspace 2_U^X . Of course the canonical copy X is contained in $B_F \cup B_{X \setminus F}$ and since $\{U_{\varepsilon_n}\}_{n \in \mathbb{N}}$ is a base of neighborhoods of X inside 2_U^X , by the choice of the sequence $\{\varepsilon_n\}$, we infer that there is an $n_0 \in \mathbb{N}$ such that $X \subset U_{\varepsilon_m} \subset B_F \cup B_{X \setminus F}$ for any $m \geq n_0$. Moreover the connected components $\Lambda_{k_1, \varepsilon_{n_0}}$ and $\Lambda_{k_2, \varepsilon_{n_0}}$ containing Ω_1 and Ω_2 respectively are different because $B_F \supset \Lambda_{k_1, \varepsilon_{n_0}}$, $B_{X \setminus F} \supset \Lambda_{k_2, \varepsilon_{n_0}}$ and $B_F \cap B_{X \setminus F} = \emptyset$ which is impossible by the definition of X_f . Recall that $X_f = \bigcap_{n \in \mathbb{N}} X_{n, f}$.

So we have proved that X_f is just a connected component of X . Define a function $G : \text{end}(T_{(X, \varepsilon_n)}, \Lambda_{1, \varepsilon_1}) \rightarrow \square X$ by $G(f) = X_f$ as constructed before. We are going to prove that G is a homeomorphism. First of all G is injective because if $f \neq g$ are two different ends then, by construction, there is a $m \in \mathbb{N}$ such that $f(m) \neq g(m)$. Then $f(m) \cap g(m) = \emptyset$ because they are different connected components of the same space. On the other hand $X_f \subset f(m)$ and $X_g \subset g(m)$. Consequently $G(f) \neq G(g)$.

Fix now any connected component $X_0 \in \square X$, then for every $n \in \mathbb{N}$ there is a $k_n \in \mathbb{N}$ such that $X_0 \subset \Lambda_{k_n, \varepsilon_n}$ which is the corresponding connected component of U_{ε_n} . Since $U_{\varepsilon_n} \supset U_{\varepsilon_{n+1}}$ then $\Lambda_{k_n, \varepsilon_n} \supset \Lambda_{k_{n+1}, \varepsilon_{n+1}}$ and consequently the map $f_{X_0} : [0, \infty) \rightarrow T_{(X, \varepsilon_n)}$ given by $f_{X_0}(n) = \Lambda_{k_{n+1}, \varepsilon_{n+1}}$ and extended as an isometry from the interval $[n, n+1]$ to the edge $[\Lambda_{k_{n+1}, \varepsilon_{n+1}}, \Lambda_{k_{n+2}, \varepsilon_{n+2}}]$ in the tree is an isometric embedding and consequently an end. It is obvious that $G(f_{X_0}) = X_0$. So G is surjective too.

Let us prove now that G is continuous. Take a sequence of ends $\{f_k\}_{k \in \mathbb{N}}$ with $\{f_k\}_{k \in \mathbb{N}} \rightarrow f$ for the ultrametric ρ_e induced by the metric $\rho_{(X, \varepsilon_n)}$.

Fix any neighborhood A of $G(f)$ in $\square X$, we can suppose that A is open and closed because $\square X$ is 0-dimensional, and take $n_0 \in \mathbb{N}$ such that $U_{\varepsilon_{n_0}} \subset B_{p^{-1}(A)} \cup B_{p^{-1}(\square X \setminus A)}$ where $p : X \rightarrow \square X$ is the projection. So $p^{-1}(A)$ is open and closed in X . Let $\delta > 0$ be such that $\delta < \frac{1}{e^{n_0}}$. So there is $m_0 \in \mathbb{N}$ with $\rho_e(f_m, f) < \delta < \frac{1}{e^{n_0}}$ for $m \geq m_0$. Consequently $f_m(n_0) = f(n_0)$ for any $m \geq m_0$, by the definition of the ultrametric ρ_e . This means that the connected component of $U_{\varepsilon_{n_0+1}}$ containing $G(f)$, which is just $f(n_0)$, is the same as the connected component of $U_{\varepsilon_{n_0+1}}$ containing $G(f_m)$ for any $m \geq m_0$ and then $G(f_m) \subset p^{-1}(A)$ as subset of X . Consequently $G(f_m) \in A$ considered as a point in $\square X$. So we have proved the continuity of G .

In order to prove that the inverse $G^{-1} : \square X \rightarrow \text{end}(T_{(X, \varepsilon_n)}, \Lambda_{1, \varepsilon_1})$ is continuous we have only to prove that $\text{end}(T_{(X, \varepsilon_n)}, \Lambda_{1, \varepsilon_1})$ is compact because, if so, then G is closed and being bijective and continuous G is a homeomorphism.

Recall first by Hughes paper [14] that $\text{end}(T_{(X, \varepsilon_n)}, \Lambda_{1, \varepsilon_1})$ is complete as a metric space and then we only have to prove that this metric spaces is totally bounded, see [46], to get compactness.

Being totally bounded means that for every $\varepsilon > 0$ the uniform cover by open balls of radius ε admits a finite subcover. So fix $\varepsilon > 0$ and take $p \in \mathbb{N}$ such that $\frac{1}{e^p} < \varepsilon$. The formula $\rho_e(f, g) \leq \frac{1}{e^p}$ is equivalent to the fact that $f(t) = g(t)$ for $t \in [0, p]$ which implies, in particular, that $f(p) = g(p)$. Note also that the set of values $\{f(p), f \in \text{end}(T_{(X, \varepsilon_n)})\}$ is finite. Those possible values are just the different connected components $\{\Lambda_{1, \varepsilon_{p+1}}, \dots, \Lambda_{C(\varepsilon_{p+1}), \varepsilon_{p+1}}\}$. Choose now $f_1, \dots, f_{C(\varepsilon_{p+1})} \in \text{end}(T_{(X, \varepsilon_n)}, \Lambda_{1, \varepsilon_1})$ with $f_k(p) = \Lambda_{k, \varepsilon_{p+1}}$. It is obvious that the union of balls $\cup \mathcal{B}_{\rho_e}(f_k, \frac{1}{e^p}) = \text{end}(T_{(X, \varepsilon_n)}, \Lambda_{1, \varepsilon_1})$. Consequently this space is compact and the proof is finished. \square

6. HYPERSPACES AND FINITE APPROXIMATIONS TO ČECH HOMOLOGY: THE ALEXANDROFF-McCORD CORRESPONDENCE

In this section we associate to any compact metric space (X, d) some inverse sequences related to suitable families of finite approximations of X . After that we can convert these inverse sequences into inverse sequences of finite polyhedra with simplicial bonding maps using what we call the Alexandroff-McCord correspondence that is described in [16]. The use of the upper semifinite hyperspace of a compact metric space is also needed here to construct the bonding maps.

First, let us describe how is the Alexandroff-McCord correspondence for certain subsets of the upper semifinite hyperspace of a finite metric space

Proposition 16. *Let $K = \langle x_0, \dots, x_n \rangle$ be an n -simplex considered as an abstract simplicial complex. Denote by $A = \{x_0, \dots, x_n\}$ the set of its vertices. Then the finite space $\mathcal{X}(K)$ is homeomorphic to the upper semifinite hyperspace 2_U^A where A is considered with the discrete topology. Consequently $\mathcal{K}(2_U^A) = K'$ where K' represents the first barycentric subdivision of K .*

Proof. Let us recall, [16] pages 470-471, that $\mathcal{X}(K) = \{b(\sigma), \sigma \in K\}$ where $b(\sigma)$ represents the barycenter of the simplex σ . Moreover the partial order giving the topology in $\mathcal{X}(K)$ is given by $b(\sigma) \leq b(\sigma')$ if and only if σ is a face of σ' .

Let us note that the upper semifinite hyperspace 2_U^A is a T_0 space. Moreover if $C \in 2_U^A$, then the minimal open set $V_C = 2^C$. If we consider the relative topology on V_C then $V_C = 2_U^C$. Finally, the corresponding partial order \leq' is just the inclusion \subset on subsets.

It is very easy to see that there is a bijective order-preserving map $\phi : (\mathcal{X}(K), \leq) \longrightarrow (2^A, \leq')$. So the topological spaces $\mathcal{X}(K)$ and 2_U^A are homeomorphic. The remaining part follows from the formula $\mathcal{K}(\mathcal{X}(L)) = L'$ proved in [16] for any finite simplicial complex L . \square

Let us recall, [19] page 175, that if (X, d) is a metric space and $\varepsilon > 0$, then the *Vietoris-Rips complex* (also known as the *Rips complex* and denoted here by $\mathcal{R}_\varepsilon(X)$) is the abstract simplicial complex defined as follows: the vertices of $\mathcal{R}_\varepsilon(X)$ are the points of X and a q -simplex of $\mathcal{R}_\varepsilon(X)$ is a subset $\{x_0, \dots, x_q\}$ of X such that $\text{diam}(\{x_0, \dots, x_q\}) < \varepsilon$ where diam is the diameter for the metric d .

Corollary 17. *Let (A, d) be a finite metric space. For any $\varepsilon > 0$, consider $U_\varepsilon(A) = \{C \in 2^A \mid \text{diam}(C) < \varepsilon\}$. Then $\mathcal{K}(U_\varepsilon(A)) = \mathcal{R}'_\varepsilon(A)$.*

Proof. Given $\varepsilon > 0$, consider the Vietoris-Rips complex $\mathcal{R}_\varepsilon(A)$. Note that $C = \{x_0, \dots, x_q\}$ is a simplex of $\mathcal{R}_\varepsilon(A)$ if and only if $C \in U_\varepsilon(A)$. Finally, applying the above proposition to any simplex in $\mathcal{R}_\varepsilon(A)$ we get the result. \square

We also need the following:

Proposition 18. *Let (X, d) be a compact metric space and suppose that $A \subset X$ is a non-empty finite subset. For any $x \in X$ consider the set $A(x) = \{a \in A \mid d(x, a) = d(x, A)\}$. Define the function*

$$q_A : X \longrightarrow 2_U^A$$

by $q_A(x) = A(x)$. Then q_A is continuous. Moreover the map $r_A : 2_U^X \longrightarrow 2_U^A$ given by $r_A(C) = \bigcup_{c \in C} A(c)$ is a retraction onto 2_U^A which is also a continuous extension to 2_U^X of q_A interpreting its domain X as the canonical copy inside 2_U^X .

We don't need to prove anything above. In fact a more general result is contained in the proof of Proposition 2 in [9] pages 973-974. The fact that A is finite is not necessary, we only need A to be a closed subset of X .

The following corollary will be useful for our construction. Recall that $A \subset X$ is an ε -approximation of X if for any $x \in X$ there is an $a \in A$ such that $d(x, a) < \varepsilon$.

Corollary 19. *Let (X, d) be a compact metric space and consider a real number $\varepsilon > 0$. Suppose that $A \subset X$ is a finite ε -approximation of X . Then $q_A(x) \in U_{2\varepsilon}(A)$ for every $x \in X$. Moreover there is a $\delta > 0$ such that $r_A(U_\delta(X)) \subset U_{2\varepsilon}(A)$*

Proof. Let $x \in X$ and suppose that $a, a' \in A(x)$. Then $d(a, a') \leq 2d(x, A) < 2\varepsilon$ by the triangular inequality and the definition of $A(x)$. So, $A(x) \in U_{2\varepsilon}(A)$. Now r_A is continuous and restricted to the canonical copy X is just q_A . So, there is an open set $V \subset 2_V^X$ containing X such that $r_A(V) \subset U_{2\varepsilon}(A)$. Finally, using the fact that $\{U_\alpha(X)\}_{\alpha>0}$ is a base of open neighborhoods for the canonical copy X inside 2_V^X (and proved in Proposition 1 in [9] page 973) we get the result. \square

Now we have all basic tools to do our:

6.1. Main construction. From now on we are going to begin our main general construction related to finite approximations that generalizes the ideas in our example in the introduction.

Suppose (X, d) is a (non-empty) compact metric space with

$\text{diam}(X) = \sup_{x,y \in X} \{d(x, y)\} = M$. Consider a real number $\varepsilon_1 > M$. Fix any point $a(1) \in X$ and let $A_1 = \{a(1)\}$. So, A_1 is an ε_1 -approximation of X . For the next step choose a number ε_2 such that $0 < \varepsilon_2 < \text{minimum}\{\frac{\varepsilon_1 - M}{2}, \frac{M}{2}\}$. Since X is compact, then there exist a natural number m_2 and a finite subset $A_2 = \{a_1(2), \dots, a_{m_2}(2)\}$ of X which is an ε_2 -approximation of X . Now we consider both subsets A_1 and A_2 with the restricted metric. Define the unique possible map $p_{2,1} : U_{2\varepsilon_2}(A_2) \longrightarrow U_{2\varepsilon_1}(A_1)$ which is constant and then continuous.

Let us consider now the function $d_{A_2} : X \longrightarrow \mathbb{R}$ given by $d_{A_2}(x) = d(x, A_2)$. That is, $d_{A_2}(x)$ is the minimal distance from x to any point in A_2 . Since $|d_{A_2}(x) - d_{A_2}(x')| \leq d(x, x')$ for any $x, x' \in X$, then d_{A_2} is continuous. Moreover $d_{A_2}(x) < \varepsilon_2$ for each $x \in X$. If $\gamma_2 = \text{l.u.b.}\{d_{A_2}(x) \mid x \in X\}$, then $\gamma_2 < \varepsilon_2$ because of compactness of X . Here, l.u.b. means the least upper bound. Using now the previous corollary we obtain a number δ_2 such that $r_{A_2}(U_{\delta_2}(X)) \subset U_{2\varepsilon_2}(A_2)$. Fix

now a positive number ε_3 such that

$$\varepsilon_3 < \text{minimum}\left\{\frac{\varepsilon_2 - \gamma_2}{2}, \frac{\delta_2}{2}\right\}$$

As before we choose now an ε_3 -approximation of X , $A_3 = \{a_1(3), \dots, a_{m_3}(3)\}$ where m_3 is a natural number. Since $2\varepsilon_3 < \delta_2$, it follows that $U_{2\varepsilon_3}(A_3) \subset U_{\delta_2}(X)$. Then $r_{A_2}(U_{2\varepsilon_3}(A_3)) \subset U_{2\varepsilon_2}(A_2)$. Define now $p_{3,2} : U_{2\varepsilon_3}(A_3) \longrightarrow U_{2\varepsilon_2}(A_2)$ as the restriction of the map r_{A_2} . Consequently $p_{3,2}$ is continuous.

Suppose we have constructed $\varepsilon_n, A_n, \gamma_n, \delta_n$ and $p_{n,n-1} : U_{2\varepsilon_n}(A_n) \longrightarrow U_{2\varepsilon_{n-1}}(A_{n-1})$ as before. Then choose a positive number

$$\varepsilon_{n+1} < \text{minimum}\left\{\frac{\varepsilon_n - \gamma_n}{2}, \frac{\delta_n}{2}\right\}$$

and fix a finite ε_{n+1} -approximation of X , A_{n+1} . Using analogous arguments we can define $p_{n+1,n} : U_{2\varepsilon_{n+1}}(A_{n+1}) \longrightarrow U_{2\varepsilon_n}(A_n)$ by $p_{n+1,n}(D) = r_{A_n}(D)$, for every $D \in U_{2\varepsilon_{n+1}}(A_{n+1})$.

So we have constructed an inverse sequence of finite topological spaces:

$$\dots \longrightarrow U_{2\varepsilon_{n+1}}(A_{n+1}) \longrightarrow U_{2\varepsilon_n}(A_n) \dots \longrightarrow U_{2\varepsilon_2}(A_2) \longrightarrow U_{2\varepsilon_1}(A_1)$$

with the corresponding $p_{n+1,n}$ as bonding maps. Using now Theorem 2 in page 466 in [16] we have the existence of a simplicial map $|p_{n+1,n}|$ making the following diagram commutative:

$$\begin{array}{ccc} | \mathcal{K}(U_{2\varepsilon_{n+1}}(A_{n+1})) | & \xrightarrow{|p_{n+1,n}|} & | \mathcal{K}(U_{2\varepsilon_n}(A_n)) | \\ f_{n+1} \downarrow & & \downarrow f_n \\ U_{2\varepsilon_{n+1}}(A_{n+1}) & \xrightarrow{p_{n+1,n}} & U_{2\varepsilon_n}(A_n) \end{array}$$

where f_{n+1} and f_n are the corresponding McCord's weak homotopy equivalences quoted above.

So we have two inverse sequences and a level map of inverse sequences where, at any level, the connecting map is a weak homotopy equivalence as described in the following diagram:

$$\begin{array}{ccccccc} \longrightarrow & | \mathcal{K}(U_{2\varepsilon_{n+1}}(A_{n+1})) | & \xrightarrow{|p_{n+1,n}|} & | \mathcal{K}(U_{2\varepsilon_n}(A_n)) | & \longrightarrow \dots \longrightarrow & | \mathcal{K}(U_{2\varepsilon_2}(A_2)) | & \xrightarrow{|p_{2,1}|} & | \mathcal{K}(U_{2\varepsilon_1}(A_1)) | \\ & f_{n+1} \downarrow & & \downarrow f_n & & f_2 \downarrow & & \downarrow f_1 \\ \longrightarrow & U_{2\varepsilon_{n+1}}(A_{n+1}) & \xrightarrow{p_{n+1,n}} & U_{2\varepsilon_n}(A_n) & \longrightarrow \dots \longrightarrow & U_{2\varepsilon_2}(A_2) & \xrightarrow{p_{2,1}} & U_{2\varepsilon_1}(A_1) \end{array}$$

Passing at the corresponding inverse sequences of singular homology groups we have, at any square, the commutativity of:

$$\begin{array}{ccc}
H_k(| \mathcal{K}(U_{2\varepsilon_{n+1}}(A_{n+1})) |) & \xrightarrow{H_k(|p_{n+1,n}|)} & H_k(| \mathcal{K}(U_{2\varepsilon_n}(A_n)) |) \\
\downarrow H_k(f_{n+1}) & & \downarrow H_k(f_n) \\
H_k(U_{2\varepsilon_{n+1}}(A_{n+1})) & \xrightarrow{H_k(p_{n+1,n})} & H_k(U_{2\varepsilon_n}(A_n))
\end{array}$$

$H_k(f_n)$ and $H_k(f_{n+1})$ are isomorphisms because f_{n+1} and f_n are weak homotopy equivalences. Consequently the inverse sequences $(H_k(| \mathcal{K}(U_{2\varepsilon_{n+1}}(A_{n+1})) |), H_k(| p_{n+1,n} |))$ and $(H_k(U_{2\varepsilon_{n+1}}(A_{n+1})), H_k(p_{n+1,n}))$ are isomorphic in pro-Group for any $k \geq 0$. In particular the inverse limit groups of both of them are isomorphic as groups. Using the properties of the Alexandroff-McCord correspondence described above (and the fact that the geometrical realizations of simplicial complex and of its barycentric subdivision are homeomorphic) we have that the inverse sequences $(H_k(| \mathcal{R}_{2\varepsilon_{n+1}}(A_{n+1}) |), H_k(| p_{n+1,n} |))$ and $(H_k(U_{2\varepsilon_{n+1}}(A_{n+1})), H_k(p_{n+1,n}))$ are isomorphic in pro-Group for any $k \geq 0$. Note that, by the construction, $\lim(\varepsilon_n)_{n \in \mathbb{N}} = 0$ and then the sequence $(A_n)_{n \in \mathbb{N}}$ converges to X in the Hausdorff metric related to the metric d . So we can state our

General Principle: *The properties of the maps $| p_{n+1,n} |$ are sufficiently good enough as to imply that the inverse sequence $(| \mathcal{R}_{2\varepsilon_{n+1}}(A_{n+1}) |, | p_{n+1,n} |)$ reflects at least the shape of the space X . Consequently the inverse sequence of finite spaces $(U_{2\varepsilon_{n+1}}(A_{n+1}), p_{n+1,n})$ is suitable to extrapolate high-dimensional topological properties of X as, in particular, the Čech homology groups in any dimension.*

In the future we will follow developing the mathematical analysis to study this General Principle. We will try to publish this elsewhere. Anyway we are going to finish with some comments on the General Principle and the future work.

First note that the role of the number $\gamma_n = l.u.b.\{d_{A_n}(x) \mid x \in X\}$ in our main construction have been not clarified. In fact we did not use it before but it will play an important role because of the following:

Proposition 20. *For every $n \in \mathbb{N}$, the following diagram:*

$$\begin{array}{ccc}
X & \xrightarrow{1_X} & X \\
q_{A_{n+1}} \downarrow & & \downarrow q_{A_n} \\
U_{2\varepsilon_{n+1}}(A_{n+1}) & \xrightarrow{p_{n+1,n}} & U_{2\varepsilon_n}(A_n)
\end{array}$$

is commutative in the homotopy category. Where 1_X is the identity in X .

This is so because, from the choice of γ_n , we have that $\text{diam}(p_{n+1,n}(q_{A_{n+1}}(x)) \cup q_{A_n}(x)) < 2\varepsilon_n$ for every $x \in X$. Using the upper semifinite topology it is easy then to construct a homotopy from q_{A_n} to $p_{n+1,n} \circ q_{A_{n+1}}$.

Another observation about the above construction is that we can suppose always that the sequence $(\delta_n)_{n \in \mathbb{N}}$ is decreasing and tending to 0

The main technical result for the mathematical analysis of the general principle is the following:

Proposition 21. *Let (X, d) be a compact metric space. Consider the corresponding sequences (ε_n) , (A_n) , (γ_n) , (δ_n) as in our main construction. Then the sequence of maps $(q_{A_n}) : X \longrightarrow 2_U^X$ represents the identity shape morphism on X .*

The *representation* quoted in the previous proposition is in the sense described first by Sanjurjo in [47] and reinterpreted later in [9]

Another results with influences in our future work and intrinsically related to our Example 1 in the introduction are that due to J. Latschev in [48].

7. CONCLUSIONS

In [2] and [3] the authors reformulated the concept of connectedness to the computationally more tractable concept of connected under a resolution. Later, in [8] is contained an attempt to create a theoretical framework for this. Here we point out that [8] reflects what is *visible* from the inside of a more complete picture, looking from the outside, taken in some kind of *halo*, the upper semifinite hyperspace, created naturally for any compact metric space. The advantage of this point of view is that we have not to create new techniques or concepts because the established General Topology of the hyperstructure allows us to find this concepts by intersecting topological

phenomena in the ideally created world, (The hyperspace 2_U^X) with the real world (the compact metric space X). One of the main reason why it works is because the upper semifinite topology in hyperspaces seems to be specially adequate to describe observations under an ε -resolutions because the corresponding closure operator behaves, in some sense, as the human eye.

In this paper there are some coarse graining processes that correspond, in hyperspaces, to classical decompositions related to topology, concretely to different definitions of connectedness.

Another of the results in this paper, motivated by some extrapolation processes treated in [3], says that the knowledge of the connectedness structure, to different resolutions, could help in order to get the real connectedness structure of a compact metric space not only by detecting the *number* of real connected components but also their relative topological situation.

This paper also reinforces the idea of V. Robins about the convenience of using ideas and methods from shape theory to treat the important task in computational topology about the inference of high-dimensional structure from finite approximations.

One of the interesting things in this paper is that we have found close relationships between Vietoris-Rips complexes, The Alexandroff-McCord correspondence and the upper semifinite topology in hyperspaces that could be useful in future studies both on the computational and the basic aspects of Topology.

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