# MODULI SPACE OF PRINCIPAL SHEAVES OVER PROJECTIVE VARIETIES

TOMÁS L. GÓMEZ AND IGNACIO SOLS

ABSTRACT. Let $G$ be a connected reductive group. The late Ramanathan gave a notion of (semi)stable principal $G$-bundle on a Riemann surface and constructed a projective moduli space of such objects. We generalize Ramanathan's notion and construction to higher dimension, allowing also objects which we call semistable principal $G$-sheaves, in order to obtain a projective moduli space: a principal $G$-sheaf on a projective variety $X$ is a triple $(P, E, \psi)$, where $E$ is a torsion free sheaf on $X$, $P$ is a principal $G$-bundle on the open set $U$ where $E$ is locally free and $\psi$ is an isomorphism between $E|_U$ and the vector bundle associated to $P$ by the adjoint representation.

We say it is (semi)stable if all filtrations $E_\bullet$ of $E$ as sheaf of (Killing) orthogonal algebras, i.e. filtrations with $E_i^\perp = E_{-i-1}$ and $[E_i, E_j] \subset E_{i+j}{}^{\vee\vee}$, have

$$\sum (P_{E_i} \operatorname{rk} E - P_E \operatorname{rk} E_i) \,(\preceq)\, 0,$$

where $P_{E_i}$ is the Hilbert polynomial of $E_i$. After fixing the Chern classes of $E$ and of the line bundles associated to the principal bundle $P$ and characters of $G$, we obtain a projective moduli space of semistable principal $G$-sheaves. We prove that, in case $\dim X = 1$, our notion of (semi)stability is equivalent to Ramanathan's notion.

*To A. Ramanathan, in memoriam*

## INTRODUCTION

Let $X$ be a smooth projective variety of dimension $n$ over $\mathbb{C}$, with a very ample line bundle $\mathcal{O}_X(1)$, and let $G$ be a connected algebraic reductive group. A principal $\mathrm{GL}(R,\mathbb{C})$-bundle over $X$ is equivalent to a vector bundle of rank $R$. If $X$ is a curve, the moduli space was constructed by Narasimhan and Seshadri [N-S, Sesh]. If $\dim X > 1$, to obtain a projective moduli space we have to consider also torsion free sheaves, and this was done by Gieseker, Maruyama and Simpson [Gi, Ma, Si]. Ramanathan [Ra1, Ra2, Ra3] defined a notion of stability for principal $G$-bundles, and constructed the projective moduli space of semistable principal bundles on a curve.

We equivalently reformulate in terms of filtrations of the associated adjoint bundle of (Killing) orthogonal algebras the Ramanathan's notion of (semi)stability, which is essentially of slope type (negativity of the degree of some associated line bundles), so when we generalize principal bundles to higher dimension by allowing their adjoints to be torsion free sheaves we are able to just switch degrees by Hilbert polynomials as definition of (semi)stability. We then construct a projective coarse moduli space of such semistable principal $G$-sheaves. Our construction proceeds by reductions to intermediate groups, as in [Ra3], although starting the chain higher, namely

*Date*: 21 January 2003.
2000 Mathematical Subject Classification: Primary 14D22, Secondary 14D20.

in a moduli of semistable tensors (as constructed in [G-S1]). In performing these reductions we have switched the technique, in particular studying the non-abelian étale cohomology sets with values in the groups involved, which provides a simpler proof also in Ramanathan's case $\dim X = 1$. However, for the proof of properness we have been able to just generalize the idea of [Ra3].

In order to make more precise these notions and results, let $G' = [G, G]$ be the commutator subgroup, and let $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ be the Lie algebra of $G$, where $\mathfrak{g}'$ is the semisimple part and $\mathfrak{z}$ is the center. As notion of principal $G$-sheaf, it seems natural to consider a rational principal $G$-bundle $P$, i.e. a principal $G$-bundle on an open set $U$ with $\operatorname{codim} X \setminus U \geq 2$, and a torsion free extension of the form $\mathfrak{z}_X \oplus E$, to the whole of $X$, of the vector bundle $P(\mathfrak{g}) = P(\mathfrak{z} \oplus \mathfrak{g}') = \mathfrak{z}_U \oplus P(\mathfrak{g}')$ associated to $P$ by the adjoint representation of $G$ in $\mathfrak{g}$. This clearly amounts to the following

**Definition 0.1.** *A principal $G$-sheaf $\mathcal{P}$ over $X$ is a triple $\mathcal{P} = (P, E, \psi)$ consisting of a torsion free sheaf $E$ on $X$, a principal $G$-bundle $P$ on the open set $U_E$ where $E$ is locally free, and an isomorphism of vector bundles*

$$\psi : P(\mathfrak{g}') \xrightarrow{\cong} E|_{U_E} .$$

Recall that the algebra structure of $\mathfrak{g}'$ given by the Lie bracket provides $\mathfrak{g}'$ an orthogonal (Killing) structure, i.e. $\kappa : \mathfrak{g}' \otimes \mathfrak{g}' \to \mathbb{C}$ inducing an isomorphism $\mathfrak{g}' \cong \mathfrak{g}'^{\vee}$. Correspondingly, the adjoint vector bundle $P(\mathfrak{g}')$ on $U$ has a Lie algebra structure $P(\mathfrak{g}') \otimes P(\mathfrak{g}') \to P(\mathfrak{g}')$ and an orthogonal structure, i.e. $\kappa : P(\mathfrak{g}') \otimes P(\mathfrak{g}') \to \mathcal{O}_U$ inducing an isomorphism $P(\mathfrak{g}') \cong P(\mathfrak{g}')^{\vee}$. In lemma 0.24 it is shown that the Lie algebra structure uniquely extends to a homomorphism

$$[,] : E \otimes E \longrightarrow E^{\vee\vee} ,$$

where we have to take $E^{\vee\vee}$ in the target because an extension $E \otimes E \to E$ does not always exist (so the above definition of a principal $G$-sheaf is equivalent to the one given in our announcement of results [G-S2]). Analogously, the Killing form extends uniquely to

$$\kappa : E \otimes E \longrightarrow \mathcal{O}_X$$

inducing an inclusion $E \hookrightarrow E^{\vee}$. This form assigns an orthogonal $F^{\perp} = \ker(E \hookrightarrow E^{\vee} \twoheadrightarrow F^{\vee})$ to each subsheaf $F \subseteq E$.

**Definition 0.2.** *An orthogonal algebra filtration of $E$ is a filtration*

$$0 \subsetneq E_{-l} \subseteq E_{-l+1} \subseteq \cdots \subseteq E_l = E \qquad (0.1)$$

*with*

$$(1) \quad E_i^{\perp} = E_{-i-1} \quad \text{and} \quad (2) \quad [E_i, E_i] \subseteq E_{i+j}{}^{\vee\vee}$$

*for all $i$, $j$.*

We will see that, if $U$ is an open set with $\operatorname{codim} X \setminus U \geq 2$ such that $E|_U$ is locally free, a reduction of structure group of the principal bundle $P|_U$ to a parabolic subgroup $Q$ together with a dominant character of $Q$ produces a filtration of $E$, and the filtrations arising in this way are precisely the orthogonal algebra filtrations of $E$ (lemmas 5.4 and 5.10). We define the Hilbert polynomial $P_{E_\bullet}$ of a filtration $E_\bullet \subseteq E$ as

$$P_{E_\bullet} = \sum (r P_{E_i} - r_i P_E)$$

where $P_E$, $r$, $P_{E_i}$, $r_i$ always denote the Hilbert polynomials with respect to $\mathcal{O}_X(1)$ and ranks of $E$ and $E_i$. If $P$ is a polynomial, we write $P \prec 0$ if $P(m) < 0$ for

$m \gg 0$, and analogously for "$\preceq$" and "$\leq$". We also use the usual convention: whenever "(semi)stable" and "($\preceq$)" appear in a sentence, two statements should be read: one with "semistable" and "$\preceq$" and another with "stable" and "$\prec$".

**Definition 0.3.** *(See equivalent definition in lemma 0.25) A principal $G$-sheaf $\mathcal{P} = (P, E, \psi)$ is said to be (semi)stable if all orthogonal algebra filtrations $E_\bullet \subset E$ have*

$$P_{E_\bullet} (\preceq) 0$$

In proposition 1.5 we prove that this is equivalent to the condition that the associated tensor

$$(E, \phi : E \otimes E \otimes \wedge^{r-1} E \longrightarrow \mathcal{O}_X)$$

is (semi)stable (in the sense of [G-S1]).

To grasp the meaning of this definition, recall that suppressing conditions (1) and (2) in definitions 0.2 and 0.3 amounts to the (semi)stability of $E$ as a torsion free sheaf, while just requiring condition (1) amounts to the (semi)stability of $E$ as an orthogonal sheaf (cfr. [G-S2]). Now, demanding (1) and (2) is having into account both the orthogonal and the algebra structure of the sheaf $E$, i.e. considering its (semi)stability as orthogonal algebra. By corollary 0.25, this definition coincides with the one given in the announcement of results [G-S2].

Replacing the Hilbert polynomials $P_E$ and $P_{E_i}$ by degrees we obtain the notion of *slope-(semi)stability*, which in section 5 will be shown to be *equivalent to the Ramanathan's notion of (semi)stability* [Ra2, Ra3] *of the rational principal $G$-bundle $P$* (this has been written at the end just to avoid interruption of the main argument of the article, and in fact we refer sometimes to section 5 as a sort of appendix). Clearly

$$\text{slope-stable} \Longrightarrow \text{stable} \Longrightarrow \text{semistable} \Longrightarrow \text{slope-semistable}$$

Since $G/G' \cong \mathbb{C}^{*q}$, given a principal $G$-sheaf, the principal bundle $P(G/G')$ obtained by extension of structure group provides $q$ line bundles on $U$, and since $\operatorname{codim} X \setminus U \geq 2$, these line bundles extend uniquely to line bundles on $X$. Let $d_1, \ldots, d_q \in H^2(X, \mathbb{C})$ be their Chern classes. The rank $r$ of $E$ is clearly the dimension of $\mathfrak{g}'$. Let $c_i$ be the Chern classes of $E$.

**Definition 0.4** (Numerical invariants)**.** *We call the data $\tau = (d_1, \ldots, d_q, c_i)$ the numerical invariants of the principal $G$-sheaf $(P, E, \psi)$.*

**Definition 0.5** (Family of principal $G$-sheaves)**.** *A family of (semi)stable principal $G$-sheaves parametrized by a complex scheme $S$ is a triple $(P_S, E_S, \psi_S)$ where $E_S$ is a torsion free sheaf on $X \times S$, flat over $S$, $P_S$ is a principal $G$-bundle on the open set $U_{E_S}$ where $E_S$ is locally free, and $\psi : P_S(\mathfrak{g}') \to E_S|_{U_{E_S}}$ is an isomorphism of vector bundles.*

*Furthermore, it is asked that for all closed points $s \in S$ the corresponding principal $G$-sheaf is (semi)stable with numerical invariants $\tau$.*

An isomorphism between two such families $(P_S, E_S, \psi_S)$ and $(P'_S, E'_S, \psi'_S)$ is a pair

$$(\beta : P_S \xrightarrow{\cong} P'_S, \gamma : E_S \xrightarrow{\cong} E'_S)$$

such that the following diagram is commutative

$$\begin{array}{ccc} P_S(\mathfrak{g}') & \xrightarrow{\psi} & E_S|_{U_{E_S}} \\ {\scriptstyle \beta(\mathfrak{g}')}\downarrow & & \downarrow{\scriptstyle \gamma|_{U_{E_S}}} \\ P'_S(\mathfrak{g}') & \xrightarrow{\psi'} & E'_S|_{U_{E_S}} \end{array}$$

where $\beta(\mathfrak{g}')$ is the isomorphism of vector bundles induced by $\beta$. Given an $S$-family $\mathcal{P}_S = (P_S, E_S, \psi_S)$ and a morphism $f : S' \to S$, the pullback is defined as $\widetilde{f}^*\mathcal{P}_S = (\widetilde{f}^*P_S, \overline{f}^*E_S, \widetilde{f}^*\psi_S)$, where $\overline{f} = \mathrm{id}_X \times f : X \times S \to X \times S'$ and $\widetilde{f} = i^*(\overline{f}) : U_{\overline{f}^*E_S} \to U_{E_S}$, denoting $i : U_{E_S} \to X \times S$ the inclusion of the open set where $E_S$ is locally free.

**Definition 0.6.** *The functor $\widetilde{F}^\tau_G$ is the sheafification of the functor*

$$F^\tau_G : (\mathrm{Sch}/\mathbb{C}) \longrightarrow (\mathrm{Sets})$$

*sending a complex scheme $S$, locally of finite type, to the set of isomorphism classes of families of semistable principal $G$-sheaves with numerical invariants $\tau$, and it is defined on morphisms as pullback.*

Let $\mathcal{P} = (P, E, \psi)$ be a semistable principal $G$-sheaf on $X$. An orthogonal algebra filtration $E_\bullet$ of $E$ which is *admissible*, i.e. having $P_{E_\bullet} = 0$, provides a reduction $P^Q$ of $P|_U$ to a parabolic subgroup $Q \subset G$ (lemma 5.4) on the open set $U$ where it is a bundle filtration. Let $Q \twoheadrightarrow L$ be its Levi quotient, and $L \hookrightarrow Q \subset G$ a splitting. We call the semistable principal $G$-sheaf

$$\big(P^Q(Q \twoheadrightarrow L \hookrightarrow G), \oplus E_i/E_{i-1}, \psi'\big)$$

the associated *admissible deformation* of $\mathcal{P}$, where $\psi'$ is the natural isomorphism between $P^Q(Q \twoheadrightarrow L \hookrightarrow G)(\mathfrak{g}')$ and $\oplus E_i/E_{i-1}|_U$. This principal $G$-sheaf is semistable. If we iterate this process, it stops after a finite number of steps, i.e. a semistable $G$-sheaf $\mathrm{grad}\,\mathcal{P}$ (only depending on $\mathcal{P}$) is obtained such that all its admissible deformations are isomorphic to itself (cfr. proposition 4.3).

**Definition 0.7.** *Two semistable $G$-sheaves $\mathcal{P}$ and $\mathcal{P}'$ are said $S$-equivalent if* $\mathrm{grad}\,\mathcal{P} \cong \mathrm{grad}\,\mathcal{P}'$.

When $\dim X = 1$ this is just Ramanathan's notion of S-equivalence of semistable principal $G$-bundles. Our main result generalizes Ramanathan's [Ra3] to arbitrary dimension:

**Theorem 0.8.** *For a polarized projective variety $X$ there is a coarse projective moduli space of S-equivalence classes of semistable $G$-sheaves on $X$ with fixed numerical invariants.*

Principal $\mathrm{GL}(R)$-sheaves are not objects equivalent to torsion free sheaves of rank $R$, but only in the case of bundles. As we show in section 6, even in this case, the (semi)stability of both objects do not coincide. The philosophy is that, just as Gieseker changed in the theory of stable vector bundles both the objects (torsion free sheaves instead of vector bundles) and the condition of (semi)stability (involving Hilbert polynomials instead of degrees) in order to make $\dim X$ a parameter of the theory, it is now needed to change again the objects (principal sheaves) and the condition of (semi)stability (as that of the adjoint sheaf of orthogonal algebras) in order to make the group $G$ a parameter of the theory (such variations of the conditions of stability and semistability are in both generalizations very slight, as these are implied by slope stability and imply slope semistability, and the slope conditions do not vary). The deep reason is that what we intend to do is not generalizing the notion of vector bundle of rank $R$ (which was the task of Gieseker and Maruyama), but that of principal $\mathrm{GL}(R)$-bundle, and although both notions happen to be extensionally the same, i.e. happen to define equivalent objects, they are essentially different. This subtle fact is recognized by the very sensitive condition of existence of a moduli space, i.e. by stability.

The results of this article where announced in [G-S2]. There is independent work by Hyeon [Hy], who constructs, for higher dimensional varieties, the moduli space of principal bundles whose associated adjoint is a Mumford stable vector bundle, using the techniques of Ramanathan [Ra3], and also by Schmitt [Sch] who chooses a faithful representation of $G$ in order to obtain and compactify a moduli space of principal $G$-bundles.

**Acknowledgments.** We would like to thank M.S. Narasimhan for suggesting this problem in a conversation with the first author in ICTP (Trieste) in August 1999 and for discussions. We would also like to thank J.M. Ancochea, O. Campoamor, N. Fakhruddin, S. Ilangovan, J. M. Marco, V. Mehta, A. Nair, N. Nitsure, S. Ramanan, T.N. Venkataramana and A. Vistoli for comments and fruitful discussions.

The authors are members of VBAC (Vector Bundles on Algebraic Curves), which is partially supported by EAGER (EC FP5 Contract no. HPRN-CT-2000-00099) and by EDGE (EC FP5 Contract no. HPRN-CT-2000-00101). T.G. was supported by a postdoctoral fellowship of Ministerio de Educación y Cultura (Spain), and wants to thank the Tata Institute of Fundamental Research (Mumbai, India), where this work was done while he was a postdoctoral student. I.S. wants to thank the very warm hospitality of the members of the Institute during his visit to Mumbai.

## Preliminaries

**Notation.** We denote by $(\mathrm{Sch}/\mathbb{C})$ the category of schemes over $\operatorname{Spec}\mathbb{C}$, locally of finite type. All schemes considered will belong to this category. If $f : Y \to Y'$ is a morphism, we denote $\overline{f} = \mathrm{id}_X \times f : X \times Y \to X \times Y'$. If $E_S$ is a coherent sheaf on $X \times S$, we denote $E_S(m) := E_S \otimes p_X^* \mathcal{O}_X(m)$. An open set $U \subseteq Y$ of a scheme $Y$ will be called *big* if $\operatorname{codim} Y \setminus U \geq 2$. Recall that in the étale topology, a cover of a scheme $U$ is a finite collection of morphisms $\{f_i : U_i \to U\}_{i \in I}$ such that each $f_i$ is étale, and $U$ is the union of the images of the $f_i$.

Given a principal $G$-bundle $P \to Y$ and a left action $\sigma$ of $G$ in a scheme $F$, we denote

$$P(\sigma, F) \;:=\; P \times_G F \;=\; (P \times F)/G,$$

the associated fiber bundle. If the action $\sigma$ is clear from the context, we will write $P(F)$. If $\rho : G \to H$ is a group homomorphism, let $\sigma$ be the action of $G$ on $H$ defined by left multiplication $h \mapsto \rho(g)h$. Then the associated fiber bundle is a principal $G'$-bundle, and it is denoted $\rho_* P$. If $\sigma$ is a character of the group, we will denote by $P(\sigma)$ the corresponding line bundle.

Let $\rho : H \to G$ be a homomorphism of groups, and let $P$ be a principal $G$-bundle on a scheme $Y$. A reduction of structure group of $P$ to $H$ is a pair $(P^H, \zeta)$, where $P^H$ is a principal $H$-bundle on $Y$ and $\zeta$ is an isomorphism between $\rho_* P^H$ and $P$.

Two reductions $(P^H_T,\zeta_T)$ and $(Q^H_T,\theta_T)$ are isomorphic if there is an isomorphism $\alpha$ giving a commutative diagram

$$
(0.2)\qquad
\begin{array}{ccccc}
P^H_T & \quad & \rho_* P^H_T & \xrightarrow{\zeta_T} & P_T \\
\cong\downarrow\alpha & & \downarrow \rho_*\alpha & & \| \\
Q^H_T & & \rho_* Q^H_T & \xrightarrow{\theta_T} & P_T
\end{array}
$$

Let $p:Y\to S$ be a morphism of schemes, and let $P_S$ be a principal $G$-bundle on the scheme $Y$. Define the functor of families of reductions

$$
\begin{array}{rcl}
\Gamma(\rho,P_S):(\mathrm{Sch}/S) & \longrightarrow & (\mathrm{Sets}) \\
(t:T\longrightarrow S) & \longmapsto & \big\{(P^H_T,\zeta_T)\big\}/\text{isomorphism}
\end{array}
$$

where $(P^H_T,\zeta_T)$ is a reduction of structure group of $P_T:=P_S\times_S T$ to $H$.

If $\rho$ is injective, then $\Gamma(\rho,P_S)$ is a sheaf, and it is in fact representable by a scheme $S'\to S$, locally of finite type [Ra3, lemma 4.8.1]. If $\rho$ is not injective, this functor is not necessarily a sheaf, so we denote by $\widetilde{\Gamma}(\rho,P_S)$ its sheafification with respect to the étale topology on $(\mathrm{Sch}\,/S)$.

**Lemma 0.9.** *Let $Y$ be a scheme, and let $f:\mathcal{K}\to\mathcal{F}$ be a homomorphism of sheaves on $X\times Y$. Assume that $\mathcal{F}$ is flat over $Y$. Then there is a unique closed subscheme $Z$ satisfying the following universal property: given a Cartesian diagram*

$$
\begin{array}{ccc}
X\times S & \xrightarrow{\overline{h}} & X\times Y \\
\downarrow p_S & & \downarrow p \\
S & \xrightarrow{h} & Y
\end{array}
$$

*$\overline{h}^*f=0$ if and only if $h$ factors through $Z$.*

*Proof.* Uniqueness is clear. To show existence, assume that $\mathcal{O}_X(1)$ is very ample (taking a multiple if necessary). Recall that if $\mathcal{G}$ is a coherent sheaf on $X\times Y$, we denote $\mathcal{G}(m)=\mathcal{G}\otimes p_X^*\mathcal{O}_X(m)$. Since $\mathcal{F}$ is $Y$-flat, taking $m'$ large enough, $p_*\mathcal{F}(m')$ is locally free. The question is local on $Y$, so we can assume, shrinking $Y$ if necessary, that $Y=\operatorname{Spec}A$ and $p_*\mathcal{F}(m')$ is given by a free $A$-module. Now, since $Y$ is affine, the homomorphism

$$p_*f(m'):p_*\mathcal{K}(m')\longrightarrow p_*\mathcal{F}(m')$$

of sheaves on $Y$ is equivalent to a homomorphism of $A$-modules

$$M\overset{(f_1,\dots,f_n)}{\longrightarrow}A\oplus\cdots\oplus A$$

The zero locus of $f_i$ is defined by the ideal $I_i\subset A$ image of $f_i$, thus the zero scheme of $(f_1,\dots,f_n)$ is given by the ideal $I=\sum I_i$, hence $Z'_{m'}$ is a closed subscheme.

Since $\mathcal{O}_X(1)$ is very ample, if $m''>m'$ we have an injection $p_*\mathcal{F}(m')\hookrightarrow p_*\mathcal{F}(m'')$ (and analogously for $\mathcal{K}$), hence $Z_{m''}\subset Z_{m'}$, and since $Y$ is noetherian, there exists $N'$ such that, if $m'>N'$, we get a scheme $Z$ independent of $m'$.

To check the universal property first we will show that if $\overline{h}^*f=0$ then $h$ factors through $Z$. Since the question is local on $S$, we can take $S=\operatorname{Spec}(B)$, $Y=\operatorname{Spec}(A)$, and the morphism $h$ is locally given by a ring homomorphism $A\to B$. Since $\mathcal{F}$ is flat over $Y$, for $m'$ large enough the natural homomorphism $\alpha:h^*p_*\mathcal{F}(m')\to p_{S*}\overline{h}^*\mathcal{F}(m')$ (defined as in [Ha, Th. III 9.3.1]) is an isomorphism. Indeed, for $m'$

sufficiently large, $H^i(X, \mathcal{F}_y(m')) = 0$ and $H^i(X, \overline{h}^*(\mathcal{F}(m'))_s) = 0$ for all points $y \in Y$, $s \in S$ and $i > 0$, and since $\mathcal{F}$ is flat, this implies that $h^* p_* \mathcal{F}(m')$ and $p_{S*}\overline{h}^*\mathcal{F}(m')$ are locally free. Then, to prove that the homomorphism $\alpha$ is an isomorphism, it is enough to check it at the fiber of every $s \in S$, but this follows from [Ha, Th. III 12.11] or [Mu2, II §5 Cor. 3].

Hence the commutativity of the diagram

$$\begin{array}{ccc} p_{S*}\overline{h}^*\mathcal{K}(m') & \xrightarrow{p_{S*}\overline{h}^* f(m')=0} & p_{S*}\overline{h}^*\mathcal{F}(m') \\ \uparrow & & \uparrow \cong \\ h^*p_*\mathcal{K}(m') & \xrightarrow{h^*p_*f(m')} & h^*p_*\mathcal{F}(m') \end{array}$$

implies that $h^*p_*f(m') = 0$. This means that for all $i$, in the diagram

$$\begin{array}{ccccccc} M & \xrightarrow{f_i} & A & \longrightarrow & A/I_i & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ M \otimes_A B & \xrightarrow{f_i \otimes B} & B & \longrightarrow & A/I_i \otimes_A B & \longrightarrow & 0 \end{array}$$

it is $f_i \otimes B = 0$. Hence the image $I_i$ of $f_i$ is in the kernel $J$ of $A \to B$. Therefore $I \subset J$, hence $A \to B$ factors through $A \to A/I$, which means that $h : S \to Y$ factors through $Z$.

Now we show that if we take $S = Z$ and $h : Z \hookrightarrow Y$ the inclusion, then $\overline{h}^* f = 0$. By definition of $Z$ we have $h^*p_*f(m') = 0$ for any $m'$ with $m' > N'$. Showing that $\overline{h}^* f = 0$ is equivalent to showing that

$$\overline{h}^* f(m') : \overline{h}^*\mathcal{K}(m') \longrightarrow \overline{h}^*\mathcal{F}(m')$$

is zero for some $m'$. Take $m'$ large enough so that $\mathrm{ev} : p^*p_*\mathcal{K}(m') \to \mathcal{K}(m')$ is surjective. By the right exactness of $\overline{h}^*$ the homomorphism $\overline{h}^*\mathrm{ev}$ is still surjective. The commutative diagram

$$\begin{array}{ccc} \overline{h}^*\mathcal{K}(m') & \xrightarrow{\overline{h}^* f(m')} & \overline{h}^*\mathcal{F}(m') \\ \overline{h}^*\mathrm{ev} \twoheaduparrow & & \uparrow \\ \overline{h}^*p^*p_*\mathcal{K}(m') & \xrightarrow{\overline{h}^*p^*p_*f(m')} & \overline{h}^*p^*p_*\mathcal{F}(m') \\ \| & & \| \\ p_S^*h^*p_*\mathcal{K}(m') & \xrightarrow{p_S^*h^*p_*f(m')=0} & p_S^*h^*p_*\mathcal{F}(m') \end{array}$$

implies $\overline{h}^* f(m') = 0$, hence $\overline{h}^* f = 0$.

□

The following two lemmas and corollary will help to relate the three main objects that will be introduced in this section.

**Lemma 0.10.** *Let $E$ and $F$ be coherent sheaves on a scheme $Y$, and $L$ a locally free sheaf on $Y$. There is a natural isomorphism*

$$\mathrm{Hom}(E \otimes F, L) \cong \mathrm{Hom}(E, \mathcal{H}om(F, L)) \cong \mathrm{Hom}(E, F^\vee \otimes L)$$

*Proof.* Given a homomorphism $\varphi : E\otimes F \to L$, we define $\psi : E \to \mathcal{H}om(F,L)$ by sending a local section $e$ of $E$ to $\varphi(e,\cdot)$. Conversely, to a homomorphism $\psi : E \to \mathcal{H}om(F,L)$, we associate the homomorphism

$$E\otimes F \stackrel{\psi\otimes F}{\longrightarrow} \mathcal{H}om(F,L)\otimes F \longrightarrow L$$

where the second map is the natural pairing. It is easy to check that both constructions are inverse to each other. Finally, since $L$ is locally free, $\mathcal{H}\mathrm{om}(F,L) = F^\vee \otimes L$. □

**Lemma 0.11.** *Let $E$ be a torsion free sheaf of rank $r$ on a scheme $Y$. Then there is a canonical isomorphism*

$$E^{\vee\vee} \cong (\bigwedge{}^{r-1}E)^\vee \otimes \det E\,.$$

*Proof.* This isomorphism is obvious if we restrict to the open set $U_E$ where $E$ is locally free. Since both sheaves are reflexive and $\operatorname{codim} X\setminus U_E \geq 2$, it uniquely extends to an isomorphism on the whole of $X$ ([Ha2, Prop. 1.6(iii)]). □

Combining lemmas 0.10 and 0.11 we obtain the following

**Corollary 0.12.** *Let $E$ be a torsion free sheaf of rank $r$ on a scheme $Y$, and $L$ a line bundle on $Y$. Then, giving a homomorphism*

$$\varphi : E\otimes E \longrightarrow E^{\vee\vee}\otimes L$$

*is equivalent to giving a homomorphism*

$$\phi : E\otimes E\otimes E^{\otimes r-1} \;= E^{\otimes r+1} \longrightarrow \det E\otimes L$$

*which is skew-symmetric in the last $r-1$ entries, i.e. which induces a homomorphism on $E\otimes E\otimes \bigwedge^{r-1}E$.*

Now we introduce the progressively richer concepts of tensor, $\mathfrak{g}'$-sheaf, and principal $G$-sheaf, defining them relative to a scheme $S$. As usual, if no mention to the base scheme $S$ is made, it will be understood $S = \operatorname{Spec}\mathbb{C}$. For each of these three concepts we give compatible notions of (semi)stability, leading in each case to a projective coarse moduli space.

**Definition 0.13** (Tensor)**.** *A family of tensors parametrized by a scheme $S$ is a triple $(E_S,\phi_S,N)$ consisting of a torsion free sheaf $E_S$ on $X\times S$, flat over $S$, which restricts to a torsion free sheaf with trivial determinant and fixed Hilbert polynomial $P$ on every slice $X\times s$, a line bundle $N$ on $S$ and a homomorphism $\phi_S$*

$$\phi_S : {E_S}^{\otimes a} \;\longrightarrow p_S^* N, \tag{0.3}$$

*A tensor is called a Lie tensor if $a = r+1$ for $r$ the rank of $E_S$, and*

1. *$\phi_S$ is skew-symmetric in the last $r-1$ entries, i.e. it induces a homomorphism on $E_S\otimes E_S\otimes \bigwedge^{r-1}E_S$,*
2. *the homomorphism $\varphi_S : E_S\otimes E_S \to E_S^{\vee\vee}\otimes N$ associated to $\phi_S$ by corollary 0.12 is antisymmetric,*
3. *$\varphi_S$ satisfies the Jacobi identity*

To give a precise definition of the Jacobi identity, first define a homomorphism

$$[[\cdot,\cdot],\cdot] : E_S\otimes E_S\otimes E_S \stackrel{\varphi_S\otimes E_S}{\longrightarrow} E_S^{\vee\vee}\otimes N\otimes E_S \stackrel{E_S^{\vee\vee}\otimes N\otimes\varphi_S}{\longrightarrow} E_S^{\vee\vee}\otimes E_S^{\vee}\otimes E_S^{\vee\vee}\otimes N^2 \longrightarrow E_S^{\vee\vee}\otimes N^2$$

where the last map comes from the natural pairing of the first two factors. Then define

$$(0.4)\qquad \begin{array}{rcl} J : E_S \otimes E_S \otimes E_S & \longrightarrow & E_S^{\vee\vee} \otimes N^2 \\ (u,v,w) & \longmapsto & [[u,v],w] + [[v,w],u] + [[w,u],v] \end{array}$$

and require $J = 0$.

An isomorphism between two families of tensors $(E_S, \phi_S, N)$ and $(E'_S, \phi'_S, N')$ parametrized by $S$ is a pair of isomorphisms $\alpha : E_S \to E'_S$ and $\beta : N \to N'$ such that the induced diagram

$$\begin{array}{ccc} {E_S}^{\otimes r+1} & \xrightarrow{\phi_S} & p_S^* N \\ \big\downarrow \alpha^{\otimes r+1} & & \big\downarrow \beta \\ {E'_S}^{\otimes r+1} & \xrightarrow{\phi'_S} & p_S^* N' \end{array}$$

commutes. In particular, $(E, \phi)$ and $(E, \lambda\phi)$ are isomorphic for $\lambda \in \mathbb{C}^*$. Given an $S$-family of tensors $(E_S, \phi_S, N)$ and a morphism $F : S' \to S$, the pullback is the $S'$-family defined as $(\widetilde{f}^* E_S, \widetilde{f}^* \phi_S, f^* N_S)$.

Since we will work with GIT (Geometric invariant theory, [Mu1]), the notion of filtration $E_\bullet$ of a sheaf is going to be essential for us. By this we always understand a $\mathbb{Z}$-indexed filtration

$$\ldots \subseteq E_{i-1} \subseteq E_i \subseteq E_{i+1} \subseteq \ldots$$

starting with 0 and ending with $E$. Of course, only a finite number of inclusions can be strict

$$0 \subsetneq E_{\lambda_1} \subsetneq E_{\lambda_2} \subsetneq \cdots \subsetneq E_{\lambda_t} \subsetneq E_{\lambda_{t+1}} = E \qquad \lambda_1 < \cdots < \lambda_{t+1}$$

where we have deleted, from 0 onward, all the non-strict inclusions. Reciprocally, from $E_{\lambda_\bullet}$ we recover $E_\bullet$ by defining $E_m = E_{\lambda_{i(m)}}$, *where* $i(m)$ *is the maximum index with* $\lambda_{i(m)} \leq m$.

**Definition 0.14** (Balanced filtration)**.** *A filtration* $E_\bullet \subseteq E$ *of a torsion free sheaf* $E$ *is called balanced if* $\sum i \operatorname{rk} E^i = 0$ *for* $E^i = E_i/E_{i-1}$*. In terms of* $E_{\lambda_\bullet}$*, this is* $\sum_{i=1}^{t+1} \lambda_i \operatorname{rk}(E^{\lambda_i}) = 0$ *for* $E^{\lambda_i} = E_{\lambda_i}/E_{\lambda_{i-1}}$*.*

**Remark 0.15.** The notion of balanced filtration appeared naturally in the Gieseker-Maruyama construction of the moduli space of (semi)stable sheaves, the condition of (semi)stability of a sheaf $E$ being that all balanced filtrations of $E$ have negative (nonpositive) Hilbert polynomial. In this case the condition "balanced" can be suppressed, since $P_{E_\bullet} = P_{E_{\bullet+l}}$ for any shift $l$ in the indexing (and furthermore it is enough to consider filtrations of one element, i.e. just subsheaves).

Let $\mathcal{I}_a = \{1, \ldots, t+1\}^{\times a}$ be the set of all multi-indexes $I = (i_1, \ldots, i_a)$ of cardinality $a$. Define

$$(0.5)\qquad \mu_{\mathrm{tens}}(\phi, E_{\lambda_\bullet}) = \min_{I \in \mathcal{I}_a} \left\{ \lambda_{i_1} + \cdots + \lambda_{i_a} : \phi|_{E_{\lambda_{i_1}} \otimes \cdots \otimes E_{\lambda_{i_a}}} \neq 0 \right\}$$

**Definition 0.16** (Stability for tensors)**.** *Let* $\delta$ *be a polynomial of degree at most* $n-1$ *and positive leading coefficient. We say that* $(E, \phi)$ *is* $\delta$*-(semi)stable if* $\phi$ *is not identically zero and for all balanced filtrations* $E_{\lambda_\bullet}$ *of* $E$*, it is*

$$(0.6)\qquad \Big( \sum_{i=1}^{t} (\lambda_{i+1} - \lambda_i) \big( r P_{E_{\lambda_i}} - r_{\lambda_i} P \big) \Big) + \mu_{\mathrm{tens}}(\phi, E_{\lambda_\bullet})\, \delta \;(\preceq)\; 0$$

In this definition, *it suffices to consider saturated filtrations*, which means that the sheaves $E^i = E_i/E_{i-1}$ (i.e. the $E^{\lambda_i} = E_{\lambda_i}/E_{\lambda_{i-1}}$) are all torsion free, and thus with $\mathrm{rk}(E_{\lambda_i}) < \mathrm{rk}(E_{\lambda_{i+1}})$ for all $i$. There is a coarse moduli space of semistable tensors [G-S1].

Now we go to our second main concept, that of a $\mathfrak{g}'$-sheaf. A family of Lie algebra sheaves parametrized by $S$ is a pair

$$\big(E_S,\ \varphi_S : E_S \otimes E_S \to E_S^{\vee\vee}\big),$$

where $E_S$ is a torsion free sheaf on $X \times S$, flat over $S$, and $\varphi_S$ is a homomorphism such that

(1) $\varphi_S : E_S \otimes E_S \longrightarrow E_S^{\vee\vee}$ is antisymmetric
(2) $\varphi_S$ satisfies the Jacobi identity

The precise definition of the Jacobi identity is as in definition 0.13, but with $\mathcal{O}_{X\times S}$ instead of $N$. An isomorphism between two families is an isomorphism $\alpha : E_S \to E'_S$ with

$$\begin{array}{ccc} E_S \otimes E_S & \xrightarrow{\varphi_S} & E_S^{\vee\vee} \\ \Big\downarrow{\scriptstyle \alpha\otimes\alpha} & & \Big\downarrow{\scriptstyle \alpha^{\vee\vee}} \\ E'_S \otimes E'_S & \xrightarrow{\varphi'_S} & E_S'^{\vee\vee} \end{array}$$

Note that since conditions (1) and (2) above are closed, it is not enough to check that they are satisfied for all closed points of $S$, because $S$ could be nonreduced.

**Definition 0.17.** *The Killing form $\kappa_S$ associated to a Lie algebra sheaf $(E_S, \varphi_S)$ is the composition*

$$E_S \otimes E_S \xrightarrow{[,]\otimes[,]} (E_S^\vee \otimes E_S^{\vee\vee}) \otimes (E_S^\vee \otimes E_S^{\vee\vee}) \longrightarrow (E_S^\vee \otimes E_S^{\vee\vee}) \longrightarrow \mathcal{O}_{X\times S}$$

If the Lie algebra is semisimple, in the sense that the induced homomorphism $E_S^{\vee\vee} \to E_S^\vee$ is an isomorphism, the fiber of $E_S$ over a point $(x,s) \in X \times S$ where $E_S$ is locally free has the structure of a semisimple Lie algebra, which, because of the rigidity of semisimple Lie algebras, must be constant on connected components of $S$. This justifies the following

**Definition 0.18** ($\mathfrak{g}'$-sheaf)**.** *A family of $\mathfrak{g}'$-sheaves is a family of Lie algebra sheaves where the Lie algebra associated to each connected component of the parameter space $S$ is $\mathfrak{g}'$.*

The following is the sheaf version of the well known notion of Lie algebra filtration (see [J] for instance, recalled in section 5).

**Definition 0.19** (Algebra filtration)**.** *A filtration $E_\bullet \subseteq E$ of a Lie algebra sheaf $(E, [,])$ is called an algebra filtration if for all $i$, $j$,*

$$[E_i, E_j] \subseteq {E_{i+j}}^{\vee\vee}.$$

*In terms of $E_{\lambda_\bullet}$, this is*

$$[E_{\lambda_i}, E_{\lambda_j}] \subseteq {E_{\lambda_{k-1}}}^{\vee\vee}$$

*for all $\lambda_i$, $\lambda_j$, $\lambda_k$ with $\lambda_i + \lambda_j < \lambda_k$,*

**Definition 0.20.** *A $\mathfrak{g}'$-sheaf is (semi)stable if for all balanced algebra filtrations $E_\bullet$ it is*

$$\sum_{i=1}^{t} \big(rP_{E_i} - r_i P_E\big) \ (\preceq)\ 0$$

*or, in terms of $E_{\lambda_\bullet}$,*

$$\sum_{i=1}^{t} (\lambda_{i+1} - \lambda_i)\big(rP_{E_{\lambda_i}} - r_{\lambda_i} P_E\big) \ (\preceq)\ 0 \tag{0.7}$$

As usual, in definition 0.20 it is enough to consider saturated filtrations.

**Remark 0.21.** We will see in corollary 5.10 that for an algebra filtration of a $\mathfrak{g}'$-sheaf, the fact of being balanced and saturated is equivalent to being orthogonal, i.e. $E_{-i-1} = E_i^{\perp} = \ker(E \hookrightarrow E^{\vee\vee} \overset{\kappa}{\cong} E^{\vee} \to E_i^{\vee})$. Thus, in the previous definition we can change "balanced algebra filtration" by "orthogonal algebra filtration".

**Remark 0.22.** Observe that the condition "balanced" cannot be suppressed in this case, is it was in remark 0.15, because a shifted filtration $E_{\bullet+l}$ of an algebra filtration is no longer an algebra filtration.

**Construction 0.23** (Correspondence between Lie algebra sheaves and Lie tensors)**.** Consider a family of Lie tensors

$$(F_S, \phi_S : F_S{}^{\otimes r+1} \longrightarrow p_S^* N, N)$$

Corollary 0.12 gives

$$(F_S, \phi'_S : F_S \otimes F_S \to F_S^{\vee\vee} \otimes (\det F_S)^{-1} \otimes p_S^* N, N)$$

If we tensor $\phi'_S$ with $(\det F_S)^2 \otimes p_S^* N^{-2}$ and define $E_S = F_S \otimes \det F_S \otimes p_S^* N^{-1}$, we obtain a Lie algebra sheaf

$$(E_S, \varphi_S : E_S \otimes E_S \to E_S^{\vee\vee}) \tag{0.8}$$

such that for all $s \in S$ and $x \in U_{E_s}$, $\varphi_s(x)$ is a Lie algebra structure on the fiber $E_s(x)$.

Conversely, given a Lie algebra sheaf as in (0.8), corollary 0.12 gives a homomorphism

$$\phi_S : E_S{}^{\otimes r+1} \longrightarrow \det E_S\,.$$

Recall that we always assume that $E_S = p^* L$, where $L$ is a line bundle on the base scheme $S$, hence this is a Lie tensor.

If $S = \operatorname{Spec} \mathbb{C}$, this gives a bijection of isomorphism classes, but not for arbitrary $S$, because $E_S$ is not in general isomorphic to $F_S$. They are only locally isomorphic, in the sense that we can cover $S$ with open sets $S_i$ (where the line bundles $L$ and $N$ are trivial), so that the objects restricted to $S_i$ are isomorphic, and this suffices to provide an *isomorphism between the sheafified functors. We will show that, for $\mathfrak{g}'$-sheaves, its (semi)stability is equivalent to that of the corresponding tensor, hence there is a projective moduli space of $\mathfrak{g}'$-sheaves.* This is the key initial point of this article, allowing us to use in section 1 the results in [G-S1] to construct the moduli space of $\mathfrak{g}'$-sheaves.

Recall now, from the introduction, the notion of a principal $G$-sheaf $\mathcal{P} = (P_S, E_S, \psi_S)$ for a reductive connected group $G$ and its notion of (semi)stability. Let $\mathfrak{g}'$ be the semisimple part of its Lie algebra. We associate now to $\mathcal{P}$ a $\mathfrak{g}'$-sheaf $(E_S, \varphi_S)$ by the following

**Lemma 0.24.** *let $\mathcal{U} = U_{E_S}$ be the open set where $E_S$ is locally free. The homomorphism $\varphi_{\mathcal{U}} : E_S|_{\mathcal{U}} \otimes E_S|_{\mathcal{U}} \to E_S|_{\mathcal{U}}$, given by the Lie algebra structure of $P_S(\mathfrak{g}')$ and the isomorphism $\psi_S$, extends uniquely to a homomorphism*

$$\varphi_S : E_S \otimes E_S \longrightarrow E_S^{\vee\vee},$$

*Proof.* The homomorphism $\varphi_{\mathcal{U}}$ can be seen as a section of

$$E_S|_{\mathcal{U}}^{\vee} \otimes E_S|_{\mathcal{U}}^{\vee} \otimes E_S|_{\mathcal{U}} \cong (E_S|_{\mathcal{U}} \otimes E_S|_{\mathcal{U}} \otimes E_S|_{\mathcal{U}}^{\vee})^{\vee} \cong (E_S \otimes E_S \otimes E_S^{\vee})^{\vee}|_{\mathcal{U}},$$

Since $(E_S \otimes E_S \otimes E_S^{\vee})^{\vee}$ is a reflexive sheaf on $X \times S$, this section extends uniquely to an element of

$$\Gamma(X \times S, (E_S \otimes E_S \otimes E_S^{\vee})^{\vee}) = \mathrm{Hom}(E_S \otimes E_S \otimes E_S^{\vee}, \mathcal{O}_{X\times S}) = \mathrm{Hom}(E_S \otimes E_S, E_S^{\vee\vee}),$$

where the two equalities follow from corollary 0.12, and this element is the extended homomorphism $\varphi_S$. □

The following corollary of remark 0.21 provides an equivalent definition of (semi)stability

**Corollary 0.25.** *A principal $G$-sheaf $\mathcal{P} = (P, E, \psi)$ is (semi)stable (definition 0.3) if and only if the associated $\mathfrak{g}'$-sheaf $(E, \varphi)$ is (semi)stable (definition 0.20).*

**Remark 0.26.** Lemma 0.24 implies that there is a natural bijection between the isomorphism classes of families of $\mathfrak{g}'$-sheaves and those of principal $\mathrm{Aut}(\mathfrak{g}')$-sheaves.

**Lemma 0.27.** *Let $G$ be a connected reductive algebraic group. Let $P$ be a principal $G$-bundle on $X$ and let $E = P(\mathfrak{g}')$ be the vector bundle associated to $P$ by the adjoint representation of $G$ on the semisimple part of its Lie algebra $\mathfrak{g}'$. Then $\det E \cong \mathcal{O}_X$.*

*Proof.* We have $\mathrm{Aut}(\mathfrak{g}') \subset \mathrm{O}(\mathfrak{g}')$, where the orthogonal structure on $\mathfrak{g}'$ is given by its nondegenerate Killing form. Note that $P(\mathfrak{g}')$ is obtained by extension of structure group using the composition

$$\rho : G \longrightarrow \mathrm{Aut}(\mathfrak{g}') \hookrightarrow \mathrm{O}(\mathfrak{g}') \hookrightarrow \mathrm{GL}(\mathfrak{g}').$$

Since $G$ is connected, the image of $G$ in $\mathrm{O}(\mathfrak{g}')$ lies in the connected component of identity, i.e. in $\mathrm{SO}(\mathfrak{g}')$. Hence $P(\mathfrak{g}')$ admits a reduction of structure group to $\mathrm{SO}(\mathfrak{g}')$, and thus $\det P(\mathfrak{g}') \cong \mathcal{O}_X$. □

We end this section by extending to principal sheaves some well known definitions and properties of principal bundles and by recalling some notions of GIT [Mu1]. Let $m : H \times R \to R$ be an action of an algebraic group $H$ on a scheme $R$. Let $p_R : H \times R \to R$ be the projection to the second factor.

**Definition 0.28** (Universal family)**.** *Let $\mathcal{P}_R$ be a family of principal $G$-sheaves parametrized by $R$. Assume there is a lifting of the action of $H$ to $\mathcal{P}_R$, i.e. there is an isomorphism*

$$\Lambda : \overline{m}^* \mathcal{P}_R \xrightarrow{\cong} \overline{p}_R^* \mathcal{P}_R$$

*Assume that*

1. *Given a family $\mathcal{P}_S$ parametrized by $S$ and a closed point $s \in S$, there is an open étale neighborhood $i : S_0 \hookrightarrow S$ of $s$ and a morphism $t : S_0 \to R$ such that $\overline{i}^* \mathcal{P}_S \cong \overline{t}^* \mathcal{P}_R$.*
2. *Given two morphisms $t_1$, $t_2 : S \to R$ and an isomorphism $\beta : \overline{t_2}^* \mathcal{P} \to \overline{t_1}^* \mathcal{P}$, there is a unique $h : S \to H$ such that $t_2 = h[t_1]$ and $\overline{(h \times t_1)}^* \Lambda = \beta$.*

*Then we say that $\mathcal{P}_R$ is a universal family with group $H$ for the functor $\widetilde{F}^{\tau}_G$.*

**Definition 0.29** (Universal space)**.** *Let $F : (\mathrm{Sch}/\mathbb{C}) \to (\mathrm{Sets})$ be a functor. Let $\underline{R}/\underline{H}$ be the sheaf on $(\mathrm{Sch}/\mathbb{C})$ associated to the presheaf $S \mapsto \mathrm{Mor}(S,R)/\mathrm{Mor}(S,H)$. We say that $R$ is a universal space with group $H$ for the functor $F$ if the sheaf $F$ is isomorphic to $\underline{R}/\underline{H}$.*

The difference between these two notions can be understood as follows. Given a stack $\mathcal{M} : (\mathrm{Sch}/\mathbb{C}) \to (\mathrm{Groupoids})$, we denote by $\overline{\mathcal{M}} : (\mathrm{Sch}/\mathbb{C}) \to (\mathrm{Sets})$ the functor associated by taking the set of isomorphism classes of each groupoid. Let $[R/H]$ be the quotient stack and let $\mathcal{F}$ be the stack of semistable principal $G$-sheaves. Then $R$ is a universal space with group $H$ if $\overline{[R/H]} \cong \overline{\mathcal{F}}$, whereas it is a universal family if $[R/H] \cong \mathcal{F}$, i.e. if the isomorphism holds at the level of stacks, without taking isomorphism classes.

**Definition 0.30** (Categorical quotient)**.** *A morphism $f : R \to Y$ of schemes is a categorical quotient for an action of an algebraic group $H$ on $R$ if*

(1) *It is $H$-equivariant when we provide $Y$ with the trivial action.*
(2) *If $f' : R \longrightarrow Y'$ is another morphism satisfying (1), then there is a unique morphism $g : Y \to Y'$ such that $f' = g \circ f$.*

**Definition 0.31** (Good quotient)**.** *A morphism $f : R \to Y$ of schemes is a good quotient for an action of an algebraic group $H$ on $R$ if*

(1) *$f$ is surjective, affine and $H$-equivariant, when we provide $Y$ with the trivial action.*
(2) *$f_*(\mathcal{O}_R^H) = \mathcal{O}_Y$, where $\mathcal{O}_R^H$ is the sheaf of $H$-invariant functions on $R$.*
(3) *If $Z$ is a closed $H$-invariant subset of $R$, then $p(Z)$ is closed in $Y$. Furthermore, if $Z_1$ and $Z_2$ are two closed $H$-invariant subsets of $R$ with $Z_1 \cap Z_2 = \emptyset$, then $f(Z_1) \cap f(Z_2) = \emptyset$*

**Definition 0.32** (Geometric quotient)**.** *A geometric quotient $f : R \to Y$ is a good quotient such that $f(x_1) = f(x_2)$ if and only if the orbit of $x_1$ is equal to the orbit of $x_2$.*

Clearly, geometric quotients are good quotients, and these are categorical quotients. Assume that $R$ is projective, and the action of $H$ on $R$ has a linearization on an ample line bundle $\mathcal{O}_R(1)$. A closed point $y \in R$ is called GIT-semistable if, for some $m > 0$, there is an $H$-invariant section $s$ of $\mathcal{O}_R(m)$ such that $s(y) \neq 0$. If, moreover, every orbit of $H$ in $R_s = \{x \in R | s(x) \neq 0\}$ is closed and of the same dimension as $H$, then $y$ is called a GIT-stable point. We will use the following characterization in [Mu1] of GIT-(semi)stability: let $\lambda : \mathbb{C}^* \to H$ be a one-parameter subgroup, and $y \in R$. Then $\lim_{t\to 0} \lambda(t) \cdot y = y_0$ exists, and $y_0$ is fixed by $\lambda$. Let $t \mapsto t^a$ be the character by which $\lambda$ acts on the fiber of $\mathcal{O}_R(1)$. Defining $\mu(y,\lambda) = a$, Mumford proves that $y$ is GIT-(semi)stable if and only if, for all one-parameter subgroups, it is $\mu(y,\lambda)(\leq)0$.

**Proposition 0.33.** *Let $R^{ss}$ (respectively $R^s$) be the open subset of GIT-semistable points (respectively GIT-stable). Then there is a good quotient $R^{ss} \to R/\!\!/H$, and the restriction $R^s \to R^s/\!\!/H$ is a geometric quotient. Furthermore, $R/\!\!/H$ is projective and $R^s/\!\!/H$ is an open subset.*

**Definition 0.34.** *A scheme $Y$ corepresents a functor $F : (\mathrm{Sch}/\mathbb{C}) \to (\mathrm{Sets})$ if*

(1) *There exists a natural transformation $f : F \to \underline{Y}$ (where $\underline{Y} = \mathrm{Mor}(\cdot, Y)$ is the functor of points represented by $Y$).*

(2) *For every scheme $Y'$ and natural transformation $f' : F \to \underline{Y'}$, there exists a unique $g : \underline{Y} \to \underline{Y'}$ such that $f'$ factors through $f$.*

**Remark 0.35.** Let $R$ be a universal space with group $H$ for $F$, and let $f : R \to Y$ be a categorical quotient. It follows from the definitions that $Y$ corepresents $F$.

**Proposition 0.36.** *Let $\mathcal{P}_R = (P_R, E_R, \psi_R)$ be a universal family with group $H$ for the functor $\widetilde{F}^{\tau}_{G_1}$. Let $\rho : G_2 \to G_1$ be a homomorphism of groups, such that the center $Z_{G_2}$ of $G_2$ is applied to the center $Z_{G_1}$ of $G_1$ and the induced homomorphism*

$$\mathrm{Lie}(G_2/Z_{G_2}) \longrightarrow \mathrm{Lie}(G_1/Z_{G_1})$$

*is an isomorphism. Assume that the functor $\widetilde{\Gamma}(\rho, P_R)$ is represented by a scheme $M$. Then*

(1) *There is a natural action of $H$ on $M$, making it a universal space with group $M$ for the functor $\widetilde{F}^{\tau}_{G_2}$.*
(2) *Moreover, if $\rho$ is injective (so that $\Gamma(\rho, P_R)$ itself is representable by $M$), then the action of $H$ lifts to the family $\mathcal{P}_M$ given by $\Gamma(\rho, P_R)$, and then $\mathcal{P}_M$ becomes a universal family with group $H$ for the functor $\widetilde{F}^{\tau}_{G_2}$*

*Proof.* Analogous to [Ra3, lemma 4.10]. □

## 1. Construction of $R$ and $R_1$

Given a principal $G$-bundle, we obtain a pair $(E, \varphi : E \otimes E \to E)$, where $E = P(\mathfrak{g}')$ is the vector bundle associated to the adjoint representation of $G$ on the semisimple part $\mathfrak{g}'$ of the Lie algebra of $G$, and $\varphi$ is given by the Lie algebra structure. To obtain a projective moduli space we have to allow $E$ to become a torsion free sheaf. For technical reasons, when $E$ is not locally free, we make $\varphi$ take values in $E^{\vee\vee}$.

The first step to construct the moduli space is the construction of a scheme parameterizing semistable based $\mathfrak{g}'$-sheaves, i.e. triples $(q : V \otimes \mathcal{O}_X(-m) \twoheadrightarrow E, E, \varphi : E \otimes E \to E^{\vee\vee})$, where $V$ is a fixed vector space, $m$ is a suitable large integer depending only on the numerical invariants, and $(E, \varphi)$ is a semistable $\mathfrak{g}'$-sheaf.

We have already seen that a $\mathfrak{g}'$-sheaf can be described as a tensor in the sense of [G-S1], where a notion of (semi)stability for tensors is given, depending on a polynomial $\delta$ of degree at most $n-1$ and positive leading coefficient. *In this article we will always assume that $\delta$ has degree $n-1$.* Now we will prove, after some lemmas, that the (semi)stability of the $\mathfrak{g}'$-sheaf coincides with the $\delta$-(semi)stability of the corresponding tensor(in particular for the tensors associated to $\mathfrak{g}'$-sheaves, its $\delta$-(semi)stability does not depend on $\delta$, as long as $\deg(\delta) = n-1$), so that we can apply the results of [G-S1], and the moduli space of semistable $\mathfrak{g}'$-sheaves is a subscheme of the moduli space of $\delta$-semistable tensors.

Given a $\mathfrak{g}'$-sheaf $(E, \varphi)$ and a balanced filtration $E_{\lambda_\bullet}$, define

$$\begin{aligned} (1.1)\ \mu(\varphi, E_{\lambda_\bullet}) &= \min\left\{\lambda_i + \lambda_j - \lambda_k \ :\ 0 \neq \varphi : E_{\lambda_i} \otimes E_{\lambda_j} \longrightarrow E^{\vee\vee}/E_{\lambda_{k-1}}{}^{\vee\vee}\right\} \\ &= \min\left\{\lambda_i + \lambda_j - \lambda_k \ :\ [E_{\lambda_i}, E_{\lambda_j}] \nsubseteq E_{\lambda_{k-1}}{}^{\vee\vee}\right\} \end{aligned}$$

**Lemma 1.1.** If $(E, \phi)$ is the associated tensor, then $\mu(\varphi, E_{\lambda_\bullet})$ in (1.1) is equal to $\mu_{\mathrm{tens}}(\phi, E_{\lambda_\bullet})$ in (0.16).

*Proof.* For a general $x \in X$ let $e_1, \dots, e_r$ be a basis adapted to the flag $E_{\lambda_\bullet}(x)$, thus giving a splitting $E(x) = \oplus E^{\lambda_i}(x)$. Writing $r^{\lambda_i} = \dim E^{\lambda_i}(x)$,

$$\begin{aligned}
&\mu_{\mathrm{tens}}(\phi, E_{\lambda_\bullet}) = \\
&= \min\big\{\lambda_i + \lambda_j + \lambda_1 r^{\lambda_1} + \cdots + \lambda_k(r^{\lambda_k} - 1) + \cdots + \lambda_{t+1} r^{\lambda_{t+1}} \ : \\
&\qquad e_1 \wedge e_2 \wedge \dots \wedge e_{k'-1} \wedge \varphi_x(e_{i'} \otimes e_{j'}) \wedge e_{k'+1} \wedge \dots \wedge e_r \neq 0 \ \text{ for some} \\
&\qquad\qquad e_{i'} \in E^{\lambda_i}(x),\, e_{j'} \in E^{\lambda_j}(x),\, 1 \le k' \le r\big\} = \\
&= \min\big\{\lambda_i + \lambda_j - \lambda_k \ : \ \varphi_x(E^{\lambda_i}(x), E^{\lambda_j}(x)) \nsubseteq E_{\lambda_{k-1}}(x) \\
&\qquad \text{and } \varphi_x(E^{\lambda_i}(x), E^{\lambda_j}(x)) \subseteq E_{\lambda_k}(x)\big\} = \\
&= \min\big\{\lambda_i + \lambda_j - \lambda_k \ : \ [E_{\lambda_i}, E_{\lambda_j}] \nsubseteq E_{\lambda_{k-1}}{}^{\vee\vee} \\
&\qquad \text{and } [E_{\lambda_i}, E_{\lambda_j}] \subseteq E_{\lambda_k}{}^{\vee\vee}\big\} = \\
&= \mu(\varphi, E_{\lambda_\bullet})
\end{aligned}$$

□

We will need the following result, due to Ramanathan [Ra3, lemma 5.5.1], whose proof we recall for convenience of the reader.

**Lemma 1.2.** *Let $W$ be a vector space, and let $p \in \mathbb{P}(W^\vee \otimes W^\vee \otimes W)$ be the point corresponding to a Lie algebra structure on $W$. If the Lie algebra is semisimple, this point is GIT-semistable for the natural action of $\mathrm{SL}(W)$ and linearization in $\mathcal{O}(1)$ on $\mathbb{P}(W^\vee \otimes W^\vee \otimes W)$*

*Proof.* Define the $\mathrm{SL}(W)$-equivariant homomorphism

$$\begin{array}{rcl}
g : (W^\vee \otimes W^\vee \otimes W) = \mathrm{Hom}(W, \mathrm{End}\, W) & \longrightarrow & (W \otimes W)^\vee \\
f & \mapsto & g(f)(\cdot \otimes \cdot) = \mathrm{tr}(f(\cdot) \circ f(\cdot))
\end{array}$$

Choose an arbitrary linear space isomorphism between $W$ and $W^\vee$. This gives an isomorphism $(W \otimes W)^\vee \cong \mathrm{End}(W)$. Define the determinant map $\det : (W \otimes W)^\vee \cong \mathrm{End}(W) \to \mathbb{C}$. Then $\det \circ g$ is an $\mathrm{SL}(W)$-equivariant polynomial on $W^\vee \otimes W^\vee \otimes W$ and it is nonzero when evaluated on the point $f$ corresponding to a semisimple Lie algebra, because it is the determinant of the Killing form. Hence this point is GIT-semistable. □

**Lemma 1.3.** *If $\varphi$ is a $\mathfrak{g}'$-sheaf, then $\mu(\varphi, E_{\lambda_\bullet}) \le 0$ for any balanced filtration $E_{\lambda_\bullet}$, and $\mu(\varphi, E_{\lambda_\bullet}) = 0$ if and only if it is an algebra filtration.*

*Proof.* Since $E^{\vee\vee}$ is torsion free, the formula (1.1) is equivalent to

$$\mu(\varphi, E_{\lambda_\bullet}) = \min\big\{\lambda_i + \lambda_j - \lambda_k \ : \ [E_{\lambda_i}(x), E_{\lambda_j}(x)] \nsubseteq E_{\lambda_{k-1}}{}^{\vee\vee}(x)\big\} \tag{1.2}$$

where $x$ is a general point of $X$, so that $E_{\lambda_\bullet}$ is a vector bundle filtration near $x$. Fixing a Lie algebra isomorphism between the fiber $E(x)$ and $\mathfrak{g}'$, the filtration $E_{\lambda_\bullet}$ induces a filtration on $\mathfrak{g}'$. Consider a vector space splitting $\mathfrak{g}' = \oplus {\mathfrak{g}'}^{\lambda_i}$ of this filtration and a basis $e_l$ of $\mathfrak{g}'$ such that $e_l \in {\mathfrak{g}'}^{i(l)}$, in order to define a monoparametric subgroup of $\mathrm{SL}(\mathfrak{g}')$ given by $e_l \mapsto t^{\lambda_{i(l)}} e_l$ for all $t \in \mathbb{C}^*$ (cfr. notation introduced for definition 0.14). The Lie algebra structure on $\mathfrak{g}'$ gives a point $\langle \varphi_{\mathfrak{g}'} \rangle \in \mathbb{P}({\mathfrak{g}'}^\vee \otimes {\mathfrak{g}'}^\vee \otimes \mathfrak{g}')$. Let $a^n_{lm}$ be the homogeneous coordinates of this point, i.e. $[e_l, e_m] = \sum_n a^n_{lm} e_n$. The monoparametric subgroup acts as $t^{\lambda_{i(l)} + \lambda_{i(m)} - \lambda_{i(n)}} a^n_{lm}$ on the coordinates $a^n_{lm}$. Hence (1.2) is equivalent to

$$\mu(\varphi, E_{\lambda_\bullet}) = \min\big\{\lambda_{i(l)} + \lambda_{i(m)} - \lambda_{i(n)} \ : \ a^n_{lm} \neq 0\big\}$$

By lemma 1.2, the point $\varphi_{\mathfrak{g}'}$ is semistable under the $\mathrm{SL}(\mathfrak{g}')$ action because it corresponds to a semisimple Lie algebra, hence $\mu(\varphi, E_{\lambda_\bullet}) \leq 0$.

Now assume that $\mu(\varphi, E_{\lambda_\bullet}) = 0$. Then it follows from (1.1) that

$$[E_{\lambda_i}, E_{\lambda_j}] \subseteq E_{\lambda_{k-1}}{}^{\vee\vee}$$

for all $\lambda_i$, $\lambda_j$, $\lambda_k$ with $\lambda_i + \lambda_j - \lambda_k < 0$, i.e. $E_{\lambda_\bullet}$ is an algebra filtration of $E$.

Conversely, if $E_{\lambda_\bullet}$ is an algebra filtration of $E$, then $\mu(\varphi, E_{\lambda_\bullet}) = 0$, because if $\mu(\varphi, E_{\lambda_\bullet}) < 0$, then for some triple $(\lambda_i, \lambda_j, \lambda_k)$ with $\lambda_i + \lambda_j < \lambda_k$ it is $[E_{\lambda_i}, E_{\lambda_j}] \not\subseteq E_{\lambda_{k-1}}{}^{\vee\vee}$, contradicting that $E_{\lambda_\bullet}$ is an algebra filtration.

□

**Lemma 1.4.** *Let $(E, \varphi : E \otimes E \to E^{\vee\vee})$ be a $\mathfrak{g}'$-sheaf, and let $(E, \phi : E^{\otimes r+1} \to \mathcal{O}_X)$ be the associated Lie tensor. Assume that one of the following conditions is satisfied*

1. *$(E, \varphi)$ is a semistable $\mathfrak{g}'$-sheaf (definition 0.25)*
2. *$(E, \phi)$ is a $\delta$-semistable tensor (definition 0.16)*

*Then $E$ is a Mumford semistable sheaf.*

*Proof.* Assume $E$ is not Mumford semistable. Consider its Harder-Narasimhan filtration, i.e. the filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_t \subsetneq E_{t+1} = E \tag{1.3}$$

such that $E^i = E_i/E_{i-1}$ is Mumford semistable for all $i = 1, \ldots, t+1$, and

$$\mu_{\max}(E) := \mu(E^1) > \mu(E^2) > \cdots > \mu(E^{t+1}) =: \mu_{\min}(E), \tag{1.4}$$

where $\mu(F) := \deg(F)/\operatorname{rk}(F)$ denotes the slope of a sheaf $F$. Define

$$\lambda_i = -r!\mu(E^i) \tag{1.5}$$

(the factor $r!$ is used to make sure that $\lambda_i$ is integer). Changing the indexes $i$ by $\lambda_i$, the Harder-Narasimhan filtration becomes

$$0 \subsetneq E_{\lambda_1} \subsetneq E_{\lambda_2} \subsetneq \cdots \subsetneq E_{\lambda_t} \subsetneq E_{\lambda_{t+1}} = E$$

Since $\deg(E) = 0$ (by lemma 0.27), it follows that this filtration is balanced (definition 0.14). Now we will check that it is an algebra filtration. Given a triple $(\lambda_i, \lambda_j, \lambda_k)$, with $\lambda_i + \lambda_j < \lambda_k$, we have to show that

$$[E_{\lambda_i}, E_{\lambda_j}] \subset E_{\lambda_{k-1}}{}^{\vee\vee}.$$

Let $k'$ be the minimum integer for which

$$[E_{\lambda_i}, E_{\lambda_j}] \subset E_{\lambda_{k'-1}}{}^{\vee\vee}.$$

We have to show that $k' \leq k$. By definition of $k'$, the following composition is nonzero

$$E_{\lambda_i} \otimes E_{\lambda_j} \xrightarrow{[\cdot,\cdot]} E_{\lambda_{k'-1}}{}^{\vee\vee} \longrightarrow E_{\lambda_{k'-1}}{}^{\vee\vee}/E_{\lambda_{k'-2}}{}^{\vee\vee}$$

It is well known that, if a homomorphism $F_1 \to F_2$ between two torsion free sheaves is nonzero, then $\mu_{\min}(F_1) \leq \mu_{\max}(F_2)$, hence

$$\mu_{\min}(E_{\lambda_i} \otimes E_{\lambda_j}) \leq \mu_{\max}(E_{\lambda_{k'-1}}{}^{\vee\vee}/E_{\lambda_{k'-2}}{}^{\vee\vee}) \tag{1.6}$$

Using (1.5) and the fact that $\mu_{\min}(E_{\lambda_1} \otimes E_{\lambda_2}) = \mu_{\min}(E_{\lambda_1}) + \mu_{\min}(E_{\lambda_2})$ [A-B, Prop. 2.9]), the left hand side is

$$\mu_{\min}(E_{\lambda_i} \otimes E_{\lambda_j}) = \frac{-1}{r!}(\lambda_i + \lambda_j)$$

Since the quotient $E_{\lambda_{k'-1}}{}^{\vee\vee}/E_{\lambda_{k'-2}}{}^{\vee\vee}$ is Mumford semistable, the right hand side is

$$\mu_{\max}(E_{\lambda_{k'-1}}{}^{\vee\vee}/E_{\lambda_{k'-2}}{}^{\vee\vee}) = \mu(E_{\lambda_{k'-1}}{}^{\vee\vee}/E_{\lambda_{k'-2}}{}^{\vee\vee}) = \frac{-1}{r!}\lambda_{k'-1}$$

Hence the inequality (1.6) becomes

$$\lambda_i + \lambda_j \geq \lambda_{k'-1},$$

and then $\lambda_{k'-1} < \lambda_k$, hence $k' \leq k$, and we conclude that $E_{\lambda_\bullet}$ is a balanced algebra filtration.

If we plot the points $(r_{\lambda_i}, d_{\lambda_i}) = (\operatorname{rk} E_{\lambda_i}, \deg E_{\lambda_i})$, $1 \leq i \leq t+1$ in the plane $\mathbb{Z}\oplus\mathbb{Z}$ we get a polygon, called the Harder-Narasimhan polygon. Condition (1.4) means that this polygon is (strictly) convex. Since $d = 0$ (and $d_{\lambda_1} > 0$), this implies that $d_{\lambda_i} > 0$ for $1 \leq i \leq t$, and then

$$\sum_{i=1}^{t} r!\big(\mu(E^i) - \mu(E^{i+1})\big)(rd_{\lambda_i} - r_{\lambda_i}d) > 0. \tag{1.7}$$

Therefore

$$\sum_{i=1}^{t} (\lambda_{i+1} - \lambda_i)\big(rP_{E_{\lambda_i}} - r_{\lambda_i}P_E\big) \succ 0 \tag{1.8}$$

because the leading coefficient of (1.8) is (1.7), and then $(E, \varphi)$ is not semistable as a $\mathfrak{g}'$-sheaf. Hence, if $(E, \varphi)$ is semistable, then $E$ is Mumford semistable.

Now, since the Harder-Narasimhan filtration (1.3) of $E$ is an algebra filtration, it is, by lemma 1.3, $\mu(\varphi, E_{\lambda_\bullet}) = 0$. Now, lemma 1.1 implies $\mu_{\text{tens}}(\phi, E_{\lambda_\bullet}) = 0$, hence

$$\sum_{i=1}^{t} (\lambda_{i+1} - \lambda_i)\big(rP_{E_{\lambda_i}} - r_{\lambda_i}P_E\big) + \mu(\phi, E_{\lambda_\bullet}) = \sum_{i=1}^{t} (\lambda_{i+1} - \lambda_i)\big(rP_{E_{\lambda_i}} - r_{\lambda_i}P_E\big) \succ 0$$

and then $(E, \phi)$ is not $\delta$-semistable as a tensor. Hence, if $(E, \phi)$ is $\delta$-semistable, it follows that $E$ is Mumford semistable.

□

**Proposition 1.5.** *Let $(E, \varphi : E \otimes E \to E^{\vee\vee})$ be a $\mathfrak{g}'$-sheaf and let $(E, \phi : E^{\otimes r+1} \to \mathcal{O}_X)$ be the associated tensor. The following conditions are equivalent*

1. *$(E, \phi)$ is a $\delta$-(semi)stable tensor*
2. *$(E, \varphi)$ is a (semi)stable $\mathfrak{g}'$-sheaf*

*Proof.* Assume that $(E, \phi)$ is $\delta$-semistable. By lemma 1.4, the sheaf $E$ is Mumford semistable. Let $E_{\lambda_\bullet}$ be a balanced algebra filtration. Then $\mu_{\text{tens}}(\phi, E_{\lambda_\bullet}) = \mu(\varphi, E_{\lambda_\bullet}) = 0$ (lemmas 1.1 and 1.3), hence inequality (0.6) in definition 0.16 becomes (0.7) in definition 0.20.

Conversely, assume that the $\mathfrak{g}'$-sheaf $(E, \varphi)$ is (semi)stable, thus $E$ is again Mumford semistable, and consider a balanced filtration $E_{\lambda_\bullet}$ of $E$. We have to show that (0.6) is satisfied. If the filtration is an algebra filtration, then $\mu(\varphi, E_{\lambda_\bullet}) = 0$ by lemma 1.3, hence (0.6) holds. If it is not an algebra filtration, then $\mu(\varphi, E_{\lambda_\bullet}) < 0$ (again by lemma 1.3). Since $E$ is Mumford semistable, it is $rd_{\lambda_i} - r_{\lambda_i}d \leq 0$ for all $i$. Denote by $\tau/(n-1)!$ the coefficient of $t^{n-1}$ in $\delta$. It is $\tau > 0$ because $\deg\delta = n-1$. Then the leading coefficient of the polynomial of (0.6) becomes

$$\Big(\sum_{i=1}^{t} (\lambda_{i+1} - \lambda_i)\big(rd_{\lambda_i} - r_{\lambda_i}d\big)\Big) + \tau\mu(\varphi, E_{\lambda_\bullet}) < 0,$$

and thus (0.6) holds.

□

Now, let us recall briefly how the moduli space of tensors was constructed in [G-S1]. Start with a $\delta$-semistable tensor

$$\phi : F^{\otimes a} \longrightarrow \mathcal{O}_X$$

with $\mathrm{rk}\, F = r$, Hilbert polynomial $P_F = P$ and $\det F \cong \mathcal{O}_X$. Let $m$ be a large integer (depending only on the polarization and numerical invariants of $F$) and an isomorphism $g$ between $H^0(F(m))$ and a fixed vector space $V$ of dimension $h^0(F(m))$. This gives a quotient

$$q : V \otimes \mathcal{O}_X(-m) \longrightarrow F$$

and hence a point in the Hilbert scheme $\mathcal{H}$ of quotients of $V \otimes \mathcal{O}_X(-m)$ with Hilbert polynomial $P$. Let $l > m$ be an integer, and $W = H^0(\mathcal{O}_X(l-m))$. The quotient $q$ induces homomorphisms

$$\begin{array}{rcccl} q & : & V \otimes \mathcal{O}_X(l-m) & \twoheadrightarrow & F(l) \\ q' & : & V \otimes W & \rightarrow & H^0(F(l)) \\ q'' & : & \bigwedge^{P(l)}(V \otimes W) & \rightarrow & \bigwedge^{P(l)} H^0(F(l)) \;\cong\; \mathbb{C} \end{array}$$

If $l$ is large enough, these homomorphisms are surjective, and they give Grothendieck's embedding

$$\mathcal{H} \;\hookrightarrow\; \mathbb{P}\big(\bigwedge^{P(l)}(V^\vee \otimes W^\vee)\big).$$

and hence a very ample line bundle $\mathcal{O}_{\mathcal{H}}(1)$ on $\mathcal{H}$ (depending on $m$ and $l$). The isomorphism $g : V \xrightarrow{\cong} H^0(F(m))$ and $\varphi$ induces a linear map

$$\Phi : V^{\otimes a} \;\longrightarrow\; H^0(F(m)^{\otimes a}) \;\longrightarrow\; H^0(\mathcal{O}_X(am)) \;=:\; B,$$

and hence the tensor $\varphi$ and the isomorphism $g$ give a point in

$$\mathbb{P}\big(\bigwedge^{P(l)}(V^\vee \otimes W^\vee)\big) \times \mathbb{P}\big((V^{\otimes a})^\vee \otimes B\big) \;=\; \mathbb{P} \times \mathbb{P}'$$

Let $Z$ be the closure of the points associated to $\delta$-semistable tensors. We give $Z$ a polarization $\mathcal{O}_Z(1)$, by restricting a polarization $\mathcal{O}_{\mathbb{P}\times\mathbb{P}'}(b, b')$, where the ratio between $b$ and $b'$ depends on the polynomial $\delta$ and the integers $m$ and $l$

$$\frac{b'}{b} = \frac{P(l)\delta(m) - \delta(l)P(m)}{P(m) - a\delta(m)}$$

There is a tautological family of tensors parametrized by $Z$

$$\phi_Z : F_Z^{\otimes r+1} \longrightarrow p_{\mathbb{P}'}^* \mathcal{O}_{\mathbb{P}'}(1), \tag{1.9}$$

The scheme $Z$ has an open dense set $Z^{ss}$ representing the sheafification of the functor

$$F^b : (\mathrm{Sch}/\mathbb{C}) \longrightarrow (\mathrm{Sets}) \tag{1.10}$$

associating to a scheme $S$ the set of equivalence classes of families of $\delta$-semistable "based" tensors

$$\big(q_S : V \otimes \mathcal{O}_{X\times S}(-m) \rightarrow F_S,\ F_S,\ \phi_S : F_S^{\otimes a} \rightarrow p_S^* N,\ N\big)$$

where $q_S$ is a surjection inducing an isomorphism

$$g_S = p_{S*}(q_S(m)) : V \otimes \mathcal{O}_S \rightarrow p_{S*}(F_S(m))$$

and $(F_S, \phi_S, N)$ is a family of $\delta$-semistable tensors (definition 0.16) with fixed rank $r$, Hilbert polynomial $P$ and trivial determinant. In particular,

$$\det(F_S) \cong p_S^* L, \tag{1.11}$$

where $L$ is a line bundle on $S$. From now on, we will assume $a = r+1$, where $r$ is the rank of $F$.

**Proposition 1.6.** *There is a closed subscheme $R$ of $Z^{ss}$ representing the sheafification $\widetilde{F}^b_{\mathrm{Lie}}$ of the subfunctor of (1.10)*

$$\begin{array}{rcl} F^b_{\mathrm{Lie}} : (\mathrm{Sch}/\mathbb{C}) & \longrightarrow & (\mathrm{Sets}) \\ S & \longmapsto & F^b_{\mathrm{Lie}}(S) \subset F^b(S) \end{array} \tag{1.12}$$

*where $F^b_{\mathrm{Lie}}(S) \subset F^b(S)$ is the subset of families of based $\delta$-semistable Lie tensors.*

*A point of the closure $\overline{R}$ of $R$ in $Z$ is GIT-(semi)stable with respect to the natural $\mathrm{SL}(V)$-action and linearization on $\mathcal{O}_{\overline{R}}(1) = \mathcal{O}_Z(1)|_{\overline{R}}$ (see* [G-S1]*) if and only if the corresponding tensor is $\delta$-(semi)stable and $q$ induces an isomorphism $V \cong H^0(E(m))$. In particular the open subset of semistable points of $\overline{R}$ is $R$.*

*Proof.* Let $(q_{Z^{ss}}, F_{Z^{ss}}, \phi_{Z^{ss}} : F_{Z^{ss}}^{\otimes r+1} \to p_{Z^{ss}}^* N, N)$ be the tautological family on $Z^{ss}$ coming from (1.9). For each pair $(i,j)$ with $1 \le i < j \le r+1$, let

$$\sigma_{ij}(\phi_{Z^{ss}}) : F_{Z^{ss}}^{\otimes r+1} \longrightarrow N$$

be the homomorphism obtained from $\phi_{Z^{ss}}$ by interchanging the factors $i$ and $j$. Let $Z_{ij} \subset Z^{ss}$ be the zero subscheme defined by $\phi_{Z^{ss}} + \sigma_{ij}(\phi_{Z^{ss}})$, using lemma 0.9. Finally, define

$$Z_{\mathrm{skew}} = \bigcap_{3 \le i < j \le r+1} Z_{ij}.$$

From the universal property of $Z_{ij}$ (lemma 0.9) it follows that, for a family satisfying condition (1) of definition 0.13, the classifying morphism into $Z^{ss}$ factors through $Z_{\mathrm{skew}}$. Furthermore, the restriction of the tautological family to $Z_{\mathrm{skew}}$ satisfies condition (1), hence by corollary 0.12 we have a family parametrized by $Z_{\mathrm{skew}}$

$$(q_{Z_{\mathrm{skew}}}, F_{Z_{\mathrm{skew}}}, \varphi_{Z_{\mathrm{skew}}} : F_{Z_{\mathrm{skew}}} \otimes F_{Z_{\mathrm{skew}}} \longrightarrow F_{Z_{\mathrm{skew}}}^{\vee\vee} \otimes p_{Z_{\mathrm{skew}}}^* N, N) \tag{1.13}$$

The closed subscheme ("antisymmetric locus") $Z_{\mathrm{asym}} \subset Z_{\mathrm{skew}}$ is defined as the zero subscheme of $\varphi_{Z_{\mathrm{skew}}} + \sigma_{12}(\varphi_{Z_{\mathrm{skew}}})$ given by lemma 0.9. It follows that if a family satisfies conditions (1) and (2) of definition 0.13, then the classifying morphism factors through $Z_{\mathrm{asym}}$, and furthermore the restriction of the tautological family to $Z_{\mathrm{asym}}$ satisfies conditions (1) and (2).

Let $J$ be the homomorphism defined as in (0.4), using the tautological family parametrized by $Z_{\mathrm{asym}}$. Note that this homomorphism is zero if and only if the associated homomorphism (lemma 0.10)

$$J' : F_{Z_{\mathrm{asym}}} \otimes F_{Z_{\mathrm{asym}}} \otimes F_{Z_{\mathrm{asym}}} \otimes F_{Z_{\mathrm{asym}}}^{\vee} \longrightarrow p_{Z_{\mathrm{asym}}}^* N^2$$

is zero. Finally, define the closed subscheme $R \subset Z_{\mathrm{asym}}$ as the zero subscheme of $J'$ given in lemma 0.9. It follows that if a family satisfies conditions (1) to (3) of definition 0.13, then the classifying morphism will factor through $R$, and furthermore the restriction of the tautological family to $R$ satisfies conditions (1) to (3).

The criteria for stability follows from [G-S1]. □

Recall that a $\mathfrak{g}'$-sheaf is (semi)stable if and only if the associated Lie tensor is $\delta$-semistable (proposition 1.5).

**Proposition 1.7.** *There is a subscheme $R_1 \subset R$ representing the sheafification $\widetilde{F}^b_{\mathfrak{g}'}$ of the subfunctor of (1.12)*

$$F^b_{\mathfrak{g}'} : (\mathrm{Sch}/\mathbb{C}) \longrightarrow (\mathrm{Sets}) \qquad (1.14)$$
$$S \longmapsto F^b_{\mathfrak{g}'}(S) \subset F^b_{\mathrm{Lie}}(S)$$

*where $F^b_{\mathfrak{g}'}(S) \subset F^b_{\mathrm{Lie}}(S)$ is the subset of $S$-families of based $\delta$-semistable Lie tensors such that the homomorphism associated by construction 0.23 provides a family of based semistable $\mathfrak{g}'$-sheaves with fixed numerical invariants $\tau$.*

*Furthermore, $R_1$ is a union of connected components of $R$, hence the inclusion $R_1 \hookrightarrow R$ is proper.*

*Proof.* Consider the tautological family parametrized by $R$

$$(q_R, F_R, \phi_R : F_R^{\otimes r+1} \longrightarrow p_R^* N, N)$$

and the associated family obtained as in construction 0.23

$$(q_R, E_R, \varphi_R : E_R \otimes E_R \to E_R^{\vee\vee}) \qquad (1.15)$$

Let $\kappa$ be the Killing form (definition 0.17)

$$\kappa : E_R \otimes E_R \longrightarrow \mathcal{O}_{X\times R}.$$

This induces a homomorphism $\det \kappa' : \det E_R \to \det E_R^{\vee}$. Recall from (1.11) that $\det(F_R)$ is the pullback of a line bundle from $R$, hence the same holds for $\det(E_R)$, and then $\det \kappa'$ is constant along the fibers of $\pi : X \times R \to R$. Hence $\det \kappa'$ is nonzero on an open set of the form $X \times W$, where $W \subset R$ is an open set.

A point $(q, E, \varphi) \in R$ belongs to $W$ if and only if for all $x \in U_E$ the Lie algebra $(E(x), \varphi(x))$ is semisimple, because the Killing form is nondegenerate if and only if the Lie algebra is semisimple.

Now we show that the open set $W$ is in fact equal to $R$. Let $(q, E, \varphi : E \otimes E \to E^{\vee\vee})$ be a based algebra sheaf corresponding to a point in $R \setminus W$. Then its Killing form $\kappa : E \otimes E \to \mathcal{O}_X$ is degenerate. Let $E_1$ be the kernel of the homomorphism induced by $\kappa$

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E^{\vee}.$$

By lemma 1.4, $E$ is Mumford semistable, thus $E^{\vee}$ is Mumford semistable, and being both of degree 0, the sheaf $E_1$ is also of degree 0 and Mumford semistable. Note that $E_1$ is a solvable ideal of $E$, i.e. the fibers of $E_1$ are solvable ideals of the fibers of $E$ (over points where both sheaves are locally free) [Se2, proof of Th. 2.1 in chp VI]. Since $E_1 \otimes E_1$ (modulo torsion) and $E_1^{\vee\vee}$ are Mumford semistable of degree zero, the image $E_2' = [E_1, E_1]$ of the Lie bracket homomorphism $\varphi : E_1 \otimes E_1 \to E_1^{\vee\vee}$, is a Mumford semistable subsheaf of $E^{\vee\vee}$ of degree zero. Define $E_2 = E_2' \cap E$. It is a Mumford semistable subsheaf of $E$ of degree zero. Similarly $E_3' = [F_2, F_2]$, $E_3$, etc... are all Mumford semistable sheaves of degree zero. Since $E_1$ is solvable, we arrive eventually to a non-zero sheaf $E'$ of degree zero, which is an abelian ideal of $E$.

We claim that the balanced filtration $E_{\lambda_1} = E' \subset E_{\lambda_2} = E$ with $\lambda_1 = \mathrm{rk}\, E' - r$ and $\lambda_2 = \mathrm{rk}\, E'$ contradicts the $\delta$-semistability of the tensor $(E, \phi)$ associated to $(E, \phi)$ by construction 0.23.

To prove this we need to calculate $\mu_{\mathrm{t}}(\phi, E_{\lambda_\bullet})$ (cfr. (0.5)). By lemma 1.1 this is equal to $\mu(\varphi, E_{\lambda_\bullet})$ (cfr. (1.1)). We need to estimate which triples $(i, j, k)$ are relevant to calculate the minimum, i.e. for which triples it is $[E_{\lambda_i}, E_{\lambda_j}] \not\subseteq E_{\lambda_{k-1}}{}^{\vee\vee}$. Since $E'$ is abelian, $[E', E'] = 0$, so $(1, 1, k)$ is not relevant. Since $E'$ is an ideal, we have $[E',E] \subset E'^{\vee\vee}$. If $E'$ is in the center, then this bracket is zero, hence $(1,2,k)$ is not relevant. On the other hand, if $E'$ is not in the center, then $[E',E] \neq 0$, hence $(1,2,1)$ is relevant, and the associated weight is $\lambda_1 + \lambda_2 - \lambda_1 = \mathrm{rk}(E') > 0$. Since $E$ is not abelian, it is $[E,E] \neq 0$. There are two possibilities: if $[E,E] \subset E'^{\vee\vee}$, then $(2,2,1)$ is relevant and $\lambda_2 + \lambda_2 - \lambda_1 = \mathrm{rk}(E') + \mathrm{rk}(E) > 0$. Otherwise $(2,2,2)$ is relevant, and $\lambda_2 + \lambda_2 - \lambda_2 = \mathrm{rk}(E') > 0$. Summing up, we obtain

$$\mu(\varphi, E_{\lambda_\bullet}) > 0.$$

Since $\deg(E') = \deg(E) = 0$, the leading coefficient of

$$\big(rP_{E'} - \mathrm{rk}(E')P_E\big) + \mu(\varphi, E_{\lambda_\bullet})\delta$$

is positive, hence $(E,\phi)$ is not $\delta$-semistable (and by proposition 1.5, $(E,\varphi)$ is not semistable), contradicting the assumption, so we have proved that $W = R$.

Now assume that we have two based $\mathfrak{g}'$-sheaves $(q,E,\varphi)$ and $(q',E',\varphi')$ belonging to the same connected component of $R$, and $x \in U_E$, $x' \in U_{E'}$. Then we have

$$(E(x),\varphi(x)) \cong (E'(x'),\varphi'(x'))$$

as Lie algebras, because of the well known rigidity of semisimple Lie algebras (see [Ri], for instance). Hence $R_1$ is the union of the connected components of $R$ with $(E(x),\varphi(x)) \cong \mathfrak{g}'$.

□

We will denote by $\mathcal{E}_{R_1}$ the tautological family of $\mathfrak{g}'$-sheaves parametrized by $R_1$ obtained by restricting (1.15)

$$\mathcal{E}_{R_1} = (E_{R_1}, \varphi_{R_1}) \tag{1.16}$$

Giving a family of (semi)stable $\mathfrak{g}'$-sheaves is equivalent to giving a family of (semi)stable principal $\mathrm{Aut}(\mathfrak{g}')$-sheaves. By lemma 0.25, the (semi)stability conditions for a $\mathfrak{g}'$-sheaf and the corresponding principal $\mathrm{Aut}(\mathfrak{g}')$-sheaf coincide, hence $(E_{R_1},\varphi_{R_1})$ can be seen as a family of semistable principal $\mathrm{Aut}(\mathfrak{g}')$-sheaves.

Recall that $\mathcal{H}$ is the Hilbert scheme classifying quotients $V \otimes \mathcal{O}_X(-m) \to F$ (of fixed Chern classes), $\mathbb{P}' = \mathbb{P}\big((V^{\otimes r+1})^\vee \otimes H^0(\mathcal{O}_X((r+1)m))\big)$ and, by construction 0.23, we have $E_{R_1} = F_{R_1} \otimes \det F_{R_1} \otimes p^*\mathcal{O}_{\mathbb{P}'}(-1)$, where $F_{R_1}$ is the restriction of (1.9) to $R_1$, and $p$ is

$$p : R_1 \hookrightarrow \mathbb{P} \times \mathbb{P}' \to \mathbb{P}'$$

Let $\tau : V \otimes \mathcal{O}_{\mathrm{GL}(V)} \to V \otimes \mathcal{O}_{\mathrm{GL}(V)}$ be the universal automorphism. Let $\pi_{\mathrm{GL}(V)}$, $\pi_{R_1}$ be the projections to the two factors of $\mathrm{GL}(V) \times R_1$. The group $\mathrm{GL}(V)$ acts on $R_1$, and this action lifts to $F_{R_1}$ ([H-L, §4.3 pg. 90]) and $p^*\mathcal{O}_{\mathbb{P}'}(1)$, giving isomorphisms $(\Lambda, \mathcal{B})$

$$\begin{array}{ccccccc}
V \otimes \mathcal{O}_{X\times R_1}(-m) & \xrightarrow{\overline{\sigma}^* q_{R_1}} & \overline{\sigma}^* F_{R_1} & \qquad & \overline{\sigma}^* F_{R_1}^{\otimes r+1} & \xrightarrow{\overline{\sigma}^* \phi_{R_1}} & \overline{\sigma}^* N \\
\Big\downarrow \pi^*_{\mathrm{GL}(V)}\tau & & \cong \Big\downarrow \Lambda & & \cong \Big\downarrow \Lambda^{\otimes r+1} & & \cong \Big\downarrow \mathcal{B} \\
V \otimes \mathcal{O}_{X\times R_1}(-m) & \xrightarrow{\overline{\pi_{R_1}}^* q_{R_1}} & \overline{\pi_{R_1}}^* F_{R_1} & & \overline{\pi_{R_1}}^* F_{R_1}^{\otimes r+1} & \xrightarrow{\overline{\pi_{R_1}}^* \phi_{R_1}} & \overline{\pi_{R_1}}^* N
\end{array} \tag{1.17}$$

between the pullbacks of the family of Lie tensors $(F_{R_1}, \phi_{R_1})$ by the action $\sigma : \mathrm{GL}(V) \times R_1 \to R_1$ and the projection $\pi_{R_1}$ to the second factor.

Since this action lifts to $F_{R_1}$ and $p^*\mathcal{O}_{\mathbb{P}'}(1)$, it also lifts to $E_{R_1}$. An element $\lambda$ in the center of $\mathrm{GL}(V)$ acts trivially on $R_1$, hence the action $\sigma$ factors through an action action $m : \mathrm{PGL}(V) \times R_1 \to R_1$ of $\mathrm{PGL}(V)$ on $R_1$. The element $\lambda$ acts as multiplication by $\lambda$ on $F_{R_1}$ and as multiplication by $\lambda^{-r-1}$ on $\mathcal{O}_{\mathbb{P}'}(-1)$, hence it acts trivially on $E_{R_1}$. Therefore the action of $\mathrm{GL}(V)$ on $E_{R_1}$ factors through $\mathrm{PGL}(V)$

$$\begin{array}{ccccc} \overline{m}^* E_{R_1} & \quad & \overline{m}^* E_{R_1} \otimes \overline{m}^* E_{R_1} & \xrightarrow{\overline{m}^* \varphi_{R_1}} & \overline{m}^* E_{R_1}^{\vee\vee} \\ \cong \big\downarrow \Lambda & & \big\downarrow \Lambda\otimes\Lambda & & \big\downarrow \Lambda^{\vee\vee} \\ \overline{p_{R_1}}^* E_{R_1} & & \overline{p_{R_1}}^* E_{R_1} \otimes \overline{p}_2^* E_{R_1} & \xrightarrow{\overline{p_{R_1}}^* \varphi_{R_1}} & \overline{p_{R_1}}^* E_{R_1}^{\vee\vee} \end{array} \tag{1.18}$$

where $p_{R_1}$ is the projection of $\mathrm{PGL}(V) \times R_1$ to the second factor. *This gives a lift of the* $\mathrm{PGL}(V)$ *action on* $R_1$ *to the family* $\mathcal{E}_{R_1}$.

**Proposition 1.8.** *With this action,* $(E_{R_1}, \varphi_{R_1})$ *becomes a universal family with group* $\mathrm{PGL}(V)$ *for the functor* $\widetilde{F}^{\tau}_{\mathrm{Aut}(\mathfrak{g}')}$ *(cfr. remark 0.26).*

*Proof.* Let $(E_S, \varphi_S)$ be a family of semistable $\mathfrak{g}'$-sheaves. Shrink $S$ if necessary, so that $\det E_S \cong \mathcal{O}_{X\times S}$. Using this isomorphism and construction 0.23 we obtain a family of $\delta$-semistable Lie tensors $(E_S, \phi_S : E_S^{\otimes r+1} \to \mathcal{O}_{X\times S})$. By proposition 1.7, after shrinking $S$ if necessary, there is a morphism $f : S \to R_1$ such that the pullback $(\overline{f}^* E, \overline{f}^* \phi)$ of the family of Lie tensors parametrized by $R_1$ is isomorphic to $(E_S, \phi_S)$, hence the families of $\mathfrak{g}'$-sheaves associated by construction 0.23 to both of them are isomorphic.

Now we are going to check the second condition in the definition of universal family. Let $t_1, t_2 : S \to R_1$ be two morphisms, and let $\alpha : E_2 \to E_1$ be an isomorphism between the two pullbacks $(E_1, \varphi_1)$ and $(E_2, \varphi_2)$ of $\mathcal{E}_{R_1}$ under $t_1$ and $t_2$. We have to find a morphism $h : S \to \mathrm{PGL}(V)$ such that $t_2 = h[t_1]$ and $\overline{(h\times t_1)}^* \Lambda = \alpha$. Since the question is local on $S$, we may shrink $S$ when needed along the proof.

By pulling back the family $(F_{R_1}, \phi_{R_1})$, these morphisms also give two families of semistable based Lie tensors $(q_1, F_1, \phi_1)$ and $(q_2, F_2, \phi_2)$. By definition of $E_{R_1}$, we have $E_i = F_i \otimes \det F_i \otimes N_i$, $i = 1, 2$. After eventually shrinking $S$, there are isomorphisms $a_i : \det F_i \otimes N_i \to \mathcal{O}_{X\times S}$. Define $\alpha'$ by

$$\begin{array}{ccccc} E_2 & = & F_2 \otimes \det F_2 \otimes N_2 & \xrightarrow{F_2\otimes a_2} & F_2 \\ \alpha \big\downarrow & & & & \big\downarrow \alpha' \\ E_1 & = & F_1 \otimes \det F_1 \otimes N_1 & \xrightarrow{F_1\otimes a_1} & F_1 \end{array}$$

and hence $\alpha = \alpha' \otimes (a_1^{-1} \circ a_2)$. Given an isomorphism $\beta : N_2^{-1} \to N_1^{-1}$, we obtain an isomorphism

$$\alpha' \otimes \det \alpha' \otimes \beta : E_2 = F_2 \otimes \det F_2 \otimes N_2 \longrightarrow E_1 = F_1 \otimes \det F_1 \otimes N_1 \,.$$

Choose $\beta$ so that $\alpha' \otimes (a_1^{-1} \circ a_2) = \alpha' \otimes \det \alpha' \otimes \beta$. Since $\alpha = \alpha' \otimes \det \alpha' \otimes \beta$, the commutativity of

$$\begin{array}{ccc} E_2 \otimes E_2 & \xrightarrow{\varphi_2} & E_2^{\vee\vee} \\ \alpha\otimes\alpha \big\downarrow & & \big\downarrow \alpha^{\vee\vee} \\ E_1 \otimes E_1 & \xrightarrow{\varphi_1} & E_1^{\vee\vee} \end{array}$$

implies the commutativity of

$$\begin{array}{ccc} F_2^{\otimes r+1} & \xrightarrow{\phi_2} & N_2 \\ {\scriptstyle \alpha'^{\otimes r+1}}\downarrow & & \downarrow{\scriptstyle \beta} \\ F_1^{\otimes r+1} & \xrightarrow{\phi_1} & N_1 \end{array}$$

and hence the pair $(\alpha', \beta)$ gives an isomorphism between $(F_1, \phi_1)$ and $(F_2, \phi_2)$. Using the based Lie tensors $(q_1, F_1, \phi_1)$ and $(q_2, F_2, \phi_2)$, let $g_i = p_{S*}(q_i(m))$, $i = 1, 2$, and define the isomorphism $h'$

$$\begin{array}{ccc} V \otimes \mathcal{O}_S & \xrightarrow{g_2} & p_{S*}(F_2(m)) \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle p_{S*}(\alpha'(m))}\ \cong \\ V \otimes \mathcal{O}_S & \xrightarrow{g_1} & p_{S*}(F_1(m)) \end{array}$$

This isomorphism can be seen as a morphism $h' : S \to \mathrm{GL}(V)$. By construction, it is $t_2 = h'[t_1]$, and $(\alpha', \beta)$ is the pullback of the isomorphism (1.17) by $h' \times t_1$. Denote by $h : S \to \mathrm{PGL}(V)$ the composition with the projection to $\mathrm{PGL}(V)$. Then we have $t_2 = h[t_1]$, and $\alpha$ is the pullback of the left arrow in (1.18) by $\overline{h \times t_1}$.

Finally, we have to check that these two properties determine $h$ uniquely. Let $h_1, h_2 : S \to \mathrm{PGL}(V)$ be two such morphisms. Define $h = h_1 h_2^{-1}$. Then $h[t_1] = t_1$, and the pullback $\overline{h \times t_1}^* \Lambda$ is the identity automorphism. Replacing $S$ by an étale cover, we can lift $h$ to a morphism $h' : S \to \mathrm{GL}(V)$, and this induces an automorphism $\alpha' = \overline{h \times t_1}^* \Lambda'$ of $F_S = \overline{t_1}^* F_{R_1}$

(1.19)
$$\begin{array}{ccc} V \otimes \mathcal{O}_{X \times S} & \xrightarrow{\overline{t_1}^* q_{R_1}} & F_S(m) \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ V \otimes \mathcal{O}_{X \times S} & \xrightarrow{\overline{t_1}^* q_{R_1}} & F_S(m) \end{array}$$

Applying $p_{S*}$ to (1.19), we obtain

$$\begin{array}{ccc} V \otimes \mathcal{O}_S & \xrightarrow[\cong]{H^0(q_1)} & p_{S*} F_S(m) \\ {\scriptstyle p_{S*}h'}\downarrow & & \downarrow{\scriptstyle p_{S*}\alpha'} \\ V \otimes \mathcal{O}_S & \xrightarrow[\cong]{H^0(q_1)} & p_{S*} F_S(m) \end{array}$$

Since $\overline{h \times t_1}^* \Lambda = \mathrm{id}$, the automorphism $\alpha'$ is a family of homotethies, i.e. $p_{S*}\alpha'$ can be seen as a morphism $S \to \mathbb{C}^*$, and considering the previous diagram, $p_{S*}h'$ can also be seen as a morphism from $S$ to $\mathbb{C}^*$, the center of $\mathrm{GL}(V)$, hence $h$ is the identity morphism from $S$ to $\mathrm{PGL}(V)$.

□

## 2. Construction of $R_2$

Recall that *all schemes considered are locally of finite type over* $\operatorname{Spec}\mathbb{C}$. In this section and the following we are going to make use of the category of complex analytic spaces. For a scheme $Y$, we denote by $Y^{\mathrm{an}}$ the associated complex analytic space ([SGA1, XII], [Ha, App. B]), and given a morphism $f$ in the category of schemes, we denote by $f^{\text{an}}$ the corresponding morphism in the category of analytic spaces. Recall that the underlying set of $Y^{\text{an}}$ is the set of closed points of $Y$, and it is endowed with the analytic topology.

**Lemma 2.1.** *Let $S$ be a scheme (not necessarily smooth). Let $\mathcal{Z} \subset X \times S$ be a closed subscheme with $\operatorname{codim}_{\mathbb{R}}(\mathcal{Z}_s^{\text{an}}, X^{\text{an}} \times s) \geq m$ for all closed points $s \in S$, and $\mathcal{U} \subset X \times S$ its complement. Let $M$ be a real manifold with $\dim_{\mathbb{R}}(M) \leq m-1$ and compact boundary, and let*

$$f = (f_X, f_S) : M \longrightarrow X^{\text{an}} \times S^{\text{an}}$$

*be a continuous map such that the image of the boundary lies in $\mathcal{U}^{\text{an}}$. Then $f$ can be modified by a homotopy, relative to its boundary to a continuous map $\widetilde{f}$ whose image lies in $\mathcal{U}^{\text{an}}$.*

*Proof.* Consider the cartesian product (in the category of topological spaces and continuous maps)

$$\begin{array}{ccc} \mathcal{Z}_M^{\text{an}} & \longrightarrow & \mathcal{Z}^{\text{an}} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f_S} & S^{\text{an}} \end{array}$$

The map $f$ factors as

$$\begin{array}{ccccc} & & \overset{f}{\frown} & & \\ M & \xrightarrow{(f_X,\text{id})} & X^{\text{an}} \times M & \xrightarrow{(\text{id},f_S)} & X^{\text{an}} \times S^{\text{an}} \\ & & \uparrow & & \uparrow \\ & & \mathcal{Z}_M^{\text{an}} & \longrightarrow & \mathcal{Z}^{\text{an}} \end{array}$$

By hypothesis $\operatorname{codim}_{\mathbb{R}}(\mathcal{Z}_s^{\text{an}}, X^{\text{an}} \times s) \geq m$ for all $s \in S$, so $\operatorname{codim}_{\mathbb{R}}(\mathcal{Z}_M^{\text{an}}, X^{\text{an}} \times M) \geq m$, and since $\dim_{\mathbb{R}}(M) \leq m-1$ and $X^{\text{an}} \times M$ is smooth, we can modify $(f_X, \text{id})$ homotopically, relative to its boundary, to a map $\widetilde{f_1}$ whose image does not intersect $\mathcal{Z}_M^{\text{an}}$. Then the image of $\widetilde{f} = (\text{id}, f_S) \circ \widetilde{f_1}$ lies in $\mathcal{U}$. □

**Lemma 2.2.** *For a scheme $S$, let $\mathcal{Z} \subset X \times S$ be a closed subscheme such that $\operatorname{codim}_{\mathbb{R}}(\mathcal{Z}_s^{\text{an}}, X^{\text{an}} \times s) \geq 4$ for all $s \in S$. Let $\mathcal{U}$ be the complement of $\mathcal{Z}$, and let $x \in \mathcal{U} \subset X \times S$ be a closed point. Then the inclusion $i^{\text{an}} : \mathcal{U}^{\text{an}} \hookrightarrow X^{\text{an}} \times S^{\text{an}}$ induces an isomorphism of topological fundamental groups*

$$\pi_1(i^{\text{an}}, x) : \pi_1(\mathcal{U}^{\text{an}}, x) \xrightarrow{\cong} \pi_1(X^{\text{an}} \times S^{\text{an}}, x).$$

*Proof.* To check that $\pi_1(i^{\text{an}})$ is injective, let $f : \mathbb{S}^1 \to \mathcal{U}^{\text{an}}$ be a continuous based loop (i.e. a continuous map from the unit interval $[0,1]$ sending 0 and 1 to the base point $x$) mapping to zero in $\pi_1(X^{\text{an}} \times S^{\text{an}}, x)$. So there is a continuous map $g$ fitting into a commutative diagram

$$\begin{array}{ccccc} \mathbb{S}^1 & \xrightarrow{f} & \mathcal{U}^{\text{an}} & \hookrightarrow & X^{\text{an}} \times S^{\text{an}} \\ \downarrow & & & & \| \\ \mathbb{D} & & \xrightarrow{g} & & X^{\text{an}} \times S^{\text{an}} \end{array}$$

where $\mathbb{D}$ denotes the unit disk (whose boundary is $\mathbb{S}^1$). By lemma 2.1 we can change $g$ by a homotopy relative to its boundary to a map whose image is in $\mathcal{U}^{\mathrm{an}}$, hence $[f] \in \pi_1(\mathcal{U}^{\mathrm{an}}, x)$ is zero.

To check that $\pi_1(i^{\mathrm{an}})$ is surjective, let

$$f : \mathbb{S}^1 \longrightarrow X^{\mathrm{an}} \times S^{\mathrm{an}}$$

be a continuous based loop. Applying lemma 2.1 we can change $f$, by a homotopy relative to the endpoints of the interval, to a based loop in $\mathcal{U}^{\mathrm{an}}$. □

**Corollary 2.3.** *With the same notation and hypothesis as in lemma 2.2, the inclusion $i$ induces an isomorphism of algebraic fundamental groups*

$$\pi^{\mathrm{alg}}(i, x) : \pi^{\mathrm{alg}}(\mathcal{U}, x) \xrightarrow{\cong} \pi^{\mathrm{alg}}(X \times S, x).$$

*Proof.* The algebraic fundamental group is canonically isomorphic to the completion of the topological fundamental group with respect to the topology of finite index subgroups (cfr. [SGA1, XII Cor. 5.2]), hence the result follows from lemma 2.2. □

The monomorphism $\rho_2 : G/Z \hookrightarrow \mathrm{Aut}(\mathfrak{g}')$ is the inclusion of the connected component of the identity of $\mathrm{Aut}(\mathfrak{g}')$. Thus $F = \mathrm{Aut}(\mathfrak{g}')/(G/Z)$ is a finite group.

Recall that the tautological family (1.16) parametrized by $R_1$ is denoted

$$\mathcal{E}_{R_1} = (E_{R_1}, \varphi_{R_1})$$

Let $\mathcal{U}_{R_1} \subset X \times R_1$ be the open set where $E_{R_1}$ is locally free. Then $\mathcal{E}_{R_1}$ gives a principal $\mathrm{Aut}(\mathfrak{g}')$-bundle $P_{R_1}$ on $\mathcal{U}_{R_1}$. Consider the functor $\Gamma(\rho_2, P_{R_1})$ of reductions defined as in (0.3).

**Proposition 2.4.** *The functor $\Gamma(\rho_2, P_{R_1})$ is represented by a scheme $R_2 \to R_1$ which is étale and finite over $R_1$, so there is a tautological family parametrized by $R_2$*

$$(q_{R_2}, P^{G/Z}_{R_2}, E_{R_2}, \psi_{R_2}) \tag{2.1}$$

*Proof.* The set of isomorphism classes of $S$-families of $\rho_2$-reductions is bijective to the set

$$\mathrm{Mor}_{\mathcal{U}_S}(\mathcal{U}_S, P_S(F)) \tag{2.2}$$

of sections of the pulled back principal $F$-bundle $P_S(F) \to \mathcal{U}_S$. Since $F$ is a finite group, giving the principal $F$-bundle $p : P_{R_1}(F) \to \mathcal{U}_{R_1}$ is equivalent to giving a representation of the algebraic fundamental group $\pi^{\mathrm{alg}}(\mathcal{U}_{R_1}, x)$ in $F$ ([SGA1, V §7]). By lemma 2.2 this fundamental group is isomorphic to $\pi^{\mathrm{alg}}(X \times R_1, x)$, so there is a unique principal $F$-bundle $\overline{P_{R_1}(F)}$ on $X \times R_1$ whose restriction to $\mathcal{U}_{R_1}$ is isomorphic to $P_{R_1}(F)$. We claim that the set (2.2) is bijective to

$$\mathrm{Mor}_{X\times S}(X \times S, \overline{P_{R_1}(F)}_S). \tag{2.3}$$

Indeed, an element of the set (2.2) corresponds to a trivialization of the principal bundle $P_S(F) \to \mathcal{U}_S$. If this is trivial, then the principal bundle $\overline{P_{R_1}(F)}_S \to X \times S$ will also be trivial, and trivializations of the former are in bijection with trivializations of the later, and these correspond to elements of (2.3), thus proving the claim.

Finally, the morphism $X \times R_1 \to R_1$ is projective and faithfully flat, $\overline{P_{R_1}(F)} \to X \times R_1$ is an étale and surjective, and $\overline{P_{R_1}(F)} \to R_1$ is projective. It follows from [Ra3, lemma 4.14.1] that the functor $\Gamma(\rho_2, P_{R_1})$ is representable by a scheme $R_2 \to R_1$ which is étale and finite over $R_1$. □

From proposition, together with proposition 0.36, we obtain the following

**Corollary 2.5.** *The family* $\mathcal{P}_{R_2} = (P^{G/Z}_{R_2}, E_{R_2}, \psi_{R_2})$ *is a universal family with group* $\mathrm{PGL}(V)$ *for the functor* $\widetilde{F}^{\tau}_{G/Z}$.

Recall $G' = [G,G]$ denotes the commutator subgroup. Clearly $G/G' \cong \mathbb{C}^{*q}$, and giving a principal $G/G'$-bundle is equivalent to giving $q$ line bundles. Note that $G/Z \times G/G' = G/Z'$, where $Z'$ is the center of $G'$. Denote the projection in the first factor by

$$\rho'_2 : G/Z' \to G/Z.$$

Let $d_1, \dots, d_q$ be $q$ fixed elements of $H^2(X,\mathbb{C})$. Define

$$R'_2 = J^{d_1}(X) \times \cdots \times J^{d_q}(X) \times R_2,$$

where $J^{d_i}(X)$ is the Jacobian parameterizing line bundles on $X$ with first Chern class equal to $d_i \in H^2(X,\mathbb{C})$. Using a Poincaré line bundle on $J^{d_i}(X) \times X$, we construct a tautological family parametrized by $R'_2$

$$(q_{R'_2}, P^{G/Z'}_{R'_2}, E_{R'_2}, \psi_{R'_2}) \tag{2.4}$$

where the principal $G/Z'$-bundle $P^{G/Z'}_{R'_2}$ is the product of the pullback of the principal $G/Z$-bundle $P^{G/Z}_{R_2}$ of the family (2.1), and the principal $\mathbb{C}^*$-bundles associated to line bundles on $X \times R'_2$ pulled back from Poincaré line bundles on $X \times J^{d_i}$.

**Lemma 2.6.** *The scheme* $R'_2$ *over* $R_2$ *represents the functor* $\Gamma(\rho'_2, P_{R_2})$.

*Proof.* It follows easily from the construction of $R'_2$. □

There is a lift of the trivial $\mathbb{C}^*$ action on the Jacobian $J(X)$ to the Poincaré bundle, providing it with a structure of a universal family with group $\mathbb{C}^*$. Using this action, we obtain from lemma 2.6 and proposition 0.36 the following

**Corollary 2.7.** *There is a natural action of* $G/G' \times \mathrm{PGL}(V)$ *on the family of principal* $G/Z'$*-sheaves* $\mathcal{P}^{G/Z'}_{R'_2} = (P^{G/Z'}_{R'_2}, E_{R'_2}, \psi_{R'_2})$, *providing it with a structure of universal family with group* $G/G' \times \mathrm{PGL}(V)$ *for the functor* $\widetilde{F}^{\tau}_{G/Z'}$.

## 3. Construction of $R_3$

Let $Z'$ be the center of the commutator subgroup $G' = [G,G]$. It is a finite abelian group. Consider the exact sequence of groups

$$1 \longrightarrow Z' \longrightarrow G \xrightarrow{\rho_3} G/Z' \longrightarrow 0. \tag{3.1}$$

Recall that the family (2.4) parametrized by $R'_2$ provides a principal $G/Z'$-bundle

$$P^{G/Z'}_{R'_2} \longrightarrow \mathcal{U}_{R'_2} \subset X \times R'_2, \tag{3.2}$$

where $\mathcal{U}_{R'_2}$ is the open set where the torsion free sheaf $E_{R'_2}$ of (2.4) is locally free.

We first recall some facts about nonabelian cohomology. For a scheme $Y$ and a group $H$, we denote by $\underline{H}$ the trivial étale sheaf on $Y$ with fiber $H$. Given a morphism $p : Y \to S$, we define $R^i p_*(\underline{H})$ the étale sheaf on $S$ associated to the presheaf

$$(u : U \to S) \longmapsto \check{H}^i_{\mathrm{et}}(Y_U, \underline{H}),$$

where $\check{H}^i_{\text{et}}$ denotes the Cech cohomology set with respect to the étale topology, and $Y_U = Y \times_S U$ For a finite abelian group $F$, let $H^i(Y^{\text{an}}; F)$ be the singular cohomology of $Y^{\text{an}}$ with coefficients in $F$. We will need the following comparison

**Theorem 3.1.** *Let $F$ be a finite abelian group, and $Y$ a scheme, locally of finite type. Then there is a canonical isomorphism*

$$\check{H}^i_{\text{et}}(Y, \underline{F}) \;\cong\; H^i(Y^{\text{an}}; F)$$

*Proof.* Follows from [SGA4, XVI Th. 4.1] ($\check{H}^i_{\text{et}}(Y, \underline{F}) \cong H^i_{\text{cl}}(Y^{\text{an}}; \underline{F})$) and the fact that étale cohomology can be calculated using Cech cohomology. □

**Lemma 3.2.** *Let $p : \mathcal{U}_{R'_2} \to R'_2$ be the projection to $R'_2$. Then, for $i \leq 2$,*

$$R^i p_* \underline{Z'} = \underline{H^i(X^{\text{an}}; Z')},$$

*i.e. $R^i p_* \underline{Z'}$ is the constant sheaf with fiber $H^i(X^{\text{an}}; Z')$, the singular cohomology group of $X^{\text{an}}$ with coefficients in $Z'$.*

*Proof.* Let $U \to R'_2$ be an étale open set of $R'_2$, and let $\mathcal{U}_U = \mathcal{U}_{R'_2} \times_{R'_2} U$. The isomorphism of the homotopy groups in lemma 2.2 provides an isomorphism of the singular homology groups

$$H_1({\mathcal{U}_U}^{\text{an}}; \mathbb{Z}) \xrightarrow{\cong} H_1(X^{\text{an}} \times U^{\text{an}}; \mathbb{Z})$$

Now we will show that

$$H_2({\mathcal{U}_U}^{\text{an}}; \mathbb{Z}) \longrightarrow H_2(X^{\text{an}} \times U^{\text{an}}; \mathbb{Z})$$

is an isomorphism. To check that it is injective, consider a class $\alpha$ in $H_2({\mathcal{U}_U}^{\text{an}}; \mathbb{Z})$ which maps to zero. This class is represented by a sum with integer coefficients $\sum n_i f_i$, where $f_i : M_i^2 \to {\mathcal{U}_U}^{\text{an}}$ are continuous maps with $M_i^2$ a polyhedron of real dimension 2. Since it maps to zero, there is a 3-dimensional singular chain $\beta$ in $X^{\text{an}} \times U^{\text{an}}$, represented by a sum with integer coefficients $\sum m_j g_j$, where the $g_j : M_j^3 \to X^{\text{an}} \times U^{\text{an}}$ are continuous maps with $M_j^3$ a polyhedron of real dimension 3, and we can assume that the boundary of $M_j^3$ is mapped to the union of the images of $f_i$. In particular, the image of this boundary is in ${\mathcal{U}_U}^{\text{an}}$.

By lemma 2.1, each map $g_j$ can be changed by a homotopy, relative to its boundary, to a map $\widetilde{g}_j$ whose image lies in ${\mathcal{U}_U}^{\text{an}}$. Then $\sum m_j \widetilde{g}_j$ is a cycle in ${\mathcal{U}_U}^{\text{an}}$ whose boundary is $\sum n_i f_i$, hence $\alpha$ is already zero in $H_2({\mathcal{U}_U}^{\text{an}}; \mathbb{Z})$.

To check surjectivity, note that a singular cocycle in $X^{\text{an}} \times U^{\text{an}}$ can be represented by a sum $\sum n_i f_i$ where, for each $i$,

$$f_i : M_i^2 \longrightarrow X^{\text{an}} \times U^{\text{an}}$$

is a continuous map from $M_i^2$, a closed manifold with real dimension 2 with a triangulation. By lemma 2.1 the map $f_i$ can modified by a homotopy to a map $\widetilde{f}_i$ whose image lies in ${\mathcal{U}_U}^{\text{an}}$. This modification does not change the homology class, so this proves surjectivity.

The inclusion $j : {\mathcal{U}_U}^{\text{an}} \hookrightarrow X^{\text{an}} \times U^{\text{an}}$ induces an isomorphism

$$j^* : H^i(X^{\text{an}} \times U^{\text{an}}; Z') \xrightarrow{\cong} H^i({\mathcal{U}_U}^{\text{an}}; Z')$$

for $i = 1$ or 2. Indeed, denoting $\mathcal{U} = \mathcal{U}_{U^{\mathrm{an}}}$ and $\mathcal{M} = X^{\mathrm{an}} \times U^{\mathrm{an}}$, the inclusion induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}^1(H_{i-1}(\mathcal{M};\mathbb{Z}),Z') & \longrightarrow & H^i(\mathcal{M};Z') & \longrightarrow & \mathrm{Hom}(H_i(\mathcal{M};\mathbb{Z}),Z') & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow j^* & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathrm{Ext}^1(H_{i-1}(\mathcal{U};\mathbb{Z}),Z') & \longrightarrow & H^i(\mathcal{U};Z') & \longrightarrow & \mathrm{Hom}(H_i(\mathcal{U};\mathbb{Z}),Z') & \longrightarrow & 0 \end{array}$$

where the exact rows are given by the universal coefficient theorem for singular cohomology ([Sp, Ch. 5 §5]), and then $j^*$ is an isomorphism by the 5-lemma.

By theorem 3.1, étale cohomology coincides with singular cohomology, hence taking sheafification we obtain

$$R^i p_* \underline{Z'} \xrightarrow{\cong} \underline{H^i(X;Z')}.$$

□

Given a scheme $Y$, the exact sequence (3.1) gives an exact sequence of pointed sets [Se1], [F-M]

$$\check{H}^1_{\mathrm{et}}(Y,\underline{G}) \longrightarrow \check{H}^1_{\mathrm{et}}(Y,\underline{G/Z'}) \longrightarrow \check{H}^2_{\mathrm{et}}(Y,\underline{Z'})$$

where the distinguished element for each set corresponds to the trivial cocycle (and exactness means that the inverse image of the distinguished element of the last set is equal to the image of the first map).

This exact sequence implies that, if $p: Y \to S$ is a morphism of schemes, there is an exact sequence of sheaves of sets on $S$

$$R^1 p_* \underline{G} \longrightarrow R^1 p_* \underline{G/Z'} \longrightarrow R^2 p_* \underline{Z'}, \tag{3.3}$$

which can be thought of as the relative version of the previous sequence.

**Lemma 3.3.** *Assume there is a reduction $(P^G,\zeta)$ to $G$ of an algebraic principal $G/Z'$-bundle $P$ on a scheme $Y$. Then the set of algebraic isomorphism classes of reductions is an $H^1(Y^{\mathrm{an}};Z')$-torsor.*

*Proof.* Recall that this means that $H^1(Y^{\mathrm{an}};Z')$ acts simply transitively on this set, i.e. it is a principal $H^1(Y^{\mathrm{an}};Z')$-bundle over a point, and hence, for each reduction $(P^G,\zeta)$, there is a natural bijection between $H^1(Y^{\mathrm{an}};Z')$ and the set of isomorphism classes of reductions, sending the zero element of $H^1(Y^{\mathrm{an}};Z')$ to $(P^G,\zeta)$.

Since $Z'$ is discrete abelian, $H^1(Y^{\mathrm{an}};Z') = \check{H}^1_{\mathrm{et}}(Y,\underline{Z'})$ (theorem 3.1). The action of this group on the set of reductions is defined as follows. Let $(P^G,\zeta)$ be an analytic reduction, and $\alpha \in \check{H}^1_{\mathrm{et}}(Y,\underline{Z'})$. Let $\{g_{ij}\}$ be a $\underline{G}$-cocycle representing the isomorphism class of $P^G$, and let $\{z_{ij}\}$ be a cocycle representing $\alpha$. Then $\{g_{ij}z_{ij}\}$ defines a principal $G$-bundle $\hat{P}^G$ and, using $\zeta$, an isomorphism $\hat{\zeta} : \rho_{3*}(\hat{P}^G) \cong P$. The action is

$$(P^G,\zeta)\cdot\alpha = (\hat{P}^G,\hat{\zeta}).$$

It is easy to check that this is well defined on the set of isomorphism classes of reductions, and the action is simply transitively. □

**Remark 3.4.** In the previous proof we have used the fact that $Z'$ is in the center of $G$. In general the set of reductions is bijective to a cohomology set with twisted coefficients.

The relative version of this bijection is as follows. Assume that we have a morphism of schemes $p : Y \to S$, and a principal $G/Z'$-bundle $P_S$ on $Y$

**Lemma 3.5.** *Let $P^G_S$ be a principal $G$-bundle on $Y$, with $\rho_{3*}P^G_S \cong P_S$. Then, for all étale open sets $U \to S$*

$$\widetilde{\Gamma}(\rho_3, P_S)(U) = R^1 p_* \underline{Z'}(U) \tag{3.4}$$

*where $\widetilde{\Gamma}(\rho_3, P_S)$ is the sheaf of reductions defined in the preliminaries.*

*Proof.* Lemma 3.3 gives a bijection (depending only on $P^G_S$)

$$\Gamma(\rho_3, P_S)(U) = \check{H}^1_{\mathrm{et}}(Y_U, \underline{Z'})$$

Sheafifying sides, we obtain the result. □

**Proposition 3.6.** *The functor $\widetilde{\Gamma}(\rho_3, P^{G/Z'}_{R'_2})$ is representable by a scheme $R'_3$ étale and finite over $R'_2$.*

*Proof.* The strategy of the proof is as follows. First we see that the subscheme $\hat{R}'_2 \subset R'_2$ corresponding to principal bundles that admit a reduction of structure group to $G$ is a union of connected components of $R'_2$. Then we show that the functor $\widetilde{\Gamma}(\rho_3, P^{G/Z'}_{R'_2})$ is a principal space over $\hat{R}'_2$, and the structure group of this principal space is the finite group $H^1(X^{\mathrm{an}}; Z')$, hence affine, and then it follows from descent theory that the functor is representable [FGA].

The principal $G/Z'$-bundle $P^{G/Z'}_{R'_2} \to \mathcal{U}_{R'_2}$ (cfr. 3.2) gives a section $\sigma'$ of $R^1 p_* \underline{G/Z'}$ over $R'_2$, and using (3.3) we obtain a section of $R^2 p_* \underline{Z'}$. The principal $G/Z'$-bundle corresponding to a point in $R'_2$ can be lifted to $G$ if and only if this section is zero at this point. By lemma 3.2 this sheaf is constant with finite fiber, hence the section is locally constant and it vanishes in a subscheme $\hat{R}'_2 \subset R'_2$, which is a union of certain connected components of $R'_2$.

By exactness of the sequence (3.3), we can cover $\hat{R}'_2$ with open sets $U_i$ (in the étale topology) such that the section $\sigma'|_{U_i}$ of $R^1 p_* \underline{G/Z'}$ over $U_i$ lifts to a section $\sigma_i$ of $R^1 p_* \underline{G}$. Refining the cover $U_i$ if necessary, we can assume that

$$\sigma_i \in H^1(\mathcal{U}_{U_i}, \underline{G}).$$

This means that there are principal $G$-bundles $P^G_i \to \mathcal{U}_{U_i}$ such that $\rho_{3*}P^G_i \cong P^{G/Z'}_{U_i}$. The action of $H^1(X^{\mathrm{an}}; Z')$ described in the proof of lemma 3.3 gives an action $\Theta$ on the functor of reductions $\widetilde{\Gamma}(\rho_3, P^{G/Z'}_{R'_2})$. By lemma 3.5, after restricting to $U_i$ we have an equality of functors

$$\widetilde{\Gamma}(\rho_3, P^{G/Z'}_{U_i}) = R^1 p_* \underline{Z'}|_{U_i} : (\mathrm{Sch}\,/U_i) \longrightarrow (\mathrm{Sets})$$

By lemma 3.2, $R^1 p_* \underline{Z'}$ is the sheaf of sections of $R'_2 \times H^1(X^{\mathrm{an}}; Z') \to R'_2$, and then $\widetilde{\Gamma}(\rho_3, P^{G/Z'}_{U_i})$ is represented by the scheme $U_i \times H^1(X^{\mathrm{an}}; Z')$, the action $\Theta$ becoming just multiplication on the right. Hence the functor $\widetilde{\Gamma}(\rho_3, P^{G/Z'}_{R'_2})$ is a principal space with group $H^1(X^{\mathrm{an}}; Z')$. Since this group is affine, by descent theory it follows that it is represented by a principal $H^1(X^{\mathrm{an}}; Z')$-bundle over $\hat{R}'_2$, and the result follows. □

Let $R_3 \subset R'_3$ be the union of components corresponding to principal $G$-sheaves with fixed numerical invariants $\tau$. The morphism $R_3 \to R'_2$ is also finite. Then, proposition 3.6 together with corollary 2.7 and proposition 0.36 conclude

**Corollary 3.7.** *The scheme $R_3$ is a universal space with group* $\mathrm{PGL}(V)$ *for the functor $\widetilde{F}_G$.*

Note that we have used the fact that the action of $G/G'$ on $R_2$ is trivial.

## 4. Construction of a quotient

Let $H$ be a reductive algebraic group acting on two schemes $T$ and $S$. We will use the following ([Ra3, lemma 5.1])

**Lemma 4.1** (Ramanathan)**.** *If $f : T \to S$ is an affine $H$-equivariant morphism and $p : S \to \hat{S}$ is a good quotient for the action of $H$, then there is a good quotient $q : T \to \hat{T}$ by $H$, and the induced morphism $\hat{f} : \hat{T} \to \hat{S}$ is affine.*

*Furthermore, if $f$ is finite, then $\hat{f}$ is finite. When $f$ is finite and $p : S \to \hat{S}$ is a geometric quotient, then $q : T \to \hat{T}$ is also a geometric quotient.*

**Theorem 4.2.** *There is a projective scheme $\mathfrak{M}^{\tau}_G$ corepresenting the functor $\widetilde{F}_G$ of families of semistable principal $G$-sheaves with numerical invariants $\tau$. There is an open subscheme $\mathfrak{M}^{\tau,s}_G$ whose closed points are in bijection with isomorphism classes of stable principal $G$-sheaves.*

*Proof.* We use the notation of proposition 1.6. Using geometric invariant theory, it is proved in [G-S1] that there is a good quotient for the action of $\mathrm{SL}(V)$ on the scheme $R$ of based $\delta$-semistable Lie tensors

$$p_R : R \longrightarrow \overline{R}/\!\!/ \mathrm{SL}(V),$$

where $\overline{R}$ is the closure of $R$ defined in proposition 1.6, and $\overline{R}/\!\!/ \mathrm{SL}(V)$ is a projective scheme, and that it is a geometric quotient on the open subscheme $R^s$ of based $\delta$-semistable Lie tensors. By proposition 1.7, the inclusion of based semistable $\mathfrak{g}'$-sheaves $R_1 \hookrightarrow R$ is proper, hence the restriction of $p_R$

$$p_{R_1} : R_1 \longrightarrow R_1/\,\mathrm{SL}(V) = \mathfrak{M}_1,$$

is also a good quotient onto a projective scheme, and it is a geometric quotient on the open set $R^s_1$ corresponding to based stable $\mathfrak{g}'$-sheaves. Since the center of $\mathrm{SL}(V)$ acts trivially on $R_1$, this is also a quotient by $\mathrm{PGL}(V)$.

For the scheme $R_3$ of based semistable principal $G$-sheaves, i.e. pairs $(q, \mathcal{P})$ where $\mathcal{P} = (P, E, \psi)$ is a semistable principal $G$-sheaf and $q : V \otimes \mathcal{O}_X(-m) \twoheadrightarrow E$ is a surjection inducing an isomorphism $V \cong H^0(E(m))$, the following composition is a finite morphism

$$f : R_3 \longrightarrow R'_2 = J^{\underline{d}} \times R_2 \longrightarrow J^{\underline{d}} \times R_1,$$

where $J^{\underline{d}} = J^{d_1}(X) \times \cdots \times J^{d_q}(X)$. Let $\mathrm{PGL}(V)$ act trivially on $J^{\underline{d}}$. Then

$$p : J^{\underline{d}} \times R_1 \longrightarrow J^{\underline{d}} \times R_1/\,\mathrm{SL}(V)$$

is a good quotient by $\mathrm{PGL}(V)$, whose restriction to $J^{\underline{d}} \times R^s_1$ is a geometric quotient. Therefore, by lemma 4.1, there exists a good quotient by $\mathrm{PGL}(V)$

$$q : R_3 \longrightarrow \mathfrak{M}^{\tau}_G$$

which is a geometric quotient on the subscheme $R_3^s$ of based stable principal $G$-sheaves. Furthermore, the induced morphism $\overline{f}: \mathfrak{M}_G^\tau \to J^{\underline{d}} \times \mathfrak{M}_1$ is finite, hence $\mathfrak{M}_G^\tau$ is projective.

By corollary 3.7, the scheme $R_3$ is a universal space with group $\mathrm{PGL}(V)$ for the functor $\widetilde{F}_G$, hence, by remark 0.35, the projective scheme $\mathfrak{M}_G^\tau$ corepresents the functor $\widetilde{F}_G$.

The last statement follows also from Ramanathan's lemma, because $f$ is finite. □

*Two semistable principal sheaves are called GIT-equivalent if they correspond to the same point in the moduli space.* Now we will show that this amounts to the notion of S-equivalence given in the introduction (definition 0.7).

Let $\mathcal{P} = (P, E, \psi)$ be a semistable principal sheaf. If it is not stable, let $E_\bullet$, or $E_{\lambda_\bullet}$ be an admissible filtration, i.e. a balanced algebra filtration with

$$\sum_{i\in\mathbb{Z}} \big(rP_{E_i} - r_i P_E\big) \;=\; \sum_{i=1}^{t} (\lambda_{i+1} - \lambda_i)\big(rP_{E_{\lambda_i}} - r_{\lambda_i} P_E\big) \;=\; 0\,. \tag{4.1}$$

Let $U'$ be the open subset of $X$ where it is a vector bundle filtration. By lemma 5.4 this bundle filtration amounts to a reduction $P^Q$ of $P|_{U'}$ to a parabolic subgroup $Q \subset G$ together with a character $\chi$ of the Lie algebra of $Q$. Let $Q \twoheadrightarrow L$ be its Levi quotient, and $L \hookrightarrow Q \subset G$ a splitting. In the introduction we called the principal $G$-sheaf

$$\big(P^Q(Q \twoheadrightarrow L \hookrightarrow G), \oplus E^i, \psi'\big)$$

the *admissible deformation of $\mathcal{P}$ associated to $E_\bullet$*, whose associated $\mathfrak{g}'$-sheaf is $\oplus[,]^{i,j}: E^i \otimes E^j \to E^{i+j\,\vee\vee}$.

**Proposition 4.3.** *Any admissible deformation of a semistable principal $G$-sheaf $\mathcal{P}$ is semistable. After a finite number of admissible deformations, a principal $G$-sheaf is obtained such that any further admissible deformation is isomorphic to itself. This principal $G$-sheaf depends only on $\mathcal{P}$, and we denote it* $\operatorname{grad}\mathcal{P}$ *(and* $\operatorname{grad}\mathcal{P} := \mathcal{P}$ *if $\mathcal{P}$ is stable).*

*Two principal sheaves $\mathcal{P}$ and $\mathcal{P}'$ are GIT-equivalent if and only if they are S-equivalent in the sense that* $\operatorname{grad}\mathcal{P} \cong \operatorname{grad}\mathcal{P}'$.

*Proof.* Let $z \in R_3$ and let $\overline{\mathrm{SL}(V)\cdot z}$ be the closure of its orbit. It is a union of orbits, and by definition of good quotient, it has a unique closed orbit $B_3(z)$, which is characterized as the unique orbit in $\overline{\mathrm{SL}(V)\cdot z}$ with minimal dimension. Thus, two points $z$ and $z'$ in $R_3$ are GIT-equivalent (i.e. mapped to the same point in the moduli space) if and only if $B_3(z) = B_3(z')$.

**Claim.** If $\mathrm{SL}(V)\cdot z$ is not closed, then there exists a one-parameter subgroup $\lambda$ of $\mathrm{SL}(V)$ with $\mu(f(z), \lambda) = 0$ such that the limit $z_0 = \lim_{t\to 0} \lambda(t)\cdot z$ is in $\overline{\mathrm{SL}(V)\cdot z} \setminus \mathrm{SL}(V)\cdot z$.

Indeed, recall that we have a finite $\mathrm{SL}(V)$ equivariant morphism

$$R_3 \xrightarrow{f} J^{\underline{d}} \times R_1 \;\subset\; J^{\underline{d}} \times \overline{R_1}$$

where $\overline{R_1}$ is the closure of $R_1$ in the projective variety $\overline{R}$ defined in proposition 1.6. Note that $J^{\underline{d}} \times R_1$ is the open subscheme of semistable points of the projective variety $J^{\underline{d}} \times \overline{R_1}$. Since $z$ is not in $B_3(z)$, the point $f(z)$ is not in $B(f(z))$ (the closed orbit in the closure of $\mathrm{SL}(V)\cdot f(z) \subset J^{\underline{d}} \times R_1$), because the morphism $f$ sends orbits to orbits and $\dim(f(\mathrm{SL}(V)\cdot z)) = \dim(\mathrm{SL}(V)\cdot f(z))$, since $f$ is equivariant and finite. By [Si, lemma 1.25], there is a one parameter subgroup $\lambda$ of $\mathrm{SL}(V)$ such that $\overline{f(z)} := \lim_{t\to 0}\lambda(t)\cdot f(z) \in B(f(z))$. Since $f(z)$ is semistable, $\mu(f(z),\lambda)\leq 0$. If this inequality were strict, then $\mu(\overline{f(z)},\lambda^{-1})>0$, which is impossible because $\overline{f(z)}$ is a semistable point. Therefore $\mu(f(z),\lambda)=0$. Since $f$ is proper, $\lim_{t\to 0}\lambda(t)\cdot z$ exists, and furthermore it belongs to $B(z)\subset \overline{\mathrm{SL}(V)\cdot z}\setminus \mathrm{SL}(V)\cdot z$, thus proving our claim.

For any one-parameter subgroup with $\mu(f(z),\lambda)=0$, $\lim_{t\to 0}\lambda(t)\cdot f(z)$ exists and is semistable [G-S, Prop. 2.14], and since $f$ is proper, $\lim_{t\to 0}\lambda(t)\cdot z$ also exists in $R_3$.

**Claim.** There is a bijection between one-parameter subgroups of $\mathrm{SL}(V)$ with $\mu(f(z),\lambda)=0$ on the one side, and admissible ($P_{E_{\lambda_\bullet}}=0$) saturated balanced algebra filtrations $E_{\lambda_\bullet}$ of $E$ together with a splitting of the induced filtration $H^0(E_{\lambda_\bullet}(m))$ in $V$ on the other side.

Indeed, in [G-S1] we established a bijection between one-parameter subgroups of $\mathrm{SL}(V)$ with $\mu(f(z),\lambda)=0$ and balanced filtrations with

$$P_{E_{\lambda_\bullet}}+\mu_{\mathrm{tens}}(E_{\lambda_\bullet},\phi)\delta=0$$

where $(E,\phi)$ is the tensor corresponding to the point $f(z)$. Therefore, the $\delta$-semistability of this tensor implies that the filtration $E_{\lambda_\bullet}$ is saturated (since the left hand side of the former equality is bigger for the saturation). The leading coefficient is

$$\sum_{i=1}^{t}(\lambda_{i+1}-\lambda_i)(\deg E_{\lambda_i}\operatorname{rk}E-\operatorname{rk}E_{\lambda_i}\deg E)+\mu_{\mathrm{tens}}(E_{\lambda_\bullet},\phi)\tau=0 \tag{4.2}$$

By lemma 0.27, $\deg E=0$. Lemma 1.4 implies $\deg E_{\lambda_i}\leq 0$, and recall $\tau>0$. Therefore lemmas 1.1 and 1.3 imply $\mu_{\mathrm{tens}}(E_{\lambda_\bullet},\phi)=\mu(E_{\lambda_\bullet},\varphi)\leq 0$. Since we have equality in (4.2), it must be $\mu(E_{\lambda_\bullet},\varphi)=0$. Hence, by lemma 1.3, the filtration $E_{\lambda_\bullet}$ is an algebra filtration, thus proving the claim.

Now, let $\mathcal{P}=(P,E,\psi)$ be a semistable principal $G$-sheaf, choose a quotient $q:V\otimes\mathcal{O}_X(-m)\to E$, and let $z\in R_3$ be the point corresponding to the based principal $G$-sheaf $(q,\mathcal{P})$. Let $\lambda:\mathbb{C}^*\to\mathrm{SL}(V)$ be the one-parameter subgroup associated to an admissible saturated algebra filtration. The action of $\lambda$ on the point $z$ define a morphism $\mathbb{C}^*\to R_3$ that extends to

$$h:T=\mathbb{C}\longrightarrow R_3\,,$$

with $h(t)=\lambda(t)\cdot z$ for $t\neq 0$ and $h(0)=\lim_{t\to 0}\lambda(t)\cdot z=z_0$. In the rest of this section we shall show that the point $z_0$ corresponds to the associated admissible deformation. Then it will follow that the limit $z_0$ fails to be in the orbit of $z$ if and only if the associated admissible deformation fails to be isomorphic to $\mathcal{P}$.

If $z_0$ is not in the orbit of $z$, since $\mathrm{SL}(V)\cdot z_0\subset\overline{\mathrm{SL}(V)\cdot z}\setminus\mathrm{SL}(V)\cdot z$, it is $\dim\mathrm{SL}(V)\cdot z_0<\dim\mathrm{SL}(V)\cdot z$, so if we iterate this process (with $z_0$ and another one-parameter subgroup as before) we get a sequence of points $z_0$, $z_0'$, $z_0''$,... that must stop giving a point in $B(z)$. Hence, the principal $G$-sheaf $\operatorname{grad}\mathcal{P}$, up to isomorphism, depends only on $\mathcal{P}$, because there is only one closed orbit in $\overline{\mathrm{SL}(V)\cdot z}$.

To finish the proof of the proposition it only remains to show that the point $z_0$ corresponds to the associated admissible deformation. This will be done constructing a based family $(q_T,\mathcal{P}_T)=(q_T,P_T,E_T,\psi_T)$ such that $(q_t,\mathcal{P}_t)$ corresponds to the point $h(t)\in R_3$ when $t\neq 0$ and $\mathcal{P}_0$ is the associated admissible deformation. Since $R_3$ is separated, it will follow that $(q_0,\mathcal{P}_0)=(q_0,P_0,E_0,\psi_0)$ corresponds to $z_0$.

First we define a based family $(q_T, E_T, \varphi_T)$ of $\mathfrak{g}'$-sheaves. For any $n \in \mathbb{Z}$, define $E_n = E_{\lambda_{i(n)}}$, where (recall from Preliminaries before definition 0.15) $i(n)$ is the maximum index with $\lambda_{i(n)} \leq n$. Let $-N$ be a negative integer such that $E_n = 0$ for $n \leq -N$, and write $V_n = H^0(E_n(m))$. Borrowing the formalism from [H-L, §4.4], define

$$E_T = \bigoplus_n E_n \otimes t^n \subset E \otimes_{\mathbb{C}} t^{-N}\mathbb{C}[t] \subset E \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$$

$$\begin{array}{rcccc} q_T : V \otimes \mathcal{O}_X(-m) \otimes \mathbb{C}[t] & \xrightarrow{\gamma} & \oplus_n V_n \otimes \mathcal{O}_X(-m) \otimes t^n & \longrightarrow & E_T \\ v^n \otimes 1 & \longmapsto & v^n \otimes t^n & \longmapsto & q(v^n) \otimes t^n \end{array}$$

$$\begin{array}{rcl} \varphi_T : (\oplus E_n \otimes t^n) \otimes (\oplus E_n \otimes t^n) & \longrightarrow & (\oplus E_n \otimes t^n)^{\vee\vee} \\ w_i \otimes t^i \otimes w_j \otimes t^j & \longmapsto & [w_i, w_j] \otimes t^{i+j} \end{array}$$

where $v^n$ is a local section of $V^n \otimes \mathcal{O}_X(-m)$, and $w_i$, $w_j$ are local sections of $E_i$ and $E_j$. Then, as in [H-L, §4.4], $(q_t, E_t, \varphi_t)$ corresponds to $f(h(t))$ (in particular, if $t \neq 0$, then $(E_t, \varphi_t)$ is canonically isomorphic to $(E, \varphi)$), and $(E_0, \varphi_0)$ is the admissible deformation associated to $E_{\lambda_\bullet}$.

Now we will define the family of principal $G$-bundles $P_T$. The saturated balanced algebra filtration $E_{\lambda_\bullet}$ provides, by lemma 5.4, a reduction $P^Q$ of $P|_{U'}$ to a parabolic subgroup $Q$ on the open set $U'$ where $E_{\lambda_\bullet}$ is a bundle filtration, together with dominant character $\chi$ of $\mathfrak{q} = \mathrm{Lie}(Q)$. Let $Q = LU$ be a Levi decomposition of the parabolic subgroup $Q$, and denote $\mathfrak{l} = \mathrm{Lie}(L)$, $\mathfrak{u} = \mathrm{Lie}(U)$. Let $\mathfrak{h} \subset \mathfrak{l}$ be a Cartan algebra. Let $v \in \mathfrak{z}_\mathfrak{l}$ be the element associated to $\chi$ by lemma 5.3, We can associate to $v$, without loss of generality, a one-parameter subgroup

$$\Psi : \mathbb{C}^* \to Z_L$$

of $Z_L$, the center of the Levi factor $L$ corresponding to $\mathfrak{l}$, such that $d\Psi(1) = v$. Indeed, on the one hand, an integer multiple $av$ provides such a subgroup (lemma 5.5), and on the other hand, if we replace the indexes $\lambda_i$ by $a\lambda_i$, the associated one-parameter subgroup $\lambda(t)$ is replaced by $\lambda(t^a)$, and $h(t)$ is replaced by $h(t^a)$, and $v$ by $av$, but this doesn't change the limit $z_0$.

The adjoint action of $\Psi(t)$ on any $x \in \mathfrak{u}$ has zero limit as $t = e^\tau \in \mathbb{C}^*$ goes to zero, since using the root decomposition $x = \sum_{\alpha \in R^+(\mathfrak{z}_\mathfrak{l})} x_\alpha$ with respect to $\mathfrak{z}_\mathfrak{l}$, this action is

$$\Psi(t) \cdot x = \sum \Psi(t) \cdot x_\alpha = \sum e^{\tau v} \cdot x_\alpha = \sum e^{\tau\alpha(v)} x_\alpha = \sum t^{\alpha(v)} x_\alpha$$

and the limit is zero because $\alpha \in R^+(\mathfrak{t})$, so that $\alpha(v) > 0$. Therefore, since the exponential map is $G$-equivariant with respect to the adjoint action, for any element $u = e^x \in U$, it is

$$\lim_{t \to 0} \Psi(t) \cdot e^x = \lim_{t \to 0} e^{\Psi(t) \cdot x} = 1$$

Thus, since $\Psi(t)$ is in the center $Z_L$ of $L$, the adjoint action $\Psi(t) \cdot lu = \Psi(t)^{-1} lu \Psi(t)$ on any $lu \in LU = Q$ has limit

$$\lim_{t \to 0} \Psi(t) \cdot lu = l \lim_{t \to 0} \Psi(t) \cdot u = l$$

Let $\{g_{\alpha\beta} : U'_{\alpha\beta} \to Q \subset G\}$ be a 1-cocycle on $U'$ describing $P^Q|_{U'}$. Denote by $P_T$ the principal $G$-bundle on $U' \times T$ described by

$$\{\Psi(t)^{-1} g_{\alpha\beta} \Psi(t) : U'_{\alpha\beta} \times T \to Q \subset G\}$$

Note that $\Psi$ is defined only on values $t \in \mathbb{C}^*$, but the previous observations show that this cocycle can be extended to $t = 0$, and for this special value it describes the principal $G$-bundle $P^Q(Q \twoheadrightarrow L \hookrightarrow G)$, thus admitting a reduction of structure group to $L$. Remark also that, for $t \neq 0$, there is a canonical isomorphism between the principal $G$-bundle $P_t$ on $U'$ and $P|_{U'}$, hence $P_T$ extends canonically to a principal $G$-bundle on $U_{E_T} \subset X \times T$ which we still denote $P_T$.

It remains to construct an isomorphism of vector bundles $\psi_T : P_T(\mathfrak{g}') \to E_T|_{U' \times T}$. Let $\mathcal{W} = \mathcal{O}_X^{\oplus r}$, and let $\mathcal{W}_n \subset \mathcal{W}$ be the trivial subbundle defined as the direct sum of the first $\operatorname{rk} E_n$ summands, and $\mathcal{W}^n = \mathcal{W}_n / \mathcal{W}_{n-1}$. Take a covering $\{U'_\alpha\}$ of $U'$ with trivializations $\psi_\alpha : \mathcal{W}|_{U'_\alpha} \to E|_{U'_\alpha}$ preserving the filtration on $E$, i.e. such that $\psi$ restricts to an isomorphism between $\mathcal{W}_n|_{U'_\alpha}$ and $E_n|_{U'_\alpha}$. Consider the $\mathfrak{g}'$-sheaf isomorphism

$$\begin{array}{rcl} \gamma : \mathcal{W}|_{U'_\alpha} \otimes \mathbb{C}[t] & \longrightarrow & \oplus \mathcal{W}_n|_{U'_\alpha} \otimes t^n \\ v^n \otimes 1 & \longmapsto & v^n \otimes t^n \end{array}$$

where $v^n$ is a local section of $\mathcal{W}^n$. The transition functions $h_{\alpha\beta} : U'_{\alpha\beta} \to \operatorname{Aut}(\mathfrak{g}') \subset \operatorname{GL}(\mathfrak{g}')$ of $E|_{U'}$ can be chosen to be block-upper triangular matrices

$$h_{\alpha\beta} = \left\{ \begin{array}{cccc} M_{\lambda_1\lambda_1} & M_{\lambda_1\lambda_2} & \cdots & M_{\lambda_1\lambda_{t+1}} \\ 0 & M_{\lambda_2\lambda_2} & \cdots & M_{\lambda_2\lambda_{t+1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{\lambda_{t+1}\lambda_{t+1}} \end{array} \right\}$$

where $M_{\lambda_i\lambda_j}$ is a matrix of dimension $\operatorname{rk} E^{\lambda_i} \times \operatorname{rk} E^{\lambda_j}$. The commutativity of the diagram

$$\begin{array}{ccccc} \mathcal{W}|_{U'_{\alpha\beta}} \otimes \mathbb{C}[t] & \xrightarrow[\cong]{\gamma} & \oplus \mathcal{W}_n|_{U'_{\alpha\beta}} \otimes t^n & \hookrightarrow & \mathcal{W}|_{U'_{\alpha\beta}} \otimes t^{-N}\mathbb{C}[t] \\ \Big\downarrow {\scriptstyle \gamma^{-1}(t)h_{\alpha\beta}\gamma(t)} & & \cong \Big\downarrow {\scriptstyle \psi\otimes\mathrm{id}} & \swarrow^{\cong}_{\psi\otimes\mathrm{id}} & \cong \Big\downarrow {\scriptstyle h_{\alpha\beta}\otimes\mathrm{id}} \\ & & \oplus E_n|_{U'_{\alpha\beta}} \otimes t^n & \hookrightarrow \; E|_{U'_{\alpha\beta}} \otimes t^{-N}\mathbb{C}[t] & \\ & & \cong \Big\uparrow {\scriptstyle \psi\otimes\mathrm{id}} & \nwarrow^{\psi\otimes\mathrm{id}}_{\cong} & \\ \mathcal{W}|_{U'_{\alpha\beta}} \otimes \mathbb{C}[t] & \xrightarrow[\cong]{\gamma} & \oplus \mathcal{W}_n|_{U'_{\alpha\beta}} \otimes t^n & \hookrightarrow & \mathcal{W}|_{U'_{\alpha\beta}} \otimes t^{-N}\mathbb{C}[t] \end{array}$$

shows that the transition functions of $E_T|_{U' \times T}$ are $\gamma^{-1}(t)h_{\alpha\beta}\gamma(t) : U'_{\alpha\beta} \times T \to \operatorname{Aut}(\mathfrak{g}') \subset \operatorname{GL}(\mathfrak{g}')$, i.e.

$$\gamma^{-1}(t)h_{\alpha\beta}\gamma(t) = \left\{ \begin{array}{cccc} M_{\lambda_1\lambda_1} & M_{\lambda_1\lambda_2}t^{\lambda_2-\lambda_1} & \cdots & M_{\lambda_1\lambda_{t+1}}t^{\lambda_{t+1}-\lambda_1} \\ 0 & M_{\lambda_2\lambda_2} & \cdots & M_{\lambda_2\lambda_{t+1}}t^{\lambda_{t+1}-\lambda_2} \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & M_{\lambda_{t+1}\lambda_{t+1}} \end{array} \right\}$$

This is well defined for $t = 0$ because $M_{\lambda_i\lambda_j} = 0$ when $\lambda_i - \lambda_j < 0$. Since the adjoint action of $\Psi(t)$ on $h_{\alpha\beta}$ is precisely $\Psi(t) \cdot h_{\alpha\beta} = \gamma^{-1}(t)h_{\alpha\beta}\gamma(t)$, we obtain an isomorphism $\psi_T : P_T(\mathfrak{g}')|_{U' \times T} \to E_T|_{U' \times T}$, hence a family $\mathcal{P}_T = (P_T, E_T, \psi_T)$. Note that, for $t \neq 0$, using the canonical isomorphisms $E_t \cong E$ and $P_t \cong P|_{U'}$, the isomorphism $\psi_t$ becomes $\psi$, hence $\psi_T$ extends to an isomorphism $P_T(\mathfrak{g}') \to E_T|_{U_{E_T}}$, which we still denote $\psi_T$. Finally, it is easy to check that $(q_t, \mathcal{P}_t)$ corresponds to $h(t)$ and $\mathcal{P}_0 \cong \operatorname{grad} \mathcal{P}$.

□

## 5. Slope (semi)stability as Ramanathan (semi)stability

In [Ra2], Ramanathan defines a *rational principal bundle* on $X$ as a principal bundle $P$ over a big open set $U \subset X$, and gives a notion of (semi)stability, which is a direct generalization of his notion of (semi)stability in [Ra3] for $\dim X = 1$.

**Definition 5.1** (Ramanathan). *A rational principal $G$ bundle $P \to U \subset X$ is (semi)stable if for any reduction $P^Q$ to a parabolic subgroup $Q$ over a big open set $U' \subset U$, and for any dominant character $\chi$ of $Q$, it is*

$$\deg P^Q(\chi)\ (\leq)\ 0.$$

Let $\mathcal{P} = (P, E, \psi)$ be a principal $G$-sheaf and let $U$ be the open set where $E$ is locally free. We will show in this section that $\mathcal{P}$ is slope-(semi)stable if and only if the rational bundle $P$ is (semi)stable in the sense of Ramanathan. In particular, we will obtain that, if $X$ is a curve, our notion of (semi)stability for principal bundles coincides with that of Ramanathan. As mentioned in the introduction, this section plays also the role of an appendix where we prove some facts that have been already used.

Recall (from [J], for instance) the well known notions of filtration and graduation of a Lie algebra $\mathfrak{g}$. An algebra filtration $\mathfrak{g}_\bullet$ is a sequence

$$\cdots \subseteq \mathfrak{g}_{i-1} \subseteq \mathfrak{g}_i \subseteq \mathfrak{g}_{i+1} \cdots$$

starting by 0 and ending by $\mathfrak{g}$, such that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j} \quad \text{for all } i,\, j \in \mathbb{Z}$$

or, deleting (from 0 onward) all nonstrict inclusions, it is $\mathfrak{g}_{\lambda_\bullet}$

$$0 \subsetneq \mathfrak{g}_{\lambda_1} \subsetneq \mathfrak{g}_{\lambda_2} \subsetneq \ \cdots\ \subsetneq \mathfrak{g}_{\lambda_{t+1}} = \mathfrak{g}, \quad (\lambda_1 < \cdots < \lambda_{t+1})$$

with

$$[\mathfrak{g}_{\lambda_i}, \mathfrak{g}_{\lambda_j}] \subseteq \mathfrak{g}_{\lambda_{k-1}} \quad \text{if } \lambda_i + \lambda_j < \lambda_k.$$

A graded structure $\mathfrak{g}^\bullet$ is a decomposition

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}^i \quad \text{with} \quad [\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j} \quad \text{for all } i,\, j \in \mathbb{Z}$$

or, deleting all zero summands,

$$\mathfrak{g} = \bigoplus_{i=1}^{t+1} \mathfrak{g}^{\lambda_i} \quad (\lambda_1 < \cdots < \lambda_{t+1}).$$

with

$$[\mathfrak{g}^{\lambda_i}, \mathfrak{g}^{\lambda_j}] \subseteq \begin{cases} \mathfrak{g}^{\lambda_k} & \text{if there is } k \text{ with } \lambda_k = \lambda_i + \lambda_j \\ 0 & \text{otherwise} \end{cases}$$

To a graded algebra $\mathfrak{g}^\bullet$ it is associated a filtered algebra $\mathfrak{g}_\bullet$ with

$$\mathfrak{g}_i = \bigoplus_{j\leq i} \mathfrak{g}^j$$

and reciprocally, to a filtered algebra $\mathfrak{g}_\bullet$ it is associated a graded algebra

$$(\mathrm{gr}\,\mathfrak{g})^i = \mathfrak{g}_i/\mathfrak{g}_{i-1}$$

with Lie algebra structure

$$[\overline{v},\overline{w}]=[v,w] \mod \mathfrak{g}_{i+j-1}$$

for $v\in\mathfrak{g}_i\setminus\mathfrak{g}_{i-1}$ and $w\in\mathfrak{g}_j\setminus\mathfrak{g}_{j-1}$.

A graded algebra $\mathfrak{g}^\bullet$ is called balanced if $\sum i\dim\mathfrak{g}^i=0$. In terms of $\mathfrak{g}^{\lambda_\bullet}$, this is $\sum\lambda_i\dim\mathfrak{g}^{\lambda_i}=0$. A filtered algebra is called balanced if the associated graded algebra is so. We start this appendix proving the following

**Lemma 5.2.** *Let $\mathfrak{g}'_\bullet$ be a balanced algebra filtration of a semisimple Lie algebra $\mathfrak{g}'$. There is a Lie algebra isomorphism between $\mathfrak{g}'$ and the associated Lie algebra* $\mathrm{gr}\,(\mathfrak{g}'_\bullet)$.

*Proof.* Let $W$ be the vector space underlying the Lie algebra $\mathfrak{g}'$. Choose a basis $e_l$ of $W$ adapted to the filtration ${\mathfrak{g}'}_{\lambda_\bullet}$. Associate the one-parameter subgroup $\lambda(t)$ of $\mathrm{GL}(W)$ expressed as $\mathrm{diag}(t^{\lambda_\bullet})$ in this basis. Since the filtration is balanced, this is in fact a one-parameter subgroup of $\mathrm{SL}(W)$. The Lie algebra structure of $W$ is a point $v=\sum a^n_{lm}e^l\otimes e^m\otimes e_n$ in the linear space $W^\vee\otimes W^\vee\otimes W$. The action of the one-parameter subgroup is

$$a^n_{lm}\longmapsto t^{\lambda_{i(l)}+\lambda_{i(m)}-\lambda_{i(n)}}a^n_{lm},$$

where $i(l)$ is the minimum integer for which $e_l\in{\mathfrak{g}'}_{\lambda_{i(l)}}$. The point $\overline{v}\in\mathbb{P}(W^\vee\otimes W^\vee\otimes W)$ is GIT-semistable with respect to the induced action of $\mathrm{SL}(W)$ on this projective space and on its polarization line bundle $\mathcal{O}_{\mathbb{P}}(1)$ (by lemma 1.2), hence the Hilbert-Mumford criterion implies

$$\mu:=\min\left\{\lambda_{i(l)}+\lambda_{i(m)}-\lambda_{i(n)}\ :\ a^n_{lm}\neq 0\right\}\leq 0$$

Furthermore, $\mu=0$ because $\lambda_\bullet$ is an algebra filtration. Indeed, if $\mu<0$ then for some triple $(\lambda_i,\lambda_j,\lambda_k)$ with $\lambda_i+\lambda_j<\lambda_k$ it would be $[{\mathfrak{g}'}_{\lambda_i},{\mathfrak{g}'}_{\lambda_j}]\nsubseteq{\mathfrak{g}'}_{\lambda_{k-1}}$, contradicting the fact that ${\mathfrak{g}'}_{\lambda_\bullet}$ is algebra filtration.

Since $\mu=0$, the following limit exists and is nonzero

$$v_0:=\lim_{t\to 0}\lambda(t)\cdot v\ \in\ W^\vee\otimes W^\vee\otimes W$$

Since the subset of points of $W^\vee\otimes W^\vee\otimes W-\{0\}$ giving $W$ a Lie algebra structure is closed, the point $v_0$ itself provides $W$ with a Lie algebra structure. By construction, the coordinates $b^n_{lm}$ of $(W,v_0)$ are

$$b^n_{lm}=\begin{cases} a^n_{lm}\,, & \lambda_{i(l)}+\lambda_{i(m)}-\lambda_{i(n)}=0\\ 0\,, & \lambda_{i(l)}+\lambda_{i(m)}-\lambda_{i(n)}\neq 0\end{cases}$$

In other words, $(W,v_0)\cong\mathrm{gr}\,({\mathfrak{g}'}_{\lambda_\bullet})$. Let $k(t):W\otimes W\to\mathbb{C}$ be the Killing form of $\lambda(t)\cdot v$. Since $\lambda(t)\in\mathrm{SL}(W)$,

$$\det(k(t))=\det(\lambda(t)^{-1}k(1)\lambda(t))=\det(k(1))\neq 0\quad\text{for all } t\in\mathbb{C}^*,$$

thus also for $t=0$. Since this determinant is nonzero, $(W,v_0)$ is semisimple. By the rigidity of semisimple Lie algebras, $(W,v_0)\cong(W,v)=\mathfrak{g}'$. □

Let $\mathfrak{a}$ be a *toral algebra* $\mathfrak{a}\subset\mathfrak{g}$, i.e. an algebra consisting of semisimple elements, thus abelian [Hum, §8.1], which is not necessarily maximal. Following [B-T, §3], we can define the set $R(\mathfrak{a})\subset\mathfrak{a}^\vee$ of $\mathfrak{a}$-roots in the following way. For $\alpha\in\mathfrak{a}^\vee$, write

$$\mathfrak{g}^\alpha=\{x\in\mathfrak{g}:[s,x]=\alpha(s)x,\ \text{for all } s\in\mathfrak{a}\} \tag{5.1}$$

Then $R(\mathfrak{a}) = \{\alpha \in \mathfrak{a}^\vee \setminus 0 : \mathfrak{g}^\alpha \neq 0\}$ For $\mathfrak{h}$ is a maximal toral algebra (i.e. Cartan algebra) containing $\mathfrak{a}$, $\mathfrak{a}$-roots can be thought of as classes of $\mathfrak{h}$-roots by saying that two $\mathfrak{h}$-roots are equivalent if their restrictions to $\mathfrak{a}$ are the same. Let $R(\mathfrak{h}) = R^+(\mathfrak{h}) \cup R^-(\mathfrak{h})$ be a decomposition into positive and negative $\mathfrak{h}$-roots. If $\beta \sim \beta' \nsim 0$, then $\beta$ is positive if and only if $\beta'$ is positive, hence there is an induced decomposition $R(\mathfrak{a}) = R^+(\mathfrak{a}) \cup R^-(\mathfrak{a})$. In particular, this gives a partial ordering among $\mathfrak{a}$-roots: $\alpha < \alpha'$ when $\alpha' - \alpha$ is a sum of positive $\mathfrak{a}$-roots.

**Lemma 5.3.** *Let $\mathfrak{q}$ be a parabolic subalgebra of a semisimple Lie algebra $\mathfrak{g}'$ and $\chi : \mathfrak{q} \to \mathbb{C}$ a character of $\mathfrak{q}$. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a Levi decomposition, and $\mathfrak{z}_\mathfrak{l}$ the center of the Levi subalgebra $\mathfrak{l}$. Then there is an element $v \in \mathfrak{z}_\mathfrak{l}$ such that*

$$\chi(\cdot) = (v, \cdot) : \mathfrak{q} \longrightarrow \mathbb{C}$$

*where $(\cdot, \cdot)$ is the Killing form of $\mathfrak{g}'$.*

*Proof.* Let $\mathfrak{l}' = [\mathfrak{l}, \mathfrak{l}]$ be the commutator subalgebra. The decomposition $\mathfrak{l} = \mathfrak{l}' \oplus \mathfrak{z}_\mathfrak{l}$ is orthogonal with respect to the Killing form $\kappa = (\cdot, \cdot)$ on $\mathfrak{g}'$. Indeed, since $\kappa$ is $\mathfrak{g}'$-invariant, if $l_1$, $l_2 \in \mathfrak{l}$ and $z \in \mathfrak{z}_\mathfrak{l}$, then

$$([l_1, l_2], z) = (l_1, [l_2, z]) = 0.$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}'$ containing $\mathfrak{z}_\mathfrak{l}$ and contained in $\mathfrak{l}$. The given decomposition of $\mathfrak{l}$ induces a decomposition $\mathfrak{h} = (\mathfrak{l}' \cap \mathfrak{h}) \oplus \mathfrak{z}_\mathfrak{l}$ which is also $\kappa$-orthogonal. Let $v \in \mathfrak{h}$ be the element in $\mathfrak{h}$, $\kappa$-dual to $\chi|_\mathfrak{h}$. The restriction $\chi|_{\mathfrak{l}' \cap \mathfrak{h}}$ is zero because $\mathfrak{l}'$ is semisimple, hence $v \in (\mathfrak{l}' \cap \mathfrak{h})^\perp = \mathfrak{z}_\mathfrak{l}$. □

For a parabolic subalgebra $\mathfrak{q}$ and split $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, let $R(\mathfrak{z}_\mathfrak{l}) = R^+(\mathfrak{z}_\mathfrak{l}) \cup R^-(\mathfrak{z}_\mathfrak{l})$ be the decomposition such that ${\mathfrak{g}'}^\alpha \subset \mathfrak{q}$ when $\alpha \in R^+(\mathfrak{z}_\mathfrak{l})$. Recall that a character $\chi$ of $\mathfrak{q}$ is then called dominant if $2(\chi, \alpha)/(\alpha, \alpha)$ is a nonnegative integer for all positive $\mathfrak{a}$-roots $\alpha$. We call it integer if $(\chi, \alpha)$ is integer for all $\mathfrak{a}$-roots $\alpha$.

**Lemma 5.4.** *Let $G'$ be a semisimple group. Let $P$ be a principal $G'$-bundle over a scheme $Y$ (not necessarily proper). There is a canonical bijection between the following sets*

1. *Isomorphism classes of reductions to a parabolic subgroup $Q$ on a big open set $U \subset Y$, together with an integer dominant character $\chi$ of $\mathfrak{q} = \mathrm{Lie}(Q)$.*
2. *Isomorphism classes of saturated balanced algebra filtrations*

$$0 \subsetneq E_{\lambda_1} \subsetneq E_{\lambda_2} \subsetneq \cdots \subsetneq E_{\lambda_t} \subsetneq E_{\lambda_{t+1}} = E \tag{5.2}$$

*of the bundle of algebras $E = P(\mathfrak{g}')$ associated to $P$ by the adjoint representation of $G'$.*

*Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a Levi decomposition, and $v \in \mathfrak{z}_\mathfrak{l}$ the element associated by lemma 5.3 to the character $\chi$ in (1). The set of integers $\{\lambda_i\}_{i=1,\ldots,t+1}$ in (2) is then just the set $\{\alpha(v)\}_{\alpha \in R(\mathfrak{a}) \cup \{0\}}$*

*Proof.* We start with a filtration (5.2). Take a point $x$ of $Y$ where the filtration is a bundle filtration. Fix an isomorphism between the fiber of $E$ at this point and $\mathfrak{g}'$. We obtain a balanced algebra filtration ${\mathfrak{g}'}_{\lambda_\bullet}$ of $\mathfrak{g}'$. By lemma 5.2, the associated graded Lie algebra $\mathrm{gr}\,({\mathfrak{g}'}^{\lambda_\bullet})$ is isomorphic to $\mathfrak{g}'$, and using this isomorphism we obtain a decomposition giving $\mathfrak{g}'$ the structure of a graded Lie algebra

$$\mathfrak{g}' = \bigoplus_{i=1}^{t+1} {\mathfrak{g}'}^{\lambda_i}, \tag{5.3}$$

such that

$$\mathfrak{g}'_{\lambda_i} = \bigoplus_{j=1}^{i} \mathfrak{g}'^{\lambda_j}, \tag{5.4}$$

Define a linear endomorphism of $\mathfrak{g}'$

$$\begin{array}{rcl} f : \bigoplus_{i=1}^{t+1} \mathfrak{g}'^{\lambda_i} & \longrightarrow & \bigoplus_{i=1}^{t+1} \mathfrak{g}'^{\lambda_i} \\ v \in \mathfrak{g}'^{\lambda_i} & \longmapsto & -\lambda_i v \end{array}$$

If $v_i \in \mathfrak{g}'^{\lambda_i}$ and $v_j \in \mathfrak{g}'^{\lambda_j}$, then $[v_i, v_j] \in \mathfrak{g}'^{\lambda_i+\lambda_j}$ so

$$f([v_i, v_j]) = [f(v_i), v_j] + [v_i, f(v_j)] \, ,$$

i.e. $f$ is a derivation. Thus, since $\mathfrak{g}'$ is semisimple, a semisimple element $v \in \mathfrak{g}'$ exists such that $f(\cdot) = [v, \cdot]$. Let $\mathfrak{z}_v$ be the center of the centralizer $\mathfrak{c}_v$ of $v$. It is a toral algebra. Consider the $\mathfrak{z}_v$-root decomposition (see (5.1) or [B-T, §3])

$$\mathfrak{g}' = \bigoplus_{\alpha \in R(\mathfrak{z}_v) \cup \{0\}} \mathfrak{g}'^{\alpha} \tag{5.5}$$

Note that $\mathfrak{g}'^{\alpha=0}$ is just the centralizer $\mathfrak{c}_v$ of $v$. This decomposition is a refinement of (5.3). Since

$$\mathfrak{g}'^{\lambda_i} = \bigoplus_{\alpha(v) = -\lambda_i} \mathfrak{g}'^{\alpha} \tag{5.6}$$

**Claim.** The direct summand $\mathfrak{g}'^{\alpha=0}$ in decomposition (5.5) is equal to the direct summand $\mathfrak{g}'^{\lambda_i=0}$ in decomposition (5.3).

To prove this claim, let $\mathfrak{z}_v$-root $\alpha$ be such that $\alpha(v) = 0$. For $x \in \mathfrak{g}'^{\alpha}$ it is $[v, x] = \alpha(v)x = 0$, i.e. $x$ is in the centralizer $\mathfrak{c}_v$ of $v$. By definition, $\mathfrak{z}_v$ is the center of $\mathfrak{c}_v$, thus $[w, x] = 0$ for all $w \in \mathfrak{z}_v$, proving the claim.

As a consequence, for all $\mathfrak{z}_v$-roots $\alpha$, it is $\alpha(v) \neq 0$, and thus $\alpha(v) > 0$ gives a set of positive $\mathfrak{z}_v$-roots $R^+(\mathfrak{z}_v)$. Using (5.4), (5.6) and the claim, we obtain for $\mathfrak{g}'_0$ in (5.4)

$$\mathfrak{g}'_0 = \bigoplus_{\beta \in R^+(\mathfrak{z}_v) \cup \{0\}} \mathfrak{g}'^{\beta},$$

hence $\mathfrak{g}'_0 \subset \mathfrak{g}'$ is a parabolic subalgebra ([B-T, §4]). Let $U$ be the big open set where $E_{\lambda_\bullet}$ is a bundle filtration. The inclusion $E_0|_U \subset E|_U$ gives a reduction of structure group $P^Q$ of the principal $G'$-bundle $P|_U$ to the parabolic subgroup $Q \subset G'$ corresponding to $\mathfrak{g}'_0 \subset \mathfrak{g}'$, because the stabilizer (under the adjoint action of a connected group) of a parabolic subalgebra is the corresponding parabolic subgroup.

Finally, the character $\chi(\cdot) = (v, \cdot)$ of the parabolic $\mathfrak{g}'_0$ is dominant, because $(\chi, \alpha) = \alpha(v)$ is a positive integer for all positive $\mathfrak{z}_v$-roots.

Reciprocally, assume we are given a reduction $P^Q$ of $P$ to a parabolic subgroup $Q$ on a big open set $U \subset Y$ and a dominant character $\chi$ of $\mathfrak{q} = \mathrm{Lie}(Q)$. Choose a decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ into a Levi and a unipotent subalgebras, and let $\mathfrak{z}_{\mathfrak{l}}$ be the center of $\mathfrak{l}$. Let $v \in \mathfrak{z}_{\mathfrak{l}}$ be the element associated to $\chi$ by lemma 5.3. Consider the $\mathfrak{z}_{\mathfrak{l}}$-root decomposition of $\mathfrak{g}'$ (see (5.1) or [B-T, §3])

$$\mathfrak{g}' = \bigoplus_{\alpha \in R(\mathfrak{z}_{\mathfrak{l}}) \cup \{0\}} \mathfrak{g}'^{\alpha}$$

By hypothesis $\alpha(v) = (\chi, \alpha)$ is an integer for all $\mathfrak{z}_{\mathfrak{l}}$-roots $\alpha$. Define a filtration ${\mathfrak{g}'}_{\lambda_\bullet}$ of $\mathfrak{g}'$ by

$$ {\mathfrak{g}'}_{\lambda_i} = \bigoplus_{-\alpha(v)\leq\lambda_i} {\mathfrak{g}'}^{\alpha} . \tag{5.7} $$

This is a balanced algebra filtration of, because $\dim {\mathfrak{g}'}^{\alpha} = \dim {\mathfrak{g}'}^{-\alpha}$ and $[{\mathfrak{g}'}^{\alpha}, {\mathfrak{g}'}^{\beta}] \subset {\mathfrak{g}'}^{\alpha+\beta}$. Clearly $\mathfrak{q} \subseteq {\mathfrak{g}'}_0$, and in fact $\mathfrak{q} = {\mathfrak{g}'}_0$ because the character $\chi$ of $\mathfrak{q}$ is dominant. It is also clear that $\mathfrak{l} \subseteq \mathfrak{c}_v$, the centralizer of $v$, and since $\chi$ is dominant, it is $\mathfrak{l} = \mathfrak{c}_v$, hence the center $\mathfrak{z}_{\mathfrak{l}}$ of $\mathfrak{l}$ is the center $\mathfrak{z}_v$ of $\mathfrak{c}_v$.

For adjoint action of $Q$ on $\mathfrak{g}'$ it is

$$ Q \cdot {\mathfrak{g}'}^{\alpha} \subset \bigoplus_{\beta\geq\alpha} {\mathfrak{g}'}^{\beta} $$

Thus the filtration (5.7) is preserved by this action: $Q \cdot {\mathfrak{g}'}_{\lambda_i} \subset {\mathfrak{g}'}_{\lambda_i}$. Since $P$ has a reduction to $Q$ on $U \subset Y$, this produces a vector bundle filtration of $E|_U$, and it extends uniquely to a saturated filtration on $Y$ as in (5.2).

It is easy to check that the two constructions are inverse to each other, and by construction $\{\lambda_i\}_{i=1,\ldots,t+1} = \{\alpha(v)\}_{\alpha\in R(\mathfrak{z}_{\mathfrak{l}})\cup\{0\}}$. □

**Lemma 5.5.** *With the same hypothesis (and notation) as in lemma 5.4, there are positive integers $a$ and $b$ such that $av$ corresponds to a one-parameter subgroup of $Z_L$ (i.e. its differential is $av$) and $b\chi$ corresponds to a character of the group $Q$.*

*Proof.* Let $\mathfrak{h}$ be a Cartan algebra of $\mathfrak{g}'$ with $\mathfrak{z}_{\mathfrak{z}} \subset \mathfrak{h} \subset \mathfrak{l}$ and let $H$ be the maximal torus of the connected group $G$ corresponding to $\mathfrak{h}$. Let $R(\mathfrak{h})$ be the set of roots with respect to $\mathfrak{h}$. The element $v \in \mathfrak{z}_{\mathfrak{l}} \subset \mathfrak{h}$ is in the coweight lattice $\mathbb{Z}(W^\vee)$, because any $\mathfrak{h}$-root gives an integer when evaluated on $v$. Indeed, the $\mathfrak{z}_{\mathfrak{l}}$-roots $\alpha : \mathfrak{z}_{\mathfrak{l}} \to \mathbb{C}$ with respect to $\mathfrak{z}_{\mathfrak{l}}$ are obtained by restricting the $\mathfrak{h}$-roots $\beta : \mathfrak{h} \to \mathbb{C}$ to $\mathfrak{z}_{\mathfrak{l}}$, but by hypothesis, $\alpha(v) \in \mathbb{Z}$ for all $\alpha \in R(\mathfrak{z}_{\mathfrak{l}})$. Let $X^\vee(H)$ be the lattice of one-parameter subgroups of $H$. Sending an element of $X^\vee(H)$ to its differential gives an embedding $X^\vee(H) \hookrightarrow \mathbb{Z}(W^\vee)$ with finite quotient, hence there is an integer $a$ such that $av$ corresponds to a one-parameter subgroup of $H$ which can be written as

$$ \begin{array}{rcl} \Psi : \mathbb{C}^* & \longrightarrow & Z_L \subset H \\ t = e^{\tau} & \longmapsto & e^{\tau a v} \end{array} $$

where $Z_L$ is the center of the Lie subgroup $L$ corresponding to $\mathfrak{l} \subset \mathfrak{g}'$.

On the other hand, the character $\chi$ of the parabolic $\mathfrak{q}$ is dominant, and in particular belongs to the weight lattice $\mathbb{Z}(W)$. Let $X(H)$ be the lattice of characters of $H$. Sending an element of $X(H)$ to its differential defines a lattice embedding $X(H) \hookrightarrow \mathbb{Z}(W)$ with finite quotient, hence there is an integer $b$ such that $b\chi$ corresponds to a character $\Xi \in X(H)$, i.e. the differential of $\Xi$ is $b\chi$.

Let $\mathfrak{l}' = [\mathfrak{l}, \mathfrak{l}]$ be the commutator subalgebra, and $L' = [L, L]$ the commutator subgroup. Recall that a character of $\mathfrak{q}$ factors as $\mathfrak{q} \twoheadrightarrow \mathfrak{l} \twoheadrightarrow \mathfrak{l}/\mathfrak{l}' \twoheadrightarrow \mathbb{C}$, hence $\chi$ vanishes on $\mathfrak{l}'$. Thus the character $\Xi$ of $H$ vanishes on $H \cap L'$, so $\Xi$ gives a group homomorphism $L/L' \cong H/(H \cap L') \to \mathbb{C}^*$. Composing with the quotient $Q \twoheadrightarrow L \twoheadrightarrow L/L'$ we obtain a character of $Q$ whose differential is $\chi$. □

**Lemma 5.6.** *Let $P$ be a principal $G'$-bundle over a big open set $U \subset X$ with a reduction $P^Q$ to a parabolic subgroup $Q \subset G'$ on a big open set $U' \subset U$. Let $\Xi$ be a* *dominant character of* $Q$ *and* $\chi$ *the associated character of* $\mathfrak{q}$*. Assume that* $(\chi,\alpha)$ *is an integer for all roots of* $\mathfrak{g}'$*. Let*

$$0 \subsetneq E_{\lambda_1} \subsetneq E_{\lambda_2} \subsetneq \cdots \subsetneq E_{\lambda_t} \subsetneq E_{\lambda_{t+1}} = E = P(\mathfrak{g}') \tag{5.8}$$

*be the balanced algebra filtration associated to it by lemma 5.4. Then*

$$\sum_{i=1}^{t+1} (\lambda_{i+1}-\lambda_i) \deg E_{\lambda_i} = \deg P^Q(\Xi) \tag{5.9}$$

*where* $P^Q(\Xi)$ *is the line bundle associated to* $P^Q$ *by the character* $\Xi$*.*

*Proof.* Let $L \subset Q$ be a Levi factor of $Q$, and $Z_L$ the center of $L$. For $\mathfrak{z}_\mathfrak{l} = \mathrm{Lie}(Z_L)$ consider the $\mathfrak{z}_\mathfrak{l}$-root decomposition of $\mathfrak{g}'$ (cfr. (5.1))

$$\mathfrak{g}' = \bigoplus_{\alpha \in R(\mathfrak{z}_\mathfrak{l}) \cup \{0\}} \mathfrak{g}'^{\alpha}.$$

Let $v \in \mathfrak{z}_\mathfrak{l}$ be the element associated to $\chi$ by lemma 5.3. Define an order $<_v$ in the set $R(\mathfrak{z}_\mathfrak{l}) \cup \{0\}$ by declaring $\alpha <_v \alpha'$ if $(\alpha-\alpha')(v) < 0$. In general, $<_v$ is not a total order, because it can happen that $(\alpha'-\alpha)(v) = 0$ even if $\alpha$ and $\alpha'$ are different. Choose a refinement of this to get a total order $\prec$. Number all the roots (including $\alpha = 0$) by $\alpha_1 \succ \alpha_2 \succ \ldots \succ \alpha_{l+1}$ in descending order, and define a filtration $\mathfrak{g}'_\bullet$

$$0 \subsetneq \mathfrak{g}'_{\alpha_1} \subsetneq \mathfrak{g}'_{\alpha_2} \subsetneq \cdots \subsetneq \mathfrak{g}'_{\alpha_l} \subsetneq \mathfrak{g}'_{\alpha_{l+1}} = \mathfrak{g}', \quad \text{with } \mathfrak{g}'_{\alpha_i} = \bigoplus_{j=1}^{i} \mathfrak{g}'^{\alpha_j}. \tag{5.10}$$

For the adjoint action of $Q$ on $\mathfrak{g}'$ it is

$$Q \cdot \mathfrak{g}'^{\alpha} \subseteq \bigoplus_{\beta \geq \alpha} \mathfrak{g}'^{\beta} \subseteq \bigoplus_{\beta \succeq \alpha} \mathfrak{g}'^{\beta}$$

This has two consequences: on the one hand, there is an induced action of $Q$ on

$$(\mathrm{gr}\, \mathfrak{g}')^{\alpha_i} := \mathfrak{g}'_{\alpha_i} / \mathfrak{g}'_{\alpha_{i-1}}$$

and on the other hand, $P^Q$ produces a vector bundle filtration of $E|_{U'}$, and this extends to a saturated filtration on $U$

$$0 \subsetneq E_{\alpha_1} \subsetneq E_{\alpha_2} \subsetneq \cdots \subsetneq E_{\alpha_l} \subsetneq E_{\alpha_{l+1}} = E \tag{5.11}$$

Note that, although as vector spaces both $\mathfrak{g}'^{\alpha}$ and $(\mathrm{gr}\, \mathfrak{g}')^{\alpha_i}$ are isomorphic, they are not isomorphic as $Q$-modules: indeed, while $Q \cdot (\mathrm{gr}\, \mathfrak{g}')^{\alpha_i} \subset (\mathrm{gr}\, \mathfrak{g}')^{\alpha_i}$, in general we only have $Q \cdot \mathfrak{g}'^{\alpha} \subseteq \bigoplus_{\beta \geq \alpha} \mathfrak{g}'^{\beta}$.

The filtration (5.11) is a refinement of (5.8), with

$$E_{\lambda_i} = E_{\alpha}, \quad \alpha = \max_{\prec} \left\{ \beta \in R(\mathfrak{z}_\mathfrak{l}) \cup \{0\} : -(\chi,\alpha) = -\alpha(v) \leq \lambda_i \right\} \tag{5.12}$$

Furthermore, $E^{\alpha_i} = E_{\alpha_i}/E_{\alpha_{i-1}}$ is isomorphic to the vector bundle associated to $P^Q$ using the action of $Q$ on $(\mathrm{gr}\, \mathfrak{g}')^{\alpha}$. Since this filtration is a refinement of (5.8), it is

$$\deg(E^{\lambda_i}) = \sum_{\alpha(v)=-\lambda_i} \deg(E^{\alpha}), \tag{5.13}$$

where $E^{\lambda_i} = E_{\lambda_i}/E_{\lambda_{i-1}}$.

For each $\mathfrak{z}_\mathfrak{l}$-root $\alpha$ the adjoint action of $Q$ on $(\mathrm{gr}\, \mathfrak{g}')^{\alpha}$ gives a character

$$\phi_\alpha : Q \xrightarrow{\mathrm{ad}} \mathrm{GL}\left((\mathrm{gr}\, \mathfrak{g}')^{\alpha}\right) \xrightarrow{\det} \mathbb{C}^*$$

Every character of a parabolic subgroup factors through its Levi quotient $L$, and two characters are equal if they coincide when restricted to its center $Z_L$. We have a commutative diagram

$$\begin{array}{ccccc}
Q & \xrightarrow{\mathrm{ad}} & \mathrm{GL}\big((\mathrm{gr}\,\mathfrak{g}')^\alpha\big) & \xrightarrow{\det} & \mathbb{C}^* \\
\downarrow & & & & \| \\
L & \xrightarrow{\mathrm{ad}} & \mathrm{GL}\big((\mathrm{gr}\,\mathfrak{g}')^\alpha\big) & \xrightarrow{\det} & \mathbb{C}^* \\
\uparrow & & & & \| \\
Z_L & \xrightarrow{\mathrm{ad}} & \mathrm{GL}\big((\mathrm{gr}\,\mathfrak{g}')^\alpha\big) & \xrightarrow{\det} & \mathbb{C}^*
\end{array}$$

It follows that

$$\phi_\alpha = \overline{(\dim \mathfrak{g}'^\alpha)\alpha},$$

where we denote by $\overline{(\dim \mathfrak{g}'^\alpha)\alpha}$ the character of $Q$ such that, after restricting to a character $Z_L \to \mathbb{C}^*$, the induced Lie algebra homomorphism $\mathfrak{z}_\mathfrak{l} \to \mathbb{C}$ is $(\dim \mathfrak{g}'^\alpha)\alpha$. Hence,

$$\det E^\alpha \cong P^Q\big(\overline{(\dim \mathfrak{g}'^\alpha)\alpha}\big). \tag{5.14}$$

Using equation (5.13), the left hand side of (5.9) is equal to the degree of the line bundle

$$\bigotimes_{i=1}^{t+1} (\det E^{\lambda_i})^{-\lambda_i} = \bigotimes_{\alpha\in R(\mathfrak{z}_\mathfrak{l})\cup\{0\}} (\det E^\alpha)^{\alpha(v)}$$

Using (5.14), this line bundle is equal to

$$P^Q\big(\overline{\sum_{\alpha\in R(\mathfrak{z}_\mathfrak{l})\cup\{0\}} \alpha(v)(\dim \mathfrak{g}'^\alpha)\alpha}\big) \tag{5.15}$$

**Claim.**

$$\sum_{\alpha\in R(\mathfrak{z}_\mathfrak{l})\cup\{0\}} \alpha(v)(\dim \mathfrak{g}'^\alpha)\alpha = \chi$$

Let $w \in \mathfrak{z}_\mathfrak{l}$. Then

$$\chi(w) = (v,w) = \mathrm{tr}[([v,\cdot])([w,\cdot])] = \sum_{\alpha\in R(\mathfrak{z}_\mathfrak{l})\cup\{0\}} (\dim \mathfrak{g}'^\alpha)\alpha(v)\alpha(w)\,,$$

and the claim follows because this holds for all $w \in \mathfrak{z}_L$.

Since $\overline{\chi} = \Xi$, it follows that the line bundle (5.15) is isomorphic to $P^Q(\Xi)$, and the lemma is proved. □

**Corollary 5.7.** *A principal $G$-sheaf $\mathcal{P} = (P, E, \psi)$ is slope-(semi)stable if and only if the associated rational principal $G$-bundle $P \to U \subset X$ is (semi)stable in the sense of Ramanathan.*

*Proof.* Without loss of generality, we can assume that $G$ is semisimple. Assume that $\mathcal{P}$ is slope-(semi)stable. Consider a reduction to a parabolic subgroup $Q$ of $P|_{U'} \to U' \subset U$, where $U'$ is a big open set, and a dominant character $\Xi$ of $Q$. This gives a dominant character $\chi$ of $\mathfrak{q} = \mathrm{Lie}(Q)$. Let $\mathfrak{q} = \mathfrak{q} \oplus \mathfrak{z}$ be a Levi decomposition and $\mathfrak{z}_\mathfrak{l}$ the center of $\mathfrak{l}$. A positive integer multiple $\widetilde{\chi} = c\chi$ has the property that $(\widetilde{\chi}, \alpha)$ is integer for all $\mathfrak{z}_\mathfrak{l}$-roots $\alpha$. Consider the balanced algebra filtration $\widetilde{E}^{U'}_{\lambda_\bullet}$ associated to $\widetilde{\chi}$ by lemma 5.4.

This filtration of $E|_{U'}$ can be extended uniquely to a saturated filtration $\widetilde{E}_{\lambda_\bullet}$ of $E$ on $X$, namely, the intersection $\widetilde{E}_{\lambda_i}$, inside $E^{\vee\vee}$, of $E$ and the reflexive sheaf $F_{\lambda_i}$ extending $\widetilde{E}^{U'}_{\lambda_\bullet}$ to $X$ (cfr. [Ha, II Ex. 5.15]). By lemma 5.6, and using the slope-(semi)stability of $\mathcal{P}$ we have

$$\deg P^Q(\Xi) = \sum_{i=1}^{t+1} \frac{\lambda_{i+1}-\lambda_i}{c} \deg E_{\lambda_i} \ (\leq)\ 0.$$

This means that $P \to U \subset X$ is Ramanathan (semi)stable.

Conversely, assume that $P \to U \subset X$ is Ramanathan (semi)stable. Consider a balanced algebra filtration of $E$. We may assume that this filtration is saturated. Let $U' \subset U \subset X$ be the big open set where this is a bundle filtration. Lemma 5.4 produces a reduction $P^Q$ on $U'$ of $P$ to a parabolic subgroup and a dominant character $\chi$ of $\mathfrak{q} = \mathrm{Lie}(Q)$. By lemma 5.5, there is a positive integer $b$ such that $b\chi$ corresponds to a character $\widetilde{\Xi}$ of $Q$. Then, by lemma 5.6 and because of the Ramanathan (semi)stability of $P$, it is

$$\sum_{i=1}^{t+1} (\lambda_{i+1}-\lambda_i) \deg E_{\lambda_i} = \frac{1}{b} \deg P^Q(\widetilde{\Xi}) \ (\leq)\ 0.$$

i.e. $\mathcal{P}$ is slope-(semi)stable. □

**Corollary 5.8.** *If $X$ is a curve, our notion of (semi)stability for principal bundles coincides with that of Ramanathan.*

Let us characterize (semi)stability in terms of the Killing form, as announced in the introduction. An orthogonal sheaf, relative to a scheme $S$, is a pair

$$(E_S, E_S \otimes E_S \longrightarrow \mathcal{O}_{X\times S})$$

such that the bilinear form induced on the fibers of $E_S$ over closed points $(x,s) \in X \times S$ where it is locally free, is nondegenerate. For instance, if $(E_S, \varphi_S)$ is a $\mathfrak{g}'$-sheaf, the Killing form gives an orthogonal structure to $E_S$.

**Definition 5.9** (Orthogonal filtration)**.** *A filtration $E_\bullet \subseteq E$ of an orthogonal sheaf is said to be orthogonal if $E_i^\perp = E_{-i-1}$ for all $i$. In terms of $E_{\lambda_\bullet}$, if the integers*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_t < \lambda_{t+1}$$

*can be denoted*

$$\gamma_{-l} < \gamma_{-l+1} < \cdots < \gamma_{l-1} < \gamma_l$$

*so that*

$$\gamma_{-i} = -\gamma_i, \quad \text{and} \quad E^\perp_{\gamma_i} = E_{\gamma_{-i-1}}$$

Observe that an orthogonal filtration is necessarily balanced and saturated. These filtrations were introduced in our former article [G-S1] in order to define the (semi)stability of an orthogonal sheaf as the condition of admitting no orthogonal filtration of negative (nonpositive) Hilbert polynomial.

**Corollary 5.10.** *Let $\mathcal{P} = (P, E, \psi)$ be a principal $G$-sheaf, or just let $(E, \varphi)$ be a $\mathfrak{g}'$-sheaf. An algebra filtration of $E$ is balanced and saturated if and only if it is orthogonal. Therefore, $\mathcal{P}$ is (semi)stable in the sense of definition 0.25 if and only if it is so in the sense of definition 0.3.*

*Proof.* We have seen that a balanced algebra filtration of $\mathfrak{g}'$-sheaves is induced from a filtration of Lie algebras as in (5.10). On the other hand, for a semisimple Lie algebra we have

$$({\mathfrak{g}'}^{\alpha})^{\perp} = \bigoplus_{\beta \neq -\alpha} {\mathfrak{g}'}^{\beta}$$

for $\alpha$, $\beta \in R(\mathfrak{h}) \cup \{0\}$. The first statement follows easily from these two facts. The second follows from the first and from the fact that it is enough to consider saturated filtrations. □

## 6. Comparison with Gieseker-Maruyama moduli space

The Gieseker-Maruyama moduli space is another natural compactification of the moduli space of principal $\mathrm{GL}(R)$-bundles. In this section we compare this with the moduli space of semistable principal $\mathrm{GL}(R)$-sheaves. We give two examples. In the first one, we show that our moduli space does not coincide, in general, with the Gieseker-Maruyama moduli space of torsion free sheaves. In the second example, we construct examples showing that, for principal $\mathrm{GL}(R)$-bundles, our notion of (semi)stability does not coincide, in general, with the Gieseker-Maruyama (semi)stability of the associated vector bundle (but recall that the slope-(semi)stability notions do coincide).

**Example 1.** Let $X = \mathbb{P}^2$ and $G = \mathrm{GL}(2)$. The Gieseker-Maruyama moduli space of semistable torsion free sheaves with rank 2, $c_1 = 1$ and $c_2 = 2$ is smooth of dimension 4. We are going to show that the moduli space of principal $\mathrm{GL}(2)$-sheaves with the corresponding numerical invariants has a component of dimension at least 16, hence the two moduli spaces are different.

Let $p \in \mathbb{P}^2$ be a point. Since $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(1) \otimes I_p, \mathcal{O}_{\mathbb{P}^2}) = \mathbb{C}$, there is a unique extension up to isomorphism

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow F \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \otimes I_p \longrightarrow 0$$

It is easy to show that $F$ is a slope-stable vector bundle, hence the associated principal $\mathrm{GL}(2)$-bundle is also slope-stable. Let $E$ be the vector bundle associated to the adjoint representation on $\mathfrak{g}' = \mathfrak{sl}_2$.

$$F^\vee \otimes F = \mathrm{ad}(F) \oplus \mathcal{O}_{\mathbb{P}^2} = E \oplus \mathcal{O}_{\mathbb{P}^2}$$

The vector bundle $E$ has rank 3, $c_1 = 0$ and $c_2 = 7$, and furthermore it is Mumford stable. To show this, note that since $E$ has rank 3 and zero degree, if it is not Mumford stable then either it has a subline bundle of nonnegative degree (hence a nonvanishing section) or it has a subsheaf of rank two of nonnegative degree. In this second case, it will have a rank one quotient of nonpositive degree. Taking the dual, this produces a subline bundle of nonnegative degree of $E^\vee \cong E$. In both cases, we conclude that if $E$ is not Mumford stable then it has a nonvanishing section. Now let $\xi$ be a section of

$$H^0(F^\vee \otimes F) = H^0(E) \oplus H^0(\mathcal{O}_{\mathbb{P}^2})$$

Since $F$ is slope stable, it is simple, then $\xi$ is a scalar multiple of identity, hence $\xi \in H^0(\mathcal{O}_{\mathbb{P}^2})$, the second summand. This shows that $E$ has no sections, hence it is Mumford stable.

Let $\mathrm{Quot}(E,4)$ be the Hilbert scheme of quotients $q : E \to T$ where $T$ is a torsion sheaf of length 4 supported on a zero-dimensional scheme. For each $q$ define $E_q$ to be the kernel

$$0 \longrightarrow E_q \xrightarrow{i} E \xrightarrow{q} T \longrightarrow 0$$

This torsion free sheaf inherits an $\mathfrak{sl}_2$-sheaf structure

$$\varphi_q : E_q \otimes E_q \xrightarrow{i\otimes i} E \otimes E \xrightarrow{\varphi} E \cong E_q^{\vee\vee}$$

If $q$ and $q'$ are two quotients corresponding to different points, then $E_q$ and $E_{q'}$ are not isomorphic. Indeed, if $\psi$ is an isomorphism between them, then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_q & \xrightarrow{i} & E & \xrightarrow{q} & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\xi} & & \\ 0 & \longrightarrow & E_{q'} & \xrightarrow{i'} & E & \xrightarrow{q'} & T & \longrightarrow & 0 \end{array}$$

where $\eta$ is induced from $\psi^{\vee\vee}$ and the isomorphisms $i'^{\vee\vee}$ and $i^{\vee\vee}$. Since $E$ is Mumford stable, it is simple, and then $\eta = \lambda\,\mathrm{id}$, a nonzero multiple of identity. Then the following diagram is commutative

$$\begin{array}{ccc} E & \xrightarrow{q} & T \\ \| & & \cong\downarrow{\scriptstyle\frac{1}{\lambda}\xi} \\ E & \xrightarrow{q'} & T \end{array}$$

and this implies that $q$ and $q'$ correspond to the same point in $\mathrm{Quot}(E,4)$.

The subscheme $\mathrm{Quot}^0(E,4)$ corresponding to quotients supported in 4 distinct points is smooth of dimension 16, and this construction provides a family of $\mathfrak{sl}_2$-sheaves parametrized by $\mathrm{Quot}^0(E,4)$ (with different points giving nonisomorphic $\mathfrak{sl}_2$-sheaves). To construct a family of principal $\mathrm{GL}(2)$-sheaves we have to consider reductions of structure group using the homomorphisms

$$\mathrm{GL}(2) \longrightarrow \mathrm{GL}(2)/(\mathbb{Z}/2\mathbb{Z}) = \mathrm{PGL}(2)\times\mathbb{C}^* \longrightarrow \mathrm{PGL}(2) \longrightarrow \mathrm{Aut}(\mathfrak{sl}_2)$$

Let $P_q^{\mathrm{Aut}(\mathfrak{sl}_2)}$ be the principal $\mathrm{Aut}(\mathfrak{sl}_2)$-bundle on $U_{E_q}$ associated to $(E_q,\varphi_q)$. Since $\mathfrak{sl}_2$ has no outer automorphisms, $\mathrm{PGL}(2) = \mathrm{Aut}(\mathfrak{sl}_2)$, and then this is a principal $\mathrm{PGL}(2)$-bundle. Now we have to consider a reduction to a principal $\mathrm{PGL}(2)\times\mathbb{C}^* = \mathrm{GL}(2)/(\mathbb{Z}/2\mathbb{Z})$-bundle with numerical invariant equal to 1, i.e. we have to give a line bundle on $U_{E_q}$ with degree 1, but since $U_{E_q}$ is a big open set, $\mathrm{Pic}(U_{E_q}) = \mathrm{Pic}(\mathbb{P}^2)$, hence there is a unique such line bundle: the restriction of $\mathcal{O}_{\mathbb{P}^2}(1)$. Finally, the reductions of a principal $\mathrm{GL}(2)/(\mathbb{Z}/2\mathbb{Z})$-bundle to $\mathrm{GL}(2)$ are parametrized by $\check{H}^1_{\mathrm{et}}(U_{E_q},\mathbb{Z}/2\mathbb{Z})$. From theorem 3.1 and the proof of lemma 3.2, this is isomorphic to the singular cohomology group $H^1(\mathbb{P}^2;\mathbb{Z}/2\mathbb{Z})$, and this cohomology group is trivial because $\mathbb{P}^2$ is simply connected. Then there is a unique reduction to a principal $\mathrm{GL}(2)$-bundle on $U_{E_q}$. Hence each $\mathfrak{sl}_2$-sheaf $(E_q,\varphi_q)$ produces a unique principal $\mathrm{GL}(2)$-sheaf, and this provides a family of principal $\mathrm{GL}(2)$-sheaves parametrized by a scheme of dimension 16, with different points giving nonisomorphic objects, hence there is component of the moduli space of principal $\mathrm{GL}(2)$-sheaves of dimension at least 16.

**Example 2.** Let $\pi : X = \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at one closed point. Let $D$ be the exceptional divisor, and $R$ the divisor class of the strict transform of a line through the blown up point, i.e. $\pi^*\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_X(D+R)$. Hence $K_X = \mathcal{O}_X(-2D-3R)$. Let $p \in X$ be a closed point outside the exceptional divisor. Then

$$H^0(\mathcal{O}_X(aD+bf)) \neq 0 \quad \Leftrightarrow \quad a \geq 0 \text{ and } b \geq 0 \tag{6.1}$$

$$H^0(\mathcal{O}_X(aD+bf)\otimes I_p) \neq 0 \quad \Leftrightarrow \quad a \geq 0 \text{ and } b > 0 \tag{6.2}$$

Let $c,\ s \in \mathbb{Z}$, $L = \mathcal{O}_X(sR)$, $M = \mathcal{O}_X(-cD+(c+s)R)$. The local to global spectral sequence for Ext gives an exact sequence

$$\mathrm{Ext}^1(M\otimes I_p, L) \xrightarrow{\alpha} H^0(\mathcal{E}xt^1(M\otimes I_p, L)) = \mathbb{C} \longrightarrow H^2(M^\vee \otimes L) = 0$$

The last group is zero by Serre duality and (6.1), hence $\alpha$ is surjective. The second group is $\mathbb{C}$ because $\mathcal{E}xt^1(M\otimes I_p, L) \cong \mathbb{C}_p$, the skyscraper sheaf at $p$. Let $\eta$ be an element in the first group with $\alpha(\eta)\neq 0$, so that the extension corresponding to $\eta$

$$0 \longrightarrow L \longrightarrow F \longrightarrow M\otimes I_p \longrightarrow 0 \tag{6.3}$$

is locally free. Fix the ample line bundle $\mathcal{O}_X(D+2R)$. All degrees, stability, etc... will be with respect to this line bundle.

**Lemma 6.1.** *The vector bundle $F$ is Mumford strictly semistable with slope $\mu(F) = s$, and the only subsheaf $L'$ with $\mu(L') = \mu(F)$ is $L$.*

*Proof.* A calculation shows $\mu(L) = \mu(F) = s$. We can assume that $L'$ is a line bundle which does not factor through $L$. Then the composition $L' \to F \to M\otimes I_p$ is nonzero, hence

$$H^0(L'^\vee \otimes M \otimes I_p) \neq 0$$

Denote $L' = \mathcal{O}_X(aD+bR)$. Using (6.2) we obtain $a \leq -c$ and $b < c+s$, and then $\mu(L') < \mu(F)$. □

**Lemma 6.2.** *The vector bundle $F$ is Gieseker-Maruyama (semi)stable if and only if*

$$3c^2 + (2s-1)c + 2\ (\leq)\ 0$$

*Proof.* By lemma 6.1, it is enough to check the subbundle $L$. A calculation shows that the following polynomial is constant

$$P_L(m) - \frac{P_E(m)}{2} = \frac{3c^2+(2s-1)c+2}{2}$$

and the result follows. □

**Lemma 6.3.** *The principal* GL(2)*-bundle associated to the vector bundle $E$ is (semi)stable in the sense of definition 0.3 if and only if*

$$3\ (\leq)\ c$$

*Proof.* By lemma 5.4, all orthogonal filtrations come from reductions to a parabolic subgroup on a big open set $U'$. Since $F$ is a rank 2 vector bundle, such a reduction can be seen as an extension

$$0 \longrightarrow F_1 \longrightarrow F \longrightarrow F_2 \longrightarrow 0 \tag{6.4}$$

where $F_1$ is a line bundle and $F_2$ is a rank one torsion free sheaf. Indeed, this gives a reduction to a maximal parabolic subgroup on the big open set $U'$ where $F_2$ is locally free.

Let $E$ be the vector bundle associated to the adjoint representation on $\mathfrak{g}' = \mathfrak{sl}_2$

$$F^\vee \otimes F = \mathrm{ad}(F) \oplus \mathcal{O}_X = E \oplus \mathcal{O}_X$$

i.e. the vector bundle $E$ is the sheaf of traceless homomorphisms $\mathcal{H}om(F,F)^0$

$$0 \longrightarrow \mathcal{H}om(F,F)^0 \longrightarrow \mathcal{H}om(F,F) \xrightarrow{\mathrm{tr}} \mathcal{O}_X \longrightarrow 0$$

Since the parabolic subgroup is maximal, there is a unique dominant character (up to scalar), and by lemma 5.4, this reduction (and character) gives a saturated balanced algebra filtration $E_\bullet$ (we take double dual $(\,\cdot\,)^{\vee\vee}$ in order to obtain a saturated filtration)

$$\begin{array}{ccccc} (F_2^\vee \otimes F_1)^{\vee\vee} & \subsetneq & (\mathcal{H}om_{F_1}(F,F)^0)^{\vee\vee} & \subsetneq & \mathcal{H}om(F,F) \\ \| & & \| & & \| \\ E_{-1} & & E_0 & & E_1 = E \end{array}$$

where $\mathcal{H}om_{F_1}(F,F)^0$ denotes the sheaf of traceless homomorphisms preserving $F_1$, i.e. it is the kernel of the homomorphism $\alpha$

$$0 \longrightarrow \mathcal{H}om_{F_1}(F,F)^0 \longrightarrow \mathcal{H}om(F,F)^0 \xrightarrow{\alpha} F_1^\vee \otimes F_2 \tag{6.5}$$

The Hilbert polynomial of the filtration $E_\bullet$ is

$$P_{E_\bullet} = (3P_{E_{-1}} - P_E) + (3P_{E_0} - 2P_E)$$

It has degree at most 1, and the coefficient of the term of degree 1 can be obtained substituting the Hilbert polynomials by degrees in the previous expression. We calculate

$$\begin{aligned} \deg(E_{-1}) &= \deg(F_2^\vee \otimes F_1) = \deg F_1 - \deg F_2 \\ \deg(E_0) &= \deg(\mathcal{H}om_{F_1}(F,F)^0) = \\ &= \deg(\mathcal{H}om(F,F)^0) - \deg(F_1^\vee \otimes F_2) = \deg F_1 - \deg F_2 \\ \deg(E) &= 0 \end{aligned}$$

The degree of $E_0$ can be calculated from the exact sequence (6.5), because the homomorphism $\alpha$ is surjective where $F_2$ is locally free, and this is a big open set. Then the coefficient of degree 1 in the Hilbert polynomial $P_{E_\bullet}$ of the filtration $E_\bullet$ is

$$3(\deg(E_{-1}) + \deg(E_0) - \deg(E)) = 6(\deg F_1 - \deg F_2) = 12(\deg F_1 - \frac{\deg F}{2})$$

By lemma 6.1, $\deg F_1 - (\deg F)/2 \leq 0$, and to check the (semi)stability of the principal bundle we can assume that the sequence (6.4) is (6.3):

$$0 \longrightarrow L \longrightarrow F \longrightarrow M \otimes I_p \longrightarrow 0$$

Taking the dual of this sequence, we obtain the short exact sequence

$$0 \longrightarrow M^\vee \longrightarrow F^\vee \longrightarrow L^\vee \otimes I_p \longrightarrow 0$$

It is easy to check that the composition

$$\mathcal{H}om(F,F)^0 \hookrightarrow F^\vee \otimes F \twoheadrightarrow L^\vee \otimes I_p \otimes M \otimes I_p$$

is surjective, hence $\mathcal{H}om_L(F,F)^0$ fits in an exact sequence

$$0 \longrightarrow \mathcal{H}om_L(F,F)^0 \longrightarrow \mathcal{H}om(F,F)^0 \longrightarrow L^\vee \otimes I_p \otimes M \otimes I_p \;=\; L^\vee \otimes M \otimes (I_Z \oplus \mathbb{C}_p) \longrightarrow 0$$

where $Z$ is the "fat point" supported at $p$, i.e. $I_Z = I_p^2$. Then

$$0 \longrightarrow (\mathcal{H}om_L(F,F)^0)^{\vee\vee} \longrightarrow \mathcal{H}om(F,F)^0 \longrightarrow L^\vee \otimes M \otimes I_Z \longrightarrow 0$$

On the other hand, $(F_2^\vee \otimes F_1)^{\vee\vee} = M^\vee \otimes L$. A calculation shows

$$P_{E_\bullet}(m) = 3(-c+3)$$

and the result follows. □

Lemmas 6.2 and 6.3 show that for a principal GL(2)-bundle, its (semi)stability in the sense of definition 0.3 does not coincide in general with the Gieseker-Maruyama (semi)stability of the associated rank 2 vector bundle. The following table gives the stability of $F$ for concrete values of the parameters $c$ and $s$, showing this fact.

| $(c,s)$ | Vector bundle | Principal bundle |
|---|---|---|
| (-1,4) | unstable | unstable |
| (-1,3) | semistable | unstable |
| (-1,2) | stable | unstable |
| (3,-4) | unstable | semistable |
| (3,-5) | stable | semistable |
| (4,-5) | unstable | stable |
| (4,-6) | stable | stable |

## References


[A-B] B. Anchouche and I. Biswas, *Einstein–Hermitian connections on polystable principal bundles over a compact Kähler manifold,* Amer. Jour. Math. **123** (2001), 207–228.

[B-T] A. Borel, J. Tits, *Groupes réductifs,* Inst. Hautes Études Sci. Publ. Math. **27** (1965), 55–150.

[FGA] A. Grothendieck, *Technique de descente et théorèmes d'existence en géométrie algébrique,* (Bourbaki exposés 190, 195, 212, 221, 232 and 236). Also in: *Fondements de la géométrie algébrique,* Secrétariat Mathematique Paris (1962).

[F-M] R. Friedman and J.W. Morgan : Holomorphic Principal Bundles over Elliptic Curves II: The Parabolic Construction. J. Differential Geom. **56** (2000), 301–379.

[Gi] D. Gieseker, *On the moduli of vector bundles on an algebraic surface,* Ann. Math., **106** (1977), 45–60.

[G-S] T. Gómez and I. Sols, *Stability of conic bundles,* Internat. J. Math., **11** (2000), 1027–1055.

[G-S1] T. Gómez and I. Sols, *Stable tensors and moduli space of orthogonal sheaves,* Preprint 2001, math.AG/0103150.

[G-S2] T. Gómez and I. Sols, *Projective moduli space of semistable principal sheaves for a reductive group,* Le Matematiche **15** (2000), 437–446, conference in honor of Silvio Greco (April 2001)

[J] N. Jacobson, Lie algebras, Dover, 1979.

[Ha] R. Hartshorne, Algebraic Geometry, Grad. Texts in Math. 52, Springer Verlag, 1977.

[Ha2] R. Hartshorne, *Stable reflexive sheaves,* Math. Ann. **254** (1980), 121–176.

[Hum] J. E. Humphreys, Introduction to Lie algebras and representation theory, Grad. Texts in Math. 9, Springer Verlag, New York-Berlin, 1972.

[H-L] D. Huybrechts and M. Lehn, The geometry of moduli spaces of sheaves, Aspects of Mathematics E31, Vieweg, Braunschweig/Wiesbaden 1997.

[Hy] D. Hyeon, *Principal bundles over a projective scheme,* Trans. Amer. Math. Soc. **354** (2002), 1899–1908.

[Ma] M. Maruyama, *Moduli of stable sheaves, I and II.* J. Math. Kyoto Univ. **17** (1977), 91–126. **18** (1978), 557–614.

[Mu1] D. Mumford, Geometric invariant theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34. Springer-Verlag, Berlin-New York, 1965.

[Mu2] D. Mumford, Abelian varieties. Oxford University Press, Bombay, 1970.

[N-S] M.S. Narasimhan and C.S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface.* Ann. of Math. (2) **82** (1965), 540–567.

[Ra1] A. Ramanathan, *Stable principal bundles on a compact Riemann surface.* Math. Ann. **213** (1975), 129–152.

[Ra2] A. Ramanathan, *Moduli for principal bundles. Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978),* pp. 527–533, Lecture Notes in Math., **732**, Springer, Berlin, 1979.

[Ra3] A. Ramanathan, *Moduli for principal bundles over algebraic curves: I and II,* Proc. Indian Acad. Sci. (Math. Sci.), **106** (1996), 301–328, and 421–449.

[Ri] R. Richardson, *Compact real forms of a complex semi-simple Lie algebra,* J. Differential Geometry **2** (1968), 411–419.

[Sch] A. Schmitt, *Singular principal bundles over higher-dimensional manifolds and their moduli spaces,* Internat. Math. Res. Notices **23** (2002), 1183–1210.

[Se1] J. P. Serre, *Espaces fibrés algébriques,* Séminaire C. Chevalley, ENS (1958)

[Se2] J. P. Serre, *Lie algebras and Lie groups, 1964 Lectures given at Harvard University*, (1965), W. A. Benjamin Inc.

[Sesh] C.S. Seshadri, *Space of unitary vector bundles on a compact Riemann surface.* Ann. of Math. (2) **85** (1967), 303–336.

[SGA1] A. Grothendieck and M. Raynaud, *Revêtements étales et groupe fondemental,* Lecture Notes in Math. **224**, Springer-Verlag (1971).

[SGA4] M. Artin, A. Grothendieck, J.L. Verdier, *Théorie des topos et cohomologie étale des schémas,* Lecture Notes in Math. **269**, **270**, **305**, Springer-Verlag (1972).

[Si] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety I,* Publ. Math. I.H.E.S. **79** (1994), 47–129.

[Sp] E.H. Spanier, Algebraic topology, Corrected reprint. Springer-Verlag, New York-Berlin, 1981.

Departamento de Algebra, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid (Spain)

*E-mail address*: tgomez@alg.mat.ucm.es,sols@mat.ucm.es