THE HAUSDORFF METRIC AND CLASSIFICATIONS OF COMPACTA.

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Abstract. In this paper we use the Hausdorff metric to prove that two compact metric spaces are homeomorphic if and only if their canonical complements are uniformly homeomorphic. So, we take one of the two steps needed to prove that the difference between the homotopical and topological classifications of compact connected ANRs depends only on the difference between continuity and uniform continuity of homeomorphisms in their canonical complements, which are totally bounded metric spaces. The more important step was provided by the Chapman complement and the Curtis-Schori-West theorems. We also improve the multivalued description of shape theory given by J. M. R. Sanjurjo but only in the class of locally connected compacta.

1. INTRODUCTION

The Hausdorff metric in the hyperspace 2^X of nonempty closed subsets of a metric compactum (X, d) depends only on the metric space (X, d). The Hausdorff distance between two nonempty closed subsets of X is the smallest number δ such that each closed ball of radius δ centered at a point of either set necessarily contains a point of the other set. From now on we will denote by $2^X_{H_d}$ the metric hyperspace of (X, d), and by 2^X_H the corresponding topological space.

It can happen that $2^X_H \equiv 2^Y_H$ in the topological category but $X \not\equiv Y$ in the same context. A special result in this direction is the so called Curtis-Schori-West Theorem,

First named author has been supported by DGI.

MSC: 54B20, 55P15, 57N25 .

see [23] and the references therein, which states that for any pair of non-degenerate Peano continua X_1, X_2 we have $2_H^{X_1} \equiv 2_H^{X_2} \equiv Q$ (the Hilbert cube). Another family of examples on this line is given in [20] . It is a classical problem in the theory of continua, of current interest, to find kinds of spaces X that are determined by their hyperspaces 2_H^X in different categories.

The hyperspace 2_H^X is a natural environment for the space X because there is a canonical way to embed (X, d) isometrically into $2_{H_d}^X$, identifying a point $x \in X$ with the closed subset $\{x\}$ of X. So the properties related to this embedding of X inside 2^X are intrinsic and extrinsic to X (in some sense). A very interesting result related to this kind of thing is obtained by joining the Curtis-Schori-West Theorem and the so called Chapman Complement Theorem in the Theory of Shape, see [9], [10] and the references therein. Putting together both deep results one obtains that in the class of non-degenerate Peano continua the shape of such space X determines, and it is determined by, the topological type of the complement of a Z-embedded copy of X inside the hyperspace 2_H^X. Furthermore, in this context, the canonical copy of X is Z-embedded in 2_H^X. In particular the homotopy type in the class of compact connected manifolds (with positive dimension), or in the class of non-degenerate connected finite polyhedra, in general in the class of connected ANR's, depends only on the topological type of the canonical complement, which is the complement of the canonical copy of X in 2_H^X. Moreover, the canonical complement is a non-compact contractible Q-manifold, see [10], that can be compactified, to obtain the Hilbert cube, adding the canonical copy.

In this paper we use the Hausdorff metric in hyperspaces for two purposes: first we use it to describe Borsuk's shape for compact locally connected metric spaces; second we characterize the topological type of X in terms of the uniform type of the canonical complement.

The first part is motivated by J. M. R. Sanjurjo [21], and the second part by T. A. Chapman [9]. In [21] the author describes Shape theory in terms of sequences of upper semicontinuous multivalued maps with decreasing diameters on its images . J. M. R. Sanjurjo asked us about the possibility of having a better description of shape following his line. In particular he inquired of the possibility of changing upper semicontinuity to continuity (upper and lower [18]) in the description given in [21]. We answer the question positively, for the realm of locally connected compacta. To do this quickly we use the so called Wojdysławski Theorem, [25], which assures that the hyperspace of a Peano continuum is an AR-space, in the sense of Borsuk [6]. This theorem can be used as a previous step to get the Curtis-Schori-West Theorem. A short and elementary proof of the Wojdysławski Theorem has been given by Sergey Antonyan in [2].

In the second part of this note we prove a general result establishing that the topological type of a compact metric space is completely determined by the uniform type of the canonical complement. In order to get this result we use Pełczynski's paper [20]. We finish the paper showing how this duality can be used to reformulate some problems in topology and to prove known results in this way. In particular we use the Freudenthal ends of the complement of a Z-set in Q to obtain a Borsuk's result in Shape Theory.

We recommend the books [4, 14, 19, 23] for information on hyperspaces; we also suggest [7, 12, 17] for Shape theory. We think that the book [5] could be of interest for further studies. We are particularly interested in the properties of uniformly continuous homeomorphisms used there.

2. HAUSDORFF METRIC AND THE SHAPE OF LOCALLY CONNECTED COMPACTA

Recall that a compact metric space (X,d) can be identified , isometrically, with the subspace , $\phi(X) \subset 2^X_{H_d}$, where

$$\begin{aligned} \phi \ &: X \longrightarrow \ 2^X_{H_d} \\ &x \longrightarrow \ \{x\} \end{aligned}$$

is the so called canonical embedding. From now on we will identify X thus. Note also that we can consider, isometrically, (X,d) as the subspace $X = \{C \in 2^X_{H_d} \mid diam(C) = 0\}$, with the induced metric, where diam represents the diameter function for the metric d.

The first thing we want to note is the following:

Proposition 1. *Let (X,d) be a compact metric space. Then the family $\mathcal{U} = \{U_\varepsilon\}_{\varepsilon>0}$ is a base of open neighborhoods of X in $2^X_{H_d}$, where $U_\varepsilon = \{C \in 2^X_{H_d} \mid diam(C) < \varepsilon\}$. Consequently $\{U_{1/n}\}_{n\in\mathbb{N}}$ is a countable base.*

Proof. First of all, each U_ε is open in 2^X_H because diam:$2^X_H \longrightarrow \mathbb{R}$ is continuous (see p. 55 in [16]). Obviuosly $X \subset U_\varepsilon$.

Now let $U \subset 2^X_H$ be an open set and $X \subset U$. For each $x \in X$ choose $\varepsilon_x > 0$ such that $V_x = B_{H_d}(x, \varepsilon_x) \subset U$, where B_{H_d} represents the open ball in the Hausdorff metric H_d, and let $V_0 = 2^X_H \setminus X$. Choose a Lebesgue number β for the open cover $\{V_x, x \in X\} \cup \{V_0\}$. Then for each $x \in X$, $B_{H_d}(\{x\}, \beta) \subset U$.

Put $U_\beta = \{C \in 2^X_{H_d} \mid diam(C) < \beta\}$, fix $C \in U_\beta$ and $c \in C$. Then $C \in B_{H_d}(\{c\}, \beta)$ and we are done.

□

Let us recall now the definition of approximative map given by Borsuk, see [7] page $87-88$:

Suppose that X, Y are compact metric spaces and $\mathcal{N}$ an AR-space with $Y \subset \mathcal{N}$, an approximative map of X towards Y is a sequence of continuous functions $\{f_k\}_{k\in\mathbb{N}} : X \longrightarrow \mathcal{N}$ such that for every neighborhood V of Y in $\mathcal{N}$ there is a $k_0 \in \mathbb{N}$ with $f_k \simeq f_{k+1}$ in V ($\simeq$ means homotopic) for every $k \geq k_0$. Two approximative maps $\{f_k\}_{k\in\mathbb{N}}$, $\{g_k\}_{k\in\mathbb{N}} : X \longrightarrow \mathcal{N}$ towards Y are said to be homotopic if for every neighborhood V of Y in $\mathcal{N}$ there is a $k_0 \in \mathbb{N}$ with $f_k \simeq g_k$ in V for every $k \geq k_0$. S. Mardešić proved later, see [17] page 333 for a proof, that the homotopy classes of approximative maps are just the shape morphisms. Note that Borsuk required the ambient space $\mathcal{N}$ to be an AR-space. It can be easily proved that it suffices that $\mathcal{N}$ be an ANR-space in the sense of Borsuk [6].

Sanjurjo in [21], Def. 1, defined a multinet $\{F_n\}_{n\in\mathbb{N}} : X \longrightarrow Y$ between two compact metric spaces as a sequence of multivalued upper semicontinuous functions with the property that for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that F_n is ε-multihomotopic to F_{n+1} for every $n \geq n_0$, where ε-multihomotopic means that there is an upper semicontinuous multivalued map $H : X \times I \longrightarrow Y$ such that for every $(x, t) \in X \times I$ we have $\operatorname{diam}(H(x, t)) < \varepsilon$ and $H(x, 0) = F_n(x)$, $H(x, 1) = F_{n+1}(x)$. He also defined, [21] page 628, the notion of homotopy between two multinets as follows: multinets $F = \{F_n\}$, $G = \{G_n\} : X \longrightarrow Y$ are said to be homotopic provided that for every $\varepsilon > 0$, F_n is ε-multihomotopic to G_n for almost all n. On the other hand, the Vietoris topology in hyperspaces coincides with the topology induced by the Hausdorff metric when the space is compact metric. So, a function into this hyperspace is continuous if and only if it is upper and lower semicontinuous in the usual sense (as multivalued maps [18]). So from now on we will say that $F = \{F_n\}$ is a *continuous multinet* if it satisfies Sanjurjo's definition changing everywhere upper semicontinuity by continuity. Also we will say that two continuous multinets are *continuously homotopic* if the homotopies between the levels are assumed to be continuous in the above sense. We can say

Proposition 2. *Let (X, d), (Y, d') be compact metric spaces and suppose that Y is locally connected. Then the shape morphisms from X to Y are in one-to-one correspondence with the continuous homotopy classes of continuous multinets from X to Y. Consequently the shape category in the realm of locally connected compacta can be described using continuity on multivaled maps not only upper semicontinuity.*

Proof. Since Y is locally connected then it is the topological sum of a finite number of Peano continua, $\{Y_i\}_{i=1,\ldots,n}$ the components of Y.

Consider now $2^Y_{H_{d'}}$. It is easy to see that $Y \subset 2^{Y_1}_{H_{d'}} \oplus \ldots \oplus 2^{Y_n}_{H_{d'}} \subset 2^Y_{H_{d'}}$ and the second inclusion is as an open and closed set. Applying the Wojdysławski Theorem to the non-degenerate components Y_k, we have $\mathcal{N} = 2^{Y_1}_{H_{d'}} \oplus \ldots \oplus 2^{Y_n}_{H_{d'}}$ is an ANR containing Y. So the shape morphisms from X to Y are represented by the homotopy classes of approximative maps $\{f_k\}_{k\in\mathbb{N}} : X \longrightarrow \mathcal{N}$ in the sense of Borsuk. Finally, by Proposition 1, $\{f_k\}_{k\in\mathbb{N}}$ can be considered as a continuous multinet $\{f_k\}_{k\in\mathbb{N}} : X \longrightarrow Y$ and the homotopy relation of approximative maps converts to continuous homotopy between continuous multinets. The second part of the proposition can be proved following Sanjurjo's procedure in [21] to define composition in homotopy classes. □

Remark 3. *Note that the above proposition can be used to characterize the main concepts in Shape Theory in terms of the Hausdorff metric in the realm of locally connected compacta.*

Any continuous multinet defines a shape morphism because it is a multinet in the sense of Sanjurjo. We do not know what classification one obtains using, among compacta, the continuous homotopy classes of continuous multinets. Anyway, this classification *is stronger than the shape one and weaker than the homotopic one and coincides with Shape theory in locally connected compacta and, of course, in the class of zero-dimensional compacta.*

3. A COMPLEMENT THEOREM IN TOPOLOGY

The main result in this section establishes a duality between the topology of compact metric spaces and the uniformity of their complements in $2^X_{H_d}$. We need the following convention. We say that metric spaces (X, d) and (Y, d') are uniformly homeomorphic if there is a homeomorphism $f : (X, d) \longrightarrow (Y, d')$ such that f and f^{-1} are both uniformly continuous functions with respect to the corresponding metrics.

To prove the next theorem we need a Pełczynski's result. We recall it here for the reader's convenience.

Pełczynski's Proposition (page 85 of [20])

Let X and X' be infinite compact metric spaces, let X_0 and X'_0 denote the sets of all isolated points of X and X', respectively. Let us suppose that X_0 is dense in X and X'_0 is dense in X'. Then every homeomorphism h from $X_1 = X \setminus X_0$ onto $X'_1 = X' \setminus X'_0$ can be extended to a homeomorphism from X onto X'.

The following **remark** on notation is also important: As we said at the beginning of the last section we identify the space X to the canonical copy $\phi(X)$ inside the hyperspace 2^X_H. So when we write X in the theorem below we refer to the canonical copy. For example, take $X = \{0, 1\}$ with the discrete topology. When we write below $2^X_H \setminus X$ we refer to the unitary subset $\{\{0, 1\}\}$ of $2^{\{0,1\}}$ because $X = \phi(X) = \{\{0\}, \{1\}\}$.

Theorem 4. *Let (X, d) and (Y, d') be compact metric spaces. Then X and Y are homeomorphic if and only if the canonical complements $(2^X_{H_d} \setminus X, H_d)$ and $(2^Y_{H_{d'}} \setminus Y, H_{d'})$ are uniformly homeomorphic.*

Proof. Suppose first that there is a homeomorphism $f : (X,d) \longrightarrow (Y,d')$ then the hyperspace map $2^f : 2^X_{H_d} \longrightarrow 2^Y_{H_{d'}}$ is a uniformly continuous homeomorphism, where $2^f(C) = f(C) = \{f(c) : c \in C\}$. In fact $2^f : (2^X_{H_d}, X) \longrightarrow (2^Y_{H_{d'}}, Y)$ is a uniformly continuous homeomorphism between these compact pairs. So $2^f : 2^X_{H_d} \setminus X \longrightarrow 2^Y_{H_{d'}} \setminus Y$ and $2^{f^{-1}} : 2^Y_{H_{d'}} \setminus Y \longrightarrow 2^X_{H_d} \setminus X$ are uniformly continuous homeomorphisms.

On the other hand let $h : 2^X_{H_d} \setminus X \longrightarrow 2^Y_{H_{d'}} \setminus Y$ be a uniformly continuous homeomorphism such that h^{-1} is also uniformly continuous. First of all note that $C \in 2^Z_H$ (Z compact metric) is an isolated point if and only if C is formed only by isolated points of Z (see [20] Lemma 1). Consequently Card(C) is finite. Let $\mathcal{A}(X) = \{x \in X : x \text{ is an isolated point}\}$. So $\mathcal{A}(X)$ is open in X and $X = \mathcal{A}(X) \cup X'$ (X' is the set of non-isolated points of X). We need to prove that the sets of isolated points $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ of X and Y, respectively, have the same cardinality. Consider the set of isolated points $\mathcal{A}(2^X_{H_d} \setminus X)$ of $2^X_{H_d} \setminus X$. So

$$\mathcal{A}(2^X_{H_d} \setminus X) = \{C \in \mathcal{P}(\mathcal{A}(X)) \text{ with } 2 \le Card(C) < \aleph_0\}.$$

Since h is a homeomorphism;

$$\mathcal{A}(2^Y_{H_{d'}} \setminus Y) = h(\mathcal{A}(2^X_{H_d} \setminus X)) = \{D \in \mathcal{P}(\mathcal{A}(Y)) \text{ with } 2 \le Card(D) < \aleph_0\}.$$

The equalities above imply that if Card($\mathcal{A}(X)$) is finite, then Card($\mathcal{A}(Y)$) is also finite and $2^n - (n+1) = 2^m - (m+1)$ if Card($\mathcal{A}(X)$) $= n$ and Card($\mathcal{A}(Y)$) $= m$. Consequently Card($\mathcal{A}(X)$)= Card ($\mathcal{A}(Y)$).

If Card($\mathcal{A}(X)$) is infinite then Card($\mathcal{A}(X)$)= $\aleph_0$ because X is a compact metric space. This implies that Card($\mathcal{A}(2^X_{H_d} \setminus X$)= $\aleph_0$ because $\mathcal{A}(2^X_{H_d} \setminus X)$ is a subset of the set of finite subsets of $\mathcal{A}(X)$. So, Card($\mathcal{A}(2^Y_{H_{d'}} \setminus Y) = \aleph_0$ and we have proved that Card($\mathcal{A}(Y)$)=$\aleph_0$. Consequently Card($\mathcal{A}(X)$)= Card($\mathcal{A}(Y)$)=$\aleph_0$

Consider the set $X \setminus \mathcal{A}(X)$. If $X \setminus \mathcal{A}(X) = \emptyset$, then X is finite and Card$(\mathcal{A}(2^X_{H_d} \setminus X))=$ Card$(2^X_{H_d} \setminus X) = 2^n - (n+1)$ where Card$(X) = n$. Since h is a homeomorphism, then Card$(2^Y_{H_{d'}} \setminus Y) = 2^n - (n+1)$. Consequently Y is also finite and obviously Card$(X)=$ Card(Y). Hence X is homeomorphic to Y because they are Hausdorff spaces.

Suppose now that $X \setminus \mathcal{A}(X) \neq \emptyset$. Take a point $x_0 \in X \setminus \mathcal{A}(X)$. So there is a decreasing sequence $\{\varepsilon_n\} \longrightarrow 0$ and a sequence of points $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that $x_n \neq x_m$ if $n \neq m$ and $x_n \neq x_0$ for all $n \in \mathbb{N}$ with $d(x_n, x_0) < \varepsilon_n$. Consequently the sequence $\{C_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(2^X_{H_d} \setminus X, H_d)$ where $C_n = \{x_n, x_0\}$. In fact $C_n \underset{H_d}{\longrightarrow} \{x_0\}$ in $2^X_{H_d}$; hence $h(C_n) \underset{H_{d'}}{\longrightarrow} D \in 2^Y_{H_{d'}}$. If $D \in 2^Y_{H'_d} \setminus Y$ we have $h^{-1}(D) \in 2^X_{H_d} \setminus X$ and $\lim_{n\to\infty} h^{-1}(h(C_n)) = h^{-1}(D) \neq \{x_0\}$ which is not possible. So $D = \{y_0\} \subset Y$ and we define $\widetilde{h}(x_0) = y_0$.

Suppose now another sequence $\{F_n\}_{n\in\mathbb{N}} \subset 2^X_{H_d} \setminus X$ that converges to $\{x_0\}$ in $2^X_{H_d}$. Take

$$G_n = \begin{cases} C_k & \text{if } n = 2k \\ F_k & \text{if } n = 2k-1 \end{cases}$$

Then $\{G_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and so is $\{h(G_n)\}_{n\in\mathbb{N}} \subset 2^Y_{H_{d'}}$. Thus $\{h(G_n)\} \underset{H_{d'}}{\longrightarrow} \{y_0\}$ because $\{h(C_n)\} \underset{H_{d'}}{\longrightarrow} \{y_0\}$ and consequently $\{h(F_n)\} \underset{H_{d'}}{\longrightarrow} \{y_0\}$.

The function $\widetilde{h}$ is then independent of the chosen sequence $\{x_n\}_{n\in\mathbb{N}}$ that converges to $\{x_0\}$. So we have defined

$$\widehat{h} : (2^X_{H_d} \setminus X) \cup X' \longrightarrow (2^Y_{H_{d'}} \setminus Y) \cup Y'$$

by

$$\widehat{h}(C) = \begin{cases} h(C) & \text{if } C \in 2^X_{H_d} \setminus X \\ \widetilde{h}(x) & \text{if } x \in X' \text{ (the set of non-isolated points of } X) \end{cases}$$

because if $x \in X'$ then it is clear that $\widetilde{h}(x) \in Y'$. In fact $(2^X_{H_d} \setminus X) \cup X' = 2^X_{H_d} \setminus \mathcal{A}(X)$ is the metric completion of $(2^X_{H_d} \setminus X, H_d)$ and the same argument says that $(2^Y_{H_{d'}} \setminus Y) \cup Y' = 2^Y_{H_{d'}} \setminus \mathcal{A}(Y)$ is the metric completion of $(2^Y_{H_{d'}} \setminus Y, H_{d'})$. Finally since h and h^{-1} are uniformly continuous functions, both of them can be extended as uniformly continuous functions to the complements. So $\widehat{h}$, and by the same construction for h^{-1}, $\widehat{h}^{-1}$, are uniformly continuous and $(\widehat{h})^{-1} = \widehat{h^{-1}}$.

Consider the set $\overline{\mathcal{A}(X)} \setminus \mathcal{A}(X)$. There are two options:

If $\overline{\mathcal{A}(X)} \setminus \mathcal{A}(X) = \emptyset$, then $\text{Card}(\mathcal{A}(X)) < \aleph_0$ and then $\mathcal{A}(X)$ is also closed in X. Since $\text{Card}(\mathcal{A}(X))= \text{Card}(\mathcal{A}(Y))$, then we have a bijection $\alpha : \mathcal{A}(X) \longrightarrow \mathcal{A}(Y)$. Define now $F : 2^X_{H_d} \longrightarrow 2^Y_{H_{d'}}$ by

$$F(C) = \begin{cases} \widehat{h}(C) & \text{if } C \in (2^X_H \setminus X) \cup X' \\ \alpha(C) & \text{if } C \in A(X). \end{cases}$$

In fact $F : (2^X_{H_d}, X) \longrightarrow (2^Y_{H_{d'}}, Y)$ is a homeomophism of compact pairs. Consequently X is homeomorphic to Y.

If $\overline{\mathcal{A}(X)} \setminus \mathcal{A}(X) \neq \emptyset$ then $\text{Card}(\mathcal{A}(X))=\aleph_0$. Take now $x_0 \in \overline{\mathcal{A}(X)} \setminus \mathcal{A}(X) \subset X'$. Then there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{A}(X)$ with $\lim_{n\to\infty} x_n = x_0$ such that $x_n \neq x_m$ if $n \neq m$ and $x_n \neq x_0$ for all $n \in \mathbb{N}$. So $\widehat{h}(x_0) = y_0 = \lim_{n\to\infty} h(\{x_n, x_0\}) = \lim_{n\to\infty} h(\{x_{2n-1}, x_{2n}\})$. But $\{x_{2n-1}, x_{2n}\}$ is an isolated point in $2^X_H \setminus X$, so $h(\{x_{2n-1}, x_{2n}\})$ is also isolated in $2^Y_{H_{d'}} \setminus Y$. Choose now, for every $n \in \mathbb{N}$, a point $y_n \in h(\{x_{2n-1}, x_{2n}\})$, then we have $y_n \underset{n\to\infty}{\longrightarrow} y_0$ in (Y, d') and y_n is an isolated point in Y for every $n \in \mathbb{N}$. This implies that if $x_0 \in \overline{\mathcal{A}(X)} \setminus \mathcal{A}(X)$ then $\widehat{h}(x_0) \in \overline{\mathcal{A}(Y)} \setminus \mathcal{A}(Y)$. Thus we have $\widehat{h}|_{(\overline{\mathcal{A}(X)} \setminus \mathcal{A}(X))} : \overline{\mathcal{A}(X)} \setminus \mathcal{A}(X) \longrightarrow \overline{\mathcal{A}(Y)} \setminus \mathcal{A}(Y)$ is a homeomorphism. Apply now Pełczynski's Proposition to get a homeomorphism $\alpha : \overline{\mathcal{A}(X)} \longrightarrow \overline{\mathcal{A}(Y)}$ which is an extension of $\widehat{h}|_{(\overline{\mathcal{A}(X)} \setminus \mathcal{A}(X))}$.

Finally, $F : (2^X_{H_d}, X) \longrightarrow (2^Y_{H_{d'}}, Y)$ defined by

$$F(C) = \begin{cases} \widehat{h}(C) & \text{if } C \in (2^X_H \setminus X) \cup X' \\ \alpha(C) & \text{if } C \in \overline{\mathcal{A}(X)}, \end{cases}$$

is a homeomorphism of pairs and consequently X is homeomorphic to Y. □

Take now the Hilbert cube Q with a fixed metric d inducing the topology. Recall that a closed set $A \subset Q$ is a Z-set if for every $\varepsilon > 0$ there is a continuous function $f_\varepsilon : Q \longrightarrow Q$ such that $d(f_\varepsilon(x), x) < \varepsilon$ for every $x \in Q$ and $f(Q) \cap A = \emptyset$. This concept was introduced by Anderson [1] in a different but equivalent way. We recommend Chapman's book [10] for getting acquainted with that. We can now obtain a result analogous to Theorem 4 in this context.

Proposition 5. *Let (Q, d) be the Hilbert cube with fixed metric and suppose that $X, Y \subset Q$ are Z-sets, then X is homeomorphic to Y if and only if $(Q \setminus X, d)$ and $(Q \setminus Y, d)$ are uniformly homeomorphic.*

Proof. Suppose that $h : X \longrightarrow Y$ is a homeomorphism then, the Extension Homeomorphism Theorem (see [10]) allows us to extend h to an onto homeomorphism $\widehat{h} : Q \longrightarrow Q$ so $\widehat{h}|_{(Q \setminus X)} : (Q \setminus X, d) \longrightarrow (Q \setminus Y, d)$ is a uniformly continuous homeomorphism (recall that this also implies that $\widehat{h^{-1}}|_{(Q \setminus Y)}$ is also uniformly continuous). The reverse implication is now easier than that in Theorem 1, because (Q, d) is the metric completion of both $(Q \setminus X, d)$ and $(Q \setminus Y, d)$. □

So, we can deduce.

Proposition 6. *a) Let (X, d) and (Y, d') be two compact connected nondegenerate (more than one point) ANR's. Then*

a_1) X is homeomorphic to Y if and only if $(2^X_{H_d} \setminus X)$ is uniformly homeomorphic to $(2^Y_{H_{d'}} \setminus Y)$

a_2) X has the same homotopy type as Y if and only if $2^X_{H_d} \setminus X$ and $2^Y_{H_{d'}} \setminus Y$ are homeomorphic.

b) Let X, Y be arbitrary compacta embedded in the metric Hilbert cube (Q, d) as Z-sets. Then

b_1) X is homeomorphic to Y if and only if $(Q \setminus X, d)$ and $(Q \setminus Y, d)$ are uniformly homeomorphic.

b_2) $Sh(X) = Sh(Y)$ (Sh means the shape) if and only if $Q \setminus X$ is homeomorphic to $Q \setminus Y$.

Remark 7. *The above proposition states that the problems on topological rigidity of manifolds, see* [22] *for a recent survey, can be reformulated to metric problems in special metric spaces. In fact, among compact connected manifolds of positive dimension, the homotopy type is determined by the topological type of the canonical complement and the topological type is determined by the uniform(metric) type of the canonical complement. Moreover, the canonical complement is metrically totally bounded and, topologically, a contractible Q-manifold admitting a boundary in the sense of* [11].

¿From the material in this paper many questions arise and many problems are reformulated. In particular, topological invariants for X can be translated to metric uniform invariants for $2^X_{H_d} \setminus X$ and (for connected ANR's) homotopical invariants of X convert to topological invariants of $2^X_{H_d} \setminus X$. Of special interest for us are the following questions:

Question 1. How can the relation $\dim(X) = n$ be characterized from the outside? (That is, from $2^X_{H_d} \setminus X$) where dim is the dimension.

Question 2. What is a manifold from the outside (even from the inside [8])?

This second question seems to be related to some works of Wilder, see [24], with $\mathbb{R}^n$ as ambient space where the concept of uniformly locally contractible space appears. Also in Kroonenberg [15] there are results related to question 1 where, again, concepts related to uniform locally contractible spaces appear.

We can also use the stated duality to reprove some known facts. An easy example, using the Freudenthal ends (see [3] and [13]) could be the following

Proposition 8. *Let (Q, d) be the Hilbert cube with a fixed metric and suppose that $X, Y \subset Q$ are two zero-dimensional Z-sets. If $f : (Q \setminus X, d) \longrightarrow (Q \setminus Y, d)$ is a homeomorphism, then it is in fact a uniformly continuous homeomorphism.*

Corollary 9. *(See* [17] *and* [7]*)*

Two zero-dimensional compact metric spaces have the same shape if and only if they are homeomorphic.

The results above are consequences of the easy fact that if X is zero-dimensional then the Freudenthal compactification of $Q \setminus X$ is obtained by simply adding the deleted X, and we know that a proper map between spaces can be extended to the Freudenthal compactifications. This implies that any homeomorphism f between the complements is a uniform homeomorphism. In fact, a more general result, due to Borsuk, can be proved

Corollary 10. *(See Borsuk* [7] *page 214) Suppose that X, Y are metric compact spaces and $\alpha : X \longrightarrow Y$ is a shape morphism. Then there is a map $\Lambda_\alpha : \mathcal{C}(X) \longrightarrow \mathcal{C}(Y)$ between the corresponding spaces of components, such that for every $X_0 \in \mathcal{C}(X)$ there is a shape morphism $\alpha_0 : X_0 \longrightarrow \Lambda_\alpha(X_0)$ such that*

$$
\begin{array}{ccc}
X_0 & \xrightarrow{\alpha_0} & \Lambda_\alpha(X_0) \\
i \downarrow & & \downarrow j \\
X & \xrightarrow{\alpha} & Y
\end{array}
$$

is commutative (in the shape category), where i and j are the corresponding inclusions. Moreover the assignment $\alpha \longrightarrow \Lambda_\alpha$ is functorial. In particular if $Sh(X) = Sh(Y)$, then there is a homeomorphism $\Lambda : \mathcal{C}(X) \longrightarrow \mathcal{C}(Y)$, such that $Sh(X_0) = Sh(\Lambda(X_0))$ for every $X_0 \in \mathcal{C}(X)$.

Proof. Consider X and Y embedded in the Hilbert cube Q as Z-sets. Let Q_X, Q_Y be the Freundenthal compactifications of $Q \setminus X$, $Q \setminus Y$ respectively. It is easy to see that Q_X is obtained by adding to $Q \setminus X$ the space of components $\mathcal{C}(X)$. In fact we have a natural projection $P_X : Q \longrightarrow Q_X$ ($P_Y : Q \longrightarrow Q_Y$) defined by

$$
P_X(x) = \begin{cases} x & \text{if } x \in Q \setminus X \\ C_x & \text{if } x \in X \end{cases}
$$

where C_x is the component of X containing x. Using Chapman's description of shape theory, [9], α can be represented by a proper map $f_\alpha : Q \setminus X \longrightarrow Q \setminus Y$. So there is a continuous extension $\widehat{f_\alpha} : Q_X \longrightarrow Q_Y$. Take $\Lambda_\alpha = \widehat{f_\alpha}|_{\mathcal{C}(X)} : \mathcal{C}(X) \longrightarrow \mathcal{C}(Y)$ between the subsets of the corresponding Freudenthal ends. Now one can proceed easily to obtain the complete statement of Borsuk's Theorem, beginning with the fact that the induced map on ends depends only on the weak proper homotopy class of the chosen f_α . $\square$

Acknowledgments

The authors wish to express their gratitude to the referee for his or her many helpful comments that improved a first version of this paper. They are also grateful to J.M.R. Sanjurjo for helpful discussions.

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