DEGREE OF THE DIVISOR OF SOLUTIONS OF A DIFFERENTIAL EQUATION ON A PROJECTIVE VARIETY

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Abstract. Using the data schemes from [5] we give a rigorous definition of algebraic differential equations on the complex projective space $\mathbb{P}^n$. For an algebraic subvariety $S \subseteq \mathbb{P}^n$, we present an explicit formula for the degree of the divisor of solutions of a differential equation on S and give some examples of applications. We extend the technique and result to the real case.

1. Introduction

We deal with the problem of finding the degree of the divisor of solutions of a differential equation on a projective variety, which was studied by Halphen in [6] for differential equations on plane curves and on hypersurfaces of $\mathbb{P}^n$. With the notion of infinitesimal data in $\mathbb{P}^n$, introduced in the plane by A. Collino [4], used by S. Colley and G. Kennedy in [2] [3], and generalised to higher dimensions in [5], we solve this problem for algebraic subvarieties of $\mathbb{P}^n$ (the method of Halphen seems to generalise only to complete intersections). The plane data of Collino have been rediscovered by Mohamed Belghiti [1] and applied to give a modern proof of the Halphen formula for plane curves, including some explicit calculation of the involved invariants of the equation that Halphen was able to obtain in this case.

Let $\mathbb{P}^n$ the complex projective space of dimension n (although we can work over any algebraically closed field K of zero characteristic). Associated to a set of coordinates $x_1, \dots, x_k, y_1, \dots, y_{n-k}$ on an affine open set $U^0 = \mathbb{C}^n \subseteq \mathbb{P}^n$, we construct inductively an open set U^r of the data scheme $D_k^r\mathbb{P}^n$, where the partial derivatives of the y_j with respect to the x_i are understood as the canonical coordinates of U^r. In this context, a differential equation

$$f(x_i, y_j, \frac{\partial y_j}{\partial x_i}, \frac{\partial^2 y_j}{\partial x_{i_2}\partial x_{i_1}}, \cdots, \frac{\partial^r y_j}{\partial x_{i_r}\cdots\partial x_{i_1}}) = 0,$$

is understood as a algebraic equation on the open set U^r. The data satisfying the differential equation form a divisor $D_k^r\mathbb{P}^n(f)$ on $D_k^r\mathbb{P}^n$ defined as the closure of the solutions to $f = 0$.

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For the differential equation f there are naturally well defined enumerative invariants γ_s^f, $0 \leq s \leq r$.

Let $S \subseteq \mathbb{P}^n$ be an algebraic subvariety of dimension k, then $S(f)$ will be the locus of the solutions of the differential equation $f = 0$ on S. The main result of the paper (theorem 13) gives a closed formula for $\deg S(f)$ (assuming $S(f)$ is a proper subset of S), expressed as the scalar product between the γ_s^f and the cuspidal numbers γ_S^s of S. This reduces to a simple formula $\deg S(f) = \gamma_0^f \gamma_S^0 + \gamma_1^f \gamma_S^1$ when S is smooth (or normal), where γ_S^0 and γ_S^1 are the degree and class of S respectively, and γ_0^f and γ_f^1 can be computed directly, either with the use of proposition 7 or remark 9 (when f is distinguished in suitable variables, which is the general case) or by using the formula $\deg S(f) = \gamma_0^f \gamma_S^0 + \gamma_1^f \gamma_S^1$ applied to a smooth quadric and a smooth cubic subvarieties $S \subseteq \mathbb{P}^n$, and solving the system of linear equations thus obtained.

The use of theorem 13 is two-fold. On the one hand it gives the degree of $S(f)$ if we know enough information from f and S. This can be used to understand geometric properties of S as these are usually described by a suitable differential equation (e.g. flexes of plain curves, parabolic points). On the other hand the explicit computations of $\deg S(f)$ in particular cases may lead to finding the invariants γ_s^f of a differential equation f, and the cuspidal numbers γ_S^s of a subvariety S. Section 4 is devoted to examples in which these two applications of theorem 13 are worked out. In particular, we find the degree of the divisor of parabolic points of a subvariety of $\mathbb{P}^n$.

Finally in the last section we extend the results for the real projective space $\mathbb{P}^n_{\mathbb{R}}$, by studying the behaviour of the constructions involved via the conjugation involution. This is merely an introduction and not a thorough analysis of the use of data varieties in the real case. The formula thus obtained in theorem 23 is analogue to the one in the complex case, but we have to restrict ourselves to coefficients in $\mathbb{Z}/2\mathbb{Z}$. We end up with a simple application on the number of umbilical points of a real surface in $\mathbb{P}^3_{\mathbb{R}}$.

2. Differential equations on projective space

To make precise sense of a differential equation in the projective space $\mathbb{P}^n$, we need to recall from [5] the definition of the smooth compact moduli $D_k^r\mathbb{P}^n$ of infinitesimal data of dimension k at order r in $\mathbb{P}^n$.

Let Z be a scheme smooth over $\mathbb{C}$. We define schemes $D_k^r Z$ together with embeddings and projections

$$\begin{array}{ccc} D_k^r Z & \hookrightarrow & \mathrm{Gr}_k T D_k^{r-1} Z \\ \downarrow b_r & \swarrow \pi_r & \\ D_k^{r-1} Z & & \end{array}$$

inductively, as follows. We take $D_k^0Z = Z$, $D_k^1Z = \mathrm{Gr}_kTZ$ and if $r \geq 2$ and

$$\begin{array}{ccc} D_k^{r-1}Z & \hookrightarrow & \mathrm{Gr}_kTD_k^{r-2}Z \\ \Big\downarrow b_{r-1} & \swarrow \pi_{r-1} & \\ D_k^{r-2}Z & & \end{array}$$

is already defined then $D_k^rZ \subseteq \mathrm{Gr}_kTD_k^{r-1}Z$ is the Grassmannian $\mathrm{Gr}_k\mathcal{F}_{r-1}$ of the (Semple) bundle $\mathcal{F}_{r-1} \subseteq TD_k^{r-1}Z$ obtained as pull-back

(1)
$$\begin{array}{ccccccccc} & & & & \Lambda_{r-1} & = & \Lambda_{r-1} & & \\ & & & & \Big\uparrow & & \Big\uparrow & & \\ 0 & \longrightarrow & TD_k^{r-1}Z/D_k^{r-2}Z & \longrightarrow & TD_k^{r-1}Z & \xrightarrow{Tb_{r-1}} & b_{r-1}^*TD_k^{r-2}Z & \longrightarrow & 0 \\ & & \Big\| & & \Big\uparrow \cup & \text{pull-back} & \Big\uparrow \cup & & \\ 0 & \longrightarrow & TD_k^{r-1}Z/D_k^{r-2}Z & \longrightarrow & \mathcal{F}_{r-1} & \longrightarrow & \Sigma_{r-1} & \longrightarrow & 0 \end{array}$$

where the right hand sequence is the restriction to $D_k^{r-1}Z$ of the universal sequence on $\mathrm{Gr}_kTD_k^{r-2}Z$.

On each $D_k^rZ = \mathrm{Gr}_k\mathcal{F}_{r-1}$ there is a natural divisor C_k^rZ, namely the Schubert special cycle of k-planes of $\mathcal{F}_{r-1}$ meeting $TD_k^{r-1}Z/D_k^{r-2}Z$, whose elements are called cuspidal. The elements in the complement are called non-cuspidal.

If Z' is a subvariety of Z of dimension k and Z'_{reg} is the open set of smooth points, then

$$Z'_{\mathrm{reg}} \cong D_k^rZ'_{\mathrm{reg}} \subseteq D_k^rZ$$

by the obvious functoriality of our construction. We then define $D_k^rZ' \subseteq D_k^rZ$ as the closure of this subset.

From now on $Z = \mathbb{P}^n = \mathbb{P}(V)$ and $1 \leq k \leq n-1$. Fix a trivialisation $V \cong \mathbb{C}^{n+1}$ or projective reference, with corresponding hyperplane C^0 at infinity and affine part $U^0 = \mathbb{P}^n - C^0 = \mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^{n-k}$, with affine coordinates which we name

(2)
$$x_1, \dots, x_i, \dots, x_k, y_1, \dots, y_j, \dots, y_{n-k}.$$

In order to make sense of a differential equation, we need to introduce formal partial derivative symbols of the variables y_j with respect to the variables x_i. For any $s \geq 1$, $1 \leq j \leq n-k$, and $1 \leq i_1, \dots, i_s \leq k$ we introduce

(3)
$$y^j_{i_1,\dots,i_s} = \frac{\partial^s y_j}{\partial x_{i_s} \cdots \partial x_{i_1}}.$$

At the present stage these are to be understood merely as symbols. Later we will identify them as coordinates of an affine open set of the data scheme $D_k^r\mathbb{P}^n$. Note that we have introduced non-commuting derivative symbols (see remark 5 below).

On each point $P \in U^0$ we consider the k-space $X(P) = P + \mathbb{C}^k \subseteq TU^0$ and the $(n-k)$-space $Y(P) = P + \mathbb{C}^{n-k} \subseteq TU^0$ providing subbundles X, Y of $TU^0 = X \oplus Y$, all of them trivial. The complement U^1 in $D_k^1 U^0 = \mathrm{Gr}_k TU^0$ of the special Schubert cycle of k-subbundles of TU^0 meeting Y is thus

$$U^1 = \mathrm{Hom}(X,Y) = U^0 \times \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^{n-k}) = U^0 \times V^1, \tag{4}$$

where $V^1 = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^{n-k})$ is a cartesian power of $\mathbb{C}$ with coordinates y_i^j (standing for $\frac{\partial y_j}{\partial x_i}$). On U^1, the lifted k-subbundle $X_{U^1} \subseteq (TU^0)_{U^1}$ provides a split of the universal sequence

$$0 \to \Sigma_1 \to (TU^0)_{U^1} \to \Lambda_1 \to 0$$

on U^1. The universal subbundle Σ_1 on U^1 becomes then isomorphic to the bundle $X_{U^1} \cong U^1 \times \mathbb{C}^k$.

We shall denote C^1 the closure in $D_k^1\mathbb{P}^n$ of the complement of U^1 in $D_k^1 U^0$. Also for all $r \geq 2$ we let $C^r = C_k^r\mathbb{P}^n$ be the cuspidal divisor of $D_k^r\mathbb{P}^n$. Denote by C_s^r for all $0 \leq s \leq r$ the divisor of $D_k^r\mathbb{P}^n$ counterimage by

$$b_{r,s} = b_{s+1} \circ \cdots \circ b_{r-1} \circ b_r : D_k^r\mathbb{P}^n \to D_k^s\mathbb{P}^n \tag{5}$$

of the divisor C^s of $D_k^s\mathbb{P}^n$ (in particular $C_r^r = C^r$). Denote by U^r the open subset of $D_k^r\mathbb{P}^n$

$$U^r = D_k^r\mathbb{P}^n \backslash (C_0^r \cup \ldots \cup C_s^r \cup \ldots \cup C_r^r)$$

consisting of data which are not in C_0^r (i.e. which are "finite"), are not in C_1^r (i.e. which are nowhere "vertical") and are not in C_s^r for any $s \geq 2$ (i.e. which are "nonsingular"). It is important to note that C_s^r is intrinsically defined for $s \geq 2$, but is dependent on the choice of trivialisation for $s = 0, 1$. The data in U^r are the data for which the symbols (3), for $s = 1, \ldots, r$, are going to acquire a precise meaning.

Lemma 1. *$C_0^r, C_1^r, \ldots, C_s^r, \ldots, C_r^r$ is a basis for $A^1 D_k^r\mathbb{P}^n$.*

Proof. We can see this by induction on $r \geq 0$. For $r = 0$ it is obvious. In general, we have to show that the natural map

$$\mathrm{Pic}\, D_k^{r-1}\mathbb{P}^n \times \mathbb{Z}C^r \to \mathrm{Pic}\, D_k^r\mathbb{P}^n = \mathrm{Pic}\,\mathrm{Gr}_k(\mathcal{F}_{r-1}) \tag{6}$$

is an isomorphism. We need only to remark that for any point $P \in D_k^{r-1}\mathbb{P}^n$, $(C^r)_P$ is a basis of the Picard group of the Grassmannian $\mathrm{Gr}_k((\mathcal{F}_{r-1})_P)$. Surjectivity of (6) now follows easily from [7, Ex. III.12.4] and injectivity by restricting to a fibre.

Alternatively, the lemma follows from the explicit descriptions given below. The complement of the union of all C_s^r is a cartesian power of $\mathbb{C}$ (proposition 2) and C_s^r are all irreducible, therefore they generate $A^1 D_k^r\mathbb{P}^n$. On the other hand, by the proof of step 2 of proposition 7, they are linearly independent. So they form a basis for $A^1 D_k^r\mathbb{P}^n$. □

Proposition 2. *The given trivialisation* (2) *of* U^0 *induces trivialisations*

$$U^r \cong U^{r-1} \times Hom((\mathbb{C}^k)^{\otimes r}, \mathbb{C}^{n-k})$$

of U^r *with cartesian powers of* $\mathbb{C}$. *The coordinates of* U^r *are given by* (2) *and* (3), $1 \leq s \leq r$.

Proof. Starting from the trivialisation of U^0, we get an induced trivialisation (4) of U^1. Suppose we have already trivialisations $U^s = U^{s-1} \times V^s$ with $V^s \cong \mathrm{Hom}((\mathbb{C}^k)^{\otimes s}, \mathbb{C}^{n-k})$, which is isomorphic to a cartesian power of $\mathbb{C}$, for $1 \leq s \leq r$. Then $(\Sigma_r)_{U^r} \cong (\Sigma_{r-1})_{U^r} \cong \cdots \cong (\Sigma_1)_{U^r} \cong X_{U^r} \cong U^r \times \mathbb{C}^k$. The Semple sequence on U^r is split by

$$\mathcal{F}_r \hookrightarrow TU^r \twoheadrightarrow TU^r/U^{r-1}$$

for all r, so that $\mathcal{F}_r = \Sigma_r \oplus TU^r/U^{r-1}$. Consequently,

$$U^{r+1} \cong \mathrm{Hom}((\Sigma_r)_{U^r}, TU^r/U^{r-1}) = U^r \times \mathrm{Hom}(\mathbb{C}^k, V^r),$$

i.e. $U^{r+1} \cong U^r \times V^{r+1}$ with $V^{r+1} \cong \mathrm{Hom}((\mathbb{C}^k)^{\otimes(r+1)}, \mathbb{C}^{n-k})$, as required.

As for naming the coordinates corresponding to this chart U^r of $D^r_k\mathbb{P}^n$, these are given by (2) for U^0, y^j_i for V^1, and in general, assuming $y^j_{i_1,\dots,i_r}$ are coordinates for V^r, then the space $V^{r+1} \cong \mathrm{Hom}(\mathbb{C}^k, V^r)$ is described by coordinates which we denote $\frac{\partial y^j_{i_1,\dots,i_r}}{\partial x_{i_{r+1}}}$, i.e. $y^j_{i_1,\dots,i_r,i_{r+1}}$. □

For later use, there is a zero section $0_r : U^{r-1} \to U^r$ of $b_r : U^r \to U^{r-1}$ and composing, a zero section $0_{s,r} : U^s \to U^r$ of $b_{r,s}$, for all $s \leq r$. Now we are in the position of defining differential equations on $\mathbb{P}^n$.

Definition 3. A differential equation on $\mathbb{P}^n$ relative to a reference x_i, y_j is a non-zero algebraic equation

$$f(x_i, y_j, y^j_i, y^j_{i_1,i_2}, \dots, y^j_{i_1,\dots,i_r}) = 0,$$

on U^r, for some $r \geq 1$.

Let $U^r(f)$ be the divisor of U^r defined by the differential equation f, and let $H_f = D^r_k\mathbb{P}^n(f)$ denote its closure in $D^r_k\mathbb{P}^n$, i.e. the data satisfying the differential equation. In this way, any differential equation f gives a hypersurface H_f in $D^r_k\mathbb{P}^n$ not containing any of $C^r_2, \dots, C^r_r$. Conversely, given a hypersurface H in $D^r_k\mathbb{P}^n$ not containing any of $C^r_2, \dots, C^r_r$, we can find a suitable reference (2) such that H does not contain C^r_0 and C^r_1, thus being defined as $D^r_k\mathbb{P}^n(f)$ for a suitable polynomial f on U^r. So we can give a more intrinsic definition

Definition 4. A differential equation on $\mathbb{P}^n$ is a hypersurface $H \subseteq D^r_k\mathbb{P}^n$ not containing any of $C^r_2, \dots, C^r_r$.

This second viewpoint allows us to forget coordinates. Nonetheless, in practice, we need to work with coordinates. Whenever we say that f is a differential equation on the projective space, we shall understand that there is a projective reference implicit. There are two caveats. First, if we change the reference, the polynomial f will change in general.

Second, there might be some forbidden references that we cannot choose (namely, those in which H_f contains either of C_0^r, C_1^r).

Remark 5. The formal partial derivative symbols we have defined are non-commuting. This simply means that, for instance, $y^j_{i_1,i_2}$ and $y^j_{i_2,i_1}$ ($i_1 \neq i_2$) are independent coordinates. We can solve this difficulty by restricting to the subset of U^r given by the equations $y^j_{i_1,\dots,i_s} = y^j_{\sigma(i_1),\dots,\sigma(i_s)}$, $1 \leq j \leq n-k$, $1 \leq s \leq r$, σ any permutation of indices, and considering the closure of such subset in $D^r_k\mathbb{P}^n$.

More intrinsically, this subvariety of symmetric data $S^r_k Z \subseteq D^r_k Z$ has been extracted in [5], for any smooth scheme Z. In our case, dealing with $D^r_k Z$ is simpler and will suffice.

Associated to a differential equation we have naturally defined enumerative invariants, thanks to lemma 1.

Definition 6. For a differential equation f on $\mathbb{P}^n$, define $\gamma^f_0, \dots, \gamma^f_r \in \mathbb{Z}$ by

$$[D^r_k\mathbb{P}^n(f)] = \gamma^f_0 C^r_0 + \cdots + \gamma^f_r C^r_r$$

in $A^1 D^r_k\mathbb{P}^n$.

It is important to note that γ^f_s do not depend on the trivialisation, as the divisor classes of C^r_0 and C^r_1 are independent of the trivialisation as well. The following proposition allows us to compute some of γ^f_s in the general case.

Proposition 7. *Let f be a differential equation on $\mathbb{P}^n$ relative to a reference x_i, y_j.*

- *Suppose f is distinguished in the variable $y^1_{1,\overset{(r)}{\dots},1}$ (i.e. for $d = \deg(f)$ the monomial $(y^1_{1,\overset{(r)}{\dots},1})^d$ appears in f with nonzero coefficient), then the leading γ^f_r is*

$$\gamma^f_r = \deg f(0, \dots, y^1_{1,\overset{(r)}{\dots},1}, \dots, 0).$$

- *Suppose f is distinguished in the variable y^1_1, then*

$$\gamma^f_1 = \deg f(0, \dots, y^1_1, \dots, 0).$$

- *Suppose f is distinguished in the variable x_1, then*

$$\gamma^f_0 = \deg f(x_1, \dots, 0).$$

Proof. **Step 1.** We need first to exhibit a suitable basis $c^r_0, \dots, c^r_s, \dots, c^r_r$ of $A_1 D^r_k\mathbb{P}^n$ (we name equally closed subsets, their associated cycles, and their rational classes). These will be the closure in $D^r_k\mathbb{P}^n$ of the one dimensional subschemes $\mathring{c}^r_s \subseteq U^r$ defined by the vanishing of all coordinates but $y^1_{1,\overset{(s)}{\dots},1}$ (if $s = 0$, this is the coordinate x_1). That they form a basis will be proved in Step 2.

To give an intrinsic description of $\mathring{c}^r_s$, we define the subspace $\mathbb{C}^{k-1} \subseteq \mathbb{C}^k$ in the given affine plane $\mathbb{C}^n \subseteq U^0$ as the space corresponding to coordinates $x_2, \dots, x_k$, and the rank $(k-1)$-subbundle $\tilde{X}$ of the bundle X on this plane by $\tilde{X}(P) = P + \mathbb{C}^{k-1}$, and correspondingly the trivial rank $(k-1)$-subbundle $(\tilde{\Sigma}_r)_{U^r} \cong \tilde{X}_{U^r}$ of $(\Sigma_r)_{U^r} \cong X_{U^r}$. Analogously, if $\mathbb{C} \subseteq \mathbb{C}^{n-k}$ is the y_1-axis, we define the rank 1 subbundle $\tilde{Y}$ of Y by $\tilde{Y}(P) = P + \mathbb{C}$.

Furthermore, we define inductively $\tilde{U}^r = \tilde{U}^{r-1} \times \mathbb{C}$ inside $U^r = U^{r-1} \times V^r$ by taking $\tilde{U}^0 = U^0$,

$$\tilde{U}^1 = \operatorname{Hom}(X/\tilde{X}, \tilde{Y}) \subseteq \operatorname{Hom}(X, Y) = U^1,$$

and assuming that $\tilde{U}^r$ is already defined, by taking

$$\tilde{U}^{r+1} = \operatorname{Hom}((\Sigma_r)_{U^r}/(\tilde{\Sigma}_r)_{U^r}, T\tilde{U}^r/U^{r-1}) = \operatorname{Hom}(U^r \times \mathbb{C}, \tilde{U}^r \times \mathbb{C}) = \tilde{U}^r \times \mathbb{C}$$

inside $U^{r+1} = \operatorname{Hom}((\Sigma_r)_{U^r}, TU^r/U^{r-1})$.

Let 0 be the origin of $\mathbb{C}^n$, and let $0_{s-1} = 0_{0,s-1}(0)$ be the origin of U^{s-1}. Note that $0_{s-1} \in \tilde{U}_{s-1}$. Define $\mathring{c}^0 = 0 + \mathbb{C} \subseteq U^0$, where $\mathbb{C} \subseteq \mathbb{C}^k$ is the x_1-axis. For $s \geq 1$, let $\mathring{c}^s \subseteq U^s$ be the 1-dimensional subvariety $\tilde{U}^s \cap b_s^{-1}(0_{s-1})$ and c^s the closure of $\mathring{c}^s$ in $D_k^s\mathbb{P}^n$. For $0 \leq s \leq r$, the above $\mathring{c}_s^r \subseteq U^r$ is just $0_{s,r}(\mathring{c}^s)$, thus $b_{r,s}(\mathring{c}_s^r) = \mathring{c}^s$ and $b_{r,s}(c_s^r) = c^s$.

Step 2. Now we want to relate the elements $c_0^r, \dots, c_s^r, \dots, c_r^r$ of $A_1 D_k^r \mathbb{P}^n$ with the elements $C_0^r, \dots, C_s^r, \dots, C_r^r$ of $A^1 D_k^r \mathbb{P}^n$. One has $\mathring{c}_s^r \cong \mathbb{C}$ and therefore $c_s^r \cap (C_0^r \cup \dots \cup C_r^r) = c_s^r \setminus U^r = c_s^r \setminus \mathring{c}_s^r \cong \mathbb{P} \setminus \mathbb{C}$ consists, set-theoretically, of just one point, which we want now to show not to be in any C_t^r for $t < s$ and that this point is in fact the *schematic* intersection $c_s^r \cap C_s^r$, i.e. that $c_s^r \cdot C_s^r = 1$, as rational classes. On the one hand, this gives an alternative proof that the C_s^r are linearly independent in lemma 1. On the other hand, this proves that c_s^r are a basis for $A_1 D_k^r \mathbb{P}^n$, since the intersection matrix $(C_t^r \cdot c_s^r)$ is lower-triangular.

Let $0 \leq t < s \leq r$. Since $C_t^r = b_{r,t}^{-1}(C^t)$, in order to show that $c_s^r \cap C_t^r = \emptyset$ it is enough to show that $b_{r,t}(c_s^r) \cap C^t$ is empty. By construction, $b_{r,t}(\mathring{c}_s^r) = 0 \in U^t$, since $t < s$, thus $b_{r,t}(c_s^r) = 0$ is disjoint with C^t. This proves that, as rational classes, $c_s^r \cdot C_t^r = 0$.

Again, since $C_s^r = b_{r,s}^{-1}(C^s)$, in order to show that $c_s^r \cdot C_s^r = 1$, it is enough to show that $b_{r,s}(c_s^r) \cdot C^s = c^s \cdot C^s = 1$. For $s = 0$ this is true since $c^0 \subseteq \mathbb{P}^n$ is a line and C^0 is the hyperplane. For $s \geq 1$, we note that since $c^s \subseteq b_s^{-1}(0_{s-1})$, it is enough to show for $C_{0_{s-1}}^s = C^s \cap b_s^{-1}(0_{s-1})$ that

$$c^s \cdot C_{0_{s-1}}^s = 1$$

in the Grassmannian $b_s^{-1}(0_{s-1})$. Suppose now that $s \geq 2$ (the case $s = 1$ is similar and is left to the reader). Then $b_s^{-1}(0_{s-1}) = \operatorname{Gr}_k((\mathcal{F}_{s-1})_{0_{s-1}})$, where $(\mathcal{F}_{s-1})_{0_{s-1}}$ is the Semple vector space

$$(\mathcal{F}_{s-1})_{0_{s-1}} = (TU^{s-1}/U^{s-2})_{0_{s-1}} \oplus (\Sigma_{s-1})_{0_{s-1}}.$$

In this Grassmannian, $C_{0_{s-1}}^s$ is the Schubert cycle of k-spaces meeting $(TU^{s-1}/U^{s-2})_{0_{s-1}}$, i.e. the base of its Picard group A^1. On the other hand, the 1-dimensional subvariety

$$\mathring{c}^s = \operatorname{Hom}((\Sigma_{s-1})_{0_{s-1}}/(\tilde{\Sigma}_{s-1})_{0_{s-1}}, (T\tilde{U}^{s-1}/U^{s-2})_{0_{s-1}})$$

of the open subset $\operatorname{Hom}((\Sigma_{s-1})_{0_{s-1}}, (TU^{s-1}/U^{s-2})_{0_{s-1}})$ of $\operatorname{Gr}_k((\mathcal{F}_{s-1})_{0_{s-1}})$ is an open subset of the Grassmannian of k-subspaces of the $(k+1)$-subspace $(T\tilde{U}^{s-1}/U^{s-2})_{0_{s-1}} \oplus (\Sigma_{s-1})_{0_{s-1}} \subseteq (\mathcal{F}_{s-1})_{0_{s-1}}$ which contain the $(k-1)$-subspace $(\tilde{\Sigma}_{s-1})_{0_{s-1}} \subseteq (\Sigma_{s-1})_{0_{s-1}} \subseteq (\mathcal{F}_{s-1})_{0_{s-1}}$. Thus c^s is the Grassmannian of such k-spaces, i.e. the Schubert cycle base of A_1 dual to the base $C_{0_{s-1}}^s$ of A^1, thus $c^s \cdot C_{0_{s-1}}^s = 1$.

Step 3. Now we can find the coefficient γ_r^f in the expression of definition 6. By the above, this is

$$\gamma_r^f = [D_k^r\mathbb{P}^n(f)]\cdot c_r^r = \text{length } U^r(f)\cap \mathring{c}_r^r = \deg f(0,\dots,y^1_{1,\overset{(r)}{\dots},1},\dots,0).$$

Indeed, the second equality is due to

$$(D_k^r\mathbb{P}^n(f)\backslash U^r(f))\cap c_r^r = D_k^r\mathbb{P}^n(f)\cap (C_0^r\cup\dots\cup C_r^r)\cap c_r^r=\emptyset$$

since $(C_0^r\cup\dots\cup C_r^r)\cap c_r^r$ is schematically one point, say the point $(0,\dots,\infty,\dots,0)$ at infinity of $\mathring{c}_r^r=\{(0,\dots,y^1_{1,\overset{(r)}{\dots},1},\dots,0)\}\cong\mathbb{C}$, and this point is not in $D_k^r\mathbb{P}^n(f)$, i.e.

$$0\neq f(0,\dots,\infty,\dots,0)=\lim_{y^1_{1\dots1}\to\infty} f(0,\dots,y^1_{1,\overset{(r)}{\dots},1},\dots,0),$$

because f is distinguished in the variable $y^1_{1,\overset{(r)}{\dots},1}$. This finishes the proof of the first item in the statement of the proposition.

Step 4. To get the coefficient γ_1^f, we prove first that $c_1^r\cdot C_s^r=0$ for $s>1$. For this we consider the (flat) family of subvarieties $\{S_\lambda\}_{\lambda\in\mathbb{P}^1}$ of $\mathbb{P}^n$ such that for $\lambda\in\mathbb{C}$, S_λ is given by the equations $y_1-\lambda x_1=0$, $y_2=0$, $\dots$, $y_{n-k}=0$, and S_∞ has equations $x_1=0$, $y_2=0$, $\dots$, $y_{n-k}=0$. Clearly the points of $\mathring{c}_1^r$ are parametrized by $D_k^rS_\lambda\cap b_{r,0}^{-1}(0)$, $\lambda\in\mathbb{C}$. Therefore $c_1^r\backslash\mathring{c}_1^r$ is the point $D_k^rS_\infty\cap b_{r,0}^{-1}(0)$. But S_∞ is smooth, so $D_k^rS_\infty$ is disjoint from $C_s^r=0$ for $s>1$ (see remark 11). This proves that $c_1^r\cdot C_s^r=0$ for $s>1$.

Now $\gamma_1^f=[D_k^r\mathbb{P}^n(f)]\cdot c_1^r=\deg f(0,\dots,y_1^1,\dots,0)$, when f is distinguished in the variable y_1^1, arguing as in step 3.

Step 5. To get the coefficient γ_0^f, we need to prove first that $c_0^r\cdot C_s^r=0$ for $s>0$. This time we fix the subvariety S with equations $y_1=0$, $y_2=0$, $\dots$, $y_{n-k}=0$. Recall $\mathring{c}^0=0+\mathbb{C}\subseteq U^0$, where $\mathbb{C}\subseteq\mathbb{C}^k$ is the x_1-axis and c^0 is its closure in $\mathbb{P}^n$. The points of c_0^r are parametrized by $D_k^rS\cap b_{r,0}^{-1}(\lambda)$, $\lambda\in c^0\subseteq\mathbb{P}^n$. Now D_k^rS is disjoint from C_s^r, for $s\geq 1$, so $c_0^r\cdot C_s^r=0$ for $s>0$. The rest of the argument is as in steps 3 and 4. □

Remark 8. The explicit computation of the other γ_s^f, $2\leq s<r$, requires the full knowledge of the intersection matrix $c_s^r\cdot C_t^r$, which is a more delicate issue (see remark 14). Nonetheless if we are only interested in applications of differential equations on smooth varieties, γ_0^f and γ_1^f will suffice (see corollary 15).

Remark 9. Suppose that $f(x_i,y_j,y_i^j,y^j_{i_1,i_2},\cdots,y^j_{i_1,\dots,i_r})$ is a differential equation on $\mathbb{P}^n$ relative to a reference x_i,y_j with $d=\deg(f)$. Suppose that

$$\deg f(0,\dots,0,y^1_{1,\overset{(r)}{\dots},1},\dots,y^{n-k}_{k,\overset{(r)}{\dots},k})=d.$$

Then $\gamma_r^f=d$. The proof is similar to steps 2 and 3 in the proof of proposition 7. Choose a generic line $\mathring{l}\subseteq V^r$ and identify it with $0_{r-1}\times\mathring{l}\subseteq U^r=U^{r-1}\times V^r$. Let $l\subseteq D_k^r\mathbb{P}^n$ be its closure. Then it is easy to see that $l\cdot C_s^r=0$ for $s<r$ and $l\cdot C_r^r=1$. Now

$$\gamma_r^f=[D_k^r\mathbb{P}^n(f)]\cdot l=\deg f(0,\dots,0,y^1_{1,\overset{(r)}{\dots},1},\dots,y^{n-k}_{k,\overset{(r)}{\dots},k})=d,$$

as in step 3 in the proof of proposition 7.

3. Degree of the divisor of solutions

Let S be a subvariety of $\mathbb{P}^n$ of dimension k. We have defined $D^r_kS \subseteq D^r_k\mathbb{P}^n$ as the closure of $D^r_kS_{\text{reg}} \subseteq D^r_k\mathbb{P}^n$. There is a natural map

$$D^r_kS \to S,$$

which is an isomorphism over the smooth part. Therefore if S is smooth then $S \cong D^r_kS$. In general, we have the following invariants associated to S

Definition 10. The cuspidal numbers γ^s_S, $s \geq 0$, of S are defined as

- γ^0_S is the degree of $S \subseteq \mathbb{P}^n$.
- γ^1_S is the class of S, i.e. the degree of the divisor of S consisting of the points of tangency of tangent k-planes to S that can be drawn from a generic $(n-k-1)$-plane of $\mathbb{P}^n$.
- For $s \geq 2$, γ^s_S is the degree of the push-forward $b_{s,0}(C^sS) \subseteq \mathbb{P}^n$ of $C^sS = D^s_kS \cap C^s \subseteq D^s_k\mathbb{P}^n$ under $b_{s,0} : D^s_k\mathbb{P}^n \to \mathbb{P}^n$.

Remark 11. As remarked in [5], if S is smooth then $\gamma^s_S = 0$ for $s \geq 2$. Indeed, in this case, D^r_kS is disjoint from the cuspidal divisors C^r_s, $s \geq 2$. Moreover if S has singularities only in codimensions 2 or more (e.g. S normal), the cycle $b_{s,0}(C^sS) \subseteq \mathbb{P}^n$, $s \geq 2$, has dimension less or equal than $k-2$ and hence $\gamma^s_S = 0$ for $s \geq 2$. If S has no cuspidal singularities (for instance, if the only singularities are normal double crossings along a smooth subvariety) then $\gamma^s_S = 0$ for $s \geq 2$ as well. The cuspidal numbers measure in some sense how complicated the singularities (on codimension 1) of S are.

Lemma 12. *For any $0 \leq s \leq r$, it is $\gamma^s_S = D^r_kS \cap C^r_s \cap H^{k-1}$, where $H = C^r_0 = b^{-1}_{r,0}(H)$ is the hyperplane in $D^r_k\mathbb{P}^n$.*

Proof. Using that $D^r_kS \cap C^r_s \cap b^{-1}_{r,0}(H^{k-1}) = D^s_kS \cap C^s \cap b^{-1}_{s,0}(H^{k-1})$, we reduce to the case $r = s$. Now for $r = 0$ is obvious, for $r = 1$ follows from the definition, and for $r \geq 2$, $D^r_kS \cap C^r \cap b^{-1}_{r,0}(H^{k-1}) = C^rS \cdot b^{-1}_{r,0}(H^{k-1}) = b_{r,0}(C^rS) \cdot H^{k-1} = \gamma^r_S$. □

Given a differential equation f on $\mathbb{P}^n$, we define the divisor of solutions of f on S as follows. First, $D^r_kS(f)$ is the schematic intersection in $D^r_k\mathbb{P}^n$

$$D^r_kS(f) = D^r_kS \cap D^r_k\mathbb{P}^n(f).$$

Unless $D^r_kS(f) = D^r_kS$, we have that $D^r_kS(f)$ is a divisor in D^r_kS. The divisor of solutions $S(f) \subset S$ of f on S is the push-forward of $D^r_kS(f)$ under $b_{r,0} : D^r_k\mathbb{P}^n \to \mathbb{P}^n$, i.e. $S(f) = b_{r,0}(D^r_kS(f))$.

Theorem 13. *Let f be a differential equation on $\mathbb{P}^n$ with γ^f_s, $0 \leq s \leq r$. Let S be a subvariety of $\mathbb{P}^n$ of dimension k. Suppose that $S(f)$ is a proper subset of S. Then the degree of $S(f)$ is*

$$\gamma^f_0\gamma^0_S + \cdots + \gamma^f_s\gamma^s_S + \cdots + \gamma^f_r\gamma^r_S.$$

Proof. The numbers γ_s^f are defined by the condition

$$[D_k^r\mathbb{P}^n(f)] = \gamma_0^f C_0^r + \cdots + \gamma_s^f C_s^r + \cdots + \gamma_r^f C_r^r$$

in $A^1 D_k^r \mathbb{P}^n$. Now the degree of $S(f)$ is $S(f)\cdot H^{k-1}$ i.e.

$$D_k^r S(f)\cdot b_{r,0}^{-1}(H^{k-1}) = D_k^r S \cap D_k^r\mathbb{P}^n(f) \cap H^{k-1} = [D_k^r S\cap H^{k-1}]\cdot[D_k^r\mathbb{P}^n(f)].$$

Lemma 12 says that the s^{th}-cuspidal degree of S is given as $\gamma_S^s = [D_k^r S\cap H^{k-1}]\cdot C_s^r$. Therefore we have

$$\deg S(f) = \gamma_0^f\gamma_S^0 + \cdots + \gamma_s^f\gamma_S^s + \cdots + \gamma_r^f\gamma_S^r.$$

□

Remark 14. Theorem 13 can be used for computing γ_s^f for a given differential equation f, by using a standard set of subvarieties whose cuspidal numbers are known (or easily obtainable). This method is used in the examples of section 4.

In the smooth case we get the following

Corollary 15. *Let f be a differential equation on $\mathbb{P}^n$ and let S be a smooth (or just normal) subvariety of $\mathbb{P}^n$ of dimension k. Suppose that $S(f)$ is a proper subset of S. Then*

$$\deg S(f) = \gamma_0^f\gamma_S^0 + \gamma_1^f\gamma_S^1.$$

Let us work out the values of γ_S^0 and γ_S^1 for a smooth S. In the case $k=1$, $S\subseteq\mathbb{P}^n$ is a smooth curve. If d is its degree and g is genus, then $\gamma_S^0 = d$ and $\gamma_S^1 = 2g-2+2d$. Note that when $n=2$, i.e. $S\subseteq\mathbb{P}^2$ is a smooth plane curve, the class of S is $\gamma_S^1 = d(d-1)$, which can be obtained by using the adjunction formula $2g-2 = d(d-3)$.

For $k>1$, let $S\subseteq\mathbb{P}^n$ be a smooth (or just normal) subvariety of degree d and let g be the genus of the generic section $C = S\cap H^{k-1}$ (which is a smooth curve). Then the degree of C is d and its class is $\gamma_C^1 = 2g-2+2d$. Now it is easy to see that the class of S equals that of C, $\gamma_S^1 = \gamma_C^1$ (for instance take a reference x_i, y_j such that the H^{k-1} has equations $y_1 = 0, \ldots, y_{n-k} = 0, x_1 = 0$ and use lemma 12).

Corollary 16. *Let f be a differential equation on $\mathbb{P}^n$ and let $S\subseteq\mathbb{P}^n$ be a smooth (or just normal) subvariety of degree d whose generic section $S\cap H^{k-1}$ has genus g. Suppose that $S(f)$ is a proper subset of S. Then $\deg S(f) = \gamma_0^f\, d + \gamma_1^f\,(2g-2+2d)$. Furthermore, if S is a hypersurface, then the formula is reduced to $\deg S(f) = \gamma_0^f\, d + \gamma_1^f\, d(d-1)$.*

4. Examples

Theorem 13 can be used in two directions, either to compute the degree of the divisor of solutions of a differential equations on a subvariety of $\mathbb{P}^n$ or to extract information about a differential equation f and a subvariety $S\subseteq\mathbb{P}^n$ once we have computed the degree of the divisor of solutions of f on S.

Computation of deg $S(f)$. Given a differential equation f and a k-dimensional subvariety $S\subseteq\mathbb{P}^n$, to find the divisor $S(f)$ we proceed as follows. Take a reference x_i, y_j. When S is smooth we find explicitly $U^r(f)\cap D_k^r S$ by considering the ideal generated by the equations defining S, their formal derivatives up to order r together with the equation $f = 0$. If S is non-smooth we have to be a little bit more careful, as $D_k^r S$ is defined as the closure of $D_k^r S_{\mathrm{reg}}$, and hence it may be smaller than the set defined by the above ideal (see example 18).

Using different references x_i, y_j, we find with this method all the points in $D_k^r S(f)$ not lying in

$$(C_2^r \cup \ldots \cup C_r^r) \cap D_k^r \mathbb{P}^n(f) \cap D_k^r S, \tag{7}$$

i.e. $(D_k^r S(f))_{\mathrm{nc}} = D_k^r S(f) - \bigcup_{s=2}^{r} C_s^r$. When S is smooth (7) is empty and hence $D_k^r S(f) = (D_k^r S(f))_{\mathrm{nc}}$. In general, $\deg S(f)$ equals the degree of the closure of $(D_k^r S(f))_{\mathrm{nc}}$ unless (7) has $(k-1)$-dimensional components (we expect it to be $(k-2)$-dimensional).

Note that to compute $\deg S(f)$ we have to take a linear section H^{k-1} of $S(f)$. Therefore we look for the number of solutions of the differential equation in a general H^{k-1} section of S, which is a curve.

Example 17. Let $n = 2$ and $k = 1$, i.e. the case of curves in $\mathbb{P}^2$. Consider the smooth cubic S given by $x^3 + y^2 = 1$. To compute $\deg S(y')$, where $y' = \frac{\partial y}{\partial x}$, we compute the number of points in

$$\begin{cases} x^3 + y^2 = 1 \\ 3x^2 + 2yy' = 0 \\ y' = 0 \end{cases}$$

which is 4. Now S passes through the point $P = (0, \infty)$ at infinity C^0 and is tangent to C^0 there. Therefore this point counts with multiplicity 2 (otherwise compute in a different reference). So $\deg S(y') = 4 + 2 = 6$.

Now let us compute $\deg S(y'')$, where $y'' = \frac{\partial^2 y}{\partial x^2}$. The number of points in

$$\begin{cases} x^3 + y^2 = 1 \\ 3x^2 + 2yy' = 0 \\ 6x + 2(y')^2 + 2yy'' = 0 \\ y'' = 0 \end{cases}$$

is 8. The point P at infinity counts once, since it is a simple flex of S. Hence $\deg S(y'') = 8 + 1 = 9$.

Example 18. Now consider the singular cubic T given by $x^3 + y^2 = 0$. Now the closure of $D_1^1 T_{\mathrm{reg}}$ is the irreducible component of

$$\begin{cases} x^3 + y^2 = 0 \\ 3x^2 + 2yy' = 0 \end{cases}$$

given by the equations $x = \frac{2}{3}(y')^2$, $y = -\frac{2}{3}(y')^3$. Hence $U^0 \cap T(y')$ consists of 1 point. Also T passes through $P = (0, \infty)$ at infinity with multiplicity 2, so $\deg T(y') = 1 + 2 = 3$.

For computing $\deg T(y'')$ we study

$$\begin{cases} 3x - 2(y')^2 = 0 \\ 3y + 2(y')^3 = 0 \\ 3 - 4y'y'' = 0 \\ 3y' + 6(y')^2 y'' = 0 \\ y'' = 0 \end{cases}$$

which is empty. Counting the point at infinity, $\deg T(y'') = 1$.

Computation of γ_s^f. Apart from proposition 7 (and in some cases the extension given in remark 9), we can compute γ_s^f by coupling f with some simple examples of varieties $S \subseteq \mathbb{P}^n$. Many differential equations f have a geometrical meaning. For instance for curves in the plane, $y' = 0$ gives the points of a curve with tangent parallel to the x-axis, and $y'' = 0$ gives the flexes of a curve. Let us work out these two examples.

For $f = y'$, we have $\gamma_1^{y'} = 1$. Let $C \subseteq \mathbb{P}^2$ be a smooth conic, which has degree $\gamma_C^0 = 2$ and class $\gamma_C^1 = 2$. Hence $2 = \deg C(y') = 2\gamma_0^{y'} + 2$, so $\gamma_0^{y'} = 0$.

For $f = y''$, proposition 7 says that $\gamma_2^{y''} = 1$. A conic has no flexes, so $0 = \deg C(y'') = 2\gamma_0^{y''} + 2\gamma_1^{y''}$. The smooth cubic S of example 17 gives first $\gamma_1^S = \deg S(y') = 6$ and then $9 = \deg S(y'') = 3\gamma_0^{y''} + 6\gamma_1^{y''}$. Therefore $\gamma_0^{y''} = -3$ and $\gamma_1^{y''} = 3$.

Computation of γ_S^s. For a smooth (or normal) variety $S \subseteq \mathbb{P}^n$, we have that γ_S^0 is its degree and γ_S^1 its class. This can be computed by taking a general section $C = H^{k-1} \cap S$, which is a curve of degree $\gamma_C^0 = \gamma_S^0$ and class $\gamma_C^1 = \gamma_S^1$. So $\gamma_S^1 = 2g - 2 + 2d$, where g is the genus of C and d its degree. Also $\gamma_S^s = 0$ for $s \geq 2$.

In the non-smooth case, things are a little bit more complicated. For instance, for the smooth cubic S of example 17, $\gamma_S^0 = 3$, $\gamma_S^1 = 6$ and $\gamma_S^2 = 0$. Instead for the non-smooth cubic T of example 18, $\gamma_T^0 = 3$, $\gamma_T^1 = \deg T(y') = 3$ and $\gamma_T^2 = \deg T(y'') - \gamma_0^{y''} 3 - \gamma_1^{y''} 3 = 1$.

Degree of the divisor of parabolic points. As an application, we shall determine the degree of the divisor of parabolic points of a subvariety $S \subseteq \mathbb{P}^n$. Suppose first that $k = n-1$, i.e. S is a hypersurface of $\mathbb{P}^n$. Parabolic points are those points of S with higher contact with the tangent space than expected. In terms of a reference $x_1, \ldots, x_{n-1}, y$, they are the solutions to the differential equation

$$f = \det(\frac{\partial^2 y}{\partial x_i \partial x_j}).$$

To compute the invariants of f we work as follows. By remark 9, $\gamma_2^f = n - 1$. Now for a smooth quadric $C \subseteq \mathbb{P}^n$ there are no parabolic points, so $0 = \deg C(f) = 2\gamma_0^f + 2\gamma_1^f$, and $\gamma_1^f = -\gamma_0^f$. Now let $S \subseteq \mathbb{P}^n$ be the smooth cubic given by $x_1^3 + \cdots + x_{n-1}^3 + y^2 + 1 = 0$. It is easy to compute

$$\begin{cases} \frac{\partial^2 y}{\partial x_i^2} = -\frac{3x_i}{y} - \frac{9x_i^4}{4y^3}, & 1 \leq i \leq n-1 \\ \frac{\partial^2 y}{\partial x_i \partial x_j} = \frac{-9x_i^2 x_j^2}{4y^3}, & 1 \leq i, j \leq n-1,\ i \neq j \end{cases}$$

so that $\det(\frac{\partial^2 y}{\partial x_i \partial x_j}) = (-3)^{n-1} x_1 \cdots x_{n-1}(y^2-3)/4y^{n+1}$ and hence $3(n+1) = \deg S(f) = 3\gamma_0^f + 6\gamma_1^f$. This yields $\gamma_1^f = -\gamma_0^f = n+1$. Our conclusion is that for a hypersurface $S \subseteq \mathbb{P}^n$, the degree of the divisor of parabolic points is

$$\deg S(f) = -(n+1)\gamma_S^0 + (n+1)\gamma_S^1 + (n-1)\gamma_S^2.$$

If S is a smooth hypersurface of degree d, it reduces to

$$\deg S(f) = -(n+1)d + (n+1)d(d-1) = (n+1)d(d-2).$$

This agrees with the following alternative argument (only applies to the smooth case): if S is given by the equation $F(x_0, \ldots, x_n) = 0$, then the parabolic points are the intersection of $F = 0$ and $\det(\frac{\partial^2 F}{\partial X_i \partial X_j}) = 0$, and so form a divisor of degree $(n+1)d(d-2)$.

For the general case $0 < k < n$, we consider a reference x_i, y_j and fix $1 \leq r \leq n-k$. Then the parabolic points in the y_r-direction are the solutions to the differential equation

$$f_r = \det(\frac{\partial^2 y_r}{\partial x_i \partial x_j}).$$

Working as above we have the formula

$$\deg S(f_r) = -(n+1)\gamma_S^0 + (n+1)\gamma_S^1 + (n-1)\gamma_S^2.$$

5. The real case

In this section, our purpose is to extend theorem 13 to the case of real projective varieties inside the real projective space $\mathbb{P}^n_{\mathbb{R}}$. We are not going to develop the general theory of data schemes for smooth real algebraic varieties, but only to outline the construction of $D^r_k\mathbb{P}^n_{\mathbb{R}}$.

First of all fix a trivialisation on $\mathbb{P}^n$ as in section 2, so $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$. Again $1 \leq k \leq n-1$. There is an anti-holomorphic involution $\sigma : \mathbb{P}^n \to \mathbb{P}^n$ coming from conjugation on $\mathbb{C}^{n+1}$, whose fixed point set is $\mathbb{P}^n_{\mathbb{R}}$. Inductively, σ induces anti-holomorphic involutions $\sigma_r : D^r_k\mathbb{P}^n \to D^r_k\mathbb{P}^n$, $r \geq 1$. We define $D^r_k\mathbb{P}^n_{\mathbb{R}}$ to be the fixed point set of σ_r. The maps $b_{r,s}$ of (5) restrict to $D^r_k\mathbb{P}^n_{\mathbb{R}}$ and so there are well-defined maps $b_{r,s} : D^r_k\mathbb{P}^n_{\mathbb{R}} \to D^s_k\mathbb{P}^n_{\mathbb{R}}$. It is easy to see that we can construct $D^r_k\mathbb{P}^n_{\mathbb{R}}$ as follows. We take $D^0_k\mathbb{P}^n_{\mathbb{R}} = \mathbb{P}^n_{\mathbb{R}}$, $D^1_k\mathbb{P}^n_{\mathbb{R}} = \mathrm{Gr}_k T\mathbb{P}^n_{\mathbb{R}}$, the real Grassmannian of k-planes in the tangent bundle $T\mathbb{P}^n_{\mathbb{R}}$, and for $r \geq 2$, $D^r_k\mathbb{P}^n_{\mathbb{R}} = \mathrm{Gr}_k\mathcal{F}^{\mathbb{R}}_{r-1}$, where the real (Semple) bundle $\mathcal{F}^{\mathbb{R}}_{r-1}$ is defined by a diagram as (1), where the relative tangent bundles and the universal sequence of the right hand side are understood to be real. These $D^r_k\mathbb{P}^n_{\mathbb{R}}$ are also smooth compact differentiable manifolds.

The involution σ_r takes C^r to itself and the fixed point set will be called $C^r_{\mathbb{R}}$. This is a smooth 1-codimensional real algebraic subvariety of $D^r_k\mathbb{P}^n_{\mathbb{R}}$. Clearly $C^0_{\mathbb{R}} \subseteq \mathbb{P}^n_{\mathbb{R}}$ is the hyperplane. The other $C^r_{\mathbb{R}}$ can be also defined as the cuspidal locus of $D^r_k\mathbb{P}^n_{\mathbb{R}}$, namely the Schubert special cycle of real k-planes of $\mathcal{F}^{\mathbb{R}}_{r-1}$ meeting $TD^{r-1}_k\mathbb{P}^n_{\mathbb{R}}/D^{r-2}_k\mathbb{P}^n_{\mathbb{R}}$. Again $C^r_{\mathbb{R},s}$ are defined either as the fixed point set of σ_r on C^r_s or as $b^{-1}_{r,s}(C^s_{\mathbb{R}})$.

We can parallel the discussion in section 2 to see that

$$U^r_{\mathbb{R}} = D^r_k\mathbb{P}^n_{\mathbb{R}} \backslash (C^r_{\mathbb{R},0} \cup \ldots \cup C^r_{\mathbb{R},s} \cup \ldots \cup C^r_{\mathbb{R},r})$$

are cartesian powers of $\mathbb{R}$. Indeed, $U^0_{\mathbb{R}} = \mathbb{R}^k \times \mathbb{R}^{n-k}$ has real coordinates (the restriction to $U^0_{\mathbb{R}} \subseteq U^0$ of) $x_1, \dots, x_k, y_1, \dots, y_{n-k}$. In general

$$U^r_{\mathbb{R}} \cong U^{r-1}_{\mathbb{R}} \times V^r_{\mathbb{R}},$$

where $V^r_{\mathbb{R}} = \mathrm{Hom}((\mathbb{R}^k)^{\otimes r}, \mathbb{R}^{n-k})$, for any $r \geq 1$. Alternatively, σ_r restricts to U^r and the fixed point locus is $U^r_{\mathbb{R}}$. The coordinates for $U^r_{\mathbb{R}}$ will be

$$x_1, \dots, x_k, y_1, \dots, y_{n-k} \qquad \text{and} \qquad y^j_{i_1,\dots,i_s} = \frac{\partial^s y_j}{\partial x_{i_s} \cdots \partial x_{i_1}}, \tag{8}$$

for any $1 \leq s \leq r$, $1 \leq j \leq n-k$, and $1 \leq i_1, \dots, i_s \leq k$.

Lemma 19. *$C^r_{0,\mathbb{R}}, \dots, C^r_{s,\mathbb{R}}, \dots, C^r_{r,\mathbb{R}}$ form a basis for $H^1(D^r_k\mathbb{P}^n_{\mathbb{R}}; \mathbb{Z}/2\mathbb{Z})$, where we name equally the algebraic subvarieties and the cohomology classes they represent through Poincaré duality.*

Proof. Given the description of the real Grassmannian as a homogeneous space

$$\mathrm{Gr}_k\mathbb{R}^n = O(n)/O(k) \times O(n-k),$$

it is easy to prove that the fundamental group $\pi_1(\mathrm{Gr}_k\mathbb{R}^n) = \mathbb{Z}/2\mathbb{Z}$ for $n > 2$ and $\pi_1(\mathrm{Gr}_1\mathbb{R}^2) = \pi_1(\mathbb{RP}^1) = \mathbb{Z}$. So $H^1(\mathrm{Gr}_k\mathbb{R}^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and it is generated by the cuspidal subvariety. The Serre spectral sequence of the fibration

$$\mathrm{Gr}_k\mathbb{R}^\bullet \to D^r_k\mathbb{P}^n_{\mathbb{R}} = \mathrm{Gr}_k\mathcal{F}^{\mathbb{R}}_{r-1} \to D^{r-1}_k\mathbb{P}^n_{\mathbb{R}}$$

implies that $H^1(D^r_k\mathbb{P}^n_{\mathbb{R}}; \mathbb{Z}/2\mathbb{Z})$ is generated by $C^r_{0,\mathbb{R}}, \dots, C^r_{r,\mathbb{R}}$. Now the 1-cycles $c^r_{s,\mathbb{R}}$ are defined as in the proof of proposition 7 (alternatively as the fixed locus of σ_r on c^r_s) and the homology classes they represent in $H_1(D^r_k\mathbb{P}^n_{\mathbb{R}}; \mathbb{Z}/2\mathbb{Z})$ satisfy $c^r_{t,\mathbb{R}} \cdot C^r_{s,\mathbb{R}} = 0$, for $t < s$ and $c^r_{s,\mathbb{R}} \cdot C^r_{s,\mathbb{R}} = 1 \pmod 2$. This shows that $C^r_{s,\mathbb{R}}$ are linearly independent. ◻

Recall that the cohomology ring of $\mathbb{P}^n_{\mathbb{R}}$ (with $\mathbb{Z}/2\mathbb{Z}$-coefficients) is $(\mathbb{Z}/2\mathbb{Z})[H]/H^{n+1}$, where H stands for the hyperplane class. Again $H = C^r_0 = b^{-1}_{r,0}(H)$ is the hyperplane in $D^r_k\mathbb{P}^n$. Now we are in the position of defining differential equations on $\mathbb{P}^n_{\mathbb{R}}$.

Definition 20. A differential equation on $\mathbb{P}^n_{\mathbb{R}}$ relative to a reference x_i, y_j is a non-zero algebraic equation with real coefficients

$$f(x_i, y_j, y^j_i, y^j_{i_1,i_2}, \cdots, y^j_{i_1,\dots,i_r}) = 0,$$

on $U^r_{\mathbb{R}}$, for some $r \geq 1$.

This time the zero locus of such f is not necessarily a hypersurface of $D^r_k\mathbb{P}^n_{\mathbb{R}}$ (it might even be empty). But we may still define the numbers $\gamma^f_s \in \mathbb{Z}/2\mathbb{Z}$ by looking at the component in $H^1(D^r_k\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z})$ defined by the closure of the zero locus of f in $U^r_{\mathbb{R}} \subseteq D^r_k\mathbb{P}^n_{\mathbb{R}}$. So

$$[D^r_k\mathbb{P}^n_{\mathbb{R}}(f)]_1 = \gamma^f_0 C^r_{0,\mathbb{R}} + \cdots + \gamma^f_r C^r_{r,\mathbb{R}}.$$

We leave the following analogue of proposition 7 to the reader.

Proposition 21. *Let f be a differential equation on $\mathbb{P}^n_{\mathbb{R}}$ relative to a reference x_i, y_j.*

- *If f is distinguished in the variable $y^1_{1,\overset{(r)}{\ldots},1}$, then $\gamma^f_r = \deg f(0,\ldots,y^1_{1,\overset{(r)}{\ldots},1},\ldots,0)$.*
- *If f is distinguished in the variable y^1_1, then $\gamma^f_1 = \deg f(0,\ldots,y^1_1,\ldots,0)$.*
- *If f is distinguished in the variable x_1, then $\gamma^f_0 = \deg f(x_1,\ldots,0)$.*

Let $S \subseteq \mathbb{P}^n_\mathbb{R}$ be a real algebraic k-dimensional subvariety (this is the zero locus of polynomial equations with real coefficients such that it has a dense open subset that is a smooth differentiable manifold of dimension k). Considering the same equations in $\mathbb{P}^n$, we get a complex subvariety $S_\mathbb{C} \subseteq \mathbb{P}^n$ on which σ acts with fixed point set S. We define $D^r_k S \subseteq D^r_k \mathbb{P}^n_\mathbb{R}$ as the fixed point set of σ_r on $D^r_k S_\mathbb{C} \subseteq D^r_k \mathbb{P}^n$. There is a natural map $D^r_k S \to S$, which is an isomorphism over the smooth part. When S is smooth then $S \cong D^r_k S$ and $D^r_k S$ is disjoint from the cuspidal divisors $C^r_{s,\mathbb{R}}$, $s \geq 2$. By analogy with lemma 12, we define the cuspidal numbers of S as follows

Definition 22. For any $0 \leq s \leq r$, we define the s^{th}-cuspidal number of S as $\gamma^s_S = D^r_k S \cap C^r_s \cap H^{k-1} \in \mathbb{Z}/2\mathbb{Z}$ (computed in the cohomology ring of $D^r_k \mathbb{P}^n_\mathbb{R}$).

Note also that when $S \subseteq \mathbb{P}^n_\mathbb{R}$ is a smooth differentiable k-dimensional manifold, $D^r_k S \subseteq \mathbb{P}^n_\mathbb{R}$ is also defined by mimicking the algebraic construction and $D^r_k S \to S$ is a diffeomorphism. In this case γ^0_S and γ^1_S are defined as above.

Given a differential equation f on $\mathbb{P}^n$, we define $S(f)$ as the homology class in $H_{k-1}(\mathbb{P}^n_\mathbb{R};\mathbb{Z}/2\mathbb{Z})$ given as the push-forward of

$$D^r_k S(f) = D^r_k S \cap D^r_k \mathbb{P}^n_\mathbb{R}(f) \in H_{k-1}(D^r_k \mathbb{P}^n_\mathbb{R};\mathbb{Z}/2\mathbb{Z}).$$

Its degree is

$$\deg S(f) = S(f) \cap H^{k-1} \in \mathbb{Z}/2\mathbb{Z}.$$

The following result is proved much in the same way as theorem 13.

Theorem 23. *Let f be a differential equation on $\mathbb{P}^n_\mathbb{R}$ with $\gamma^f_s \in \mathbb{Z}/2\mathbb{Z}$, $0 \leq s \leq r$. Let S be a real algebraic subvariety of $\mathbb{P}^n_\mathbb{R}$ of dimension k. Then*

$$\deg S(f) = \gamma^f_0 \gamma^0_S + \cdots + \gamma^f_s \gamma^s_S + \cdots + \gamma^f_r \gamma^r_S \in \mathbb{Z}/2\mathbb{Z}.$$

If S is smooth or S is just a differentiable manifold of dimension k, we have

$$\deg S(f) = \gamma^f_0 \gamma^0_S + \gamma^f_1 \gamma^1_S \in \mathbb{Z}/2\mathbb{Z}.$$

Umbilical points. We shall compute the parity of the degree of the subset of umbilical points for surfaces in $\mathbb{P}^3_\mathbb{R}$. For a reference x_1, x_2, y, these are the points where the Hessian $\left(\frac{\partial^2 y}{\partial x_i \partial x_j}\right)$ is diagonal. Therefore we look for the solutions to

$$f = \left(\frac{\partial^2 y}{\partial x_1^2} - \frac{\partial^2 y}{\partial x_2^2}\right)^2 + 4\left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right)^2.$$

Clearly $\gamma^f_2 = 2 = 0$ (we work over $\mathbb{Z}/2\mathbb{Z}$). For the cubic S given by $x_1^3 + y^2 + 1 = 0$ with $\gamma^0_S = 1$ and $\gamma^1_S = 0$ (see example 17), the equation f reduces to $(\frac{\partial^2 y}{\partial x_1^2})^2$, so $\gamma^f_0 = \deg S((\frac{\partial^2 y}{\partial x_1^2})^2) = 2\deg(\frac{\partial^2 y}{\partial x_1^2}) = 0$. For the cubic T given by $x_1^3 + y^2 = 0$, with $\gamma^0_S = 1$ and

$\gamma^1_S = 1$ (see example 18), the equation f reduces to $(\frac{\partial^2 y}{\partial x_1^2})^2$, so $\gamma^f_0 + \gamma^f_1 = \deg S((\frac{\partial^2 y}{\partial x_1^2})^2) = 0$. Therefore $\gamma^f_0 = \gamma^f_1 = \gamma^f_2 = 0$ and the number of umbilical points of any surface $S \subseteq \mathbb{P}^3_{\mathbb{R}}$ (smooth or not) is always even.

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