GROUPS OF TRANSFORMATIONS OF A G-STRUCTURE WHICH LEAVE INVARIANT A SUBSTRUCTURE

by

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1. INTRODUCTION.

Troughout this paper our manifolds will be Hausdorff, infinitely differentiable, and second countable. We fix the following objects: A m-dimensional manifold M, two closed subgroups G, H of GL (m; R), with $G \supset H$, and a G-structure $p: A \to M$. Let Σ be a Lie group of transformations of A. We ask: Is there an H-structure $B \subset A$ such that $\Sigma \subset \mathrm{Aut}(B)$? If the answer is affirmative we say that Σ is inessential. Thus, inessential groups are groups of transformations of a substructure of A. In order to avoid trivial cases we will assume that there are H-structures contained in A. The aim of this paper is to impose conditions on M, G, H, A, and Σ so that Σ be inessential.

2. AUXILIARY RESULTS.

We will write $G/H = L$. The group G acts canonically on L on the left, and we denote by E the bundle associated to A with fibre L. There is a projection $A \times L \to E$ which we write $(a, z) \to az$. The group Σ acts on the left on E by $(\sigma_o, az) \to (\sigma_o(a))z$, where σ_o is the bundle isomorphism of A induced by σ. There are canonical bijections between the set of H-structures contained in A, the set of sections of E, and the set of maps $\Phi: A \to L$ such that $\Phi(ag) = g^{-1}\Phi(a)$ for all $a \in A$, $g \in G$. In fact, the H-structure B corresponds to Φ if and only if $B = \Phi^{-1}(H)$, and Φ corresponds to the section s if and only if for all

$$a \in A,\ s(p(a)) = a\,\Phi(a).$$

The following lemma is easy.

*2.1. **Lemma.*** Let $\sigma \in \mathrm{Aut}(A)$, and let B be an H-structure contained in A determined by the map Φ: $A \to L$, or by a section s of E. Then, the following statements are equivalent

a) $\sigma \in \mathrm{Aut}(B)$

b) $\Phi \circ \sigma_o = \Phi$

c) For all $x \in M$, we have $\sigma(s(x)) = s(\sigma(x))$.

*2.2. **Lemma.*** Let Σ be a Lie group acting on the manifolds M and N, and $f: N \to M$ an equivariant map; i.e. $f(\sigma(x)) = \sigma(f(x))$ for all $x \in M$ and $\sigma \in \Sigma$. Suppose there is a submanifold P of M such that

a) The map $\Sigma \times P \to M$, $(\sigma, x) \to \sigma(x)$ is a surjective submersion.

b) There is a map s': $P \to N$ such that $f \circ s' = id_P$ and for all $\sigma \in \Sigma$ and $x \in P$ with $\sigma(x) \in P$ we have $s'(\sigma(x)) = \sigma(s'(x))$.

Then the map s' can be extended to a unique map $s: M \to N$ such that $f \circ s = id_M$, and $s(\sigma(x)) = \sigma(s(x))$ for all $\sigma \in \Sigma$ and $x \in M$.

Proof: Define h: $\Sigma \times P \to N$ and g: $\Sigma \times P \to M$ by $h(\sigma, x) = \sigma(s'(x))$ and $g(\sigma, x) = \sigma(x)$. It is clear from the hypothesis that h is constant on the fibres of g. Hence there is a s: $M \to N$ such that $s \circ g = h$, and one checks easily that it is the required extensión.

We will give without proofs some results about a Lie group acting properly on a manifold.

*2.3. **Lemma.*** If Σ acts freely and properly on M, the orbit space is a quotient manifold of M. In fact, M is a principal Σ-bundle with base the orbit space.

(It follows from proposition (1.2.3), th (1.1.3), and th. (4.1) in [4]).

*2.4. **Lemma.*** Let Σ be a Lie group acting properly on a manifold M. There exists an open set U and a closed set C such that $C \subset U$, $\Sigma C = M$, and for each compact K, $\{\sigma \in \Sigma \mid U \cap \sigma(K) \neq \varphi\}$ is relatively compact
(See [3] I. lemma 2).

3. Case of a free proper action.

*3.1. **Theorem.*** If $\Sigma \subset \mathrm{Aut}(A)$ is diffeomorphic to R^k for some k and acts feely and properly on M, then Σ is inessential.

Proof: If Q is the orbit space of Σ, we know by (2.3) that $M \to Q$ is a principal bundle with group Σ. The fibre of this bundle is diffeomorphic to R^k. Hence, this bundle admits a global section. Being a principal bundle it must be trivial. Therefore there is a submanifold $P \subset M$ such that the map $\Sigma \times P \to M, (\sigma, x) \to \sigma(x)$ is a diffeomorphism. Let s' be the associated section to an H-structure contained in A. We still denote by s' the restriction to P. If $x \in P$ and $\sigma(x) \in P$, then $\sigma = \mathrm{id}_M$. Therefore we may apply (2.2) getting a section s of E. If B is the H-structure associated to s, we get from (2.1) and (2.2) that $\Sigma \subset \mathrm{Aut}(B)$.

Example: We take $G = GL(m; R)$ and $H = O(m; R)$. Thus, any free proper action of R on M induces a group Σ to which (3.1) can be applied, and we get that Σ can be considered as a group of isometries of a certain Riemannian metric on M. The vector field induced by the R-action is a Killing vector field.

4. Case of a transitive action.

We will denote by Σ_x the isotropy group of Σ at $x \in M$. Consider the property: Any Lie homomorphism $h: \Sigma_x \to G$ has its image contained in a conjugate of H.

*4.1. **Theorem:*** If $\Sigma \subset \mathrm{Aut}(A)$ acts transitively and the property above holds, then Σ is inessential.

Proof: Choose a frame $a \in p^{-1}(x)$. For each $\sigma \in \Sigma_x$, $\sigma_o(a) \in p^{-1}(x)$. Therefore there is a unique element $h(\sigma)$ of G such that $\sigma_o(a) = a\,h(\sigma)$. It is clear that $h: \Sigma_x \to G$ is a Lie group homomorphism, and that if $a' \in p^{-1}(x)$ is written $a' = ag$, the corresponding h' is related to h by $h'(\sigma) = g^{-1} h(\sigma) g$. We may assume then that a has been chosen with the condition $h(\Sigma_x) \subset H$. We take in (2.2) $P = \{x\}$. Hypothesis (a) holds clearly because the action is transitive. We define $s': P \to E$ by $s'(x) = aH$. Then $\sigma(x) \in P$ if and only if $\sigma \in \Sigma_x$, and we have

$$\sigma s'(x) = \sigma(aH) = (a\,h(\sigma))H = a(h(\sigma)H) = aH = s'(x) = s'(\sigma(x)).$$

If B is the H-structure associated to the extension s of s', it follows from (2.1) that $\Sigma \subset \mathrm{Aut}(B)$.

We point out some cases in which the property of the isotropy group holds.

(4.2) Suppose Σ_x is compact and H is a normal subgroup such that G/H is isomorphic to R^k for some k. If h: $\Sigma_x \to G$ is a continuous homomorphism, then h $(\Sigma_x) \subset H$. If this were not true the homomorphism h': $\Sigma_x \to R^k$, composition of h, the projection $G \to G/H$, and the isomorphism $G/H \to R^k$ would not be constant. Then h (Σ_x) cannot be bounded and Σ_x is not compact.

(4.3) Let G have a finite number of connected components. There is a compact subgroup H of G having the following property: If K is a compact subgroup of G, then K is contained in a conjugate of H. (See [2] (XV.3.1)). Clearly the property holds for G and H. It is well known that if G = GL (m; R), we can take H = O (m, R). Analogously, if G = GL (n; C), we can take H to be the unitary group. We get then for example, that if Σ is an isotropy compact Lie group of transformations acting transitively on M, there is a Riemannian metric for which Σ is a subgroup of its group of isometries.

5. Case of a proper action.

(5.1) Theorem: Suppose there is a vector space V and a linear action of G on V such that: (a) For a certain $v_0 \in V$, H is the isotropy group at v_0. (b) The orbit W of v_o is an open cone; i.e. if $w \in W$ and $r > 0$, $r\,w \in W$. If $\Sigma \subseteq \mathrm{Aut}(A)$ acts properly, then Σ is inessential.

Proof: Let i: $G/H \to W$ be the canonical diffeomorphism, and j its inverse. Let $\Phi: A \to G/H$ be a map corresponding to an M-structure $B \subset A$. Take C and U as in (2.4) and let f: $M \to R$ be a map which is 1 on C and 0 outside U. For each $a \in A$ define $h_a: \Sigma \to V$ by $h_a(\sigma) = f(p(\sigma_o(a)))(i\Phi(\sigma_o(a)))$. Now, define

$$\Phi': A \to G/H \quad , \quad \Phi'(a) = j \int h_a(\sigma)\, d\sigma,$$

where the integral is the left invariant Haar integral on Σ. Since one-point sets are compact, $\Phi'(a)$ is defined and $\Phi'(\sigma(a)) = \Phi'(a)$, for the integral is invariant under left translations. Given $g \in G$ one checks that $h_{ag} = g^{-1} h_a$. Then, since the integral commutes with linear maps we obtain $\Phi'(ag) = g^{-1}\ \Phi'(a)$. If B' is the H-structure associated to Φ' we have $\Sigma \subset \mathrm{Aut}(B')$.

Compact groups act properly. Therefore.

*(5.2) **Corollary:*** If G and H verify the hypothesis of (5.1), any compact group $\Sigma \subset \mathrm{Aut}(\Lambda)$ is inessential.

*(5.3) **Example:*** Let G be the group of matrices with positive determinant, and $H = SL(m; R)$. We take as V the space of alternating m-multilinear maps on R^m, and as v_0 the determinant. We get that if M is an oriented manifold and Σ a Lie group of transformations preserving the orientation, there is a volume element inducing the orientation such that all elements of Σ are volume preserving.

*(5.4) **Example:*** Let G be the conformal group and H the orthogonal group of R^m. We take as V the space of multiples of the standard inner product v_0. The conclusion of (5.1) is that for any Lie group of conformal transformations of a metric g, there is a metric g′ of the form $g' = hg$, with h a positive map, such that Σ is a group of isometries of g.

*(5.5) **Example:*** Let $G = GL(m; R)$, and $H = O(m; R)$. We take as V the space of bilinear symmetric forms on R^m on which G acts by pull-back, and v_0 is the standard inner product. The conclusion of (5.1) is that for any Lie group of transformations Σ acting properly on M there is a Riemannian metric g such that Σ is contained in its group of isometries.

There is a similar example for $G = GL(n;C)$ and $H = U(n;C)$, where $m = 2n$.

Examples (5.4) and (5.5) are known, although the result is proved by a different method (See [1] ths. 1 and 4, and [4] (4.3.1)). Besides showing other examples, our theorem allows us to give a simpler treatment with the help of (2.4).

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