MODULI SPACES OF CONNECTIONS ON A RIEMANN SURFACE

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Abstract. Let E be a holomorphic vector bundle over a compact connected Riemann surface X. The vector bundle E admits a holomorphic projective connection if and only if for every holomorphic direct summand F of E of positive rank, the equality $\text{degree}(E)/\text{rank}(E) = \text{degree}(F)/\text{rank}(F)$ holds. Fix a point x_0 in X. There is a logarithmic connection on E, singular over x_0 with residue $-\frac{d}{n}\,\text{Id}_{E_{x_0}}$ if and only if the equality $\text{degree}(E)/\text{rank}(E) = \text{degree}(F)/\text{rank}(F)$ holds. Fix an integer $n \geq 2$, and also fix an integer d coprime to n. Let $\mathcal{M}(n, d)$ denote the moduli space of logarithmic $\text{SL}(n, \mathbb{C})$–connections on X singular of x_0 with residue $-\frac{d}{n}\,\text{Id}$. The isomorphism class of the variety $\mathcal{M}(n, d)$ determines the isomorphism class of the Riemann surface X.

1. Introduction

Let X be a compact connected Riemann surface. Let E be a holomorphic vector bundle over X of rank n. A holomorphic connection on E is given by locally defined holomorphic trivializations of E such that all the transition functions are locally constant maps to $\text{GL}(n, \mathbb{C})$. We recall that if $\{U_i\}_{i\in I}$ is a covering of X by open subsets and

$$\varphi_i : E_{U_i} \longrightarrow U_i \times \mathbb{C}^n$$

are holomorphic isomorphisms, then for an ordered pair $(i, j) \in I \times I$, the transition function

$$g_{i,j} : U_i \cap U_j \longrightarrow \text{GL}(n, \mathbb{C})$$

is the unique function satisfying the identity

$$g_{i,j} \circ \varphi_i = \varphi_j .$$

Therefore, if E is equipped with a holomorphic connection, then it makes sense to talk of locally constant holomorphic sections of E.

A holomorphic projective connection on E is given by locally defined holomorphic trivializations of E such that all the transition functions $g_{i,j}$ project to locally constant maps under the projection $\text{GL}(n, \mathbb{C}) \longrightarrow \text{PGL}(n, \mathbb{C})$. Therefore, if E is equipped with a holomorphic projective connection, then the projective bundle $\mathbb{P}(E)$ has locally defined holomorphic trivializations such that all the transition functions are locally constant maps to $\text{PGL}(n, \mathbb{C})$.

Fix a point $x_0 \in X$. Set $X' := X \setminus \{x_0\}$ to be the complement. A logarithmic connection on E singular at x_0 is a holomorphic connection on $E|_{X'}$ which, locally around x_0, has a holomorphic connection matrix with a single pole at x_0. Given a logarithmic

2000 *Mathematics Subject Classification.* 14H60, 14C30.

connection on E singular at x_0, the behavior of the flat sections near x_0 are captured by what is called the residue of the connection. The residue is a linear endomorphism of the fiber E_{x_0}. If the residue of a logarithmic connection on E singular at x_0 is a scalar multiple of the identity automorphism of E_{x_0}, then the logarithmic connection gives a holomorphic projective connection on E. Here we shall consider logarithmic connections whose residue is a scalar multiple of the identity automorphism.

We investigate some properties of holomorphic projective connections and logarithmic connections. Especially we address the question of existence of such connections on a given holomorphic vector bundle.

In the last section we explain a recent result of the authors on the moduli spaces of logarithmic connections.

2. Connections and projective connections

Let X be a compact connected Riemann surface; alternatively, X is an irreducible smooth projective curve defined over the field of complex numbers. The complexified tangent bundle $T^{\mathbb{R}}X \bigotimes_{\mathbb{R}} \mathbb{C}$ decomposes as

$$T^{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$$

into $(1\,,0)$ and $(0\,,1)$ components. The dual line bundle $(T^{1,0}X)^*$ will also be denoted by $\Omega^{1,0}_X$ or K_X, and the complex line bundle $(T^{0,1}X)^*$ will also be denoted by $\Omega^{0,1}_X$.

A holomorphic vector bundle over X is a C^∞ vector bundle of rank n whose transition functions are given by holomorphic maps to $\mathrm{GL}(n, \mathbb{C})$. The usual Dolbeault operator $\overline{\partial}$-operator on locally defined smooth functions over X give a Dolbeault operator $\overline{\partial}_E$ on a holomorphic vector bundle E. This operator $\overline{\partial}_E$ is a first order differential operator that sends smooth sections of E to those of $E \bigotimes \Omega^{0,1}_X$.

We note that the line bundles $T^{1,0}X$ and K_X have natural holomorphic structures. The trivial holomorphic line bundle $X \times \mathbb{C}$ will also be denoted by $\mathcal{O}_X$.

Definition 2.1. *Let E be a holomorphic vector bundle over X of rank n. A holomorphic connection on E is a first order holomorphic differential operator*

$$\mathcal{D} \,:\, E \longrightarrow E \otimes K_X$$

satisfying the Leibniz identity, which says that $\mathcal{D}(fs) = f\mathcal{D}(s) + \mathrm{d}f \bigotimes s$, where f (respectively, s) is a locally defined holomorphic function (respectively, holomorphic section of E).

Associated to a holomorphic connection $\mathcal{D}$ there is a C^∞ connection

$$\nabla \,:\, C^\infty(E) \longrightarrow C^\infty(E \otimes T^*X)\,,$$

where $C^\infty(E)$ denotes the sheaf of smooth sections of the vector bundle E. Using the isomorphism

$$\Gamma(E) = \mathcal{O}(E) \otimes_{\mathcal{O}} C^\infty,$$

where $\mathcal{O}(E)$ is the sheaf of holomorphic sections of E, we have

$$\nabla(fs) = f\mathcal{D}(s) + \mathrm{d}f \otimes s\,,$$

for s a holomorphic section of E and $f \in C^\infty(X)$. In other terms, we define $\nabla = \mathcal{D} + \overline{\partial}_E$ on holomorphic sections, and then we extend to C^∞ sections.

Let $\mathcal{D}$ be a holomorphic connection on E. The curvature $\mathcal{D} \circ \mathcal{D}$ of $\mathcal{D}$ is a holomorphic section of the vector bundle $\mathrm{End}(E) \bigotimes \Omega^{2,0}_X$. Since $\dim_{\mathbb{C}} X = 1$, we have $\Omega^{2,0}_X = 0$. Hence a holomorphic connection on a Riemann surface is automatically flat.

If E has a holomorphic connection, then $\mathrm{degree}(E) = 0$ [At]. Actually, the connection ∇ is also flat, since its curvature is $F_\nabla = \mathcal{D}\overline{\partial}_E + \overline{\partial}_E\mathcal{D}$, which is zero on holomorphic sections, hence $F_\nabla = 0$. Considering a local flat trivialization, the constant sections (in the local trivialization) are automatically holomorphic, hence there are local flat holomorphic trivializations of E.

Note also that any flat connection ∇ on a C^∞ vector bundle E gives rise to a holomorphic structure on E together with a holomorphic connection. For this, we only need to take flat trivializations and declare them to be holomorphic.

Let E be a holomorphic vector bundle over X and F a direct summand of E. This means that F is a holomorphic subbundle of E, and there exists a holomorphic subbundle F' of E such that the natural homomorphism $F \bigoplus F' \longrightarrow E$ is an isomorphism. Let $\iota : F \hookrightarrow E$ be the inclusion map. Fix a complement F' of F as above. Using the natural isomorphism $F \bigoplus F' \longrightarrow E$ we get a projection

$$q : E \longrightarrow F\,.$$

If $\mathcal{D}$ is a holomorphic connection on E, then it is easy to see that the composition

$$F \stackrel{\iota}{\hookrightarrow} E \stackrel{\mathcal{D}}{\longrightarrow} E \otimes K_X \stackrel{q\otimes \mathrm{Id}}{\longrightarrow} F \otimes K_X$$

is a holomorphic connection on F. Hence if E admits a holomorphic connection, then each direct summand of E also admits a holomorphic connection. So, if E admits a holomorphic connection, then the degree of each direct summand of E is zero. A theorem due to Atiyah and Weil says that the converse is also true.

Theorem 2.2 ([At], [We]). *A holomorphic vector bundle E over X admits a holomorphic connection if and only if the degree of each direct summand of E is zero.* □

In particular, any indecomposable bundle E admits a holomorphic connection.

A holomorphic vector bundle E of rank n over X admits a holomorphic connection if and only if we can choose (holomorphic) local trivializations of E over a covering of

X (by open sets) such that all the transition functions are locally constant functions to $\mathrm{GL}(n,\mathbb{C})$. This follows from the flatness of ∇.

Recall that the Atiyah bundle $\mathrm{At}(E)$ associated to a holomorphic vector bundle E is the holomorphic vector bundle associated to the sheaf of first order holomorphic differential operators on E. The *Atiyah exact sequence* for E is

$$0 \longrightarrow \mathrm{End}(E) \longrightarrow \mathrm{At}(E) \longrightarrow T^{1,0}X \longrightarrow 0 \tag{2.1}$$

where the last arrow is the symbol map. Note that $\mathrm{End}(E)$ is identified with the zeroth order differential operators. Therefore, a holomorphic connection on E corresponds to a holomorphic splitting of the Atiyah exact sequence in (2.1).

Now let P_E be the holomorphic principal $\mathrm{GL}(n,\mathbb{C})$–bundle over X defined by E. So P_E is the space of all bases in the fibers of E. Let $\mathcal{O}_X \subset \mathrm{End}(E)$ be the line subbundle given by the sheaf of endomorphisms of E of the type $s \longmapsto f\cdot s$. Therefore, from (2.1) we have the exact sequence of vector bundles

$$0 \longrightarrow \mathrm{End}(E)/\mathcal{O}_X \longrightarrow \mathrm{At}(E)/\mathcal{O}_X \longrightarrow T^{1,0}X \longrightarrow 0 \tag{2.2}$$

over X. This exact sequence is the Atiyah bundle for the principal $\mathrm{PGL}(n,\mathbb{C})$–bundle over X corresponding to E. We note that this principal $\mathrm{PGL}(n,\mathbb{C})$ is the quotient space $P_E/\mathbb{C}^*$, where $\mathbb{C}^*$ is considered as a subgroup of $\mathrm{GL}(n,\mathbb{C})$ through scalar multiplications of $\mathbb{C}^n$. Also, note that the quotient vector bundle $\mathrm{End}(E)/\mathcal{O}_X$ is identified with the subbundle of $\mathrm{ad}(E) \subset \mathrm{End}(E)$ defined by trace zero endomorphisms. Indeed, the natural projection of the subbundle $\mathrm{ad}(E)$ to the quotient bundle $\mathrm{End}(E)/\mathcal{O}_X$ is an isomorphism.

Definition 2.3. *A holomorphic projective connection on E is a holomorphic splitting of the exact sequence in* (2.2). *By a projective connection we will always mean a holomorphic projective connection.*

It follows that a holomorphic vector bundle E admits a projective connection if we can choose holomorphic local trivializations of E over a covering of X (by open sets) such that images of all the transition functions are locally constant under the projection $\mathrm{GL}(n,\mathbb{C}) \longrightarrow \mathrm{PGL}(n,\mathbb{C})$. This means that they are of the form $f\cdot \mathrm{Id}_{n\times n}$ times a locally constant function, with f being some nonzero holomorphic function.

To see this, consider the splitting of the exact sequence (2.2) given by the projective connection, and take a covering by open sets U_α. Over each U_α, take an arbitrary splitting of $\mathrm{At}(E) \longrightarrow \mathrm{At}(E)/\mathcal{O}_X$. This produces a holomorphic connection $\mathcal{D}_\alpha$ on U_α, and hence a flat trivialization of $E|_{U_\alpha}$. Since $\mathcal{D}_\alpha - \mathcal{D}_\beta = f_{\alpha\beta}\cdot \mathrm{Id}_{n\times n}$ on $U_\alpha \bigcap U_\beta$, we have that the transition functions of E (with respect to the chosen trivializations) are of the form $f\cdot \mathrm{Id}_{n\times n}$ times a locally constant function.

Assume that the vector bundle E admits a projective connection. Let E_{PGL} be the principal $\mathrm{PGL}(n,\mathbb{C})$–bundle given by E. We noted earlier that $E_{\mathrm{PGL}} = P_E/\mathbb{C}^*$, where P_E is the principal $\mathrm{GL}(n,\mathbb{C})$–bundle given by E. Giving a projective connection on E is equivalent to giving a holomorphic connection on the principal $\mathrm{PGL}(n,\mathbb{C})$–bundle E_{PGL}, that is, a family of flat trivializations of E_{PGL} such that the transition functions are locally constant functions with values in $\mathrm{PGL}(n,\mathbb{C})$. Note that this also can be expressed in terms of giving trivializations of the projective bundle $\mathbb{P}(E)$ whose transition functions are locally constant functions in $\mathrm{PGL}(n,\mathbb{C})$.

Using the commutative diagram of vector bundles

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{End}(E) & \longrightarrow & \mathrm{At}(E) & \longrightarrow & T^{1,0}X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{End}(E)/\mathcal{O}_X & \longrightarrow & \mathrm{At}(E)/\mathcal{O}_X & \longrightarrow & T^{1,0}X & \longrightarrow & 0 \end{array}$$

over X, a holomorphic connection on E induces a projective connection on E. However, the converse is not true in the sense that a vector bundle admitting a projective connection need not admit a holomorphic connection.

If E admits a projective connection and $\mathrm{degree}(E) = 0$ then the projective connection is induced by a holomorphic connection. To see this, take a covering by open sets U_α with a holomorphic connection $\mathcal{D}_\alpha$ on each U_α, as before. Since $\mathcal{D}_\alpha - \mathcal{D}_\beta = f_{\alpha\beta} \cdot \mathrm{Id}_{n\times n}$ on $U_\alpha \bigcap U_\beta$, we have that $f_{\alpha\beta} \in \Omega^1_X(U_\alpha \bigcap U_\beta)$ form a cocycle, and hence define a class in $H^1(X, K_X) = \mathbb{C}$. Therefore there exists $g_\alpha \in \Omega^1_X(U_\alpha)$ such that if we modify $\mathcal{D}_\alpha$ to $\widetilde{\mathcal{D}}_\alpha = \mathcal{D}_\alpha + g_\alpha \cdot \mathrm{Id}_{n\times n}$, then the corresponding $\widetilde{\mathcal{D}}_\alpha - \widetilde{\mathcal{D}}_\beta$ is a constant multiple of the identity. This gives transition functions for E which are locally constant functions in $\mathrm{GL}(n,\mathbb{C})$.

The following theorem classifies all holomorphic vector bundles over X that admit a projective connection.

Theorem 2.4. *Let E be a holomorphic vector bundle over X of rank n and degree d. Then the following two statements are equivalent:*

1. *The vector bundle E admits a projective connection.*
2. *For any direct summand $F \subset E$, the equality*
$$\frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} = \frac{d}{n}$$
holds.

Proof. Assume that the vector bundle E admits a projective connection. Let E_{PGL} be the principal $\mathrm{PGL}(n,\mathbb{C})$–bundle given by E. The projective connection on E gives a holomorphic connection on E_{PGL}. The adjoint vector bundle over X for the principal $\mathrm{PGL}(n,\mathbb{C})$–bundle E_{PGL} is $\mathrm{ad}(E)$. Therefore $\mathrm{ad}(E)$ admits a family of trivializations with transitions functions locally constant functions on $\mathrm{PGL}(n,\mathbb{C})$. This means that $\mathrm{ad}(E)$ admits a projective connection. Since the degree of $\mathrm{ad}(E)$ is zero, the vector bundle $\mathrm{ad}(E)$ admits a holomorphic connection.

Let $\mathcal{D}$ be a holomorphic connection on $\mathrm{ad}(E)$. Assume that

$$E = F_1 \oplus F_2 .$$

We will show that $\mathcal{D}$ induces a holomorphic connection on the vector bundle $\text{Hom}(F_1, F_2)$.

For this, first note that $\text{Hom}(F_1, F_2)$ is a direct summand of $\text{ad}(E)$. Indeed, we have a holomorphic decomposition

$$\text{ad}(E) = \text{Hom}(F_1, F_2) \oplus \text{Hom}(F_2, F_1) \oplus (\text{End}(F_1) \oplus \text{End}(F_2))/\mathcal{O}_X \tag{2.3}$$

of $\text{ad}(E)$ into a direct sum of vector bundles; here $\mathcal{O}_X$ is considered as a subbundle of $\text{End}(F_1) \bigoplus \text{End}(F_2)$ by sending any locally defined holomorphic function f to $f(\text{Id}_{F_1} + \text{Id}_{F_2})$. $\text{Hom}(F_1, F_2)$ is a direct summand of the bundle $\text{ad}(E)$, which admits a holomorphic connection. Hence Theorem 2.2 implies that

$$\text{degree}(\text{Hom}(F_1, F_2)) = n_1 d_2 - d_1 n_2 = 0, \tag{2.4}$$

where n_i (respectively, d_i) is the rank (respectively, degree) of F_i, $i = 1, 2$. Since

$$\text{degree}(E) = d_1 + d_2 = d,$$

from (2.4), it follows immediately that

$$\frac{\text{degree}(F_i)}{\text{rank}(F_i)} = \frac{d}{n}.$$

Therefore, the first statement in the theorem implies the second statement.

To prove the converse, we begin with a lemma.

Lemma 2.5. *Let E_1 and E_2 are two holomorphic vector bundles over X, both admitting projective connections. If*

$$\frac{\text{degree}(E_1)}{\text{rank}(E_1)} = \frac{\text{degree}(E_2)}{\text{rank}(E_2)},$$

then $E_1 \bigoplus E_2$ also admits a projective connection.

Proof. Since both E_1 and E_2 admit projective connections, the vector bundle $E_1^* \bigotimes E_2$ also admits a projective connection. On the other hand, the condition that

$$\frac{\text{degree}(E_1)}{\text{rank}(E_1)} = \frac{\text{degree}(E_2)}{\text{rank}(E_2)}$$

implies that $\text{degree}(E_1^* \bigotimes E_2) = 0$. This condition together with the condition that $E_1^* \bigotimes E_2$ admits a projective connection imply that $E_1^* \bigotimes E_2$ admits a holomorphic connection. Similarly, $E_2^* \bigotimes E_1$ admits a holomorphic connection. Now using the natural decomposition

$$\text{End}(E_1 \oplus E_2) = \text{End}(E_1) \oplus \text{End}(E_2) \oplus (E_1^* \otimes E_2) \oplus (E_2^* \otimes E_1)$$

it follows that $E_1 \bigoplus E_2$ also admits a projective connection, and the proof of the lemma is complete. □

Continuing with the proof of the theorem, assume that

$$\frac{\text{degree}(F)}{\text{rank}(F)} = \frac{d}{n}$$

for each direct summand F of E.

We will prove that E admits a projective connection. This will be done by imitating the Atiyah's proof in [At] of the criterion for the existence of a holomorphic connection.

As before, let $E_{\rm PGL}$ denote the holomorphic principal $\text{PGL}(n, \mathbb{C})$–bundle over X defined by E. Let

$$0 \longrightarrow \text{ad}(E) \longrightarrow \text{At}(E_{\rm PGL}) \longrightarrow T^{1,0}X \longrightarrow 0 \tag{2.5}$$

be the Atiyah exact sequence for the principal $\text{PGL}(n, \mathbb{C})$–bundle $E_{\rm PGL}$ (see [At, page 187, Theorem 1]), where $\text{At}(E_{\rm PGL})$ is the Atiyah bundle for $E_{\rm PGL}$. The exact sequence of vector bundles in (2.5) coincides with the exact sequence in (2.2). A projective connection on E is equivalent to a holomorphic connection on the principal $\text{PGL}(n, \mathbb{C})$–bundle $E_{\rm PGL}$.

Let

$$a(E) \in H^1(X,\, K_X \otimes \text{ad}(E)) \tag{2.6}$$

be the obstruction class for holomorphic splitting of (2.5). We note that the vector bundle $\text{ad}(E)$ is self-dual, that is, $\text{ad}(E)^* = \text{ad}(E)$. Indeed, the trace map

$$\text{ad}(E) \otimes \text{ad}(E) \longrightarrow \mathcal{O}_X$$

that sends $s \otimes t$ to $\text{trace}(s \circ t)$ identifies $\text{ad}(E)^*$ with $\text{ad}(E)$. Therefore, the Serre duality says that

$$H^1(X,\, K_X \otimes \text{ad}(E)) = H^0(X,\, \text{ad}(E))^* .$$

Let

$$b(E) \in H^0(X,\, \text{ad}(E))^* \tag{2.7}$$

be the linear form corresponding to the cohomology class $a(E)$, constructed in (2.6), by the Serre duality.

In view of Lemma 2.5, it suffices to prove the theorem under the assumption that the vector bundle E is indecomposable.

Assume that E is indecomposable. Then any section

$$s \in H^0(X,\, \text{ad}(E))$$

is actually nilpotent [At, page 201, Proposition 16]. In other words, the endomorphism s gives a holomorphic filtration of subbundles

$$0 = E_0 \subset E_1 \subset E_2 \cdots \subset E_k = E$$

such that $s(E_i) = E_{i-1}$ for all $1 \leq i \leq k$. More precisely, E_i is the subbundle of E generated by the kernel of the i-fold composition $s^i = s \circ \cdots \circ s$.

Now, for a nilpotent endomorphism s of E we know that

$$b(E)(s) = 0 ,$$

where $b(E)$ is constructed in (2.7) [At, page 202, Proposition 18(ii)]. Since the functional $b(E)$ vanishes on $H^0(X,\, \text{ad}(E))$, we have $b(E) = 0$. Therefore, the obstruction class $a(E)$

in (2.6) vanishes, and hence E_{PGL} admits a holomorphic connection. This completes the proof of the theorem. □

3. Logarithmic connections

As before, X will denote a compact connected Riemann surface of genus g, where $g \geq 2$. Fix a point $x_0 \in X$. The holomorphic line bundle over X defined by the divisor x_0 will be denoted by $\mathcal{O}_X(x_0)$.

Definition 3.1. *Let E be a holomorphic vector bundle over X. A logarithmic connection on E singular over x_0 is a holomorphic differential operator*

$$\mathcal{D} \,:\, E \,\longrightarrow\, E \otimes \mathcal{O}_X(x_0) \otimes K_X$$

satisfying the Leibniz identity

$$\mathcal{D}(fs) \,=\, f\mathcal{D}(s) + \mathrm{d}f \otimes s\,, \tag{3.1}$$

where f (respectively, s) is a locally defined holomorphic function (respectively, holomorphic section of E).

Note that a logarithmic connection on E singular over x_0 produces a holomorphic connection on E over $X \setminus \{x_0\}$. Let U be a chart around x_0 with local holomorphic coordinate z. Then

$$\mathcal{D}(s) = ds + A(z)s\,\frac{dz}{z}\,, \tag{3.2}$$

for any local holomorphic section s of $\mathrm{End}(E)$ and a holomorphic connection matrix $A(z)$.

The curvature of a holomorphic connection on E is a holomorphic two–form with values in $\mathrm{End}(E)\bigotimes\mathcal{O}_X(x_0)$. Since a Riemann surface does not have nonzero holomorphic two forms, any logarithmic connection on a Riemann surface is flat.

A differential operator that satisfies the Leibniz identity (3.1) is clearly of order one. The above condition that a logarithmic connection $\mathcal{D}$ satisfies the Leibniz identity is evidently equivalent to the condition that the symbol of the first order differential operator $\mathcal{D}$ coincides with the section

$$\mathrm{Id}_E \,\in\, H^0(X,\, \mathcal{O}_X(x_0) \otimes \mathrm{End}(E))\,,$$

where Id_E denotes the identity automorphism of E.

The Poincaré adjunction formula says the following: Let D be a smooth hypersurface on a smooth variety Z, and let L denote the line bundle over Z defined by the divisor D. Then the restriction of the line bundle L to D is canonically identified with the normal bundle of D. See [GH] for a proof.

Therefore, the Poincaré adjunction formula says that the fiber over x_0 of the line bundle $\mathcal{O}_X(x_0)$ is identified with the holomorphic tangent space $T_{x_0}X$. Using this isomorphism between $\mathcal{O}_X(x_0)|_{x_0}$ and $T_{x_0}X$, the fiber $(K_X\bigotimes\mathcal{O}_X(x_0))_{x_0}$ is identified with $\mathbb{C}$.

We will now recall the definition of the important notion of residue of a logarithmic connection.

Let $\mathcal{D}$ be a logarithmic connection on a vector bundle E over X, which is singular over x_0. Let $v \in E_{x_0}$ be any vector in the fiber of E over the point x_0. Let $\widehat{v}$ be a holomorphic section of E defined around x_0 such that $\widehat{v}(x_0) = v$. Consider

$$\mathcal{D}(\widehat{v})(x_0) \in (K_X \otimes \mathcal{O}_X(x_0))_{x_0} \otimes_{\mathbb{C}} E_{x_0} = \mathbb{C} \otimes_{\mathbb{C}} E_{x_0} = E_{x_0} .$$

Note that if $v = 0$, then $\mathcal{D}(\widehat{v})$ is a (locally defined) section of the subsheaf

$$E \otimes K_X \subset E \otimes \mathcal{O}_X(x_0) \otimes K_X .$$

So, in that case the above evaluation $\mathcal{D}(\widehat{v})(x_0) \in E_{x_0}$ vanishes. Consequently, we have a well–defined endomorphism

$$\text{Res}(\mathcal{D}, x_0) \in \text{End}(E_{x_0})$$

that sends any $v \in E_{x_0}$ to $\mathcal{D}(\widehat{v})(x_0) \in E_{x_0}$. This endomorphism $\text{Res}(\mathcal{D}, x_0)$ is called the *residue* of the logarithmic connection $\mathcal{D}$ at the point x_0. See [De1] for more details.

If $\mathcal{D}$ is a logarithmic connection on E singular over x_0 and $\theta \in H^0(X, \text{End}(E) \otimes K_X)$, then the differential operator $\mathcal{D} + \theta$ is also a logarithmic connection on E singular over x_0. Furthermore, we have

$$\text{Res}(\mathcal{D}, x_0) = \text{Res}(\mathcal{D} + \theta, x_0) .$$

Conversely, if $\mathcal{D}$ and $\mathcal{D}'$ are two logarithmic connections on E regular on $X \setminus \{x_0\}$ with

$$\text{Res}(\mathcal{D}, x_0) = \text{Res}(\mathcal{D}', x_0) , \tag{3.3}$$

then $\mathcal{D}' = \mathcal{D} + \theta$, where $\theta \in H^0(X, \text{End}(E) \bigotimes K_X)$. In this way the space of all logarithmic connections $\mathcal{D}'$ on the given vector bundle E, regular on $X \setminus \{x_0\}$ and satisfying (3.3), with $\mathcal{D}$ fixed, is an affine space for the vector space $H^0(X, K_X \bigotimes \text{End}(E))$.

Let E be a holomorphic vector bundle over X of rank n, and let $\mathcal{D}$ be a logarithmic connection on E, regular on $X \setminus \{x_0\}$, satisfying the residue condition

$$\text{Res}(\mathcal{D}, x_0) = -\frac{d}{n} \text{Id}_{E_{x_0}} , \tag{3.4}$$

where $d \in \mathbb{C}$. Consider the nonsingular flat connection on the complement $X \setminus \{x_0\}$ defined by $\mathcal{D}$. The above condition on the residue implies that the monodromy around x_0 of this flat connection is the $n \times n$ diagonal matrix with $\exp(2\pi\sqrt{-1}d/n)$ as the diagonal entries [De1, page 79, Proposition 3.11]. From the expression of the degree of E in terms of the residue of $\mathcal{D}$ it follows immediately that $\text{degree}(E) = d$; the last sentence of Corollary B.3 in [EV, page 186] gives the expression of the first Chern class in terms of the trace of the residue. In particular, d in (3.4) must be an integer.

We will now relate logarithmic connections with projective connections discussed in Section 2.

Let E be a holomorphic vector bundle over X of rank n and degree d. Let $\mathcal{D}$ be a logarithmic connection on E, regular on $X \setminus \{x_0\}$, satisfying the residue condition in

(3.4). The logarithmic connection $\mathcal{D}$ gives a holomorphic connection on the vector bundle $E|_{X\setminus\{x_0\}}$ over the open subset $X \setminus \{x_0\} \subset X$. This connection on $E|_{X\setminus\{x_0\}}$ induces a projective connection on the $\mathrm{PGL}(n,\mathbb{C})$–principal bundle $E_{\mathrm{PGL}}|_{X\setminus\{x_0\}}$ over $X\setminus\{x_0\} \subset X$, understood as a family of (holomorphic) trivializations with transition functions which are locally constant in $\mathrm{PGL}(n,\mathbb{C})$. The residue of $\mathcal{D}$ is in the center of $\mathrm{M}(n,\mathbb{C})$ (the Lie algebra of $\mathrm{GL}(n,\mathbb{C})$); its projection to $\mathrm{sl}(n,\mathbb{C})$ vanishes. Therefore, the projective connection of $E_{\mathrm{PGL}}|_{X\setminus\{x_0\}}$ extends across x_0 as a regular projective connection.

Hence we have proved the following lemma:

Lemma 3.2. *Let E be a holomorphic vector bundle over X such that E admits a logarithmic connection $\mathcal{D}$ regular on $X \setminus \{x_0\}$ such that*

$$\mathrm{Res}(\mathcal{D}, x_0) \;=\; -\frac{d}{n}\,\mathrm{Id}_{E_{x_0}}\,.$$

Then the vector bundle E admits a projective connection. □

The following theorem is the analog of Theorem 2.4 for logarithmic connections.

Theorem 3.3. *Let E be a holomorphic vector bundle over X of rank n and degree d. Then the following two statements are equivalent:*

1. *The vector bundle E admits a logarithmic connection regular on $X \setminus \{x_0\}$ whose residue at x_0 is $-\frac{d}{n}\mathrm{Id}_{E_{x_0}}$.*
2. *For any direct summand $F \subset E$, the equality*

$$\frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} \;=\; \frac{d}{n}$$

holds.

Proof. Let $\mathcal{D}$ be a logarithmic connection on the vector bundle E, regular on $X\setminus\{x_0\}$, with residue $\mathrm{Res}(\mathcal{D}, x_0) \;=\; -\frac{d}{n}\,\mathrm{Id}_{E_{x_0}}$. From Lemma 3.2 we know that E admits a projective connection. Therefore, from Theorem 2.4 it follows that

$$\frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} \;=\; \frac{d}{n}$$

for every direct summand F of E.

To prove the converse, assume that

$$\frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} \;=\; \frac{d}{n} \tag{3.5}$$

for every direct summand F of E. Therefore, E admits a projective connection (see Theorem 2.4). Fix a projective connection $\mathcal{D}^P$ on E.

Let

$$\gamma \,:\, \widetilde{X} \,\longrightarrow\, X \tag{3.6}$$

be a ramified Galois covering of degree n with the property that that the map γ is totally ramified over the point $x_0 \in X$. (The map γ is allowed to have ramifications over points in $X \setminus \{x_0\}$.) The condition that γ is totally ramified over x_0 means that there is a unique point $y_0 \in \widetilde{X}$ such that $\gamma(y_0) = x_0$.

Consider the vector bundle

$$V := \mathcal{O}_{\widetilde{X}}(-dy_0) \otimes_{\mathcal{O}_{\widetilde{X}}} \gamma^* E \tag{3.7}$$

over $\widetilde{X}$, where γ is the map in (3.6), and y_0 is the unique point with $\gamma(y_0) = x_0$. The projective bundle $\mathbb{P}(V)$ over $\widetilde{X}$ is canonically identified with the projective bundle $\gamma^*\mathbb{P}(E)$. Therefore, the projective connection $\mathcal{D}^P$ on E gives a projective connection $\gamma^*\mathcal{D}^P$ on V.

Note that degree$(V) = 0$. Therefore, the condition that V admits a projective connection implies that V admits a holomorphic connection. Fix a holomorphic connection $\mathcal{D}'$ on V.

Let Γ denote the Galois group for the Galois covering γ in (3.6). Note that the Galois action of Γ on $\widetilde{X}$ has a canonical lift to the vector bundle V. Consider

$$\mathcal{D}'' = \frac{1}{\#\Gamma} \sum_{h \in \Gamma} h^* \mathcal{D}'$$

which is a holomorphic connection on V (as the space of holomorphic connections on V is an affine space, the average of $\mathcal{D}'$ over the action of the elements of Γ is again a holomorphic connection on V).

It is now straight–forward to check that the Galois invariant connection $\mathcal{D}''$ on V descends to a logarithmic connection on E. This descended logarithmic connection is regular outside x_0, and its residue at x_0 is $-\frac{d}{n}\,\mathrm{Id}_{E_{x_0}}$. This completes the proof of the theorem. □

Theorem 2.4 and Theorem 3.3 together have the following corollary:

Corollary 3.4. *Let E be a holomorphic vector bundle over X of rank n and degree d. Then the following three statements are equivalent:*

(1) *The vector bundle E admits a logarithmic connection regular on $X \setminus \{x_0\}$ whose residue at x_0 is $-\frac{d}{n}\, Id_{E_{x_0}}$.*
(2) *The vector bundle E admits a projective connection.*
(3) *For any direct summand $F \subset E$, the equality*

$$\frac{\text{degree}(F)}{\text{rank}(F)} = \frac{d}{n}$$

holds.

□

4. Moduli space of logarithmic connections

Henceforth, we will assume the rank n and the degree d are mutually coprime.

Let $\mathcal{M}_D(n)$ denote the moduli space of all logarithmic connections $(E\,,\mathcal{D})$ over X, where E is any holomorphic vector bundle of rank n and degree d and $\mathcal{D}$ is a logarithmic connection on E, regular on $X\setminus\{x_0\}$, satisfying the residue condition

$$\text{Res}(\mathcal{D},x_0) \,=\, -\frac{d}{n}\,\text{Id}_{E_{x_0}}\,.$$

(See [Si1], [Si2], [Ni] for the construction of this moduli space.)

So $\mathcal{M}_D(n)$ parametrizes isomorphism classes of pairs of the form $(E\,,\mathcal{D})$, where E is any rank n holomorphic vector bundle of degree d and $\mathcal{D}$ is a logarithmic connection on E regular on $X\setminus\{x_0\}$ with $\text{Res}(\mathcal{D},x_0) \,=\, -\frac{d}{n}\,\text{Id}_{E_{x_0}}$. We note that every logarithmic connection over X occurring in the moduli space $\mathcal{M}_D(n)$ is irreducible. Indeed, for $(E\,,\mathcal{D})\,\in\,\mathcal{M}_D(n)$, if $F\,\subset\,E$ is a holomorphic subbundle invariant by the connection $\mathcal{D}$, i.e. $\mathcal{D}(F)\subset F\bigotimes\mathcal{O}(x_0)\bigotimes K_X$, then $\mathcal{D}$ induces a logarithmic connection $\mathcal{D}_F$ on F which is regular on $X\setminus\{x_0\}$ and

$$\text{Res}(\mathcal{D}_F,x_0) \,=\, -\frac{d}{n}\,\text{Id}_{F_{x_0}}\,.$$

Therefore,

$$\frac{\text{degree}(F)}{\text{rank}(F)} \,=\, \frac{d}{n}.$$

This contradicts the assumption that d and n are mutually coprime if F is a proper subbundle of E.

Since every logarithmic connection over X occurring in the moduli space $\mathcal{M}_D(n)$ is irreducible, the variety $\mathcal{M}_D(n)$ is smooth; singular points of a moduli space of connections correspond to reducible connections. The variety $\mathcal{M}_D(n)$ is known to be irreducible. The dimension of $\mathcal{M}_D(n)$ is $2(n^2(g-1)+1)$.

Consider the holomorphic line bundle $L\,:=\,\mathcal{O}_X(d\,x_0)$ over X. The de Rham differential $f\,\longmapsto\,\mathrm{d}f$, defines a logarithmic connection $\mathcal{D}_L$ on L which is regular on $X\setminus\{x_0\}$, and

$$\text{Res}(\mathcal{D}_L,x_0) \,=\, -d\,\text{Id}_{L_{x_0}}\,. \tag{4.1}$$

Let

$$\mathcal{M}_D(L)\,\subset\,\mathcal{M}_D(n) \tag{4.2}$$

be the subvariety parametrizing isomorphism classes of logarithmic connections $(E\,,\mathcal{D})\,\in\,\mathcal{M}_D(n)$ such that

- $\bigwedge^n E\,\cong\,L$, and
- the logarithmic connection on L induced by $\mathcal{D}$ using an isomorphism $\bigwedge^n E\,\longrightarrow\,L$ coincides with the logarithmic connection $\mathcal{D}_L$ in (4.1).

Note that a logarithmic connection $\mathcal{D}$ on E induces a logarithmic connection on $\bigwedge^n E$. Since any two holomorphic isomorphisms between $\bigwedge^n E$ and L differ by a constant scalar, the connection of L given by the induced connection on $\bigwedge^n E$ using an isomorphism $\bigwedge^n E \longrightarrow L$ is independent of the choice of the isomorphism.

The subset $\mathcal{M}_D(L)$ in (4.2) is an irreducible smooth closed subvariety of dimension $2(n^2-1)(g-1)$.

We will show that the biholomorphism class of both $\mathcal{M}_D(L)$ and $\mathcal{M}_D(n)$ are independent of the complex structure of X.

Let $X' := X \setminus \{x_0\}$ be the complement. Fix a point $x' \in X'$. The point x_0 gives a conjugacy class in the fundamental group $\pi_1(X', x')$ as follows. Let

$$f : \mathbb{D} \longrightarrow X$$

be an orientation preserving embedding of the closed unit disk $\mathbb{D} := \{z \in \mathbb{C} \,|\, ||z||^2 \leq 1\}$ into the Riemann surface X such that $f(0) = x_0$. The free homotopy class of the map $\partial\mathbb{D} = S^1 \longrightarrow X'$ obtained by restricting f to the boundary of $\mathbb{D}$ is independent of the choice of f. The orientation of $\partial\mathbb{D}$ coincides with the anti–clockwise rotation around x_0. Any free homotopy class of oriented loops in X' gives a conjugacy class in $\pi_1(X', x')$. Let γ denote the orbit in $\pi_1(X', x')$, for the conjugation action of $\pi_1(X', x')$ on itself, defined by the above free homotopy class of oriented loops associated to x_0.

Let

$$\text{Hom}^0(\pi_1(X', x')\,, \text{GL}(n, \mathbb{C})) \subset \text{Hom}(\pi_1(X', x')\,, \text{GL}(n, \mathbb{C})) \tag{4.3}$$

be the space of all homomorphisms from the fundamental group $\pi_1(X', x')$ to $\text{GL}(n, \mathbb{C})$ satisfying the condition that the image of γ (the free homotopy class defined above) is $\exp(2\pi\sqrt{-1}d/n) \cdot I_{n\times n}$. It may be noted that since $\exp(2\pi\sqrt{-1}d/n) \cdot I_{n\times n}$ is in the center of $\text{SL}(n, \mathbb{C})$, a homomorphism sends the orbit γ in $\pi_1(X', x')$ (for the adjoint action of $\pi_1(X', x')$ on itself) to $\exp(2\pi\sqrt{-1}d/n) \cdot I_{n\times n}$ if and only if there is an element in the orbit which is mapped to $\exp(2\pi\sqrt{-1}d/n) \cdot I_{n\times n}$.

Take any homomorphism $\rho \in \text{Hom}^0(\pi_1(X', x')\,, \text{GL}(n, \mathbb{C}))$. Let $(V\,, \nabla)$ be the flat vector bundle or rank n over X' given by ρ. Therefore V is a holomorphic vector bundle on X'. The monodromy of ∇ along the oriented loop γ is $\exp(2\pi\sqrt{-1}d/n) \cdot I_{n\times n}$. Using the logarithm $2\pi\sqrt{-1}d/n \cdot I_{n\times n}$ of the monodromy, the vector bundle V over X' extends to a holomorphic vector bundle $\overline{V}$ over X, and furthermore, the connection ∇ on V extends to a logarithmic connection $\overline{\nabla}$ on the vector bundle $\overline{V}$ over X such that $(\overline{V}\,, \overline{\nabla}) \in \mathcal{M}_D(n)$, where $\mathcal{M}_D(n)$ is the moduli space of logarithmic connections defined earlier (see [Ma, p. 159, Theorem 4.4]).

Since $\text{GL}(n, \mathbb{C})$ is an algebraic group defined over the field of complex numbers, and $\pi_1(X', x')$ is a finitely presented group, the representation space $\text{Hom}(\pi_1(X', x')\,, \text{SL}(n, \mathbb{C}))$ is a complex algebraic variety in a natural way. The conjugation action of $\text{GL}(n, \mathbb{C})$ on itself induces an action of $\text{GL}(n, \mathbb{C})$ on $\text{Hom}^0(\pi_1(X', x')\,, \text{GL}(n, \mathbb{C}))$. The action of any $T \in \text{GL}(n, \mathbb{C})$ on $\text{Hom}^0(\pi_1(X', x')\,, \text{GL}(n, \mathbb{C}))$ sends any homomorphism ρ to the homomorphism $\pi_1(X', x') \longrightarrow \mathrm{GL}(n, \mathbb{C})$ defined by $\beta \longmapsto T^{-1}\rho(\beta)T$. Let

$$\mathcal{R}_g := \mathrm{Hom}^0(\pi_1(X', x'), \mathrm{GL}(n, \mathbb{C}))/\mathrm{GL}(n, \mathbb{C}) \tag{4.4}$$

be the quotient space for this action.

The algebraic structure of $\mathrm{Hom}^0(\pi_1(X', x'), \mathrm{GL}(n, \mathbb{C}))$ induces an algebraic structure on the quotient $\mathcal{R}_g$. The scheme $\mathcal{R}_g$ is an irreducible smooth quasiprojective variety of dimension $2(n^2 - 1)(g - 1) + 2$ defined over $\mathbb{C}$.

The isomorphism class of this variety $\mathcal{R}_g$ is independent of the complex structure of the topological surface X. The isomorphism class depends only on the integers g, n and d. On the other hand, the moduli space $\mathcal{M}_D(n)$ is canonically biholomorphic to $\mathcal{R}_g$. Therefore, the biholomorphism class of the moduli space $\mathcal{M}_D(n)$ is independent of the complex structure of X; the biholomorphism class depends only on the integers g, n and d.

Replacing $\mathrm{GL}(n, \mathbb{C})$ by the algebraic subgroup $\mathrm{SL}(n, \mathbb{C})$ above, we have an algebraic irreducible smooth quasiprojective variety

$$\mathcal{S}_g := \mathrm{Hom}^0(\pi_1(X', x'), \mathrm{SL}(n, \mathbb{C}))/\mathrm{SL}(n, \mathbb{C}),$$

which is biholomorphic to $\mathcal{M}_D(L)$. We conclude that the biholomorphism class of the moduli space $\mathcal{M}_D(L)$ defined in (4.2) is also independent of the complex structure of X.

5. The second intermediate Jacobian of the moduli space

In this section we recall some results of [BM].

A holomorphic vector bundle E over X is called stable if for every nonzero proper subbundle $F \subset E$, the inequality

$$\frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} < \frac{\mathrm{degree}(E)}{\mathrm{rank}(E)}$$

holds.

Let $\mathcal{N}_X$ denote the moduli space parametrizing all stable vector bundles E over X with $\mathrm{rank}(E) = n$ and $\bigwedge^n E \cong L = \mathcal{O}_X(d\, x_0)$. The moduli space $\mathcal{N}_X$ is an irreducible smooth projective variety of dimension $(n^2 - 1)(g - 1)$ defined over $\mathbb{C}$.

Let $\mathcal{M}_D(L)$ be the moduli space defined in (4.2). Let

$$\mathcal{U} \subset \mathcal{M}_D(L) \tag{5.1}$$

be the Zariski open subset parametrizing all $(E, \mathcal{D})$ such that the underlying vector bundle E is stable. The openness of this subset follows from [Ma]. Let

$$\Phi : \mathcal{U} \longrightarrow \mathcal{N}_X \tag{5.2}$$

denote the forgetful map that sends any $(E, \mathcal{D})$ to E.

By Theorem 3.3, any $E \in \mathcal{N}_X$ admits a logarithmic connection $\mathcal{D}$ such that $(E, \mathcal{D}) \in \mathcal{M}_D(L)$, since any $E \in \mathcal{N}_X$ is indecomposable. Therefore, the projection Φ in (5.2) is surjective. Furthermore, Φ makes $\mathcal{U}$ an affine bundle over $\mathcal{N}_X$. More precisely, $\mathcal{U}$ is a torsor over $\mathcal{N}_X$ for the holomorphic cotangent bundle $T^*\mathcal{N}_X$. This means that the fibers of the vector bundle $T^*\mathcal{N}_X$ act freely transitively on the fibers of Φ [BR, p. 786]. Since $\mathcal{M}_D(L)$ is irreducible, and $\mathcal{U}$ is nonempty, the subset $\mathcal{U} \subset \mathcal{M}_D(L)$ is Zariski dense.

The following lemma is proved in [BM]:

Lemma 5.1. *Let $\mathcal{Z} := \mathcal{M}_D(L) \setminus \mathcal{U}$ be the complement of the Zariski open dense subset. The codimension of the Zariski closed subset $\mathcal{Z}$ in $\mathcal{M}_X$ is at least $(n-1)(g-2)+1$.* $\square$

For any $i \geq 0$, the i–th cohomology of a complex variety with coefficients in $\mathbb{Z}$ is equipped with a mixed Hodge structure [De2], [De3].

The following proposition is proved in [BM].

Proposition 5.2. *The mixed Hodge structure $H^3(\mathcal{M}_D(L), \mathbb{Z})$ is pure of weight three. More precisely, the mixed Hodge structure $H^3(\mathcal{M}_D(L), \mathbb{Z})$ is isomorphic to the Hodge structure $H^3(\mathcal{N}_X, \mathbb{Z})$, where $\mathcal{N}_X$ is the moduli space of stable vector bundles introduced at the beginning of this section.* $\square$

Let

$$J^2(\mathcal{M}_D(L)) := H^3(\mathcal{M}_D(L), \mathbb{C})/(F^2H^3(\mathcal{M}_D(L), \mathbb{C}) + H^3(\mathcal{M}_D(L), \mathbb{Z})) \tag{5.3}$$

be the intermediate Jacobian of the mixed Hodge structure $H^3(\mathcal{M}_D(L))$ (see [Ca, p. 110]). The intermediate Jacobian of any mixed Hodge structure is a generalized torus [Ca, p. 111]. Let

$$J^2(\mathcal{N}_X) := H^3(\mathcal{N}_X, \mathbb{C})/(F^2H^3(\mathcal{N}_X, \mathbb{C}) + H^3(\mathcal{N}_X, \mathbb{Z}))$$

be the intermediate Jacobian for $H^3(\mathcal{N}_X, \mathbb{Z})$, which is a complex torus.

The following proposition is proved in [BM].

Proposition 5.3. *The intermediate Jacobian $J^2(\mathcal{M}_D(L))$ is isomorphic to $J^2(\mathcal{N}_X)$, which is isomorphic to the Jacobian $\mathrm{Pic}^0(X)$ of the Riemann surface X.* $\square$

The homology $H_1(J^2(\mathcal{M}_D(L)), \mathbb{Z})$ has a natural skew–symmetric pairing which we will describe below.

We first note that from Proposition 5.3, and the fact that $H^3(\mathcal{N}_X, \mathbb{Z})$ is torsionfree, it follows that

$$H_1(J^2(\mathcal{M}_D(L)), \mathbb{Z}) = H^3(\mathcal{M}_D(L), \mathbb{Z}) = H^3(\mathcal{N}_X, \mathbb{Z}). \tag{5.4}$$

Also, we have

$$H^2(\mathcal{M}_D(L), \mathbb{Z}) = H^2(\mathcal{N}_X, \mathbb{Z}) = \mathbb{Z}.$$

Fix a generator

$$\alpha \in H^2(\mathcal{M}_D(L), \mathbb{Z}).$$

For any

$$\theta \in H_{2(n^2-1)(g-1)}(\mathcal{M}_D(L), \mathbb{Z}), \tag{5.5}$$

we have a skew–symmetric pairing

$$B_\theta : \bigwedge^2 H^3(\mathcal{M}_D(L), \mathbb{Z}) \longrightarrow \mathbb{Z} \tag{5.6}$$

defined by

$$B_\theta(\omega_1, \omega_2) = (\omega_1 \cup \omega_2 \cup \alpha^{(n^2-1)(g-1)-3}) \cap \theta \in \mathbb{Z}.$$

The following theorem is proved in [BM].

Theorem 5.4. *There exists a homology class θ as in* (5.5) *such that the pairing B_θ in* (5.6) *is nonzero.*

Take any θ such that B_θ is nonzero. Using the isomorphism in (5.4)*, B_θ defines a nonzero pairing*

$$\widetilde{B}_\theta : \bigwedge^2 H_1(J^2(\mathcal{M}_D(L)), \mathbb{Z}) \longrightarrow \mathbb{Z}.$$

The pair $(J^2(\mathcal{M}_D(L)), \widetilde{B}_\theta)$ is isomorphic to $\mathrm{Pic}^0(X)$ *equipped with a multiple of the canonical principal polarization on* $\mathrm{Pic}^0(X)$ *given by the class of a theta line bundle.* □

Given a multiple of a principal polarization on an abelian variety, there is a unique way to recover the principal polarization from it. Therefore, from Theorem 5.4 it follows immediately that the isomorphism class of $(\mathrm{Pic}^0(X), \Theta)$, where Θ is the canonical polarization on $\mathrm{Pic}^0(X)$, is determined by the isomorphism class of the variety $\mathcal{M}_D(L)$.

The Torelli theorem says that the isomorphism class of the principally polarized abelian variety $(\mathrm{Pic}^0(X), \Theta)$ determines the curve X up to an isomorphism. Thus Theorem 5.4 give the following corollary:

Corollary 5.5. *The isomorphism class of the variety $\mathcal{M}_D(L)$ uniquely determines the isomorphism class of the curve X.* □

This is in contrast with the fact that the biholomorphism class of the complex manifold $\mathcal{M}_D(L)$ is independent of the complex structure of the Riemann surface X (see the end of Section 4).

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