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SOME QUESTIONS CONCERNING THE CUBIC NUMBER FIELD $Q(\theta)$ GENERATED BY A ROOT OF $X^3 + abX + b = 0$

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We have solved the problem proposed by M. Scarowsky about this number field [4] ;see [1,2,3] for definitions.It is easy to see that we may assume w.l.o.g. that $b = h^2k$,where $(h,k) = 1$ and h,k are both square-free.

LEMMA:The following statements hold:

(1) $disc(\theta) = -h^4k^2(4a^3b+27)$

(2) $\theta_1 = \theta^2/h$ *is an algebraic integer*

(3) $disc(1,\theta,\theta_2) = -h^2k^2(4a^3b+27)$

(4) $\theta_2 = (-4a^2h^2k+9\theta-6a\theta^2)/d$ *is an algebraic integer,where* $4a^3b+27 = d^2q$ *and* q *is square-free.*

(5) $disc(1,\theta_1,\theta_2) = -3^4h^2k^2q$

(6) *When* $p \neq 3$ *is any rational prime dividing* hk , p *is minimal in* $disc(1,\theta_1,\theta_2)$*,that is,* p *does not divide* $|R/\mathbf{Z}(1,\theta_1,\theta_2)|$ *if* R *is the ring of algebraic integers of* $Q(\theta)$*.In particular,the only probably non-minimal rational prime in* $disc(1,\theta_1,\theta_2)$ *is* 3.

THEOREM 1: If $hk \equiv 0 \pmod 3$,then:

(1) *In case that* $ah \equiv 0 \pmod 3$, $disc(R) = -3^2h^2k^2q$ *and* $\{1,\theta_1,\theta_2/3\}$ *is an integral basis of* R.

(2) *In case that* $k \equiv 0 \pmod 3$ *but* $a \not\equiv 0 \pmod 3$, $disc(R) = -h^2k^2q$ *and* $\{1,\theta_1,(A+B\theta_1+\theta_2/3)/3\}$ *is an integral basis of* R *,where* $A,B = 0,1,-1$, $A \neq 0$ *satisfy the following system of congruences:*

$$9-12ahkAB+hkdqB-ah^2kq \equiv 0 \pmod{27}$$

$$-27A+54ahkB-9hkdqAB-27a^2h^2k^2AB^2+9ah^2kqA - -27hk^2B+9ah^2k^2dqB^2-12a^2h^3k^2qB+h^2kdq^2 \equiv 0 \pmod{729}$$

COROLLARY: (M. Scarowsky) *Suppose that* θ *is a root of* $X^3+108A^2X-12 = 0$ *and* $6^4A^6+1 = B^2Q$ *,where* Q *is square-free. Then,*$disc(R) = -2^23^5Q$ *and* $\{1,\theta^2/2,(2^43^3A^4+\theta+6^2A^2\theta^2)/B\}$ *is an integral basis of the ring of integers of* $Q(\theta)$.

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THEOREM 2: If $ahk \not\equiv 0 \pmod 3$, then $disc(R) = -h^2k^2q$ *and an integral basis of* R *is available by two applications of Harvey-Cohn's algorithm to* $\{1,\theta_1,\theta_2\}$.

COROLLARY: If $ahk \not\equiv 0 \pmod 3$, then one and only one of the following congruences are satisfied:

$$h^2kdq^2+3ah^2kq \equiv 1 \pmod{27}$$
$$h^2kdq^2-3ah^2kq \equiv -1 \pmod{27}$$

THEOREM 3: If $a \equiv 0 \pmod 9$ and $hk \not\equiv 0 \pmod 3$, then :

(1) $disc(R) = -h^2k^2q$ *if and only if* $h \equiv \pm k \pmod 9$

(2) $disc(R) = -3^2h^2k^2q$ *if and only if* $h \equiv \pm k \pmod 9$

Moreover, in case (2) $\{1,\theta_1,\theta_2/3\}$ *is an integral basis of* R *; in case (1) there are* $A,B = 0,3,-3$ *such that an integral basis of* R *can be expressed as* $\{1,\theta_1,(A+B\theta_1+\theta_2)/9\}$ *by just one application of Harvey-Cohn's algorithm to* $\{1,\theta_1,\theta_2\}$.

REMARK: From last theorem one can deduce the discriminat and an integral basis of pure cubic fields.

THEOREM 4: If $a \equiv 0 \pmod 3$, $a \not\equiv 0 \pmod 9$ and $hk \not\equiv 0 \pmod 3$, then:

(1) *If* $(ahk,hk^2) \equiv (3,23),(12,14),(21,5) \pmod{27}$, *then* $\{1,(1+\theta_1)/3,\theta_2/3\}$ *is an integral basis of* R *and* $dis(R) = -h^2k^2q$.

(2) *If* $(ahk,hk^2) \equiv (6,22),(15,13),(24,4) \pmod{27}$, *then* $\{1,(-1+\theta_1)/3,\theta_2/3\}$ *is an integral basis of* R *and* $disc(R) = -h^2k^2q$.

(3) *If* $(ahk,hk^2) \equiv (3,1),(3,4),(3,7),(3,5) \pmod 9$, *except the three cases considered in* (1), *then* $\{1,\theta_1,\theta_2/3\}$ *is an integral basis of* R *and* $disc(R) = -3^2h^2k^2q$.

(4) *If* $(ahk,hk^2) \equiv (6,2),(6,5),(6,8),(6,4) \pmod 9$, *except the three cases considered in* (2), *then* $\{1,\theta_1,\theta_2/3\}$ *is an integral basis of* R *and* $disc(R) = -3^2h^2k^2q$.

(5) *In any other case, that is, when* $(ahk,hk^2) \equiv (3,2),(3,8),(6,1),(6,7) \pmod 9$, $\{1,\theta_1,\theta_2\}$ *is an integral basis of* R *and* $disc(R) = -3^4h^2k^2q$.

Furthermore, $q \equiv 0 \pmod 3$ *in cases* (3)-(4) *and* $q \not\equiv 0 \pmod 3$ *in last case.*

REFERENCES

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