Characterization of Veronese varieties via projection in Grassmannians

Enrique Arrondo and Raffaella Paoletti

A Giacomo Paoletti, amico e padre

Abstract. We characterize for any d the d-uple Veronese embedding of $\mathbb{P}^n$ as the only variety that, under certain general conditions, can be projected from the Grassmannian of $(d-1)$-planes in $\mathbb{P}^{nd+d-1}$ to the Grassmannian of $(d-1)$-planes in $\mathbb{P}^{n+2d-3}$ in such a way that two $(d-1)$-planes meet at most in one point. We also study the relation of this problem with the Steiner bundles over $\mathbb{P}^n$.

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Introduction

In [2], the first author characterized, under certain conditions, the embedding of $\mathbb{P}^n$ in $G(1,2n+1)$ via $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2}$ as the only n-dimensional subvariety of $G(1,2n+1)$ that is isomorphically projectable into a subvariety of $G(1,n+1)$ (under a projection induced by a linear projection from $\mathbb{P}^{2n+1}$ to $\mathbb{P}^{n+1}$). In this paper we deal with the analogous problem for the embedding of $\mathbb{P}^n$ in $G(d-1,nd+d-1)$ via $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus d}$.

One first problem is that the naive generalization to arbitrary d of the case $d=2$ does not work (see Remark 2.2). As we will see in Section 2, the right notion for a subvariety of a Grassmann variety of $(d-1)$-planes to be special under projection is to be projectable into a subvariety of $G(d-1,n+2d-3)$ in such a way that any two $(d-1)$-planes meet at most in one point.

With this notion in mind, in Section 3 we prove our main projectability result (Theorem 3.1) by essentially repeating most of the steps in the proof of [2]. The only step we cannot reproduce is the result that a projectable variety has positive defect, so that we will need to add this as a hypothesis.

Finally, in Section 4 we observe that the projectability of $\mathbb{P}^n$ in Grassmannians is closely related to Steiner bundles. However, not all of them appear. We will see for instance that, if $n \geq 3$, we never obtain the most particular case, namely the Schwarzenberger bundles.

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1. Preliminaries and projectability in Grassmannians of lines

We will work over an algebraically closed field k of characteristic zero. We will denote by $G(r,m)$ the Grassmann variety of r-linear spaces in $\mathbb{P}^m$. A linear projection $G(r,m) \dashrightarrow G(r,m')$ will mean the natural (rational) map induced by the corresponding linear projection $\mathbb{P}^m \dashrightarrow \mathbb{P}^{m'}$.

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The main example we are going to consider is the following.

Example 1.1. Consider the natural embedding of $\mathbb{P}^n$ in $G(d-1, nd+d-1)$ defined by $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus d}$ and call V its image. In coordinates, it can be described by associating to each $(x_0 : \ldots : x_n) \in \mathbb{P}^n$ the $(d-1)$-plane spanned by the rows of the matrix

$$\begin{pmatrix} x_0 \ldots x_n & 0 \ldots 0 & \ldots & 0 \ldots 0 \\ 0 \ldots 0 & x_0 \ldots x_n & \ldots & 0 \ldots 0 \\ \vdots & & \ddots & \vdots \\ 0 \ldots 0 & 0 \ldots 0 & \ldots & x_0 \ldots x_n \end{pmatrix}.$$

We consider now the linear projection $\mathbb{P}^{nd+d-1} \dashrightarrow \mathbb{P}^{n+d-1}$ defined by

$$(z_{10} : z_{11} : \ldots : z_{1n} : z_{20} : \ldots : z_{d,n-1} : z_{dn}) \mapsto$$

$$(z_{10} : z_{11} + z_{20} : \ldots : z_{1,d-1} + \ldots + z_{d0} : \ldots : z_{1n} + \ldots$$

$$\ldots + z_{d,n+1-d} : \ldots : z_{d-1,n} + z_{d,n-1} : z_{dn}).$$

This projection induces a projection from $G(d-1, nd+d-1)$ to $G(d-1, n+d-1)$ under which the image of the above $\mathbb{P}^n$ consists of the $(d-1)$-planes spanned by the rows of the matrix

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_n & 0 & \ldots & 0 \\ 0 & x_0 & x_1 & \ldots & x_n & \ldots & 0 \\ \vdots & & \ddots & \ddots & & \ddots & \vdots \\ 0 & \ldots & 0 & x_0 & x_1 & \ldots & x_n \end{pmatrix}. \tag{1}$$

This is still an embedding of $\mathbb{P}^n$ in $G(d-1, n+d-1)$, since the maximal minors of the above matrix (which give the image of $\mathbb{P}^n$ after the Plücker embedding of

$G(d-1, n+d-1)$ in $\mathbb{P}^{\binom{n+d}{d}-1})$ define the d-uple Veronese embedding of $\mathbb{P}^n$ in $\mathbb{P}^{\binom{n+d}{d}-1}$. By this reason we will call this subvariety $V \subset G(d-1, nd+d-1)$ (or any of its isomorphic projections) the *n-dimensional Veronese variety.*

Following [2] we will use the following definitions:

Definition 1.2. *We will say that a subvariety $X \subset G(d-1, N)$ of dimension n is* nondegenerate *if the union of all the $(d-1)$-planes parametrized by X is not contained in a hyperplane of $\mathbb{P}^N$. We will also say that X is* uncompressed *if the union in $\mathbb{P}^N$ of all of the $(d-1)$-planes parametrized by X has the expected dimension $n+d-1$. Otherwise, if the dimension is smaller, we will say that X is* compressed.

The main result of [2] is the following:

Theorem 1.3. *The only nondegenerate uncompressed n-dimensional subvariety of $G(1, 2n+1)$ that can be isomorphically projected to $G(1, n+1)$ is the Veronese variety (i.e. the one of Example 1.1 for $d=2$).*

The main tool to prove that result is the use of the generalization to the Grassmannians context of the notions of secant variety and secant defect. The precise definitions are as follows:

Definition 1.4. *The k-secant variety of a variety $X \subset G(1, N)$ is the variety $S^kX \subset G(r_k, N)$ consisting of the closure of the set of linear spans of $k+1$ general lines of X (observe that, although one should expect r_k to be $2k+1$, its value could be actually smaller). The k-*secant defect *of X is the dimension δ_k of the set of lines of X contained in a general space of S^kX. It follows that S^kX has dimension $(k+1)(n-\delta_k)$.*

The two main ingredients in the proof of Theorem 1.3 are the fact that a projectable variety has positive first defect δ_1 (actually in this case one can prove that $\delta_1 = 1$), and the fact that more precisely the set of lines contained in a general 3-plane of S^1X is a conic (specifically, the set of lines meeting two skew lines). It is thus easy to prove that a projectable variety in $G(1, 2n+1)$ has defects $\delta_k = k$ for $k = 1, \ldots, n$ and that the divisors of X obtained for $k = n-1$ yield an isomorphism of X to $\mathbb{P}^n$ whose inverse is the double Veronese embedding.

2. Projectability in arbitrary Grassmannians

The obvious natural generalizations of the notions of k-secant varieties and k-secant defects in arbitrary Grassmannians are the following:

Definition 2.1. *The* k-secant variety *of* $X \subset G(d-1,N)$ *is the closure* S^kX *in* $G(r_k,N)$ *of the set of linear spans of* $k+1$ *general* $(d-1)$*-planes of* X*. The* k-secant defect *of* X *is the dimension* δ_k *of the set of* $(d-1)$*-planes of* X *contained in a general space of* S^kX*. It follows that* S^kX *has dimension* $(k+1)(n-\delta_k)$*. In the case* $k=1$*, we will just speak about secant variety* SX *and secant defect* δ*. In general, one should expect* $r_1 = 2d-1$*.*

Example 2.2. The Veronese variety V defined in Example 1.1 has first secant variety of dimension $2n-2$, i.e. less than expected. Indeed, let $(z_{10} : z_{11} : \ldots : z_{1n} : z_{20} : \ldots : z_{dn})$ be homogeneous coordinates on $\mathbb{P}^{nd+d-1}$ and for each $k \in \{1,\ldots,d\}$ consider the n-dimensional linear subspaces Π_k defined by the equations: $z_{ij} = 0$ for $j = 1,\ldots,d$ and $i \neq k$. Then each point of V can be represented as the span of d corresponding points $P_1,\ldots,P_d$ s.t. $P_k \in \Pi_k$. Let $\varphi : V \times V \dashrightarrow G(2d-1, nd+d-1)$ defined by $\varphi((\Lambda_1,\Lambda_2)) = \langle \Lambda_1, \Lambda_2 \rangle$. Then $SV = \overline{Im\ \varphi}$. Let $\pi \in Im\ \varphi$, $\pi = \langle \Lambda_1,\Lambda_2\rangle$ with $\Lambda_1 = \langle P_1,\ldots,P_d\rangle$ and $\Lambda_2 = \langle Q_1,\ldots,Q_d\rangle$. Then $\pi = \langle \overline{P_1Q_1},\ldots,\overline{P_dQ_d}\rangle$ where $\overline{P_kQ_k}$ is the line through P_k and Q_k. Thus

$$\varphi^{-1}(\pi) = \left\{ (\langle \widetilde{P}_1,\ldots,\widetilde{P}_d\rangle,\ \langle \widetilde{Q}_1,\ldots,\widetilde{Q}_d\rangle) \ / \ \widetilde{P}_1 \neq \widetilde{Q}_1 \in \overline{P_1Q_1} \right\}$$

has dimension 2 and $\dim SV = 2n-2$. In the same way, it is easy to see that V has secant defects $\delta_k = k$ for $k = 1,\ldots,n$.

Remark 2.3. Let us see that the Veronese varieties cannot be characterized by means of the projectability to $G(d-1,n+d-1)$ that we have seen in Example 1.1. In fact, let $X \subset G(d-1,N)$ be an irreducible nondegenerate n-dimensional variety and let us analyze when it is possible to project it isomorphically to $G(d-1,n+d-1)$. A projection will be induced by a linear projection $\pi_A : \mathbb{P}^N \to \mathbb{P}^{n+d-1}$ with center a linear space A of dimension $N-n-d$. The fact that π_A induces an isomorphism between X and its image in $G(d-1,n+d-1)$ is equivalent to the following property:

for any $\Lambda_1,\Lambda_2 \in X$ *(maybe infinitely close),*
$\dim(\langle \Lambda_1,\Lambda_2\rangle \cap A) < \dim(\langle \Lambda_1,\Lambda_2\rangle) - d.$

Actually this condition says that $\dim \pi_A(\langle \Lambda_1,\Lambda_2\rangle) \geq d$, so that $\pi_A(\Lambda_1)$, $\pi_A(\Lambda_2)$ represent distinct points in $G(d-1,n+d-1)$. In particular, if Λ_1,Λ_2 are skew the above condition states that $\dim(\langle \Lambda_1,\Lambda_2\rangle \cap A) < d-1$. Consider the following incidence variety:

$$I = \{(A,\Pi) \mid \dim(A \cap \Pi) \geq d-1\} \subset G(N-n-d,N) \times SX$$

and let p,q be the corresponding projections. The elements of $p(I)$ represent "bad centers" of projections. A dimensional count on the fibers of q shows that $\dim(I) = \dim(SX) + \dim G(N-n-d,N) - nd$. Since $\dim(SX) \leq 2n$ (and in fact one expects to have an equality), it follows that $\dim(I) \leq n(2-d) + \dim G(N-n-d,N)$. Therefore, only if $d = 2$ one can expect p to be surjective, which means that all the possible centers of projection should be bad. On the contrary, if $d \geq 3$, one should be able to always find a good center of projection. Hence only for $d = 2$ the Veronese varieties would be special by means of its projectability to $G(d-1, n+d-1)$ as in Example 1.1. This is in fact the content of Theorem 1.3.

The right notion of projectability will be the following:

Definition 2.4. *Let $X \subset G(d-1, N)$ a smooth irreducible variety and let k be an integer such that $0 \leq k \leq d-1$. We will say that X is k-*projectable *to $G(d-1, M)$ if there exists a projection from $G(d-1, N)$ to $G(d-1, M)$ such that any two $(d-1)$-planes of the image of X (maybe infinitely close) do not meet along a linear space of dimension greater than or equal to k.*

Remark 2.5.
1) If X is k-projectable, then any two $(d-1)$-planes of X itself do not meet along a linear space of dimension greater than or equal to k.
2) If $k = d-1$, then X is $(d-1)$-projectable to $G(d-1, M)$ if and only if X is isomorphically projectable to $G(d-1, M)$.
3) If $k = 0$, then X is 0-projectable to $G(d-1, M)$ if and only if any two $(d-1)$-planes of the image of X are skew. In other words, the union of the $(d-1)$-planes of X is a smooth scroll and it is isomorphic to its image in $\mathbb{P}^M$.
4) If X is k-projectable to $G(d-1, M)$, it is also $(k+1)$-projectable to $G(d-1, M-1)$. In particular, if X is 1-projectable to $G(d-1, M)$ it is also isomorphically projectable to $G(d-1, M-d+2)$.
5) If X is uncompressed, then it is certainly not $(d-1)$-projectable to $G(d-1, n+d-2)$. Therefore, X is not 0-projectable to $G(d-1, n+2d-3)$.

Example 2.6. We have seen (Example 1.1) that the Veronese variety is isomorphically projectable to $G(d-1, n+d-1)$, but this was not a sufficiently special property to characterize it (Remark 2.3). By the above fourth remark, 1-projectability to $G(d-1, n+2d-3)$ is a stronger condition. Let us see that this is now a very restrictive property satisfied by the Veronese variety.

Let us repeat first the dimension count of Remark 2.3 considering now 1-projectability to $G(d-1, n+2d-3)$ instead of $(d-1)$-projectability to $G(d-1, n+d-1)$ (obviously the two notions coincide if $d = 2$).

So let $X \subset G(d-1, N)$ be an irreducible nondegenerate n-dimensional variety. A projection to $G(d-1, n+2d-3)$ will be induced by a linear projection $\pi_A : \mathbb{P}^N \to \mathbb{P}^{n+2d-3}$ with center a linear space A of dimension $N-n-2d+2$. For any $\Lambda_1, \Lambda_2 \in X$ (maybe infinitely close), the fact that $\pi_A(\Lambda_1)$ and $\pi_A(\Lambda_2)$ meet at most in one point is equivalent to $\dim(\langle \Lambda_1, \Lambda_2 \rangle \cap A) < 1$. Consider now the following incidence variety:

$$I = \{(A, \Pi) \mid \dim(A \cap \Pi) \geq 1\} \subset G(N-n-2d+2, N) \times SX$$

and let p, q be the corresponding projections. The elements of $p(I)$ represent "bad centers" for 1-projectability. A new dimensional count on the fibers of q shows that this time $\dim(I) = \dim(SX) + \dim G(N-n-d, N) - 2n$. Therefore,

$\dim(I) \leq \dim G(N-n-d, N)$ and in general one expects to have an equality. This shows that 1-projectability to $G(d-1, n+2d-3)$ is the right property to study (observe also that the fifth remark above shows that we can never have 0-projectability unless X is compressed).

On the other hand, the Veronese varieties V are 1-projectable to $G(d-1, n+2d-3)$: this follows immediately from the above dimensional count and the fact that the dimension of SV is smaller than expected (Example 2.2).

We thus strengthen the conjecture in [2] to the following:

Conjecture 2.7. *The only smooth irreducible nondegenerate n-dimensional variety of $G(d-1, nd+d-1)$ that is 1-projectable to $G(d-1, n+2d-3)$ is the Veronese variety.*

3. Characterization of Veronese varieties in arbitrary Grassmannians.

In this section we will prove the following evidence of Conjecture 2.7:

Theorem 3.1. *Let $X \subset G(d-1, nd+d-1)$ a smooth irreducible nondegenerate n-dimensional variety such that any two (possibly infinitely close) $(d-1)$-planes of X do not meet. If X has positive defect and is 1-projectable to $G(d-1, n+2d-3)$, then X is the Veronese variety.*

The case $n = 1$ of Conjecture 2.7 is very easy to prove. Before proving it, we state without proof an easy technical lemma that we will need.

Lemma 3.2. *The set $\Omega \subset G(d-1, 2d-1)$ of $(d-1)$-planes of $\mathbb{P}^{2d-1}$ meeting a fixed $(d-1)$-plane Λ is (after the Plücker embedding) a hyperplane section of $G(d-1, 2d-1)$ having a point of multiplicity d at the point represented by Λ. The intersection of $G(d-1, 2d-1)$ with the tangent cone of Ω at this singular point consists of the set of $(d-1)$-planes meeting Λ in a space of dimension at least $d-2$.*

Proposition 3.3. *The only smooth irreducible nondegenerate curve X in $G(d-1, 2d-1)$ that is 1-projectable to $G(d-1, 2d-2)$ is the embedding of $\mathbb{P}^1$ via the vector bundle $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d}$.*

Proof. Let m be the degree of X after the Plücker embedding. For any $\Lambda \in X$, consider the set of $(d-1)$-planes of X meeting Λ. From Lemma 3.2 there are other $m-d$ of them besides Λ. Hence there are $m-d$ hyperplanes of $\mathbb{P}^{2d-1}$ containing two $(d-1)$-planes of X (one of them being Λ). Of course everything is counted with multiplicity, and in particular it could happen that some or all of the above

$(d-1)$-planes are infinitely close to Λ. But using in this case the second part of Lemma 3.2 we also get hyperplanes containing two infinitely close $(d-1)$-planes of X (and in fact we get not just hyperplanes but even linear spaces of dimension d).

As a consequence, if $m \neq d$ we would get, varying Λ, an infinite family of hyperplanes containing two $(d-1)$-planes of X. Therefore, through any point $p \in \mathbb{P}^{2d-1}$ we find a hyperplane containing two (maybe infinitely close) $(d-1)$-planes of X. This implies that the projection of X from any point would produce in the image two $(d-1)$-planes meeting along a line, which contradicts our hypothesis on X. We thus get that X has degree d.

We then have that the union of the $(d-1)$-planes of X is a d-dimensional scroll in $\mathbb{P}^{2d-1}$ of degree d, and hence it is a rational normal scroll. Since two $(d-1)$-planes of the scroll do not meet, the splitting type is necessarily $(1,\ldots,1)$, i.e. X is the embedding of $\mathbb{P}^1$ via the vector bundle $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d}$. □

The proof of Theorem 3.1 is based, following the steps in [2], on two main lemmas that we will state and prove first.

Lemma 3.4. *Let $X \subset G(d-1,N)$ a smooth irreducible nondegenerate variety of dimension n with positive defect and such that any two (maybe infinitely close) $(d-1)$-planes are skew. Then, for general $\Lambda_1, \Lambda_2 \in X$, the set Y_Π of $(d-1)$-planes of X contained in $\Pi := \langle \Lambda_1, \Lambda_2 \rangle$ is the one-dimensional family (or one of the two families if $d = 2$) of the $\mathbb{P}^{d-1}$'s of the Segre variety $\mathbb{P}^1 \times \mathbb{P}^{d-1} \subset \Pi$. In particular, the secant defect δ of X is equal to 1.*

Proof. Since the defect is positive, Y_Π has dimension at least one. In the Grassmann variety of $(d-1)$-planes of Π, the set of all those meeting Λ_1 is a hyperplane section (under the Plücker embedding). Therefore, if Y_Π had dimension bigger than or equal to two, then it would meet that hyperplane section in at least a curve. This implies that there would be infinitely many $(d-1)$-planes of X meeting Λ_1. This contradicts the assumption that any two $(d-1)$-planes of X are skew.

Hence Y_Π is a curve, and the result follows now from [3], Theorem 5.1 or from (the proof of) Proposition 3.3. □

Lemma 3.5. *Let $X \subset G(d-1,N)$ a smooth irreducible nondegenerate variety of dimension n with positive defect and such that any two (maybe infinitely close) $(d-1)$-planes are skew. Then, for any integer $k > 1$, $\delta_k \geq \min\{\delta_{k-1}+1,\ n\}$.*

Proof. Since clearly $\delta_k \geq \delta_{k-1}$, there is nothing to prove if $\delta_{k-1} = n$. We hence assume $\delta_{k-1} < n$. Take $k+1$ general $(d-1)$-planes $\Lambda_0, \ldots, \Lambda_k$ of X and write $\Pi' = \langle \Lambda_0, \ldots, \Lambda_{k-1} \rangle$ and $\Pi = \langle \Lambda_0, \ldots, \Lambda_k \rangle = \langle \Pi', \Lambda_k \rangle$. By the generality of these $(d-1)$-planes, $Y_{\Pi'}$ has dimension δ_{k-1} and Λ_k is not contained in Π'. Consider the incidence variety

$$I = \{(\Lambda', \Lambda) \mid \Lambda \subset \langle \Lambda', \Lambda_k \rangle\} \subset Y_{\Pi'} \times Y_\Pi$$

and let p, q be the corresponding projections. Consider a general element $\Lambda' \in Y_{\Pi'}$; then $p^{-1}(\Lambda') = G(d-1, \langle\Lambda', \Lambda_k\rangle) \cap X = Y_{\langle\Lambda',\Lambda_k\rangle}$ and this has dimension $\delta_1 \geq 1$ by hypothesis. Therefore, I has dimension $\delta_{k-1}(X) + \delta_1$, so it is enough to show that q is generically finite over its image. We thus take a general element $\Lambda \in Im\ q$. Then $q^{-1}(\Lambda) = \{\Lambda' \in Y_{\Pi'} \mid \Lambda' \subset \langle\Lambda, \Lambda_k\rangle\} = \{\Lambda' \in X \mid \Lambda' \subset \Pi',\ \Lambda' \subset \langle\Lambda, \Lambda_k\rangle\} = X \cap G(d-1, \Pi' \cap \langle\Lambda, \Lambda_k\rangle)$. Since $\langle\Lambda, \Lambda_k\rangle$ is not contained in Π' (because Λ_k is not) then $\Pi' \cap \langle\Lambda, \Lambda_k\rangle$ has dimension at most $2d-2$. Therefore any two $(d-1)$-planes inside $\Pi' \cap \langle\Lambda, \Lambda_k\rangle$ should meet. Since two $(d-1)$-planes of X cannot meet, it follows that $X \cap G(d-1, \Pi' \cap \langle\Lambda, \Lambda_k\rangle)$ consists of at most one element. This completes the proof. □

Proof of Theorem 3.1*:* By Lemma 3.4, we have $\delta_1 = 1$ and iterating Lemma 3.5 we get $\delta_k \geq \min\{k, n\}$. This implies in particular that $\delta_m = n$ for some $m \leq n$. Therefore all the $(d-1)$-planes of X are contained in the span of $m+1$ general elements of X, which has dimension at most $md+d-1$. Since X is nondegenerate in $G(d-1, nd+d-1)$, it follows that $m = n$ and the span of $n+1$ general elements of X is the whole $\mathbb{P}^{nd+d-1}$, i.e. the elements are in general position. Moreover, we get that, for $k = 1, \ldots, n$, it holds $\delta_k = k$ and $k+1$ general elements of X span a linear space of dimension $kd + d - 1$.

In the particular case $k = n-1$ we have that n general elements of X span a linear space Π of codimension d in $\mathbb{P}^{nd+d-1}$ and Y_Π is a hypersurface of X. This hypersurface is contained in the hyperplane section H_Π of $G(d-1, nd+d-1)$ consisting of the $(d-1)$-planes meeting Π. By Lemma 3.2, this hyperplane section is singular with multiplicity d along the set of $(d-1)$-planes contained in Π. Hence, as a divisor on X, the hyperplane section H_Π can be written as $aY_\Pi + E_\Pi$, with $a \geq d$. If now Π' is the span of two general elements of X, we know from Lemma 3.4 that $Y_{\Pi'}$ is the set of $(d-1)$-planes of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{d-1}$ in Π'. This has degree (after the Plücker embedding) equal to d, i.e. the intersection product of H_Π and $Y_{\Pi'}$ is d. This means, using the above identity, that $d = a + E_\Pi \cdot Y_{\Pi'}$. Therefore $a = d$ and $E_\Pi \cdot Y_{\Pi'} = 0$. Since $Y_{\Pi'}$ "moves", we obtain as in [2] that $E_\Pi = 0$. Summing up, we got that $d \cdot Y_\Pi$ is linearly equivalent to the hyperplane section of X. Now the same reasoning as in [2] shows that $X \cong \mathbb{P}^n$ and that X is the Veronese variety. □

4. Relation with Steiner bundles

In this section we will study the relation of the projection of Veronese varieties in Grassmannians with the so-called Steiner bundles on $\mathbb{P}^n$.

Remark 4.1. Let us analyze more closely Example 1.1. The matrix 1 represents a set of $(d-1)$-planes in $\mathbb{P}^{n+d-1}$ parametrized by $\mathbb{P}^n$ or, dually, a set of $(n-1)$-planes in $\mathbb{P}^{n+d-1}$. It is well-known that this dual representation corresponds to the set of n-secant spaces to the rational normal curve. Indeed we can take the standard rational normal curve in $\mathbb{P}^{n+d-1}$

$$(u_0 : u_1 : \ldots : u_{n+d-1}) = (\lambda^{n+d-1} : \lambda^{n+d-2}\mu : \ldots : \mu^{n+d-1}).$$

Then a set of (maybe infinitely close) n points on the curve is given by the zeros of a homogeneous polynomial $x_0\lambda^n + x_1\lambda^{n-1}\mu + \ldots + x_n\mu^n$ in the variables λ, μ. On the other hand, the system of hyperplanes of $\mathbb{P}^{n+d-1}$ containing the $(n-1)$-plane spanned by this set of points is generated by the independent hyperplanes of $\mathbb{P}^{n+d-1}$:

$$\begin{gathered} x_0u_0 + x_1u_1 + \ldots + x_nu_n = 0 \\ x_0u_1 + x_1u_2 + \ldots + x_nu_{n+1} = 0 \\ \vdots \\ x_0u_{d-1} + x_1u_d + \ldots + x_nu_{n+d-1} = 0 \end{gathered}$$

whose coefficients yield the matrix 1 in Example 1.1.

Of course this is not the most general situation, in the sense that only after a very particular projection we will get such a dual representation. In the language of vector bundles, the embedding of $\mathbb{P}^n$ in $G(d-1, nd+d-1)$ is equivalent to the evaluation epimorphism $\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)d} \to \mathcal{O}_{\mathbb{P}^n}(1)^d$ (with kernel $\Omega_{\mathbb{P}^n}(1)^{\oplus d}$) while the projection to $G(d-1, n+d-1)$ is equivalent to an epimorphism $\mathcal{O}_{\mathbb{P}^n}^{\oplus n+d} \to \mathcal{O}_{\mathbb{P}^n}(1)^d$. The dual of its kernel is called a *Steiner bundle*. And the bundle corresponding to the n-secant spaces to a rational normal curve (called *Schwarzenberger bundle*) is the most particular case. We refer to [1] for a thorough study of these bundles.

Definition 4.2. *We will call* Schwarzenberger-Veronese variety *a Veronese variety in $G(d-1, n+d-1)$ that corresponds to a Schwarzenberger bundle.*

A first fact showing how special Schwarzenberger-Veronese varieties are is the following:

Proposition 4.3. *If $n \geq 3$, the Schwarzenberger-Veronese variety with $d \geq 3$ is never obtained as a projection of a Veronese variety $V' \subset G(d-1, n+2d-3)$ such that any two $(d-1)$-planes of V' meet at most in one point.*

Proof. Following the notations introduced in Example 1.1, we want to show that if $n \geq 3$ it is not possible to factorize the projection φ of the Veronese variety $V \simeq \mathbb{P}^n \hookrightarrow G(d-1, nd+d-1)$ to $G(d-1, n+d-1)$ by means of a projection ψ to $V' \subset G(d-1, n+2d-3)$ such that any two $(d-1)$-planes of V' meet at most in one point. Assume for contradiction that it were possible. Then this projection would be induced by a linear map $\mathbb{P}^{nd+d-1} \dashrightarrow \mathbb{P}^{n+2d-3}$ defined by

$$\begin{gathered} (z_{10} : z_{11} : \ldots : z_{1n} : z_{20} : \ldots : z_{d,n-1} : z_{dn}) \mapsto \\ (z_{10} : z_{11} + z_{20} : \ldots : z_{1,d-1} + \ldots + z_{d0} : \ldots : z_{1n} + \ldots \\ \ldots + z_{d,n+1-d} : \ldots : z_{dn} : L_1 : \ldots : L_{d-2}) \end{gathered}$$

where $L_1, \dots, L_{d-2}$ are linear forms in $z_{10}, \dots, z_{dn}$. Then the image under ψ of a point $(x_0 : \dots : x_n) \in \mathbb{P}^n \simeq V$ would be represented by the $(d-1)$-plane spanned by the rows of a matrix of the following type:

$$\begin{pmatrix} x_0 & x_1 & \dots & x_n & 0 & \dots & 0 & | & l_{1,1} & \dots & l_{1,d-2} \\ 0 & x_0 & x_1 & \dots & x_n & \dots & 0 & | & l_{2,1} & \dots & l_{2,d-2} \\ \vdots & & \ddots & \ddots & & \ddots & \vdots & | & & \dots & \\ 0 & \dots & 0 & x_0 & x_1 & \dots & x_n & | & l_{d,1} & \dots & l_{d,d-2} \end{pmatrix} \tag{2}$$

where $l_{1,1}, \dots, l_{d,d-2}$ are linear forms in $x_0, \dots, x_n$. We will find a contradiction by showing that there exist two of these $(d-1)$-planes meeting in more than one point.

The span in $\mathbb{P}^{n+2d-3}$ of the $(d-1)$-planes corresponding to the (clearly distinct) points $(t_0 : \dots : t_{n-1} : 0)$, $(0 : t_0 : \dots : t_{n-1})$ would be the linear space generated by the rows of the matrix

$$\begin{pmatrix} t_0 & t_1 & \dots & t_{n-1} & 0 & 0 & \dots & 0 & | & l'_{1,1} & \dots & l'_{1,d-2} \\ 0 & t_0 & t_1 & \dots & t_{n-1} & 0 & \dots & 0 & | & l'_{2,1} & \dots & l'_{2,d-2} \\ \vdots & & \ddots & \ddots & & \ddots & \ddots & \vdots & | & & \dots & \\ 0 & \dots & 0 & t_0 & t_1 & \dots & t_{n-1} & 0 & | & l'_{d,1} & \dots & l'_{d,d-2} \\ 0 & t_0 & t_1 & \dots & t_{n-1} & 0 & \dots & 0 & | & l''_{1,1} & \dots & l''_{1,d-2} \\ 0 & 0 & t_0 & t_1 & \dots & t_{n-1} & \dots & 0 & | & l''_{2,1} & \dots & l''_{2,d-2} \\ \vdots & \vdots & & \ddots & \ddots & & \ddots & \vdots & | & & \dots & \\ 0 & 0 & \dots & 0 & t_0 & t_1 & \dots & t_{n-1} & | & l''_{d,1} & \dots & l''_{d,d-2} \end{pmatrix}$$

where $l'_{i,j} = l_{i,j}(t_0, \dots, t_{n-1}, 0)$ and $l''_{i,j} = l_{ij}(0, t_0, \dots, t_{n-1})$.

Performing the obvious elementary transformations on the rows, the matrix can be reduced to the matrix:

$$\left(\begin{array}{cccccccc|ccc} t_0 & \dots & t_{n-1} & 0 & 0 & \dots & 0 & & l'_{1,1} & \dots & l'_{1,d-2} \\ 0 & t_0 & \dots & t_{n-1} & 0 & \dots & 0 & & l'_{2,1} & \dots & l'_{2,d-2} \\ \vdots & & \ddots & & \ddots & \ddots & \vdots & & & \dots & \\ 0 & \dots & 0 & t_0 & \dots & t_{n-1} & 0 & & l'_{d,1} & \dots & l'_{d,d-2} \\ 0 & 0 & \dots & 0 & t_0 & \dots & t_{n-1} & & l''_{d,1} & \dots & l''_{d,d-2} \\ \hline 0 & & \dots & & & \dots & 0 & & l''_{1,1} - l'_{2,1} & \dots & l''_{1,d-2} - l'_{2,d-2} \\ \vdots & & & & & & \vdots & & \vdots & & \vdots \\ 0 & & \dots & & & \dots & 0 & & l''_{d-1,1} - l'_{d,1} & \dots & l''_{d-1,d-2} - l'_{d,d-2} \end{array}\right).$$

Obviously, the upper-left block of the matrix has rank $d+1$. The lower-right block can be regarded as a $(d-1) \times (d-2)$ matrix of linear forms in the $\mathbb{P}^{n-1}$ of coordinates $t_0, \dots, t_{n-1}$. Thus this block has rank $\leq d-3$ on a subvariety of $\mathbb{P}^{n-1}$ of codimension at most 2. Since $n \geq 3$, it follows that there exist $t_0, \dots, t_{n-1}$ such that the rank of matrix is at most $2d-2$. In other words, the $(d-1)$-planes corresponding to the points $(t_0 : \ldots : t_{n-1} : 0)$, $(0 : t_0 : \ldots : t_{n-1})$ meet in more than one point. This yields the wanted contradiction. $\square$

Let us consider now the case $n = 2$.
Denote by $(a : b : c)$ a system of Plücker coordinates on the space $\mathbb{P}^{2*}$ of lines in $\mathbb{P}^2$: if l is the line passing through the points $(x_0 : x_1 : x_2)$ and $(y_0 : y_1 : y_2)$, then

$$a = x_0 y_1 - x_1 y_0, \; b = x_0 y_2 - x_2 y_0, \; c = x_1 y_2 - x_2 y_1. \tag{3}$$

Remark 4.4. Observe that the line trough the points $(t_0 : t_1 : 0)$ and $(0 : t_0 : t_1)$ has coordinates $(t_0^2 : t_0 t_1 : t_1^2)$ which satisfy the equation $b^2 - ac = 0$; viceversa, each line whose coordinates $(a : b : c)$ satisfy the same equation is a line generated by two points of type $(t_0 : t_1 : 0)$, $(0 : t_0 : t_1)$.

Proposition 4.5. *If $n = 2$, the Schwarzenberger-Veronese variety with $d \geq 3$ can be obtained as a projection of a Veronese variety $V' \subset G(d-1, 2d-1)$ such that any two $(d-1)$-planes of V' meet at most in one point.*

Proof. According to the proof of Proposition 4.3, we need to find linear forms $l_{1,1}, \ldots, l_{d,d-2}$ in the matrix (4.1) such that for any two (a priori possibly infinitely close) points $(x_0 : x_1 : x_2)$, $(y_0 : y_1 : y_2)$ in $\mathbb{P}^2$ the matrix

$$\left(\begin{array}{ccccccc|ccc}
x_0 & x_1 & x_2 & 0 & 0 & \ldots & 0 & l'_{1,1} & \ldots & l'_{1,d-2} \\
0 & x_0 & x_1 & x_2 & 0 & \ldots & 0 & l'_{2,1} & \ldots & l'_{2,d-2} \\
\vdots & & \ddots & \ddots & & \ddots & \vdots & & \ldots & \\
0 & \ldots & 0 & x_0 & x_1 & x_2 & 0 & l'_{d-1,1} & \ldots & l'_{d-1,d-2} \\
0 & \ldots & 0 & 0 & x_0 & x_1 & x_2 & l'_{d,1} & \ldots & l'_{d,d-2} \\
\hline
y_0 & y_1 & y_2 & 0 & 0 & \ldots & 0 & l''_{1,1} & \ldots & l''_{1,d-2} \\
0 & y_0 & y_1 & y_2 & 0 & \ldots & 0 & l''_{2,1} & \ldots & l''_{2,d-2} \\
\vdots & & \ddots & \ddots & & \ddots & \vdots & \vdots & & \vdots \\
0 & \ldots & 0 & y_0 & y_1 & y_2 & 0 & l''_{d-1,1} & \ldots & l''_{d-1,d-2} \\
0 & \ldots & 0 & 0 & y_0 & y_1 & y_2 & l''_{d,1} & \ldots & l''_{d,d-2}
\end{array}\right) \tag{4}$$

where $l'_{i,j} = l_{i,j}(x_0, x_1, x_2)$ and $l''_{i,j} = l_{ij}(y_0, y_1, y_2)$ has rank at least $2d-1$.
A long series of matrix manipulations shows that the choice

$$\begin{pmatrix} l_{1,1} & \ldots & l_{1,d-2} \\ l_{2,1} & \ldots & l_{2,d-2} \\ & \ldots & \\ l_{d,1} & \ldots & l_{d,d-2} \end{pmatrix} = \begin{pmatrix} t_2 & 0 & \ldots & 0 \\ t_0 + 2t_1 + t_2 & t_2 & & 0 \\ -t_0 & t_0 + 2t_1 + t_2 & & \vdots \\ 0 & -t_0 & \ddots & 0 \\ 0 & 0 & & t_2 \\ \vdots & \vdots & \ldots & t_0 + 2t_1 + t_2 \\ 0 & 0 & & -t_0 \end{pmatrix}$$

satisfies the required property. It is helpful for the computations to remark that the rank of the matrix 4 does not change if we substitute the points $P = (x_0 : x_1 : x_2)$ and $Q = (y_0 : y_1 : y_2)$ with any two distinct points of the line $\langle P, Q\rangle$. In particular, we should not care about the possibility that P and Q become infinitely close. □

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Enrique Arrondo
Departamento de Algebra, Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid
28040 Madrid, Spain

Email: enrique_arrondo@mat.ucm.es

Raffaella Paoletti
Dipartimento di Matematica e Applicazioni per l'Architettura
Piazza Ghiberti 23
50123 Firenze, Italy

Email: raffy@math.unifi.it