

HODGE POLYNOMIALS OF THE MODULI SPACES OF TRIPLES OF RANK $(2, 2)$

VICENTE MUÑOZ, DANIEL ORTEGA, AND MARIA-JESÚS VÁZQUEZ-GALLO

ABSTRACT. Let X be a smooth projective curve of genus $g \geq 2$ over the complex numbers. A holomorphic triple (E_1, E_2, ϕ) on X consists of two holomorphic vector bundles E_1 and E_2 over X and a holomorphic map $\phi: E_2 \rightarrow E_1$. There is a concept of stability for triples which depends on a real parameter σ . In this paper, we determine the Hodge polynomials of the moduli spaces of σ -stable triples with $\text{rk}(E_1) = \text{rk}(E_2) = 2$, using the theory of mixed Hodge structures (in the cases that they are smooth and compact). This gives in particular the Poincaré polynomials of these moduli spaces. As a byproduct, we also give the Hodge polynomial of the moduli space of even degree rank 2 stable vector bundles.

1. INTRODUCTION

Let X be a smooth projective curve of genus $g \geq 2$ over the field of complex numbers. A holomorphic triple $T = (E_1, E_2, \phi)$ on X of rank (n_1, n_2) consists of two holomorphic vector bundles E_1 and E_2 over X (of ranks n_1 and n_2 , and degrees d_1 and d_2 , respectively) and a holomorphic map $\phi: E_2 \rightarrow E_1$. There is a concept of stability for a triple which depends on the choice of a parameter $\sigma \in \mathbb{R}$. Let \mathcal{N}_σ and \mathcal{N}_σ^s denote the moduli spaces of σ -semistable and σ -stable triples, respectively. These have been widely studied in [4, 5, 15, 21].

The range of the parameter σ is an interval $I \subset \mathbb{R}$ split by a finite number of *critical values* σ_c in such a way that, when σ moves without crossing a critical value, then \mathcal{N}_σ remains unchanged, but when σ crosses a critical value, \mathcal{N}_σ undergoes a transformation which we call *flip*. The study of this process allows us to obtain information on the topology of all moduli spaces \mathcal{N}_σ , for any σ , once we know such information for one particular \mathcal{N}_σ (usually the one corresponding to the minimum or maximum possible value of the parameter).

One of the main motivations to study the topology of the moduli spaces of triples is that they appear when looking at the topology of the moduli spaces of Higgs bundles [19, 16, 15] via Morse theory techniques. Higgs bundles are pairs (E, Φ) , formed by a holomorphic vector bundle E of rank r and a holomorphic map $\Phi: E \rightarrow E \otimes K$, where K is the canonical bundle of the curve, and they are intimately related to the representation varieties of the fundamental group of the surface underlying the complex curve into the general Lie group $\text{GL}(r, \mathbb{C})$. The moduli spaces of triples, and the more general moduli spaces of chains [1, 17, 2] appear as critical sets of a natural Morse-Bott function on the moduli space of Higgs bundles [19, 16].

When the rank of E_2 is one, we have the so-called *pairs* [3, 14, 21]. The moduli spaces of pairs are smooth for any rank n_1 , and in the case of rank $n_1 = 2$ and fixed determinant, they are very well-understood thanks to the work of Thaddeus [24]. In this case, the flips have a very nice geometrical interpretation, consisting of blowing up an embedded subvariety and then blowing-down the exceptional divisor in a different way. Moreover, there are also very explicit descriptions of the moduli spaces of pairs for the minimum and maximum possible values of σ .

The flips do not have such a nice behavior for moduli spaces of triples of rank (n_1, n_2) with $n_1 + n_2 > 3$. The flip locus may have singularities, it may consist of several irreducible components intersecting in a non-transverse way, the moduli spaces themselves may have singularities for $n_1, n_2 \geq 2$, and the moduli spaces for σ large are difficult to handle in the situation when $n_1 = n_2$, since then they are described in terms of Quot schemes.

2000 *Mathematics Subject Classification*. Primary: 14F45. Secondary: 14D20, 14H60.

Key words and phrases. Moduli space, complex curve, stable triple, Hodge polynomial.

These difficulties can be overcome in two different ways. The first way is to introduce parabolic structures with generic weights. The moduli spaces of parabolic triples have been studied in [15], where the Poincaré polynomials have been given for the moduli of parabolic triples of ranks $(2, 1)$. The parabolic weights tend to prevent the singularities of the moduli spaces and flip loci. However, for obtaining information on the moduli space of non-parabolic triples, one should relate the parabolic and the non-parabolic situations.

The second route to compute the Poincaré polynomials of the moduli spaces of triples was introduced in [21]. It consists of using the theory of mixed Hodge structures of Deligne [8] to compute the Hodge polynomials of the moduli space. The Hodge polynomials recover the usual Poincaré polynomial when we deal with a smooth compact algebraic variety, but they can be defined for non-smooth and non-compact algebraic varieties as well. This allows to compute the Poincaré polynomials of the moduli spaces of triples which are smooth and compact, no matter if the flip loci have singularities.

In this paper, we use mixed Hodge theory to compute the Hodge polynomials of some of the moduli spaces of triples of rank $(2, 2)$. By the results of [5], if $d_1 - d_2 > 4g - 4$ then \mathcal{N}_σ^s is smooth. Moreover, when $d_1 + d_2$ is odd, the moduli spaces \mathcal{N}_σ only consist of σ -stable triples for non-critical values of σ , therefore \mathcal{N}_σ^s are projective varieties. Because of this, we shall compute the Hodge polynomials of the moduli spaces of triples of rank $(2, 2)$ in the case $d_1 - d_2 > 4g - 4$ and $d_1 + d_2$ odd. This gives in particular the Poincaré polynomials of these moduli spaces.

We start by reviewing the rudiments of mixed Hodge theory and the standard results on triples that we shall use throughout the paper, in Sections 2 and 3. Then Section 4 recalls the computations of the Hodge polynomials of the moduli spaces of triples of ranks $(2, 1)$ and $(1, 2)$, from [21]. In Section 5 we use the Hodge polynomial of the moduli spaces of triples to deduce the Hodge polynomials of the moduli spaces of rank 2 stable vector bundles. The case of odd degree rank 2 bundles is already known [7, 12, 21], but we do the case of even degree rank 2 stable bundles, proving the following result (see Theorem 5.2).

Theorem A. *Let $M^s(2, d)$ denote the moduli space of rank 2, degree d stable vector bundles on X . If d is even then the Hodge polynomial of $M^s(2, d)$ is*

$$e(M^s(2, d)) = \frac{1}{2(1-uv)(1-(uv)^2)} \left(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g \right. \\ \left. - (1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1} - u^2v^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2 \right).$$

Note that the moduli space $M^s(2, d)$ is smooth but non-compact.

Next we move to the study of the moduli spaces of triples of rank $(2, 2)$, which are the main focus of the paper. The critical values are computed in Section 6.

In Section 7 we compute the Hodge polynomial of the moduli space of stable triples of rank $(2, 2)$ for the smallest allowable values of the parameter σ , proving the following result (see Theorem 7.2 and Corollary 7.3).

Theorem B. *Let $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2, 2, d_1, d_2)$ be the moduli space of σ -stable triples of rank $(2, 2)$. Assume that $d_1 - d_2 > 4g - 4$ and $d_1 + d_2$ is odd. Let $\sigma_m = \frac{d_1}{2} - \frac{d_2}{2}$ be the minimum value of the parameter σ and $\sigma_m^+ = \sigma_m + \epsilon$ for $\epsilon > 0$ small. Then $\mathcal{N}_{\sigma_m^+}$ is smooth and projective, it only consists of stable triples, and its Hodge polynomial is*

$$e(\mathcal{N}_{\sigma_m^+}) = \frac{(1+u)^{2g}(1+v)^{2g}(1-(uv)^N)(u^g v^g(1+u)^g(1+v)^g - (1+u^2v)^g(1+uv^2)^g)}{(1-uv)^3(1-(uv)^2)^2} \\ \left((1+u)^g(1+v)^g(u^{g+1}v^{g+1} + u^{N+g-1}v^{N+g-1}) - (1+u^2v)^g(1+uv^2)^g(1+u^Nv^N) \right),$$

where $N = d_1 - d_2 - 2g + 2$.

Under the condition $d_1 - d_2 > 4g - 4$, the Hodge polynomial of $\mathcal{N}_{\sigma_m^+}(2, 2, d_1, d_2)$ when both d_1, d_2 are odd is easily given (see Theorem 7.1). When both d_1, d_2 are even, it may be computed with similar techniques to those of Theorem 7.2. However, to remove the condition $d_1 - d_2 > 4g - 4$ is not possible with the current techniques.

The contribution of the flips to the Hodge polynomials of the moduli spaces of σ -stable triples of rank $(2, 2)$ is computed in Section 8. This is added up to the information for the Hodge polynomial of the small parameter moduli space to get the Hodge polynomial of the moduli space of σ -stable triples of rank $(2, 2)$ for the largest values of σ in Section 9. We get the following result (see Theorem 9.2 and Corollary 9.3).

Theorem C. *Let $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2, 2, d_1, d_2)$ be the moduli space of σ -stable triples of rank $(2, 2)$. Assume that $d_1 - d_2 > 4g - 4$ and $d_1 + d_2$ is odd. Let $\sigma_M = d_1 - d_2$. Then all the moduli spaces \mathcal{N}_σ are isomorphic for $\sigma > \sigma_M$. Let $\sigma_M^+ = \sigma_M + \epsilon$ for $\epsilon > 0$. Then $\mathcal{N}_{\sigma_M^+}$ is smooth and projective, it only consists of stable triples, and*

$$e(\mathcal{N}_{\sigma_M^+}) = \frac{(1+u)^{2g}(1+v)^{2g}}{(1-uv)^3(1-(uv)^2)^2} \left[(1+u^2v)^{2g}(1+uv^2)^{2g}(1-(uv)^{2N}) \right. \\ - N(1+u^2v)^g(1+uv^2)^g(1+u)^g(1+v)^g(uv)^{N+g-1}(1-(uv)^2) \\ + (1+u)^{2g}(1+v)^{2g}(1+uv)^2(uv)^{2g-2+(N+1)/2} \left((1-(uv)^{N+1}) - \frac{N+1}{2}(1-uv)(1+(uv)^N) \right) \\ \left. - g(1+u)^{2g-1}(1+v)^{2g-1}(1-(uv)^2)^2(uv)^{2g-2+(N+1)/2}(1-(uv)^N) \right],$$

where $N = d_1 - d_2 - 2g + 2$.

The computation of the contribution of the flips to the Hodge polynomials of the moduli spaces of σ -stable triples of rank $(2, 2)$ is done under the assumptions $d_1 + d_2$ odd and $d_1 - d_2 > 2g - 2$. This can be extended to the case $d_1 + d_2$ even, keeping in mind that in this case we will find the Hodge polynomials of the moduli spaces \mathcal{N}_σ^s which are non-compact and of the moduli spaces \mathcal{N}_σ which have singularities at non-stable points. However the assumption $d_1 - d_2 > 2g - 2$ cannot be removed with the current techniques.

The Poincaré polynomials of the moduli spaces $\mathcal{N}_{\sigma_m^+}$ and $\mathcal{N}_{\sigma_M^+}$ are obtained from the Hodge polynomials, for $d_1 - d_2 > 4g - 4$ and $d_1 + d_2$ odd (see Corollaries 7.4 and 9.4), since they are smooth projective varieties.

Acknowledgements: First and third authors partially supported through grant MCyT (Spain) MTM2004-07090-C03-01/02.

2. HODGE POLYNOMIALS

2.1. Hodge-Deligne theory. Let us start by recalling the Hodge-Deligne theory of algebraic varieties over \mathbb{C} . Let H be a finite-dimensional complex vector space. A *pure Hodge structure of weight k* on H is a decomposition

$$H = \bigoplus_{p+q=k} H^{p,q}$$

such that $H^{q,p} = \overline{H^{p,q}}$, the bar denoting complex conjugation in H . We denote

$$h^{p,q}(H) = \dim H^{p,q},$$

which is called the Hodge number of type (p, q) . A Hodge structure of weight k on H gives rise to the so-called *Hodge filtration F* on H , where

$$F^p = \bigoplus_{s \geq p} H^{s,p-s},$$

which is a descending filtration. Note that $\text{Gr}_F^p H = F^p / F^{p+1} = H^{p,q}$.

Let H be a finite-dimensional complex vector space. A *(mixed) Hodge structure* over H consists of an ascending weight filtration W on H and a descending Hodge filtration F on H such that F induces a pure Hodge filtration of weight k on each $\text{Gr}_k^W H = W_k / W_{k-1}$. Again we define

$$h^{p,q}(H) = \dim H^{p,q}, \quad \text{where } H^{p,q} = \text{Gr}_F^p \text{Gr}_{p+q}^W H.$$

Deligne has shown [8] that, for each complex algebraic variety Z , the cohomology $H^k(Z)$ and the cohomology with compact support $H_c^k(Z)$ both carry natural Hodge structures. If Z is a compact smooth projective variety (hence compact Kähler) then the Hodge structure $H^k(Z)$ is pure of weight k and coincides with the classical Hodge structure given by the Hodge decomposition of harmonic forms into (p, q) types.

Definition 2.1. For *any* complex algebraic variety Z (not necessarily smooth, compact or irreducible), we define the Hodge numbers as

$$h_c^{k,p,q}(Z) = h^{p,q}(H_c^k(Z)) = \dim \operatorname{Gr}_F^p \operatorname{Gr}_{p+q}^W H_c^k(Z).$$

Introduce the Euler characteristic

$$\chi_c^{p,q}(Z) = \sum_k (-1)^k h_c^{k,p,q}(Z)$$

The *Hodge polynomial* of Z is defined [11] as

$$e(Z) = e(Z)(u, v) = \sum_{p,q} (-1)^{p+q} \chi_c^{p,q}(Z) u^p v^q.$$

If Z is smooth and projective then the mixed Hodge structure on $H_c^k(Z)$ is pure of weight k , so $\operatorname{Gr}_m^W H_c^k(Z) = H_c^k(Z) = H^k(Z)$ and the other pieces $\operatorname{Gr}_m^W H_c^k(Z) = 0$, $m \neq k$. So

$$\chi_c^{p,q}(Z) = (-1)^{p+q} h^{p,q}(Z),$$

where $h^{p,q}(Z)$ is the usual Hodge number of Z . In this case,

$$e(Z)(u, v) = \sum_{p,q} h^{p,q}(Z) u^p v^q$$

is the (usual) Hodge polynomial of Z . Note that in this case, the Poincaré polynomial of Z is

$$P_Z(t) = \sum_k b^k(Z) t^k = \sum_k \left(\sum_{p+q=k} h^{p,q}(Z) \right) t^k = e(Z)(t, t). \quad (2.1)$$

where $b^k(Z)$ is the k -th Betti number of Z .

Theorem 2.2 ([21, Theorem 2.2]). *Let Z be a complex algebraic variety. Suppose that Z is a finite disjoint union $Z = Z_1 \cup \dots \cup Z_n$, where the Z_i are algebraic subvarieties. Then*

$$e(Z) = \sum_i e(Z_i).$$

□

Note that we can assign to *any* complex algebraic variety Z (not necessarily smooth, compact or irreducible) a polynomial

$$P_Z(t) = e(Z)(t, t) = \sum_m (-1)^m \chi_c^m(Z) t^m = \sum_{k,m} (-1)^{k+m} \dim \operatorname{Gr}_m^W H_c^k(Z) t^m,$$

where

$$\chi_c^m(Z) = \sum_{p+q=m} \chi_c^{p,q}(Z).$$

This is called the *virtual Poincaré polynomial* of Z (see [13, 10]). It satisfies an additive property analogous to that of Theorem 2.2 and it recovers the usual Poincaré polynomial when Z is a smooth projective variety.

The following Hodge polynomials will be needed later:

- Let $Z = \mathbb{P}^n$, then $e(Z) = 1 + uv + (uv)^2 + \dots + (uv)^n = (1 - (uv)^{n+1}) / (1 - uv)$. For future reference, we shall denote

$$e_n := e(\mathbb{P}^{n-1}) = e(\mathbb{P}(\mathbb{C}^n)) = \frac{1 - (uv)^n}{1 - uv}. \quad (2.2)$$

- Let $\text{Jac}^d X$ be the Jacobian of (any) degree d of a (smooth, projective) complex curve X of genus g . Then

$$e(\text{Jac}^d X) = (1 + u)^g(1 + v)^g. \quad (2.3)$$

Lemma 2.3 ([21, Lemma 2.3]). *Suppose that $\pi : Z \rightarrow Y$ is an algebraic fiber bundle with fiber F which is locally trivial in the Zariski topology, then $e(Z) = e(F)e(Y)$. (In particular this is true for $Z = F \times Y$.)* \square

Lemma 2.4. *Suppose that $\pi : Z \rightarrow Y$ is a map between quasi-projective varieties which is a locally trivial fiber bundle in the usual topology, with fibers projective spaces $F = \mathbb{P}^N$ for some $N > 0$. Then $e(Z) = e(F)e(Y)$.*

Proof. This follows from [18, 9]. For completeness we provide a proof. Let H be a hyperplane section of Z . We have a morphism of Hodge structures:

$$\begin{aligned} L : H^*(\mathbb{P}^N) \otimes H_c^*(Y) &\rightarrow H_c^*(Z) \\ h^i \otimes \alpha &\mapsto H^i \cap \pi^*(\alpha), \end{aligned} \quad (2.4)$$

where h is the hyperplane class of the projective space. Note that L is not multiplicative. Let us see that L is injective. If $x = \sum H^i \cap \pi^*(\alpha_i) = 0$, let i_0 be the maximum i for which $\alpha_i \neq 0$. Then

$$0 = \pi_*(H^{N-i_0} \cap x) = \alpha_{i_0}.$$

So L must be injective. On the other hand, the Leray spectral sequence of the fibration π has E_2 -term isomorphic to $H^*(\mathbb{P}^N) \otimes H_c^*(Y)$ and converges to $H_c^*(Z)$. So $\dim H^*(\mathbb{P}^N) \otimes H_c^*(Y) \geq \dim H_c^*(Z)$ and L must be bijective. Therefore L is an isomorphism of Hodge structures, and the result follows. \square

Lemma 2.5. *The Hodge polynomial of the Grassmannian $\text{Gr}(k, N)$ is*

$$e(\text{Gr}(k, N)) = \frac{(1 - (uv)^{N-k+1}) \cdots (1 - (uv)^{N-1})(1 - (uv)^N)}{(1 - uv) \cdots (1 - (uv)^{k-1})(1 - (uv)^k)}.$$

Proof. This is well-known, but we provide a proof for completeness.

Let us review first the case of the projective space $\mathbb{P}^{N-1} = (\mathbb{C}^N - \{0\})/(\mathbb{C} - \{0\})$. Then $\mathbb{C}^N - \{0\} \rightarrow \mathbb{P}^{N-1}$ is a locally trivial fibration, since it is the restriction of the universal line bundle $U \rightarrow \mathbb{P}^{N-1}$ to the complement of the zero section. Using either Lemma 2.3 or Lemma 2.4, we have $e(\mathbb{C}^N - \{0\}) = e(\mathbb{C} - \{0\})e(\mathbb{P}^{N-1})$, i.e. $(uv)^N - 1 = (uv - 1)e(\mathbb{P}^{N-1})$, from where (2.2) is recovered. Now in the case of $k > 1$, denote

$$F(k, n) = \{(v_1, \dots, v_k) \mid v_i \text{ are linearly independent vectors of } \mathbb{C}^n\}.$$

Then $\text{Gr}(k, N) = F(k, N)/\text{GL}(k, \mathbb{C})$ and there is a locally trivial fibration $F(k, N) \rightarrow \text{Gr}(k, N)$ with fiber $\text{GL}(k, \mathbb{C}) \cong F(k, k)$ (again it is the principal bundle associated to the universal bundle $U \rightarrow \text{Gr}(k, N)$). So by Lemma 2.3, $e(\text{Gr}(k, N)) = e(F(k, N))/e(F(k, k))$. Now we use that the map

$$F(k, n) \longrightarrow F(k-1, n),$$

given by forgetting the last vector, is a locally trivial fibration, with fiber $\mathbb{C}^n - \mathbb{C}^{k-1}$. Using Lemma 2.3 and Theorem 2.2, we have $e(F(k, n)) = e(F(k-1, n))e(\mathbb{C}^n - \mathbb{C}^{k-1}) = e(F(k-1, n))((uv)^n - (uv)^{k-1})$. By recursion this gives

$$e(F(k, n)) = ((uv)^n - (uv)^{k-1}) \cdots ((uv)^n - uv)((uv)^n - 1).$$

So

$$\begin{aligned} e(\text{Gr}(k, N)) &= \frac{((uv)^N - (uv)^{k-1}) \cdots ((uv)^N - uv)((uv)^N - 1)}{((uv)^k - (uv)^{k-1}) \cdots ((uv)^k - uv)((uv)^k - 1)} \\ &= \frac{(1 - (uv)^{N-k+1}) \cdots (1 - (uv)^{N-1})(1 - (uv)^N)}{(1 - uv) \cdots (1 - (uv)^{k-1})(1 - (uv)^k)}. \end{aligned}$$

\square

Lemma 2.6. *Let M be a smooth projective variety. Consider the algebraic variety $Z = (M \times M)/\mathbb{Z}_2$, where \mathbb{Z}_2 acts as $(x, y) \mapsto (y, x)$. The Hodge polynomial of Z is*

$$e(Z) = \frac{1}{2} \left(e(M)(u, v)^2 + e(M)(-u^2, -v^2) \right).$$

Proof. The cohomology of Z is

$$H^*(Z) = H^*(M \times M)^{\mathbb{Z}_2} = (H^*(M) \otimes H^*(M))^{\mathbb{Z}_2}.$$

This is an equality of Hodge structures. The Hodge structure of M is of pure type, therefore the Hodge structure of Z is also of pure type. Moreover,

$$H^{p,q}(Z) = \left(\bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} H^{p_1,q_1}(M) \otimes H^{p_2,q_2}(M) \right)^{\mathbb{Z}_2}.$$

Therefore we have

$$h^{p,q}(Z) = \frac{1}{2} \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q \\ (p_1,q_1) \neq (p_2,q_2)}} h^{p_1,q_1}(M) h^{p_2,q_2}(M) + \epsilon_{p,q},$$

where

$$\epsilon_{p,q} = \begin{cases} 0, & p \text{ or } q \text{ odd,} \\ \dim(\text{Sym}^2 H^{p_1,q_1}(M)), & p = 2p_1, q = 2q_1, p_1 + q_1 \text{ even,} \\ \dim(\wedge^2 H^{p_1,q_1}(M)), & p = 2p_1, q = 2q_1, p_1 + q_1 \text{ odd.} \end{cases}$$

If V is a vector space of dimension n , then $\dim(\text{Sym}^2 V) = \frac{1}{2}(n^2 + n)$ and $\dim(\wedge^2 V) = \frac{1}{2}(n^2 - n)$, so

$$\epsilon_{p,q} = \begin{cases} 0, & p \text{ or } q \text{ odd,} \\ \frac{1}{2}(h^{p_1,q_1}(M)^2 + (-1)^{p_1+q_1} h^{p_1,q_1}(M)), & p = 2p_1, q = 2q_1. \end{cases}$$

This yields

$$\begin{aligned} e(Z) &= \sum h^{p,q}(Z) u^p v^q \\ &= \frac{1}{2} \sum h^{p_1,q_1}(M) h^{p_2,q_2}(M) u^{p_1+p_2} v^{q_1+q_2} + \frac{1}{2} \sum (-1)^{p_1+q_1} h^{p_1,q_1}(M) u^{2p_1} v^{2q_1} \\ &= \frac{1}{2} e(M) \cdot e(M) + \frac{1}{2} e(M)(-u^2, -v^2). \end{aligned}$$

□

3. MODULI SPACES OF TRIPLES

3.1. Holomorphic triples. Let X be a smooth projective curve of genus $g \geq 2$ over \mathbb{C} . A *holomorphic triple* $T = (E_1, E_2, \phi)$ on X consists of two holomorphic vector bundles E_1 and E_2 over X , of ranks n_1 and n_2 and degrees d_1 and d_2 , respectively, and a holomorphic map $\phi: E_2 \rightarrow E_1$. We refer to (n_1, n_2, d_1, d_2) as the *type* of T , to (n_1, n_2) as the *rank* of T , and to (d_1, d_2) as the *degree* of T .

A homomorphism from $T' = (E'_1, E'_2, \phi')$ to $T = (E_1, E_2, \phi)$ is a commutative diagram

$$\begin{array}{ccc} E'_2 & \xrightarrow{\phi'} & E'_1 \\ \downarrow & & \downarrow \\ E_2 & \xrightarrow{\phi} & E_1, \end{array}$$

where the vertical arrows are holomorphic maps. A triple $T' = (E'_1, E'_2, \phi')$ is a subtriple of $T = (E_1, E_2, \phi)$ if $E'_1 \subset E_1$ and $E'_2 \subset E_2$ are subbundles, $\phi(E'_2) \subset E'_1$ and $\phi' = \phi|_{E'_2}$. A subtriple $T' \subset T$ is called *proper* if $T' \neq 0$ and $T' \neq T$. The quotient triple $T'' = T/T'$ is given by $E''_1 = E_1/E'_1$, $E''_2 = E_2/E'_2$ and $\phi'': E''_2 \rightarrow E''_1$ being the map induced by ϕ . We usually denote by (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$, the types of the subtriple T' and the quotient triple T'' .

Definition 3.1. For any $\sigma \in \mathbb{R}$ the σ -slope of T is defined by

$$\mu_\sigma(T) = \frac{d_1 + d_2}{n_1 + n_2} + \sigma \frac{n_2}{n_1 + n_2} .$$

To shorten the notation, we define the μ -slope and λ -slope of the triple T as $\mu = \mu(E_1 \oplus E_2) = \frac{d_1 + d_2}{n_1 + n_2}$ and $\lambda = \frac{n_2}{n_1 + n_2}$, so that $\mu_\sigma(T) = \mu + \sigma\lambda$.

Definition 3.2. We say that a triple $T = (E_1, E_2, \phi)$ is σ -stable if

$$\mu_\sigma(T') < \mu_\sigma(T),$$

for any proper subtriple $T' = (E'_1, E'_2, \phi')$. We define σ -semistability by replacing the above strict inequality with a weak inequality. A triple is called σ -polystable if it is the direct sum of σ -stable triples of the same σ -slope. It is σ -unstable if it is not σ -semistable, and *strictly* σ -semistable if it is σ -semistable but not σ -stable. A σ -destabilizing subtriple $T' \subset T$ is a proper subtriple satisfying $\mu_\sigma(T') \geq \mu_\sigma(T)$.

We denote by

$$\mathcal{N}_\sigma = \mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$$

the moduli space of σ -polystable triples $T = (E_1, E_2, \phi)$ of type (n_1, n_2, d_1, d_2) , and drop the type from the notation when it is clear from the context. The open subset of σ -stable triples is denoted by $\mathcal{N}_\sigma^s = \mathcal{N}_\sigma^s(n_1, n_2, d_1, d_2)$. This moduli space is constructed in [4] by using dimensional reduction. A direct construction is given by Schmitt [23] using geometric invariant theory.

There are certain necessary conditions in order for σ -semistable triples to exist. Let $\mu_i = \mu(E_i) = d_i/n_i$ stand for the slope of E_i , for $i = 1, 2$. We write

$$\begin{aligned} \sigma_m &= \mu_1 - \mu_2 , \\ \sigma_M &= \left(1 + \frac{n_1 + n_2}{|n_1 - n_2|} \right) (\mu_1 - \mu_2) , \quad \text{if } n_1 \neq n_2 . \end{aligned}$$

Proposition 3.3. [5] *The moduli space $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$ is a complex projective variety. For $n_1, n_2 > 0$, let I denote the interval $I = [\sigma_m, \sigma_M]$ if $n_1 \neq n_2$, or $I = [\sigma_m, \infty)$ if $n_1 = n_2$. A necessary condition for $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$ to be non-empty is that $\sigma \in I$. \square*

3.2. Critical values. To study the dependence of the moduli spaces \mathcal{N}_σ on the parameter, we need to introduce the concept of critical value [4, 21].

Definition 3.4. The values of $\sigma_c \in I$ for which there exist $0 \leq n'_1 \leq n_1$, $0 \leq n'_2 \leq n_2$, d'_1 and d'_2 , with $n'_1 n_2 \neq n_1 n'_2$, such that

$$\sigma_c = \frac{(n_1 + n_2)(d'_1 + d'_2) - (n'_1 + n'_2)(d_1 + d_2)}{n'_1 n_2 - n_1 n'_2}, \tag{3.1}$$

are called *critical values*.

Given a triple $T = (E_1, E_2, \phi)$, the condition of σ -(semi)stability for T can only change when σ crosses a critical value. If $\sigma = \sigma_c$ as in (3.1) and if T has a subtriple $T' \subset T$ of type (n'_1, n'_2, d'_1, d'_2) , then $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T)$ and

- (1) if $\lambda' > \lambda$ (where λ' is the λ -slope of T'), then T is not σ -stable for $\sigma > \sigma_c$,
- (2) if $\lambda' < \lambda$, then T is not σ -stable for $\sigma < \sigma_c$.

Note that $n'_1 n_2 \neq n_1 n'_2$ is equivalent to $\lambda' \neq \lambda$.

Of course, it may happen that there is no triple T as above and hence that the moduli spaces \mathcal{N}_σ and \mathcal{N}_σ^s do not change when crossing σ_c (see Remark 6.6).

Proposition 3.5 ([5, Proposition 2.6]). *Fix (n_1, n_2, d_1, d_2) . Then*

- (1) *The critical values are a finite number of values $\sigma_c \in I$.*
- (2) *The stability and semistability criteria for two values of σ lying between two consecutive critical values are equivalent; thus the corresponding moduli spaces are isomorphic.*
- (3) *If σ is not a critical value and $\gcd(n_1, n_2, d_1 + d_2) = 1$, then σ -semistability is equivalent to σ -stability, i.e., $\mathcal{N}_\sigma = \mathcal{N}_\sigma^s$.*

□

Note that if $\gcd(n_1, n_2, d_1 + d_2) \neq 1$ then it may happen that there exists triples T which are strictly σ -semistable for non-critical values of σ .

3.3. Extensions and deformations of triples. The homological algebra of triples is controlled by the hypercohomology of a certain complex of sheaves which appears when studying infinitesimal deformations [5, Section 3]. Let $T' = (E'_1, E'_2, \phi')$ and $T'' = (E''_1, E''_2, \phi'')$ be two triples of types (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$, respectively. Let $\text{Hom}(T'', T')$ denote the linear space of homomorphisms from T'' to T' , and let $\text{Ext}^1(T'', T')$ denote the linear space of equivalence classes of extensions of the form

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0,$$

where by this we mean a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E'_1 & \longrightarrow & E_1 & \longrightarrow & E''_1 & \longrightarrow & 0 \\ & & \phi' \uparrow & & \phi \uparrow & & \phi'' \uparrow & & \\ 0 & \longrightarrow & E'_2 & \longrightarrow & E_2 & \longrightarrow & E''_2 & \longrightarrow & 0. \end{array}$$

To analyze $\text{Ext}^1(T'', T')$ one considers the complex of sheaves

$$C^\bullet(T'', T'): (E''_1^* \otimes E'_1) \oplus (E''_2^* \otimes E'_2) \xrightarrow{c} E''_2^* \otimes E'_1, \quad (3.2)$$

where the map c is defined by

$$c(\psi_1, \psi_2) = \phi' \psi_2 - \psi_1 \phi''.$$

Proposition 3.6 ([5, Proposition 3.1]). *There are natural isomorphisms*

$$\begin{aligned} \text{Hom}(T'', T') &\cong \mathbb{H}^0(C^\bullet(T'', T')), \\ \text{Ext}^1(T'', T') &\cong \mathbb{H}^1(C^\bullet(T'', T')), \end{aligned}$$

and a long exact sequence associated to the complex $C^\bullet(T'', T')$:

$$\begin{aligned} 0 &\longrightarrow \mathbb{H}^0(C^\bullet(T'', T')) \longrightarrow H^0((E''_1^* \otimes E'_1) \oplus (E''_2^* \otimes E'_2)) \longrightarrow H^0(E''_2^* \otimes E'_1) \\ &\longrightarrow \mathbb{H}^1(C^\bullet(T'', T')) \longrightarrow H^1((E''_1^* \otimes E'_1) \oplus (E''_2^* \otimes E'_2)) \longrightarrow H^1(E''_2^* \otimes E'_1) \\ &\longrightarrow \mathbb{H}^2(C^\bullet(T'', T')) \longrightarrow 0. \end{aligned}$$

□

We introduce the following notation:

$$\begin{aligned} h^i(T'', T') &= \dim \mathbb{H}^i(C^\bullet(T'', T')), \\ \chi(T'', T') &= h^0(T'', T') - h^1(T'', T') + h^2(T'', T'). \end{aligned}$$

Proposition 3.7 ([5, Proposition 3.2]). *For any holomorphic triples T' and T'' we have*

$$\begin{aligned} \chi(T'', T') &= \chi(E''_1^* \otimes E'_1) + \chi(E''_2^* \otimes E'_2) - \chi(E''_2^* \otimes E'_1) \\ &= (1 - g)(n''_1 n'_1 + n''_2 n'_2 - n''_2 n'_1) + n''_1 d'_1 - n'_1 d''_1 + n''_2 d'_2 - n'_2 d''_2 - n''_2 d'_1 + n'_1 d''_2, \end{aligned}$$

where $\chi(E) = \dim H^0(E) - \dim H^1(E)$ is the Euler characteristic of E . □

Since the space of infinitesimal deformations of T is isomorphic to $\mathbb{H}^1(C^\bullet(T, T))$, the previous results also apply to studying deformations of a holomorphic triple T .

Theorem 3.8 ([5, Theorem 3.8]). *Let $T = (E_1, E_2, \phi)$ be an σ -stable triple of type (n_1, n_2, d_1, d_2) .*

- (1) *The Zariski tangent space at the point defined by T in the moduli space of stable triples is isomorphic to $\mathbb{H}^1(C^\bullet(T, T))$.*
- (2) *If $\mathbb{H}^2(C^\bullet(T, T)) = 0$, then the moduli space of σ -stable triples is smooth in a neighbourhood of the point defined by T .*
- (3) *At a smooth point $T \in \mathcal{N}_\sigma^s(n_1, n_2, d_1, d_2)$ the dimension of the moduli space of σ -stable triples is*

$$\begin{aligned} \dim \mathcal{N}_\sigma^s(n_1, n_2, d_1, d_2) &= h^1(T, T) = 1 - \chi(T, T) \\ &= (g-1)(n_1^2 + n_2^2 - n_1 n_2) - n_1 d_2 + n_2 d_1 + 1. \end{aligned}$$

- (4) *Let $T = (E_1, E_2, \phi)$ be a σ -stable triple. If T is injective or surjective (meaning that $\phi : E_2 \rightarrow E_1$ is injective or surjective) then the moduli space is smooth at T .*

□

3.4. Crossing critical values. Fix the type (n_1, n_2, d_1, d_2) for the moduli spaces of holomorphic triples. We want to describe the differences between two spaces $\mathcal{N}_{\sigma_1}^s$ and $\mathcal{N}_{\sigma_2}^s$ when σ_1 and σ_2 are separated by a critical value. Let $\sigma_c \in I$ be a critical value and set

$$\sigma_c^+ = \sigma_c + \epsilon, \quad \sigma_c^- = \sigma_c - \epsilon,$$

where $\epsilon > 0$ is small enough so that σ_c is the only critical value in the interval (σ_c^-, σ_c^+) .

Definition 3.9. We define the *flip loci* as

$$\begin{aligned} \mathcal{S}_{\sigma_c^+} &= \{T \in \mathcal{N}_{\sigma_c^+} \mid T \text{ is } \sigma_c^- \text{-unstable}\} \subset \mathcal{N}_{\sigma_c^+}, \\ \mathcal{S}_{\sigma_c^-} &= \{T \in \mathcal{N}_{\sigma_c^-} \mid T \text{ is } \sigma_c^+ \text{-unstable}\} \subset \mathcal{N}_{\sigma_c^-}. \end{aligned}$$

and $\mathcal{S}_{\sigma_c^\pm}^s = \mathcal{S}_{\sigma_c^\pm} \cap \mathcal{N}_{\sigma_c^\pm}^s$ for the stable part of the flip loci.

Note that for $\sigma_c = \sigma_m$, $\mathcal{N}_{\sigma_m^-}$ is empty, hence $\mathcal{N}_{\sigma_m^+} = \mathcal{S}_{\sigma_m^+}$. Analogously, when $n_1 \neq n_2$, $\mathcal{N}_{\sigma_M^+}$ is empty and $\mathcal{N}_{\sigma_M^-} = \mathcal{S}_{\sigma_M^-}$.

Lemma 3.10. *Let σ_c be a critical value. Then*

- (1) $\mathcal{N}_{\sigma_c^+} - \mathcal{S}_{\sigma_c^+} = \mathcal{N}_{\sigma_c^-} - \mathcal{S}_{\sigma_c^-}$.
- (2) $\mathcal{N}_{\sigma_c^+}^s - \mathcal{S}_{\sigma_c^+}^s = \mathcal{N}_{\sigma_c^-}^s - \mathcal{S}_{\sigma_c^-}^s = \mathcal{N}_{\sigma_c}^s$.

Proof. Item (1) is an easy consequence of the definition of flip loci. Item (2) is the content of [5, Lemma 5.3]. □

Let us describe the flip loci $\mathcal{S}_{\sigma_c^\pm}$. Let σ_c be a critical value, and let (n'_1, n'_2, d'_1, d'_2) such that $\lambda' \neq \lambda$ and (3.1) holds. Put $(n''_1, n''_2, d''_1, d''_2) = (n_1 - n'_1, n_2 - n'_2, d_1 - d'_1, d_2 - d'_2)$. Denote $\mathcal{N}'_\sigma = \mathcal{N}_\sigma(n'_1, n'_2, d'_1, d'_2)$ and $\mathcal{N}''_\sigma = \mathcal{N}_\sigma(n''_1, n''_2, d''_1, d''_2)$.

Lemma 3.11 ([21, Lemma 4.7]). *Let $T \in \mathcal{S}_{\sigma_c^+}$ (resp. $T \in \mathcal{S}_{\sigma_c^-}$). Then T sits in a non-split exact sequence*

$$0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0, \tag{3.3}$$

where $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T) = \mu_{\sigma_c}(T'')$, $\lambda' < \lambda$ (resp. $\lambda' > \lambda$) and T' and T'' are both σ_c -semistable.

Conversely, if $T' \in \mathcal{N}'_{\sigma_c}$ and $T'' \in \mathcal{N}''_{\sigma_c}$ are both σ_c -stable, and $\lambda' < \lambda$ (resp. $\lambda' > \lambda$). Then for any non-trivial extension (3.3), T lies in $\mathcal{S}_{\sigma_c^+}^s$ (resp. in $\mathcal{S}_{\sigma_c^-}^s$). Moreover, such T can be written uniquely as an extension (3.3) with $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T)$.

In particular, suppose σ_c is not a critical value for the moduli spaces of triples of types (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$, $\gcd(n'_1, n'_2, d'_1 + d'_2) = 1$ and $\gcd(n''_1, n''_2, d''_1 + d''_2) = 1$. Then if $\lambda' < \lambda$ (resp. $\lambda' > \lambda$), there is a bijective correspondence between non-trivial extensions (3.3), with $T' \in \mathcal{N}'_{\sigma_c}$ and $T'' \in \mathcal{N}''_{\sigma_c}$ and triples $T \in \mathcal{S}_{\sigma_c^+}$ (resp. $\mathcal{S}_{\sigma_c^-}$). □

Theorem 3.12. *Let σ_c be a critical value with $\lambda' < \lambda$ (resp. $\lambda' > \lambda$). Assume*

- (i) σ_c is not a critical value for the moduli spaces of triples of types (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$, $\gcd(n'_1, n'_2, d'_1 + d'_2) = 1$ and $\gcd(n''_1, n''_2, d''_1 + d''_2) = 1$.
- (ii) $\mathbb{H}^0(C^\bullet(T'', T')) = \mathbb{H}^2(C^\bullet(T'', T')) = 0$, for every $(T', T'') \in \mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$.

Then $\mathcal{S}_{\sigma_c^+}$ (resp. $\mathcal{S}_{\sigma_c^-}$) is the projectivization of a bundle of rank $-\chi(T'', T')$ over $\mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$.

Proof. This is the content of [21, Theorem 4.8]. Note that by [23], the moduli spaces \mathcal{N}'_{σ_c} and \mathcal{N}''_{σ_c} are fine moduli spaces (since $\gcd(n'_1, n'_2, d'_1 + d'_2) = 1$ and $\gcd(n''_1, n''_2, d''_1 + d''_2) = 1$), so the hypothesis (iii) in [21, Theorem 4.8] is satisfied. \square

The construction of the flip loci can be used for the critical value $\sigma_c = \sigma_m$, which allows to describe the moduli space $\mathcal{N}_{\sigma_m^+}$. We refer to the value of σ given by $\sigma = \sigma_m^+ = \sigma_m + \epsilon$ as *small*.

Let $M(n, d)$ denote the moduli space of polystable vector bundles of rank n and degree d over X . This moduli space is projective. We also denote by $M^s(n, d)$ the open subset of stable bundles, which is smooth of dimension $n^2(g-1) + 1$. If $\gcd(n, d) = 1$, then $M(n, d) = M^s(n, d)$.

Proposition 3.13 ([21, Proposition 4.10]). *There is a map*

$$\pi : \mathcal{N}_{\sigma_m^+} = \mathcal{N}_{\sigma_m^+}(n_1, n_2, d_1, d_2) \rightarrow M(n_1, d_1) \times M(n_2, d_2)$$

which sends $T = (E_1, E_2, \phi)$ to (E_1, E_2) .

- (i) If $\gcd(n_1, d_1) = 1$, $\gcd(n_2, d_2) = 1$ and $\mu_1 - \mu_2 > 2g - 2$, then $\mathcal{N}_{\sigma_m^+}^s = \mathcal{N}_{\sigma_m^+}$ is a projective bundle over $M(n_1, d_1) \times M(n_2, d_2)$, whose fibers are projective spaces of dimension $n_2 d_1 - n_1 d_2 - n_1 n_2 (g-1) - 1$.
- (ii) In general, if $\mu_1 - \mu_2 > 2g - 2$, then the open subset

$$\pi^{-1}(M^s(n_1, d_1) \times M^s(n_2, d_2)) \subset \mathcal{N}_{\sigma_m^+}$$

is a projective bundle over $M^s(n_1, d_1) \times M^s(n_2, d_2)$, whose fibers are projective spaces of dimension $n_2 d_1 - n_1 d_2 - n_1 n_2 (g-1) - 1$.

\square

4. HODGE POLYNOMIALS OF THE MODULI SPACES OF TRIPLES OF RANKS (2, 1) AND (1, 2)

4.1. Moduli of triples of rank (2, 1). In this section we recall the main results of [21]. Let $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2, 1, d_1, d_2)$ denote the moduli space of σ -polystable triples $T = (E_1, E_2, \phi)$ where E_1 is a vector bundle of degree d_1 and rank 2 and E_2 is a line bundle of degree d_2 . By Proposition 3.3, σ is in the interval

$$I = [\sigma_m, \sigma_M] = [\mu_1 - \mu_2, 4(\mu_1 - \mu_2)] = [d_1/2 - d_2, 2d_1 - 4d_2], \quad \text{where } \mu_1 - \mu_2 \geq 0.$$

Otherwise \mathcal{N}_σ is empty.

Theorem 4.1 ([21, Theorem 5.1]). *For $\sigma \in I$, \mathcal{N}_σ is a projective variety. It is smooth and of (complex) dimension $3g - 2 + d_1 - 2d_2$ at the stable points \mathcal{N}_σ^s . Moreover, for non-critical values of σ , $\mathcal{N}_\sigma = \mathcal{N}_\sigma^s$ (hence it is smooth and projective).* \square

The critical values corresponding to $n_1 = 2$, $n_2 = 1$ are given by Definition 3.4 :

- (1) $n'_1 = 1$, $n'_2 = 0$. The corresponding σ_c -destabilizing subtriple is of the form $0 \rightarrow E'_1$, where $E'_1 = M$ is a line bundle of degree $\deg(M) = d_M$. The critical value is

$$\sigma_c = 3d_M - d_1 - d_2.$$

- (2) $n'_1 = 1$, $n'_2 = 1$. The corresponding σ_c -destabilizing subtriple T' is of the form $E_2 \rightarrow E'_1$, where E'_1 is a line bundle. Let $T'' = T/T'$ be the quotient bundle, which is of the form $0 \rightarrow E''_1$, where $E''_1 = M$ is a line bundle, and let $d_M = \deg(M)$ be its degree. Then $d'_2 = d_2$, $d'_1 = d_1 - d_M$ and

$$\sigma_c = -(3(d_1 - d_M + d_2) - 2(d_1 + d_2)) = 3d_M - d_1 - d_2.$$

- (3) $n'_1 = 2, n'_2 = 0$. In this case, the only possible subtriple is $0 \rightarrow E_1$. This produces the critical value

$$\sigma_c = \frac{d_1 - 2d_2}{2} = \mu_1 - \mu_2 = \sigma_m,$$

i.e., the minimum of the interval I for σ .

- (4) $n'_1 = 0, n'_2 = 1$. The subtriple T' must be of the form $E_2 \rightarrow 0$. This forces $\phi = 0$ in $T = (E_1, E_2, \phi)$. So T is decomposable, of the form $T' \oplus T'' = (0, E_2, 0) \oplus (E_1, 0, 0)$, and T is σ -unstable for any $\sigma \neq \sigma_c$, where

$$\sigma_c = \frac{2d_2 - d_1}{-2} = \mu_1 - \mu_2 = \sigma_m.$$

Lemma 4.2 ([21, Lemma 5.3]). *Let $\sigma_c = 3d_M - d_1 - d_2$ be a critical value. Then*

$$\mu_1 \leq d_M \leq d_1 - d_2, \quad (4.1)$$

and $\sigma_c = \sigma_m \Leftrightarrow d_M = \mu_1$. \square

The Hodge polynomials of the moduli spaces \mathcal{N}_σ for non-critical values of σ are given in [21, Theorem 6.2]. As this moduli space is projective and smooth, we may recover the Poincaré polynomial from the Hodge polynomial via the formula (2.1).

Theorem 4.3 ([21, Theorem 6.2]). *Suppose that $\sigma > \sigma_m$ is not a critical value. Set $d_0 = \left\lceil \frac{1}{3}(\sigma + d_1 + d_2) \right\rceil + 1$. Then the Hodge polynomial of $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2, 1, d_1, d_2)$ is*

$$e(\mathcal{N}_\sigma) = \text{coeff}_{x^0} \left[\frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2-d_0}} \left(\frac{(uv)^{d_1-d_2-d_0}}{1-(uv)^{-1}x} - \frac{(uv)^{-d_1+g-1+2d_0}}{1-(uv)^2x} \right) \right]. \quad \square$$

4.2. Moduli space of triples of rank (1, 2). Triples of rank (1, 2) are of the form $\phi : E_2 \rightarrow E_1$, where E_2 is a rank 2 bundle and E_1 is a line bundle. By Proposition 3.3, σ is in the interval

$$I = [\sigma_m, \sigma_M] = [\mu_1 - \mu_2, 4(\mu_1 - \mu_2)] = [d_1 - d_2/2, 4d_1 - 2d_2], \quad \text{where } \mu_1 - \mu_2 \geq 0.$$

Theorem 4.4. *For $\sigma \in I$, \mathcal{N}_σ is a projective variety. It is smooth and of (complex) dimension $3g - 2 + 2d_1 - d_2$ at the stable points \mathcal{N}_σ^s . Moreover, for non-critical σ , $\mathcal{N}_\sigma = \mathcal{N}_\sigma^s$ (hence it is smooth and projective).*

Proof. Given a triple $T = (E_1, E_2, \phi)$ one has the dual triple $T^* = (E_2^*, E_1^*, \phi^*)$, where E_i^* is the dual of E_i and ϕ^* is the transpose of ϕ . The map $T \mapsto T^*$ defines an isomorphism

$$\mathcal{N}_\sigma(1, 2, d_1, d_2) \cong \mathcal{N}_\sigma(2, 1, -d_2, -d_1).$$

The result now follows from Theorem 4.1. \square

Also from Lemma 4.2, we get

Lemma 4.5 ([21, Lemma 7.2]). *The critical values for $\mathcal{N}_\sigma(1, 2, d_1, d_2)$ are the numbers $\sigma_c = 3d_M + d_1 + d_2$, where $-\mu_2 \leq d_M \leq d_1 - d_2$. Also $\sigma_c = \sigma_m \Leftrightarrow d_M = -\mu_2$. \square*

Theorem 4.6 ([21, Theorem 7.3]). *Consider $\mathcal{N}_\sigma = \mathcal{N}_\sigma(1, 2, d_1, d_2)$. Let $\sigma > \sigma_m$ be a non-critical value. Set $d_0 = \left\lceil \frac{1}{3}(\sigma - d_1 - d_2) \right\rceil + 1$. Then the Hodge polynomial of \mathcal{N}_σ is*

$$e(\mathcal{N}_\sigma) = \text{coeff}_{x^0} \left[\frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2-d_0}} \left(\frac{(uv)^{d_1-d_2-d_0}}{1-(uv)^{-1}x} - \frac{(uv)^{d_2+g-1+2d_0}}{1-(uv)^2x} \right) \right].$$

Proof. We use that $e(\mathcal{N}_\sigma(1, 2, d_1, d_2)) = e(\mathcal{N}_\sigma(2, 1, -d_2, -d_1))$ and the formula in Theorem 4.3, where d_1 and d_2 are substituted by $-d_2, -d_1$ and

$$d_0 = \left\lceil \frac{1}{3}(\sigma - d_2 - d_1) \right\rceil + 1. \quad \square$$

5. HODGE POLYNOMIAL OF THE MODULI SPACE OF RANK 2 EVEN DEGREE STABLE BUNDLES

Let $M(2, d)$ denote the moduli space of polystable vector bundles of rank 2 and degree d over X . As $M(2, d) \cong M(2, d + 2k)$, for any integer k , there are two moduli spaces, depending on whether the degree is even or odd. We are going to apply the results of the Section 4 to compute the Hodge polynomials of these moduli spaces.

We first recall the Hodge polynomial of the moduli space of rank 2 odd degree stable bundles from [7, 12, 21]

Theorem 5.1 ([21, Proposition 8.1]). *The Hodge polynomial of $M(2, d)$ with odd degree d , is*

$$e(M(2, d)) = \frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^{2g}(1+v)^{2g}}{(1-uv)(1-(uv)^2)}.$$

□

Now we compute the Hodge polynomial of the moduli space of rank 2 even degree stable bundles. Note that this moduli space is smooth but non-compact. It is irreducible and of dimension $4g - 3$.

Theorem 5.2. *The Hodge polynomial of $M^s(2, d)$ with even degree d , is*

$$e(M^s(2, d)) = \frac{1}{2(1-uv)(1-(uv)^2)} \left(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1} - u^2v^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2 \right).$$

Proof. We compute this by relating $M^s(2, d)$ with the moduli space $\mathcal{N}_{\sigma_m^+} = \mathcal{N}_{\sigma_m^+}(2, 1, d, d_2)$ of triples of rank (2, 1) for small σ . Choose $(n_1, d_1) = (2, d)$ and $(n_2, d_2) = (1, d_2)$. If d_2 is very negative so that $\mu_1 - \mu_2 = d/2 - d_2 > 2g - 2$ then Proposition 3.13 (ii) applies. We shall choose the maximum possible value of d_2 for this condition to hold, i.e. $d - 2d_2 = 4g - 2$.

There is a decomposition $\mathcal{N}_{\sigma_m^+} = X_0 \sqcup X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4$ into locally closed algebraic subsets, defined by the following strata:

- (1) The open subset $X_0 \subset \mathcal{N}_{\sigma_m^+}$ consists of those triples of the form $\phi : L \rightarrow E$, where E is a stable rank 2 bundle of degree d , L is a line bundle of degree d_2 , and ϕ is a non-zero map (defined up to multiplication by non-zero scalars). Actually, by Proposition 3.13 there is a map

$$\pi : \mathcal{N}_{\sigma_m^+} \rightarrow M(2, d) \times \text{Jac}^{d_2} X,$$

and $X_0 = \pi^{-1}(M(2, d) \times \text{Jac}^{d_2} X)$. Proposition 3.13 (ii) says that X_0 is a projective bundle over $M^s(2, d) \times \text{Jac}^{d_2} X$ with fibers isomorphic to $\mathbb{P}^{d-2d_2-2g+2-1} = \mathbb{P}^{2g-1}$. By Lemma 2.4,

$$e(X_0) = e(M^s(2, d))e(\text{Jac } X)e_{2g},$$

where $e_{2g} = e(\mathbb{P}^{2g-1})$ following the notation in (2.2).

- (2) The subset X_1 parametrizes triples $\phi : L \rightarrow E$ where E is a strictly semistable bundle of degree d which sits as a non-trivial extension

$$0 \rightarrow L_1 \rightarrow E \rightarrow L_2 \rightarrow 0, \tag{5.1}$$

with $L_1 \not\cong L_2$, $L_1, L_2 \in \text{Jac}^{d/2} X$ and $L \in \text{Jac}^{d_2} X$.

Let Y_1 be the family which parametrizes such bundles E . For fixed L_1, L_2 with $L_1 \not\cong L_2$, the extensions (5.1) are determined by $\mathbb{P}\text{Ext}^1(L_2, L_1)$. As L_1, L_2 are non-isomorphic, $\dim \text{Ext}^1(L_2, L_1) = \dim H^1(L_1 \otimes L_2^*) = g - 1$, so $\mathbb{P}\text{Ext}^1(L_2, L_1) \cong \mathbb{P}^{g-2}$. Therefore Y_1 is a fiber bundle over $\text{Jac}^{d/2} X \times \text{Jac}^{d/2} X - \Delta$, where Δ is the diagonal, with fibers isomorphic to \mathbb{P}^{g-2} . Thus using Theorem 2.2 and Lemma 2.4,

$$e(Y_1) = (e(\text{Jac } X)^2 - e(\text{Jac } X))e_{g-1}. \tag{5.2}$$

Now we want to describe X_1 . For each fixed $E \in Y_1$ as in (5.1), and $L \in \text{Jac}^{d_2} X$, there is an exact sequence

$$0 \rightarrow \text{Hom}(L, L_1) \rightarrow \text{Hom}(L, E) \rightarrow \text{Hom}(L, L_2) \rightarrow 0.$$

Here $\text{Ext}^1(L, L_1) = 0$ since $\deg(L_1) - \deg(L) = d/2 - d_2 > 2g - 2$. So we may write $\text{Hom}(L, E) \cong \text{Hom}(L, L_1) \oplus \text{Hom}(L, L_2)$, non-canonically. Let us see when $\phi \in \text{Hom}(L, E)$ gives rise to a σ_m^+ -stable triple $T = (E, L, \phi)$. First note that T is σ_m -semistable, since by Section 4.1, the only possibility for not being σ_m -semistable is to have a subtriple of rank $(0, 1)$, i.e., a line subbundle $M \subset E$, which by Lemma 4.2 should have degree $d_M > \mu_1$, contradicting the semistability of E . If T is not σ_m^+ -stable then it must have a σ_m -destabilizing subtriple T' of rank $(1, 1)$ by Section 4.1. Such subtriple is of the form $\phi : L \rightarrow L'$, with $L' \subset E$. As $\mu_{\sigma_m}(T') = \mu_{\sigma_m}(T) \implies \mu(L') = \mu(E)$, L' is a destabilizing subbundle of E . But the only destabilizing subbundle of E is L_1 , so ϕ satisfies $\phi(L) \subset L_1$. Equivalently, $\phi = (\phi_1, 0) \in \text{Hom}(L, E)$ gives rise to σ_m^+ -unstable triples.

This discussion implies that given $(E, L) \in Y_1 \times \text{Jac}^{d_2} X$, the morphisms ϕ giving rise to σ_m^+ -stable triples (E, L, ϕ) are those in

$$\text{Hom}(L, E) - \text{Hom}(L, L_1). \quad (5.3)$$

By Riemann-Roch, $\dim \text{Hom}(L, E) = d - 2d_2 - 2g + 2 = 2g$ and $\dim \text{Hom}(L, L_1) = d/2 - d_2 - g + 1 = g$. So the space (5.3) is isomorphic to $\mathbb{C}^{2g} - \mathbb{C}^g$.

The isomorphism class of the triple $T = (E, L, \phi)$ is determined up to multiplication by non-zero scalar $(E, L, \phi) \mapsto (E, L, \lambda\phi)$, since $\text{Aut}(T) = \mathbb{C}^*$. This follows from the fact that $\text{Aut}(E) = \mathbb{C}^*$ (since E is a non-trivial extension (5.1)) and $\text{Aut}(L) = \mathbb{C}^*$. Taking into account the \mathbb{C}^* -action by automorphisms, the fibers of the map $\pi : X_1 \rightarrow Y_1 \times \text{Jac}^{d_2} X$ are isomorphic to the projectivization of (5.3), i.e. $\mathbb{P}^{2g-1} - \mathbb{P}^{g-1}$. Hence

$$e(X_1) = e(\text{Jac } X)e(Y_1)(e_{2g} - e_g) = e(\text{Jac } X)^2(e(\text{Jac } X) - 1)e_{g-1}(e_{2g} - e_g).$$

(For this, write $X_1 = X'_1 - X''_1$, where X'_1 is a \mathbb{P}^{2g-1} -bundle over Y_1 and X''_1 is a \mathbb{P}^{g-1} -bundle over Y_1 . By Theorem 2.2, $e(X_1) = e(X'_1) - e(X''_1)$. Now use Lemma 2.4 to compute $e(X'_1)$ and $e(X''_1)$.)

- (3) The subset X_2 parametrizes triples $\phi : L \rightarrow E$ where E is a strictly semistable bundle of degree d which sits as a non-trivial extension

$$0 \rightarrow L_1 \rightarrow E \rightarrow L_1 \rightarrow 0$$

with $L_1 \in \text{Jac}^{d/2} X$ and $L \in \text{Jac}^{d_2} X$.

The family Y_2 parametrizing such bundles E is a fiber bundle over $\text{Jac}^{d/2} X$ with fibers $\mathbb{P} \text{Ext}^1(L_1, L_1) = \mathbb{P}H^1(\mathcal{O}) = \mathbb{P}^{g-1}$ (actually, this fiber bundle is trivial, so $Y_2 = \text{Jac}^{d/2} X \times \mathbb{P}^{g-1}$). Thus by Lemma 2.3,

$$e(Y_2) = e(\text{Jac } X)e_g. \quad (5.4)$$

For each $L_1 \in \text{Jac}^{d/2} X$, there is an exact sequence

$$0 \rightarrow \text{Hom}(L, L_1) \rightarrow \text{Hom}(L, E) \rightarrow \text{Hom}(L, L_1) \rightarrow 0.$$

So we may write $\text{Hom}(L, E) \cong \text{Hom}(L, L_1) \oplus \text{Hom}(L, L_1)$, non-canonically. In order to describe X_2 , let us see when a triple $T = (E, L, \phi)$, with $E \in Y_2$, is σ_m^+ -stable. As before, the morphisms ϕ giving rise to σ_m^+ -stable triples (E, L, ϕ) are those in

$$\text{Hom}(L, E) - \text{Hom}(L, L_1) = \text{Hom}(L, L_1) \times (\text{Hom}(L, L_1) - \{0\}). \quad (5.5)$$

For a bundle E in Y_2 , the automorphism group of E is $\mathbb{C} \times \mathbb{C}^*$, where $\mathbb{C} \times \mathbb{C}^*$ acts on $\text{Hom}(L, E)$ by

$$(a, \lambda) \cdot (\phi_1, \phi_2) = (\lambda\phi_1 + a\phi_2, \lambda\phi_2).$$

Thus for any $(E, L) \in Y_2 \times \text{Jac}^{d_2} X$, the morphisms ϕ giving rise to σ_m^+ -stable triples (E, L, ϕ) are parametrized by

$$(\text{Hom}(L, L_1) \times (\text{Hom}(L, L_1) - \{0\})) / \mathbb{C} \times \mathbb{C}^*. \quad (5.6)$$

This is a fiber bundle over $\mathbb{P} \text{Hom}(L, L_1) = (\text{Hom}(L, L_1) - \{0\}) / \mathbb{C}^*$ with fibers isomorphic to $\text{Hom}(L, L_1) / \mathbb{C}\phi_2$ for every $[\phi_2] \in \mathbb{P} \text{Hom}(L, L_1)$. As $\dim \text{Hom}(L, E) = d - 2d_2 - 2g + 2 = 2g$ and $\dim \text{Hom}(L, L_1) = d/2 - d_2 - g + 1 = g$, the space (5.6) is a \mathbb{C}^{g-1} -bundle over \mathbb{P}^{g-1} .

Therefore $X_2 \rightarrow Y_2 \times \text{Jac}^{d_2} X$ is \mathbb{C}^{g-1} -bundle over a \mathbb{P}^{g-1} -bundle over $Y_2 \times \text{Jac}^{d_2} X$. So

$$e(X_2) = e(\text{Jac } X)e(Y_2)e_g(e_g - e_{g-1}) = e(\text{Jac } X)^2e_g^2(e_g - e_{g-1}).$$

(To apply Lemma 2.4, we write $X_2 \rightarrow P$, where P is the \mathbb{P}^{g-1} -bundle over $Y_2 \times \text{Jac}^{d_2} X$. Then $X_2 = X'_2 - X''_2$, where X'_2 is a \mathbb{P}^{g-1} -bundle over P and X''_2 is a \mathbb{P}^{g-2} -bundle over P .)

- (4) The subset X_3 parametrizes triples $\phi : L \rightarrow E$ where E is a decomposable bundle of the form $E = L_1 \oplus L_2$, $L_1 \not\cong L_2$, $L_1, L_2 \in \text{Jac}^{d/2} X$ and $L \in \text{Jac}^{d_2} X$. The space parametrizing such bundles E is

$$Y_3 = \tilde{Y}_3/\mathbb{Z}_2, \quad \text{where } \tilde{Y}_3 = \text{Jac}^{d/2} X \times \text{Jac}^{d/2} X - \Delta, \quad (5.7)$$

with \mathbb{Z}_2 acting by permuting the two factors.

As before, the condition for $\phi \in \text{Hom}(L, E)$ to give rise to a σ_m^+ -unstable triple is that there is a subtriple $\phi : L \rightarrow L'$ where $\mu(L') = \mu(E)$. There are only two possible such choices for L' , namely L_1 and L_2 . So given $(E, L) \in Y_3 \times \text{Jac}^{d_2} X$, the morphisms $\phi \in \text{Hom}(L, E) = \text{Hom}(L, L_1) \oplus \text{Hom}(L, L_2)$ giving rise to σ_m^+ -stable triples (E, L, ϕ) are those with both components non-zero, i.e., lying in

$$(\text{Hom}(L, L_1) - \{0\}) \times (\text{Hom}(L, L_2) - \{0\}).$$

The automorphisms of E are $\text{Aut}(E) = \mathbb{C}^* \times \mathbb{C}^*$, therefore the map $\phi \in \text{Hom}(L, E) = \text{Hom}(L, L_1) \oplus \text{Hom}(L, L_2)$ is determined up to the action of $\mathbb{C}^* \times \mathbb{C}^*$ on both factors. So ϕ are parametrized by

$$\mathbb{P} \text{Hom}(L, L_1) \times \mathbb{P} \text{Hom}(L, L_2).$$

Let $\tilde{X}_3 \rightarrow \tilde{Y}_3 \times \text{Jac}^{d_2} X$ be the fiber bundle with fiber over (L_1, L_2, L) equal to $\mathbb{P} \text{Hom}(L, L_1) \times \mathbb{P} \text{Hom}(L, L_2)$. Then $X_3 = \tilde{X}_3/\mathbb{Z}_2$, where \mathbb{Z}_2 acts by permuting $(\phi_1, \phi_2) \mapsto (\phi_2, \phi_1)$. This covers the action of \mathbb{Z}_2 on \tilde{Y}_3 . Now $\tilde{X}_3 = X'_3 - X''_3$, where $\pi : X'_3 \rightarrow \text{Jac}^{d/2} X \times \text{Jac}^{d/2} X \times \text{Jac}^{d_2} X$ is a $\mathbb{P}^{g-1} \times \mathbb{P}^{g-1}$ -bundle and $X''_3 = \pi^{-1}(\Delta \times \text{Jac}^{d_2} X)$. If $A \rightarrow \text{Jac}^{d/2} X \times \text{Jac}^{d_2} X$ is the \mathbb{P}^{g-1} -bundle with fiber over L_1 equal to $\mathbb{P} \text{Hom}(L, L_1)$, then $X'_3 = A \times_{\text{Jac}^{d_2} X} A$. We apply Lemma 2.6 fiberwise: $A \rightarrow \text{Jac}^{d_2} X$ is a fibration whose fiber is A_L , which in turn is a fibration over $\text{Jac}^{d/2} X$ with fibers $\mathbb{P} \text{Hom}(L, L_1)$. Then X'_3 fibers over $\text{Jac}^{d_2} X$ with fibers $(A_L \times A_L)/\mathbb{Z}_2$. Now

$$\begin{aligned} H^*((A \times_{\text{Jac}^{d_2} X} A)/\mathbb{Z}_2) &= H^*(A \times_{\text{Jac}^{d_2} X} A)^{\mathbb{Z}_2} \\ &= (H^*(A_L \times A_L) \otimes H^*(\text{Jac}^{d_2} X))^{\mathbb{Z}_2} \\ &= (H^*(A_L \times A_L))^{\mathbb{Z}_2} \otimes H^*(\text{Jac}^{d_2} X). \end{aligned}$$

So

$$\begin{aligned} e(X'_3/\mathbb{Z}_2) &= e((A_L \times A_L)/\mathbb{Z}_2)e(\text{Jac} X) \\ &= \frac{1}{2} \left(e(\text{Jac} X)^2 e_g^2 + (1-u^2)^g (1-v^2)^g \frac{1-(uv)^{2g}}{1-u^2v^2} \right) e(\text{Jac} X). \end{aligned}$$

On the other hand, X''_3 is a $\mathbb{P}^{g-1} \times \mathbb{P}^{g-1}$ -bundle over $\Delta \times \text{Jac}^{d_2} X$, the action of \mathbb{Z}_2 is trivial on the base, and acts by permutation on the fibers. So X''_3/\mathbb{Z}_2 is a bundle over $\Delta \times \text{Jac}^{d_2} X$ with fibers

$$(\mathbb{P} \text{Hom}(L, L_1) \times \mathbb{P} \text{Hom}(L, L_1))/\mathbb{Z}_2 = (\mathbb{P}^{g-1} \times \mathbb{P}^{g-1})/\mathbb{Z}_2.$$

This fibration is locally trivial in the Zariski topology, since it is associated to a locally trivial (in the Zariski topology) vector bundle over $\Delta \times \text{Jac}^{d_2} X$. Hence by Lemma 2.3 and Lemma 2.4,

$$e(X''_3/\mathbb{Z}_2) = e(\text{Jac} X)^2 e(\mathbb{P}^{g-1} \times \mathbb{P}^{g-1}/\mathbb{Z}_2) = \frac{1}{2} e(\text{Jac} X)^2 \left(e_g^2 + \frac{1-(uv)^{2g}}{1-u^2v^2} \right).$$

Finally using Theorem 2.2,

$$\begin{aligned} e(X_3) &= e(\tilde{X}_3/\mathbb{Z}_2) = e(X'_3/\mathbb{Z}_2) - e(X''_3/\mathbb{Z}_2) \\ &= \frac{1}{2} \left(e(\text{Jac} X)^2 e_g^2 + (1-u^2)^g (1-v^2)^g \frac{1-(uv)^{2g}}{1-u^2v^2} \right) e(\text{Jac} X) - \frac{1}{2} e(\text{Jac} X)^2 \left(e_g^2 + \frac{1-(uv)^{2g}}{1-u^2v^2} \right). \end{aligned}$$

- (5) The subset X_4 parametrizes triples $\phi : L \rightarrow E$, where E is a decomposable bundle of the form $E = L_1 \oplus L_1$, $L_1 \in \text{Jac}^{d/2} X$ and $L \in \text{Jac}^{d_2} X$. Such bundles E are parametrized by $Y_4 = \text{Jac}^{d/2} X$. The morphism ϕ lives in

$$\text{Hom}(L, E) = \text{Hom}(L, L_1) \oplus \text{Hom}(L, L_1) = \text{Hom}(L, L_1) \otimes \mathbb{C}^2. \quad (5.8)$$

The condition for a triple $T = (E, L, \phi)$ to be σ_m^+ -unstable is that there is a destabilizing subbundle $L' \subset E$. A destabilizing subbundle of E is necessarily isomorphic to L_1 and there exists $(a, b) \neq (0, 0)$ such that $L' \cong L_1 \hookrightarrow E$ is given by $x \mapsto (ax, bx)$. This means that $\phi = (a\psi, b\psi) \in \text{Hom}(L, L_1) \otimes \mathbb{C}^2$, for some $\psi \in \text{Hom}(L, L_1)$. All this discussion implies that the set of ϕ giving rise to σ_m^+ -stable triples are those of the form $\phi = (\phi_1, \phi_2) \in \text{Hom}(L, L_1) \otimes \mathbb{C}^2$, with ϕ_1, ϕ_2 linearly independent.

The automorphisms of $T = (E, L, \phi)$ are $\text{Aut}(T) \cong \text{Aut}(E) = GL(2, \mathbb{C})$. This acts on (5.8) via the standard representation of $GL(2, \mathbb{C})$ on \mathbb{C}^2 . So the morphisms ϕ are parametrized by the grassmannian $\text{Gr}(2, \text{Hom}(L, L_1))$. As $\dim \text{Hom}(L, L_1) = g$, we have that $\text{Gr}(2, \text{Hom}(L, L_1)) \cong \text{Gr}(2, g)$.

Moreover $X_4 \rightarrow Y_4 \times \text{Jac}^{d_2} X$ is a locally trivial fibration in the Zariski topology since it is associated to the (locally trivial in the Zariski topology) vector bundle over $Y_4 \times \text{Jac}^{d_2} X$ with fibers $\text{Hom}(L, L_1)$. Using Lemma 2.5,

$$e(X_4) = e(\text{Jac } X)^2 e(\text{Gr}(2, g)) = e(\text{Jac } X)^2 \frac{(1 - (uv)^{g-1})(1 - (uv)^g)}{(1 - (uv)^2)(1 - uv)}.$$

Putting all together,

$$\begin{aligned} e(\mathcal{N}_{\sigma_m^+}) &= e(X_0) + e(X_1) + e(X_2) + e(X_3) + e(X_4) \\ &= e(M^s(2, d))e(\text{Jac } X)e_{2g} + e(\text{Jac } X)^2(e(\text{Jac } X) - 1)e_{g-1}(e_{2g} - e_g) + e(\text{Jac } X)^2 e_g^2(e_g - e_{g-1}) \\ &\quad + \frac{1}{2} \left(e(\text{Jac } X)^2 e_g^2 + (1 - u^2)^g (1 - v^2)^g \frac{1 - (uv)^{2g}}{1 - u^2 v^2} \right) e(\text{Jac } X) - \frac{1}{2} e(\text{Jac } X)^2 \left(e_g^2 + \frac{1 - (uv)^{2g}}{1 - u^2 v^2} \right) \\ &\quad + e(\text{Jac } X)^2 e(\text{Gr}(2, g)). \end{aligned} \tag{5.9}$$

To compute the left hand side, we use Theorem 4.3 for $\sigma = \sigma_m^+ = \mu_1 - \mu_2 + \epsilon$, $\epsilon > 0$ small. It gives

$$d_0 = \left[\frac{1}{3}(\mu_1 - \mu_2 + \epsilon + 2\mu_1 + \mu_2) \right] + 1 = [\mu_1] + 1 = \frac{d}{2} + 1.$$

Substiting into the formula for $e(\mathcal{N}_\sigma)$ with $d_1 = d/2$ and $d - 2d_2 = 4g - 2$, the Hodge polynomial of $\mathcal{N}_{\sigma_m^+}$ equals

$$e(\mathcal{N}_{\sigma_m^+}) = \text{coeff}_{x^0} \left[\frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{2g-2}} \left(\frac{(uv)^{2g-2}}{1-(uv)^{-1}x} - \frac{(uv)^{g+1}}{1-(uv)^2x} \right) \right].$$

Using the following equality (see the proof of [21, Proposition 8.1])

$$\text{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-ax)(1-bx)(1-cx)x^{2g-2}} = \frac{(a+u)^g(a+v)^g}{(a-b)(a-c)} + \frac{(b+u)^g(b+v)^g}{(b-a)(b-c)} + \frac{(c+u)^g(c+v)^g}{(c-a)(c-b)},$$

one gets the following expression

$$\begin{aligned} e(\mathcal{N}_{\sigma_m^+}) &= \frac{(1+u)^{2g}(1+v)^{2g}}{(1-uv)^2(1-(uv)^2)} \left[(1+u^2v)^g(1+v^2u)^g(1-(uv)^{2g}) + \right. \\ &\quad \left. + (1+u)^g(1+v)^g((uv)^{3g-1} + (uv)^{2g+1} - (uv)^{2g-1} - (uv)^{g+1}) \right]. \end{aligned}$$

Finally we substitute this into (5.9) to get the Hodge polynomial $e(M^s(2, d))$ as in the statement. \square

Corollary 5.3. *The Hodge polynomial of the moduli space of polystable rank 2 even degree d vector bundles is*

$$\begin{aligned} e(M(2, d)) &= \frac{1}{2(1-uv)(1-(uv)^2)} \left(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g \right. \\ &\quad \left. - (1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1} - u^2v^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2 \right) \\ &\quad + \frac{1}{2} \left((1+u)^{2g}(1+v)^{2g} + (1-u^2)^g(1-v^2)^g \right). \end{aligned}$$

Proof. We only need to compute $e(M^{ss}(2, d))$, where $M^{ss}(2, d) = M(2, d) - M^s(2, d)$ is the locus of non-stable and polystable rank 2 bundles of degree d . Such bundles are of the form $L_1 \oplus L_2$, where $L_1, L_2 \in \text{Jac}^{d/2} X$. Therefore $M^{ss}(2, d) \cong (\text{Jac } X \times \text{Jac } X)/\mathbb{Z}_2$. By Lemma 2.6 and (2.3),

$$e((\text{Jac } X \times \text{Jac } X)/\mathbb{Z}_2) = \frac{1}{2} \left((1+u)^{2g}(1+v)^{2g} + (1-u^2)^g(1-v^2)^g \right).$$

Adding this to $e(M^s(2, d))$ in Theorem 5.2 we get the result. \square

For instance, the formula of Corollary 5.3 for $g = 2$ gives

$$e(M(2, 0)) = (1+u)^2(1+v)^2(1+uv+u^2v^2+u^3v^3).$$

This formula agrees with [20, Remark 4.11]. Note that the moduli space $M(2, 0)$ is smooth for $g = 2$ (see [22]).

6. CRITICAL VALUES FOR TRIPLES OF RANK (2, 2)

Now we move to the analysis of the moduli spaces of σ -polystable triples of rank (2, 2). Let $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2, 2, d_1, d_2)$. By Proposition 3.3, σ takes values in the interval

$$I = [\sigma_m, \infty) = [\mu_1 - \mu_2, \infty), \quad \text{where } d_1 - d_2 \geq 0.$$

Otherwise \mathcal{N}_σ is empty.

Theorem 6.1. *For $\sigma \in I$, \mathcal{N}_σ is a projective variety. It is smooth of dimension $4g + 2d_1 - 2d_2 - 3$ at any σ -stable point for $\sigma \geq 2g - 2$, or at any σ -stable injective triple. Moreover, if $d_1 + d_2$ is odd then $\mathcal{N}_\sigma = \mathcal{N}_\sigma^s$ for non-critical σ .*

Proof. Projectiveness follows from Proposition 3.3. The smoothness at injective triples follows from Theorem 3.8(4); the dimension follows from Theorem 3.8(3); the smoothness result for $\sigma \geq 2g - 2$ comes from [5, Theorem 3.8(6)]. If $d_1 + d_2$ is odd then $\gcd(2, 2, d_1 + d_2) = 1$ and so, for non-critical σ , $\mathcal{N}_\sigma = \mathcal{N}_\sigma^s$, by Proposition 3.5 (3). On the other hand, if $d_1 + d_2$ is even, then it may happen that there are strictly σ -semistable triples for non-critical values of σ . \square

Let us now compute the critical values for $\mathcal{N}_\sigma(2, 2, d_1, d_2)$. According to (3.1) we have the following possibilities for $n_1 = 2, n_2 = 2$:

- (1) $n'_1 = 1, n'_2 = 0$. The corresponding σ_c -destabilizing subtriple is of the form $0 \rightarrow E'_1$ where $E'_1 = L$ is a line bundle of degree d_L . The critical value is

$$\sigma_c = \frac{4d_L - (d_1 + d_2)}{2} = 2d_L - \mu_1 - \mu_2.$$

- (2) $n'_1 = 1, n'_2 = 2$. The σ_c -destabilizing subtriple T' is of the form $E_2 \rightarrow E'_1$ where E'_1 is a line bundle. The quotient triple $T'' = T/T'$ is of the form $0 \rightarrow E''_1$, where $E''_1 = L$ is a line bundle of degree d_L , and $d'_1 = d_1 - d_L$. Note that $\phi : E_2 \rightarrow E_1$ is not injective. The critical value is

$$\sigma_c = \frac{4(d_1 - d_L + d_2) - 3(d_1 + d_2)}{-2} = 2d_L - \mu_1 - \mu_2.$$

- (3) $n'_1 = 2, n'_2 = 1$. The σ_c -destabilizing subtriple T' is of the form $E'_2 \rightarrow E_1$, where E'_2 is a line bundle. Then the quotient triple $T'' = T/T'$ is of the form $E''_2 \rightarrow 0$, where $E''_2 = F$ is a line bundle of degree d_F , and $d'_2 = d_2 - d_F$.

$$\sigma_c = \frac{4(d_1 + d_2 - d_F) - 3(d_1 + d_2)}{2} = \mu_1 + \mu_2 - 2d_F.$$

- (4) $n'_1 = 0, n'_2 = 1$. The σ_c -destabilizing subtriple is of the form $E'_2 \rightarrow 0$, where $E'_2 = F$ is a line bundle of degree d_F . Again in this case ϕ is not injective. The corresponding critical value is

$$\sigma_c = \frac{4d_F - (d_1 + d_2)}{-2} = \mu_1 + \mu_2 - 2d_F.$$

- (5) $n'_1 = 2, n'_2 = 0$. The subtriple is of the form $0 \rightarrow E_1$. The corresponding critical value is $\sigma_c = \mu_1 - \mu_2 = \sigma_m$.

- (6) $n'_1 = 0$, $n'_2 = 2$. The subtriple is of the form $E_2 \rightarrow 0$. This only happens if $\phi = 0$, and so $T = (0, E_2, 0) \oplus (E_1, 0, 0)$. The critical value is $\sigma_c = \mu_1 - \mu_2 = \sigma_m$, and the triple is σ -unstable for any $\sigma \neq \sigma_m$.

Note that the case $n'_1 = 1$, $n'_2 = 1$ does not appear, since $\lambda' = \lambda$ and therefore this does not give a critical value. In the Cases (1), (3) and (5), we have $\lambda' < \lambda$, so the corresponding triples are σ -unstable for $\sigma < \sigma_c$. In the Cases (2), (4) and (6), we have $\lambda' > \lambda$, so the corresponding triples are σ -unstable for $\sigma > \sigma_c$.

Proposition 6.2. (i) *Let $\sigma_c = 2d_L - \mu_1 - \mu_2$ be a critical value corresponding to the Cases (1) or (3). Then $\mu_1 \leq d_L \leq (3\mu_1 - \mu_2)/2$. Also $d_L = \mu_1 \iff \sigma_c = \sigma_m$.*
 (ii) *Let $\sigma_c = \mu_1 + \mu_2 - 2d_F$ be a critical value corresponding to the Cases (2) or (4). Then $(3\mu_2 - \mu_1)/2 \leq d_F \leq \mu_2$. Also $d_F = \mu_2 \iff \sigma_c = \sigma_m$.*

Proof. We shall do the first item, since the second is analogous. Fix the critical value $\sigma_c = 2d_L - \mu_1 - \mu_2$ and suppose that there is a strictly σ_c -semistable triple T in either Case (1) or (3) above. Then the subtriple T' and quotient triple T'' are both σ_c -semistable by Lemma 3.11. In either case, there exists a σ_c -semistable triple of type $(1, 2, d_1 - d_L, d_2)$. By Proposition 3.3 applied to this situation, we get

$$d_1 - d_L - \frac{d_2}{2} \leq \sigma_c = 2d_L - \frac{d_1}{2} - \frac{d_2}{2} \leq 4 \left(d_1 - d_L - \frac{d_2}{2} \right).$$

We can write this inequality in the equivalent form

$$\frac{d_1}{2} \leq d_L \leq \frac{3d_1 - d_2}{4}.$$

□

Theorem 6.3. *Let $\sigma_M = 2(\mu_1 - \mu_2)$. For $\sigma > \sigma_M$ the moduli spaces of σ -(semi)stable triples do not change, and all σ -semistable triples $T = (E_1, E_2, \phi)$ are injective, i.e., T defines an exact sequence of the form*

$$0 \rightarrow E_2 \xrightarrow{\phi} E_1 \rightarrow S \rightarrow 0,$$

where S is a torsion sheaf of degree $d_1 - d_2$.

Proof. If we are in the first situation in Proposition 6.2, then $\sigma_c = 2d_L - \mu_1 - \mu_2 \leq 3\mu_1 - \mu_2 - \mu_1 - \mu_2 = 2(\mu_1 - \mu_2)$. In the second situation, $\sigma_c = \mu_1 + \mu_2 - 2d_F \leq \mu_1 + \mu_2 - (3\mu_2 - \mu_1) = 2(\mu_1 - \mu_2)$.

Now let T be a σ -semistable triple for $\sigma > 2(\mu_1 - \mu_2)$. If $\phi : E_2 \rightarrow E_1$ were not injective, then T has a subtriple $T' = (0, \ker \phi, 0)$ with $\lambda' > \lambda$. This forces $\mu_\sigma(T') > \mu_\sigma(T)$ for σ large, and hence for σ bigger than the last critical value. □

Remark 6.4. Note that for any critical value σ_c , all the triples in $\mathcal{S}_{\sigma_c^-}$ are not injective.

Remark 6.5. By [5, Proposition 6.5] there is a value σ_0 such that all σ -semistable triples for $\sigma > \sigma_0$ are injective. By [5, Theorem 8.6] there is a value σ_L such that the moduli spaces \mathcal{N}_σ are isomorphic for all $\sigma > \sigma_L$. In our case, $n_1 = n_2 = 2$, both numbers are $2(\mu_1 - \mu_2)$.

Remark 6.6. In Proposition 6.2 we see that, for the triples of rank $(2, 2)$, there are critical values for which the moduli spaces do not change (those corresponding to $d_L > (3\mu_1 - \mu_2)/2$ and those corresponding to $d_F < (3\mu_2 - \mu_1)/2$).

Remark 6.7. If we have simultaneously $\sigma_c = 2d_L - \mu_1 - \mu_2$ and $\sigma_c = \mu_1 + \mu_2 - 2d_F$, then $2d_L - \mu_1 - \mu_2 = \mu_1 + \mu_2 - 2d_F \implies d_1 + d_2 = 2d_L + 2d_F$ is an even number.

Therefore, if $d_1 + d_2 \notin 2\mathbb{Z}$, then Cases (1) and (3) (resp. Cases (2) and (4)) do not happen simultaneously (for the same critical value). So the flip locus $\mathcal{S}_{\sigma_c^+}$ (resp. $\mathcal{S}_{\sigma_c^-}$) will consist only of triples of one type for any $\sigma_c > \sigma_m$. In this situation the critical values $\sigma_c \in (\mu_1 + \mu_2 + \mathbb{Z}) \cap [\mu_1 - \mu_2, 2(\mu_1 - \mu_2)]$.

If $d_1 + d_2 \in 2\mathbb{Z}$, then Cases (1) and (3) (resp. Cases (2) and (4)) do happen simultaneously. The flip locus $\mathcal{S}_{\sigma_c^+}$ (resp. $\mathcal{S}_{\sigma_c^-}$) consists of two types of triples, which yields two components that must be considered independently. In this situation the critical values $\sigma_c \in (\mu_1 + \mu_2 + 2\mathbb{Z}) \cap [\mu_1 - \mu_2, 2(\mu_1 - \mu_2)]$.

In the next section, it will be useful to have a vanishing result for the hypercohomology \mathbb{H}^2 to find the flip loci $\mathcal{S}_{\sigma_c^\pm}$ for the moduli spaces of triples of type $(2, 2, d_1, d_2)$.

Proposition 6.8. *Let $T = (E_1, E_2, \phi)$ be a strictly σ_c -semistable triple of type $(2, 2, d_1, d_2)$ with $\sigma_c > \sigma_m$, $T' = (E'_1, E'_2, \phi')$ a destabilizing subtriple and $T'' = T/T' = (E''_1, E''_2, \phi'')$ the corresponding quotient triple.*

- (1) *If $T \in \mathcal{S}_{\sigma_c^-}$ then $\mathbb{H}^2(C^\bullet(T'', T')) = 0$.*
- (2) *If $T \in \mathcal{S}_{\sigma_c^+}$ then $\mathbb{H}^2(C^\bullet(T'', T')) = 0$, if $d_1 - d_2 > 2g - 2$.*

Proof. By Proposition 3.6 and Serre duality, the vanishing $\mathbb{H}^2(C^\bullet(T'', T')) = 0$ is equivalent to the injectivity of the map

$$\begin{array}{ccc} H^0(E_1'^* \otimes E_2'' \otimes K) & \xrightarrow{P} & H^0(E_1'^* \otimes E_1'' \otimes K) \oplus H^0(E_2'^* \otimes E_2'' \otimes K) \\ \psi & \longmapsto & ((\phi'' \otimes Id) \circ \psi, \psi \circ \phi'). \end{array}$$

- (1) If $T \in \mathcal{S}_{\sigma_c^-}$, then $H^0(E_1'^* \otimes E_2'' \otimes K)$ is trivial because either we are in Case (4) and so $E'_1 = 0$ or we are in Case (2) and so $E_2'' = 0$.
- (2) If $T \in \mathcal{S}_{\sigma_c^+}$, we may have two cases:
 - (a) If we are in Case (3), then $E'_1 = E_1$ and $E_1'' = 0$. The map P is

$$\begin{array}{ccc} H^0(E_1^* \otimes E_2'' \otimes K) & \xrightarrow{P} & H^0(E_2'^* \otimes E_2'' \otimes K) \\ \psi & \longmapsto & \psi \circ \phi'. \end{array}$$

If P is not injective, let $\psi : E_1 \rightarrow E_2'' \otimes K$ be a non-trivial homomorphism in $\ker P$. Then, as $\phi' : E'_2 \rightarrow E_1$, ψ must factor through the quotient E_1/E'_2 . Both E_1/E'_2 and $E_2'' \otimes K$ are line bundles, hence $\deg(E_1/E'_2) = d_1 - d'_2 \leq \deg(E_2'' \otimes K) = d_2'' + 2g - 2$. This yields $d_1 - d_2 \leq 2g - 2$.

- (b) If we are in Case (1), then $E'_2 = 0$ and $E_2'' = E_2$. Then the map P is

$$\begin{array}{ccc} H^0(E_1'^* \otimes E_2'' \otimes K) & \xrightarrow{P} & H^0(E_1'^* \otimes E_1'' \otimes K) \\ \psi & \longmapsto & (\phi'' \otimes Id) \circ \psi. \end{array}$$

If P is not injective, let $\psi : E'_1 \rightarrow E_2 \otimes K$ be a non-trivial homomorphism in $\ker P$. Denote by Q the kernel of $\phi'' : E_2 \rightarrow E_1''$, so ψ must factor through $Q \otimes K$. As E'_1 and $Q \otimes K$ are line bundles, we have $\deg(E'_1) = d'_1 \leq \deg(Q \otimes K) = d_2 - d_1'' + 2g - 2$, which is rewritten as $d_1 - d_2 \leq 2g - 2$.

In both cases, if P is not injective then $d_1 - d_2 \leq 2g - 2$. Therefore, if $d_1 - d_2 > 2g - 2$, then P must be injective. □

Remark 6.9. This result is a sort of improvement of [5, Proposition 3.6] for the case of triples of rank $(2, 2)$. Here we prove the vanishing of \mathbb{H}^2 for *any* critical value σ_c under the condition $\sigma_m = \mu_1 - \mu_2 > g - 1$, whereas in [5, Proposition 3.6] it is proved the vanishing of \mathbb{H}^2 only for critical values $\sigma_c > 2g - 2$ (but without condition in σ_m).

7. HODGE POLYNOMIAL OF THE MODULI OF TRIPLES OF RANK $(2, 2)$ AND SMALL σ

In this section we want to compute the Hodge polynomial of the moduli space

$$\mathcal{N}_{\sigma_m^\pm} = \mathcal{N}_{\sigma_m^\pm}(2, 2, d_1, d_2)$$

of σ -stable triples of types $(2, 2, d_1, d_2)$ for σ small, under the assumption $\mu_1 - \mu_2 > 2g - 2$. The study of $\mathcal{N}_{\sigma_m^+}$ is simpler when both d_1 and d_2 are odd, since in this case the bundles are automatically stable. However in this case $d_1 + d_2$ is even and hence $\gcd(2, 2, d_1 + d_2) \neq 1$. So there may be strictly σ -semistable triples in \mathcal{N}_σ for non-critical values of σ , making the moduli space \mathcal{N}_σ^s non-compact and the moduli space \mathcal{N}_σ singular (this does not happen for $\sigma = \sigma_m^+$; see Theorem 7.1).

Theorem 7.1. *Suppose that d_1 and d_2 are odd and that $\mu_1 - \mu_2 > 2g - 2$. Then $\mathcal{N}_{\sigma_m^+} = \mathcal{N}_{\sigma_m^+}^s$, it is smooth, compact and*

$$e(\mathcal{N}_{\sigma_m^+}) = \left(\frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^{2g}(1+v)^{2g}}{(1-uv)(1-(uv)^2)} \right)^2 \frac{1 - (uv)^{2d_1-2d_2-4g+4}}{1-uv}.$$

Proof. The equality $\mathcal{N}_{\sigma_m^+} = \mathcal{N}_{\sigma_m^+}^s$ is a consequence of Proposition 3.13 (i). Next, since $\sigma_m = \mu_1 - \mu_2 > 2g - 2$, Theorem 6.1 implies that the moduli $\mathcal{N}_{\sigma_m^+}$ is smooth and compact. By Proposition 3.13 (i), it is the projectivization of a fiber bundle over $M(2, d_1) \times M(2, d_2)$ of rank $2d_1 - 2d_2 - 4g + 4$. Therefore

$$e(\mathcal{N}_{\sigma_m^+}) = e(M(2, d_1))e(M(2, d_2))e_{2d_1-2d_2-4g+4}.$$

The result follows now applying Theorem 5.1. \square

The case where d_1 is odd and d_2 is even is more involved, since we have to deal with the presence of strictly semistable bundles in $M(2, d_2)$.

Theorem 7.2. *Suppose that d_1 is odd and d_2 is even and that $\mu_1 - \mu_2 > 2g - 2$. Then $\mathcal{N}_{\sigma_m^+} = \mathcal{N}_{\sigma_m^+}^s$, it is smooth and compact and*

$$e(\mathcal{N}_{\sigma_m^+}) = \frac{(1+u)^{2g}(1+v)^{2g}(1-(uv)^N)(u^g v^g(1+u)^g(1+v)^g - (1+u^2v)^g(1+uv^2)^g)}{(1-uv)^3(1-(uv)^2)^2} \cdot \left((1+u)^g(1+v)^g(u^{g+1}v^{g+1} + u^{N+g-1}v^{N+g-1}) - (1+u^2v)^g(1+uv^2)^g(1+u^N v^N) \right),$$

where $N = d_1 - d_2 - 2g + 2$.

Proof. As $d_1 + d_2$ is odd, Theorem 6.1 implies that $\mathcal{N}_{\sigma_m^+} = \mathcal{N}_{\sigma_m^+}^s$, and it is smooth and compact, since $\sigma_m = \mu_1 - \mu_2 > 2g - 2$. To compute $e(\mathcal{N}_{\sigma_m^+})$ we decompose $\mathcal{N}_{\sigma_m^+} = X_0 \sqcup X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4$, where:

- (1) The open subset $X_0 \subset \mathcal{N}_{\sigma_m^+}$ consists of those triples of the form $\phi : E_2 \rightarrow E_1$, where E_1 and E_2 are both stable bundles, and ϕ is a non-zero map defined up to multiplication by scalars. By Proposition 3.13 (ii), $X_0 \rightarrow M(2, d_1) \times M^s(2, d_2)$ is a projective fibration whose fibers are projective spaces of dimension $2d_1 - 2d_2 - 4g + 4 - 1 = 2N - 1$. Therefore, and using the notation (2.2),

$$e(X_0) = e(M(2, d_1))e(M^s(2, d_2))e_{2N}.$$

- (2) The subset X_1 parametrizes σ_m^+ -stable triples of the form $\phi : E_2 \rightarrow E_1$ where E_2 is a strictly semistable bundle of degree d_2 which is a non-split extension

$$0 \rightarrow L_1 \rightarrow E_2 \rightarrow L_2 \rightarrow 0,$$

where $L_1, L_2 \in \text{Jac}^{d_2/2} X$ are non-isomorphic and E_1 is a stable bundle. The space Y_1 parametrizing such bundles E_2 was described in (2) of the proof of Theorem 5.2 and its Hodge polynomial is given in (5.2).

Now in order to describe X_1 , we must characterize when a triple $T = (E_1, E_2, \phi)$, with $E_2 \in Y_1$, is σ_m^+ -stable. As T is σ_m -semistable, then the only possibility for T being σ_m^+ -unstable is that it has a subtriple T' of rank (1, 2) or (0, 1), corresponding to Cases (2) or (4) of Section 6, respectively. If T' is of rank (1, 2), then it is of the form $E_2 \rightarrow L$, where L is a line bundle of degree $d_L = \mu_1$, by Proposition 6.2. But this is impossible, since d_1 is odd. If T' is of rank (0, 1), then it is of the form $F \rightarrow 0$, where F is a line bundle of degree $d_F = \mu_2$, by Proposition 6.2. Therefore F is a destabilizing subbundle for E_2 . Since the only destabilizing subbundle of E_2 is L_1 , we have $F = L_1$. So it must be $\phi(L_1) = 0$. Any such ϕ lies in the image of the inclusion $\text{Hom}(L_2, E_1) \hookrightarrow \text{Hom}(E_2, E_1)$, under the natural projection $E_2 \rightarrow L_2$. This discussion implies that given $(E_1, E_2) \in M(2, d_1) \times Y_1$, the morphisms ϕ giving rise to σ_m^+ -stable triples (E_1, E_2, ϕ) are those in

$$\text{Hom}(E_2, E_1) - \text{Hom}(L_2, E_1).$$

Note that since the group of automorphisms of E_1 and E_2 are both equal to \mathbb{C}^* , ϕ is defined up to multiplication by non-zero scalars. So the map $\pi : X_1 \rightarrow M(2, d_1) \times Y_1$ is a fibration with fiber over (E_1, E_2) equal to

$$\mathbb{P} \operatorname{Hom}(E_2, E_1) - \mathbb{P} \operatorname{Hom}(L_2, E_1) . \quad (7.1)$$

By Riemann-Roch, $\dim \operatorname{Hom}(E_2, E_1) = 2d_1 - 2d_2 - 4g + 4 = 2N$, since $\mu_1 - \mu_2 > 2g - 2$ implies that $H^1(E_2^* \otimes E_1) = H^0(E_1 \otimes E_2^* \otimes K) = 0$, E_1 and E_2 being both semistable bundles. Also $\dim \operatorname{Hom}(L_2, E_1) = d_1 - 2(d_2/2) - 2g + 2 = d_1 - d_2 - 2g + 2 = N$, since $\mu_1 - \deg L_2 = \mu_1 - d_2/2 > 2g - 2$. Hence (7.1) is isomorphic to $\mathbb{P}^{2N-1} - \mathbb{P}^{N-1}$. Therefore as in (2) of the proof of Theorem 5.2,

$$\begin{aligned} e(X_1) &= e(M(2, d_1))e(Y_1)(e_{2N} - e_N) \\ &= e(M(2, d_1))e(\operatorname{Jac} X)(e(\operatorname{Jac} X) - 1)e_{g-1}(e_{2N} - e_N) . \end{aligned}$$

- (3) The subset X_2 parametrizes σ_m^+ -stable triples of the form $\phi : E_2 \rightarrow E_1$ where E_2 is a strictly semistable bundle of degree d_2 which is non-split extension

$$0 \rightarrow L_1 \rightarrow E_2 \rightarrow L_1 \rightarrow 0,$$

where $L_1 \in \operatorname{Jac}^{d_2/2} X$ and E_1 is a stable bundle. The space Y_2 parametrizing such bundles E_2 was described in (3) of the proof of Theorem 5.2 and its Hodge polynomial is given in (5.4).

To describe X_2 , we must characterize when a triple $T = (E_1, E_2, \phi)$, with $E_2 \in Y_2$, is σ_m^+ -stable. As before, given $(E_1, E_2) \in M(2, d_1) \times Y_2$, the morphisms ϕ giving rise to σ_m^+ -stable triples (E_1, E_2, ϕ) are those in

$$\operatorname{Hom}(E_2, E_1) - \operatorname{Hom}(L_1, E_1) .$$

For a triple $T = (E_1, E_2, \phi) \in X_2$, $\operatorname{Aut}(E_1) = \mathbb{C}^*$, so $\operatorname{Aut}(T) \cong \operatorname{Aut}(E_2) = \mathbb{C} \times \mathbb{C}^*$. There is an exact sequence

$$0 \rightarrow \operatorname{Hom}(L_1, E_1) \rightarrow \operatorname{Hom}(E_2, E_1) \rightarrow \operatorname{Hom}(L_1, E_1) \rightarrow 0$$

Under the (non-canonical) decomposition $\operatorname{Hom}(E_2, E_1) \cong \operatorname{Hom}(L_1, E_1) \oplus \operatorname{Hom}(L_1, E_1)$, $\operatorname{Aut}(E_2)$ acts as $(a, \lambda)(x, y) \mapsto (\lambda x + ay, \lambda y)$. So the fiber of $\pi : X_2 \rightarrow M(2, d_1) \times Y_2$ is

$$(\operatorname{Hom}(E_2, E_1) - \operatorname{Hom}(L_1, E_1)) / \mathbb{C} \times \mathbb{C}^* \cong (\mathbb{C}^{2N} - \mathbb{C}^N) / \mathbb{C} \times \mathbb{C}^* ,$$

which is a \mathbb{C}^{N-1} -bundle over \mathbb{P}^{N-1} . Therefore as in (3) of the proof of Theorem 5.2,

$$\begin{aligned} e(X_2) &= e(M(2, d_1))e(Y_2)(e_N - e_{N-1})e_N \\ &= e(M(2, d_1))e(\operatorname{Jac} X)e_g(e_N - e_{N-1})e_N . \end{aligned}$$

- (4) The subset X_3 parametrizes σ_m^+ -stable triples of the form $\phi : E_2 \rightarrow E_1$ where E_1 is a stable bundle and $E_2 = L_1 \oplus L_2$, $L_1 \not\cong L_2$, are two line bundles of degree $d_2/2$. The space Y_3 parametrizing such bundles is described in (5.7).

As above, the condition for $\phi \in \operatorname{Hom}(E_2, E_1)$ to give rise to a σ_m^+ -unstable triple is that there is a subtriple T' of the form $F \rightarrow 0$, with F a line bundle of degree $d_F = \mu_2$. Then it must be either $F = L_1$ or $F = L_2$. This means that $\phi \in (\operatorname{Hom}(L_1, E_1) \oplus \{0\}) \cup (\{0\} \oplus \operatorname{Hom}(L_2, E_1)) \subset \operatorname{Hom}(E_2, E_1)$. Therefore, given $(E_1, E_2) \in M(2, d_1) \times Y_3$, the morphisms ϕ giving rise to σ_m^+ -stable triples (E_1, E_2, ϕ) are those in

$$(\operatorname{Hom}(L_1, E_1) - \{0\}) \times (\operatorname{Hom}(L_2, E_1) - \{0\}) .$$

The group of automorphisms of E_2 is $\mathbb{C}^* \times \mathbb{C}^*$ acting on $L_1 \oplus L_2$ by diagonal matrices. Therefore $\phi \in (\operatorname{Hom}(L_1, E_1) - \{0\}) \times (\operatorname{Hom}(L_2, E_1) - \{0\})$ is defined up to the action of $\mathbb{C}^* \times \mathbb{C}^*$, where each \mathbb{C}^* acts by multiplication on each of the two summands. So the map $\pi : X_3 \rightarrow M(2, d_1) \times Y_3$ has fiber

$$\mathbb{P} \operatorname{Hom}(L_1, E_1) \times \mathbb{P} \operatorname{Hom}(L_2, E_1) . \quad (7.2)$$

By Riemann-Roch, $\dim \operatorname{Hom}(L_1, E_1) = \dim \operatorname{Hom}(L_2, E_1) = d_1 - d_2 - 2g + 2$. Therefore (7.2) is isomorphic $\mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$. To compute $e(X_3)$ we work as in (4) of the proof of Theorem 5.2. Write $X_3 = \tilde{X}_3 / \mathbb{Z}_2 = X'_3 / \mathbb{Z}_2 - X''_3 / \mathbb{Z}_2$, where X'_3 is a fibration over $M(2, d_1)$ with fiber $(A_{E_1} \times A_{E_1}) / \mathbb{Z}_2$, where A_{E_1} is a projective bundle over $\operatorname{Jac}^{d_2/2} X$ with fibers $\mathbb{P} \operatorname{Hom}(L, E_1) \cong$

\mathbb{P}^{N-1} , and \mathbb{Z}_2 acts by permutation. X_3'' is a fibration over $M(2, d_1) \times \text{Jac}^{d_2/2} X$ with fibers $(\mathbb{P}^{N-1} \times \mathbb{P}^{N-1})/\mathbb{Z}_2$. So using Theorem 2.2,

$$\begin{aligned} e(X_3) &= e(\tilde{X}_3/\mathbb{Z}_2) = e(X_3'/\mathbb{Z}_2) - e(X_3''/\mathbb{Z}_2) \\ &= \frac{1}{2}e(M(2, d_1)) \left(\left(e(\text{Jac } X)^2 e_N^2 + (1-u^2)^g (1-v^2)^g \frac{1-(uv)^{2N}}{1-u^2v^2} \right) \right. \\ &\quad \left. - e(\text{Jac } X) \left(e_N^2 + \frac{1-(uv)^{2N}}{1-u^2v^2} \right) \right). \end{aligned}$$

- (5) The subset X_4 parametrizes triples $\phi : E_2 \rightarrow E_1$, where E_1 is a stable bundle and $E_2 = L_1 \oplus L_1$, $L_1 \in \text{Jac}^{d_2/2} X$. Such bundles E_2 are parametrized by $Y_4 = \text{Jac}^{d_2/2} X$. The map ϕ lies in

$$\text{Hom}(E_2, E_1) = \text{Hom}(L_1, E_1) \oplus \text{Hom}(L_1, E_1) \cong \text{Hom}(L_1, E_1) \otimes \mathbb{C}^2. \quad (7.3)$$

The condition for a triple $T = (E_1, E_2, \phi)$ to be σ_m^+ -unstable is that there is a line subbundle $F \subset E_2$ of degree $d_F = \mu_2$ such that $\phi(F) = 0$. A destabilizing subbundle of E_2 is necessarily isomorphic to L and there exists $(a, b) \neq (0, 0)$ such that $F \cong L \hookrightarrow E_2$ is given by $x \mapsto (ax, bx)$. So $\phi = (a\psi, b\psi) \in \text{Hom}(L_1, E_1) \otimes \mathbb{C}^2$, for some $\psi \in \text{Hom}(L, E_1)$. Therefore $T = (E_1, E_2, \phi)$ is σ_m^+ -stable if $\phi = (\phi_1, \phi_2) \in \text{Hom}(L_1, E_1) \otimes \mathbb{C}^2$ satisfies that ϕ_1, ϕ_2 are linearly independent.

On the other hand, a triple $(E_1, E_2, \phi) \in X_4$ is determined up to the action of $\text{Aut}(E_2) = \text{GL}(2, \mathbb{C})$. This acts on (7.3) via the standard representation on \mathbb{C}^2 . Thus for $(E_1, E_2) \in M(2, d_1) \times Y_4$, the morphisms ϕ giving rise to σ_m^+ -stable triples (E_1, E_2, ϕ) are parametrized by $\text{Gr}(2, \text{Hom}(L_1, E_1))$. But $\dim \text{Hom}(L_1, E_1) = d_1 - d_2 - 2g + 2 = N$, so this fiber is isomorphic to $\text{Gr}(2, N)$. So

$$e(X_4) = e(M(2, d_1))e(Y_4)e(\text{Gr}(2, N)) = e(M(2, d_1))e(\text{Jac } X)e(\text{Gr}(2, N)).$$

Adding up all contributions together we get

$$\begin{aligned} e(\mathcal{N}_{\sigma_m^+}) &= e(X_0) + e(X_1) + e(X_2) + e(X_3) + e(X_4) \\ &= e(M(2, d_1)) \left(e(M^s(2, d_2))e_{2N} + e(\text{Jac } X)(e(\text{Jac } X) - 1)e_{g-1}(e_{2N} - e_N) \right. \\ &\quad \left. + e(\text{Jac } X)e_g(e_N - e_{N-1})e_N + \frac{1}{2} \left(e(\text{Jac } X)^2 e_N^2 + (1-u^2)^g (1-v^2)^g \frac{1-(uv)^{2N}}{1-u^2v^2} \right) \right. \\ &\quad \left. - \frac{1}{2} e(\text{Jac } X) \left(e_N^2 + \frac{1-(uv)^{2N}}{1-u^2v^2} \right) + e(\text{Jac } X)e(\text{Gr}(2, N)) \right) \\ &= \frac{(1+u)^{2g}(1+v)^{2g}(1-(uv)^N)(u^g v^g (1+u)^g (1+v)^g - (1+u^2v)^g (1+uv^2)^g)}{(1-uv)^3(1-(uv)^2)^2} \\ &\quad \left((1+u)^g (1+v)^g (u^{g+1}v^{g+1} + u^{N+g-1}v^{N+g-1}) - (1+u^2v)^g (1+uv^2)^g (1+u^N v^N) \right). \end{aligned}$$

□

Corollary 7.3. *Suppose that d_1 is even and d_2 is odd and that $\mu_1 - \mu_2 > 2g - 2$. Then $\mathcal{N}_{\sigma_m^+} = \mathcal{N}_{\sigma_m^+}^s$, it is smooth and compact and its Hodge polynomial has the same formula as that of Theorem 7.2, where $N = d_1 - d_2 - 2g + 2$.*

Proof. We use the isomorphism $\mathcal{N}_\sigma(2, 2, d_1, d_2) \cong \mathcal{N}_\sigma(2, 2, -d_2, -d_1)$. Note that

$$d_1 - d_2 = (-d_2) - (-d_1),$$

so that the small value $\sigma_m^+ = \mu_1 - \mu_2$ and the condition on the slopes $\mu_1 - \mu_2 > 2g - 2$ is the same for both moduli spaces $\mathcal{N}_\sigma(2, 2, d_1, d_2)$ and $\mathcal{N}_\sigma(2, 2, -d_2, -d_1)$. Now we apply Theorem 7.2 to get the stated formula where $N = -d_2 - (-d_1) - 2g + 2$. □

Corollary 7.4. *Suppose that $d_1 + d_2$ is odd and $\mu_1 - \mu_2 > 2g - 2$. Then the Poincaré polynomial of $\mathcal{N}_{\sigma_m^+}$ is*

$$P_t(\mathcal{N}_{\sigma_m^+}) = \frac{(1+t)^{4g}(1-t^{2N})(t^{2g}(1+t)^{2g} - (1+t^3)^{2g})((1+t)^{2g}(t^{2g+2} + t^{2N+2g-2}) - (1+t^3)^{2g}(1+t^{2N}))}{(1-t^2)^3(1-t^4)^2},$$

where $N = d_1 - d_2 - 2g + 2$. \square

Proof. $\mathcal{N}_{\sigma_m^+}$ is smooth and projective, so $P_t(\mathcal{N}_{\sigma_m^+}) = e(\mathcal{N}_{\sigma_m^+})(t, t)$. The result follows from Theorem 7.2 and Corollary 7.3. \square

We could deal also with the case when d_1 and d_2 are both even and $d_1 - d_2 > 4g - 4$. This is similar to the case just treated in Theorem 7.2, with the further complication that there are semistable loci for both E_1 and E_2 .

However, dealing with the case $d_1 - d_2 \leq 4g - 4$ is more complicated, since Proposition 3.13 does not apply as there is a Brill-Noether problem consisting on determining the loci of those (E_1, E_2) where $\dim \text{Hom}(E_2, E_1 \otimes K)$ is constant.

8. CONTRIBUTION OF THE FLIPS TO THE HODGE POLYNOMIALS

In this section, we shall compute the change in the Hodge polynomial of $\mathcal{N}_\sigma(2, 2, d_1, d_2)$ when we cross a critical value σ_c . We restrict to the case $d_1 + d_2$ is odd, since in the case $d_1 + d_2$ even there may be strictly σ -semistable triples for non-critical values of σ (and in this case \mathcal{N}_σ^s is non-compact and \mathcal{N}_σ is non-smooth). For $d_1 + d_2$ odd, Theorem 6.1 guarantees that \mathcal{N}_σ is compact and smooth for any non-critical $\sigma \geq 2g - 2$. The critical values are given in Proposition 6.2. These are of two types. The following two propositions treat them separately.

Proposition 8.1. *Let $\sigma_c = 2d_L - \mu_1 - \mu_2$ be a critical value for triples of type $(2, 2, d_1, d_2)$ with $d_1 + d_2$ odd, such that $\sigma_c > \sigma_m$. Suppose that $\mu_1 - \mu_2 > g - 1$. Then*

$$e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-}) = \text{coeff}_{x^0} \left[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g((uv)^{g-1-d_1+2d_L} - (uv)^{1-g+d_1-d_2})}{(1-uv)^2(1-x)(1-uvx)x^{[3\mu_1-\mu_2]-2d_L}} \right. \\ \left. \left(\frac{(uv)^{(3d_1-d_2-1)/2-2d_L}}{1-(uv)^{-1}x} - \frac{(uv)^{2d_L-d_1+g}}{1-(uv)^2x} \right) \right].$$

Proof. Theorem 6.1 implies that $\mathcal{N}_{\sigma_c^\pm} = \mathcal{N}_{\sigma_c^\pm}^s$. Then Lemma 3.10 and the properties of the Hodge polynomials give

$$e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-}) = e(\mathcal{S}_{\sigma_c^+}) - e(\mathcal{S}_{\sigma_c^-}).$$

Let us start by studying $\mathcal{S}_{\sigma_c^+}$. By Lemma 3.11, any $T \in \mathcal{S}_{\sigma_c^+}$ sits in a non-split extension

$$0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0 \quad (8.1)$$

in which T' and T'' are σ_c -semistable, $\lambda' < \lambda$ and $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T) = \mu_{\sigma_c}(T'')$. Since T corresponds to Case (1) in Section 6, we have $T' \in \mathcal{N}'_{\sigma_c}$ and $T'' \in \mathcal{N}''_{\sigma_c}$, where

$$\mathcal{N}'_{\sigma_c} = \mathcal{N}_{\sigma_c}(1, 0, d_L, 0) \cong \text{Jac}^{d_L} X, \\ \mathcal{N}''_{\sigma_c} = \mathcal{N}_{\sigma_c}(1, 2, d_1 - d_L, d_2).$$

The moduli space of triples of rank $(1, 0)$ has no critical values; and for the moduli space of triples of rank $(1, 2)$, the critical values are of the form $3d_M + d'_1 + d'_2$, by Lemma 4.5, and are in particular integers. But $\sigma_c = 2d_L - \frac{d_1+d_2}{2} \notin \mathbb{Z}$, so σ_c is not a critical value for \mathcal{N}''_{σ_c} .

By [5, Proposition 3.5], $\mathbb{H}^0(T'', T') = 0$ and by Proposition 6.8 (2), $\mathbb{H}^2(T'', T') = 0$. So Theorem 3.12 implies that $\mathcal{S}_{\sigma_c^+}$ is the projectivization of a bundle over $\mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$ of rank

$$-\chi(T'', T') = 1 - g + d_1 - d_2.$$

Therefore

$$e(\mathcal{S}_{\sigma_c^+}) = e(\text{Jac}^{d_L} X) e(\mathcal{N}_{\sigma_c}(1, 2, d_1 - d_L, d_2)) e_{1-g+d_1-d_2}.$$

The case of $\mathcal{S}_{\sigma_c^-}$ is similar. Any $T \in \mathcal{S}_{\sigma_c^-}$ sits in an exact sequence (8.1) with $T' \in \mathcal{N}'_{\sigma_c}$ and $T'' \in \mathcal{N}''_{\sigma_c}$, where

$$\begin{aligned}\mathcal{N}'_{\sigma_c} &= \mathcal{N}_{\sigma_c}(1, 2, d_1 - d_L, d_2), \\ \mathcal{N}''_{\sigma_c} &= \mathcal{N}_{\sigma_c}(1, 0, d_L, 0) \cong \text{Jac}^{d_L} X,\end{aligned}$$

corresponding to the Case (2) in Section 6. The hypothesis of Theorem 3.12 are satisfied and so $\mathcal{S}_{\sigma_c^-}$ is the projectivization of a bundle over $\mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$ of rank

$$-\chi(T'', T') = g - 1 - d_1 + 2d_L.$$

Therefore

$$e(\mathcal{S}_{\sigma_c^-}) = e(\text{Jac}^{d_L} X) e(\mathcal{N}_{\sigma_c}(1, 2, d_1 - d_L, d_2)) e_{g-1-d_1+2d_L}.$$

Subtracting, we get

$$\begin{aligned}e(\mathcal{S}_{\sigma_c^+}) - e(\mathcal{S}_{\sigma_c^-}) &= (e_{1-g+d_1-d_2} - e_{g-1-d_1+2d_L})(1+u)^g(1+v)^g e(\mathcal{N}_{\sigma_c}(1, 2, d_1 - d_L, d_2)) = \\ &= \frac{(uv)^{g-1-d_1+2d_L} - (uv)^{1-g+d_1-d_2}}{1-uv} (1+u)^g(1+v)^g e(\mathcal{N}_{\sigma_c}(1, 2, d_1 - d_L, d_2)).\end{aligned}$$

Being σ_c a non-critical value for the moduli of triples of rank (1, 2), we can apply Theorem 4.6 to compute the Hodge polynomial of $\mathcal{N}_{\sigma_c}(1, 2, d_1 - d_L, d_2)$. First,

$$\begin{aligned}d_0 &= \left\lceil \frac{1}{3}(2d_L - \mu_1 - \mu_2 - (d_1 - d_L) - d_2) \right\rceil + 1 \\ &= d_L + [-\mu_1 - \mu_2] + 1.\end{aligned}$$

So $e(\mathcal{N}_{\sigma_c}(1, 2, d_1 - d_L, d_2))$ equals

$$\text{coeff}_{x^0} \left[\frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2-d_L-d_0}} \left(\frac{(uv)^{d_1-d_2-d_L-d_0}}{1-(uv)^{-1}x} - \frac{(uv)^{d_2+g-1+2d_0}}{1-(uv)^2x} \right) \right],$$

where $d_1 - d_2 - d_L - d_0 = [3\mu_1 - \mu_2] - 2d_L = (3d_1 - d_2 - 1)/2 - 2d_L$ and $d_2 + 2d_0 = 2d_L - d_1 + 1$. The result follows from this. \square

Proposition 8.2. *Let $\sigma_c = \mu_1 + \mu_2 - 2d_F$ be a critical value for triples of type $(2, 2, d_1, d_2)$ with $d_1 + d_2$ odd, such that $\sigma_c > \sigma_m$. Suppose that $\mu_1 - \mu_2 > g - 1$. Then*

$$\begin{aligned}e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-}) &= \text{coeff}_{x^0} \left[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g ((uv)^{g-1+d_2-2d_F} - (uv)^{1-g+d_1-d_2})}{(1-uv)^2(1-x)(1-uvx)x^{2d_F-[3\mu_2-\mu_1]-1}} \right. \\ &\quad \left. \left(\frac{(uv)^{2d_F+(d_1-3d_2-1)/2}}{1-(uv)^{-1}x} - \frac{(uv)^{d_2-2d_F+g}}{1-(uv)^2x} \right) \right].\end{aligned}$$

Proof. This is very similar to the proof of Proposition 8.1. Again

$$e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-}) = e(\mathcal{S}_{\sigma_c^+}) - e(\mathcal{S}_{\sigma_c^-}).$$

We start with $\mathcal{S}_{\sigma_c^+}$. Any $T \in \mathcal{S}_{\sigma_c^+}$ sits in a non-split extension like (8.1), with $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T) = \mu_{\sigma_c}(T'')$, $T' \in \mathcal{N}'_{\sigma_c}$ and $T'' \in \mathcal{N}''_{\sigma_c}$, where

$$\begin{aligned}\mathcal{N}'_{\sigma_c} &= \mathcal{N}_{\sigma_c}(2, 1, d_1, d_2 - d_F), \\ \mathcal{N}''_{\sigma_c} &= \mathcal{N}_{\sigma_c}(0, 1, 0, d_F) \cong \text{Jac}^{d_F} X,\end{aligned}$$

corresponding to the Case (3) in Section 6. The moduli space of triples of rank (0, 1) has no critical values; and for the moduli space of triples of rank (2, 1), the critical values are of the form $3d_M - d'_1 - d'_2 \in \mathbb{Z}$, whilst $\sigma_c = \frac{d_1+d_2}{2} - 2d_F \notin \mathbb{Z}$, so σ_c is not a critical value for \mathcal{N}'_{σ_c} . The other conditions of Theorem 3.12 are checked as before. So $\mathcal{S}_{\sigma_c^+}$ is the projectivization of a bundle over $\mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$ of rank

$$-\chi(T'', T') = 1 - g + d_1 - d_2.$$

Therefore

$$e(\mathcal{S}_{\sigma_c^+}) = e(\text{Jac}^{d_F} X) e(\mathcal{N}_{\sigma_c}(2, 1, d_1, d_2 - d_F)) e_{1-g+d_1-d_2}.$$

Moving to $\mathcal{S}_{\sigma_c^-}$, any $T \in \mathcal{S}_{\sigma_c^-}$ sits in an exact sequence (8.1) with $T' \in \mathcal{N}'_{\sigma_c}$ and $T'' \in \mathcal{N}''_{\sigma_c}$, where

$$\begin{aligned}\mathcal{N}'_{\sigma_c} &= \mathcal{N}_{\sigma_c}(0, 1, 0, d_F) \cong \text{Jac}^{d_F} X, \\ \mathcal{N}''_{\sigma_c} &= \mathcal{N}_{\sigma_c}(2, 1, d_1, d_2 - d_F),\end{aligned}$$

corresponding to the Case (4) in Section 6. Arguing as before, we have that $\mathcal{S}_{\sigma_c^-}$ is the projectivization of a bundle over $\mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$ of rank

$$-\chi(T'', T') = g - 1 + d_2 - 2d_F.$$

Therefore

$$e(\mathcal{S}_{\sigma_c^-}) = e(\text{Jac}^{d_F} X) e(\mathcal{N}_{\sigma_c}(2, 1, d_1, d_2 - d_F)) e_{g-1+d_2-2d_F}.$$

Subtracting, we get

$$\begin{aligned}e(\mathcal{S}_{\sigma_c^+}) - e(\mathcal{S}_{\sigma_c^-}) &= (e_{1-g+d_1-d_2} - e_{g-1+d_2-2d_F})(1+u)^g(1+v)^g e(\mathcal{N}_{\sigma_c}(2, 1, d_1, d_2 - d_F)) = \\ &= \frac{(uv)^{g-1+d_2-2d_F} - t^{1-g+d_1-d_2}}{1-uv} (1+u)^g(1+v)^g e(\mathcal{N}_{\sigma_c}(2, 1, d_1, d_2 - d_F)).\end{aligned}$$

Being σ_c a non-critical value for the moduli of triples of rank $(2, 1)$, we can apply Theorem 4.3 to compute the Hodge polynomial of $\mathcal{N}_{\sigma_c}(2, 1, d_1, d_2 - d_F)$. First,

$$\begin{aligned}d_0 &= \left\lceil \frac{1}{3}(\mu_1 + \mu_2 - 2d_F + d_1 + d_2 - d_F) \right\rceil + 1 \\ &= \lceil \mu_1 + \mu_2 \rceil - d_F + 1\end{aligned}$$

So $e(\mathcal{N}_{\sigma_c}(2, 1, d_1, d_2 - d_F))$ equals

$$\text{coeff}_{x^0} \left[\frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2+d_F-d_0}} \left(\frac{(uv)^{d_1-d_2+d_F-d_0}}{1-(uv)^{-1}x} - \frac{(uv)^{-d_1+g-1+2d_0}}{1-(uv)^2x} \right) \right],$$

where $d_1 - d_2 + d_F - d_0 = 2d_F - [3\mu_2 - \mu_1] - 1 = 2d_F + (d_1 - 3d_2 - 1)/2$ and $-d_1 + 2d_0 = d_2 - 2d_F + 1$. The result follows from this. \square

We gather together Propositions 8.1 and 8.2 in a single result.

Corollary 8.3. *The critical values $\sigma_c > \sigma_m$ for triples of type $(2, 2, d_1, d_2)$ with $d_1 + d_2$ odd are of the form $\sigma_c = \mu_1 - \mu_2 + n$, $1 \leq n \leq [\mu_1 - \mu_2]$, $n \in \mathbb{Z}$. Suppose that $\mu_1 - \mu_2 > g - 1$. Then*

$$\begin{aligned}e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-}) &= \text{coeff}_{x^0} \left[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g ((uv)^{g-1+n} - (uv)^{1-g+d_1-d_2})}{(1-uv)^2(1-x)(1-uvx)x^{[\mu_1-\mu_2]-n}} \right. \\ &\quad \left. \left(\frac{(uv)^{(d_1-d_2-1)/2-n}}{1-(uv)^{-1}x} - \frac{(uv)^{g+n}}{1-(uv)^2x} \right) \right].\end{aligned}$$

Proof. For simplicity let us assume that d_1 is odd and d_2 is even (the other case is analogous). We have the following possibilities:

- (a) If $\sigma_c = 2d_L - \mu_1 - \mu_2$, write $d_L = \mu_1 + \frac{1}{2} + m$ with m integer. Then $\sigma_c = \mu_1 - \mu_2 + 2m + 1$. As $\mu_1 < d_L \leq \frac{3\mu_1 - \mu_2}{2}$ by Proposition 6.2 (i), we have $0 \leq m \leq (\mu_1 - \mu_2 - 1)/2$. Substituting the values $3d_1 - d_2 - 1 - 4d_L = d_1 - d_2 - 1 - 4m - 2$, $2d_L - d_1 + g = g + 2m + 1$, $[3\mu_1 - \mu_2] - 2d_L = [\mu_1 - \mu_2] - 2m - 1$ and $g - 1 - d_1 + 2d_L = g + 2m$ into the formula of Proposition 8.1, one gets

$$\begin{aligned}e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-}) &= \text{coeff}_{x^0} \left[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g ((uv)^{g+2m} - (uv)^{1-g+d_1-d_2})}{(1-uv)^2(1-x)(1-uvx)(1-(uv)^{-1}x)x^{[\mu_1-\mu_2]-2m-1}} \right. \\ &\quad \left. \left(\frac{(uv)^{(d_1-d_2-1)/2-2m-1}}{1-(uv)^{-1}x} - \frac{(uv)^{g+2m+1}}{1-(uv)^2x} \right) \right].\end{aligned}$$

- (b) If $\sigma_c = \mu_1 + \mu_2 - 2d_F$, write $d_F = \mu_2 - m - 1$ with m an integer. Then $\sigma_c = \mu_1 - \mu_2 + 2m + 2$. As $\frac{3\mu_2 - \mu_1}{2} \leq d_F < \mu_2$ by Proposition 6.2 (i), we have $0 \leq m \leq (\mu_1 - \mu_2)/2 - 1$. Substituting the values $4d_F + d_1 - 3d_2 - 1 = d_1 - d_2 - 1 - 4m - 4$, $d_2 - 2d_F + g = g + 2m + 2$, $2d_F - [3\mu_2 - \mu_1] - 1 = [\mu_1 - \mu_2] - 2m - 2$ and $g - 1 + d_2 - 2d_F = g + 2m + 1$ into the formula of Proposition 8.2, we have

$$e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-}) = \text{coeff}_{x^0} \left[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g((uv)^{g+2m+1} - (uv)^{1-g+d_1-d_2})}{(1-uv)^2(1-x)(1-uvx)(1-(uv)^{-1}x)x^{[\mu_1-\mu_2]-2m-2}} \right. \\ \left. \left(\frac{(uv)^{(d_1-d_2-1)/2-2m-2}}{1-(uv)^{-1}x} - \frac{(uv)^{g+2m+2}}{1-(uv)^2x} \right) \right].$$

Case (a) corresponds to $n = 2m + 1$ odd, and Case (b) to $n = 2m + 2$ even in the formula in the statement. The range for n is $1 \leq n \leq \mu_1 - \mu_2$. But, since $\mu_1 - \mu_2$ is not an integer, this range is actually $1 \leq n \leq [\mu_1 - \mu_2]$. \square

9. HODGE POLYNOMIAL OF THE MODULI OF TRIPLES OF RANK $(2, 2)$ AND LARGE σ

Now we use all the information in Sections 6–8 to compute the Hodge polynomial of the $\mathcal{N}_\sigma(2, 2, d_1, d_2)$, for any non-critical $\sigma > \sigma_m$. Recall that by Theorem 6.3, there is a value $\sigma_M = 2(\mu_1 - \mu_2)$ such that for $\sigma > \sigma_M$ all the moduli spaces \mathcal{N}_σ are isomorphic. We refer to

$$\mathcal{N}_{\sigma_M^+} = \mathcal{N}_{\sigma_M^+}(2, 2, d_1, d_2)$$

as the *large* σ moduli space.

Proposition 9.1. *Suppose that d_1 is even and d_2 is odd and that $\mu_1 - \mu_2 > g - 1$. Let $\sigma > \sigma_m$ be a non-critical value. Set $n_0 = \min\{\sigma - \mu_1 + \mu_2, [\mu_1 - \mu_2]\}$. Then*

$$e(\mathcal{N}_\sigma) - e(\mathcal{N}_{\sigma_m^+}) = \text{coeff}_{x^0} \left[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g}{(1-uv)^2(1-x)(1-uvx)x^{[\mu_1-\mu_2]}} \right. \\ \left(\frac{(uv)^{g-1+(d_1-d_2-1)/2}x(1-x^{n_0})}{(1-(uv)^{-1}x)(1-x)} - \frac{(uv)^{(3d_1-3d_2-1)/2-g}x(1-(uv)^{-n_0}x^{n_0})}{(1-(uv)^{-1}x)^2} \right. \\ \left. \left. - \frac{(uv)^{2g+1}x(1-(uv)^{2n_0}x^{n_0})}{(1-(uv)^2x)^2} + \frac{(uv)^{d_1-d_2+2}x(1-(uv)^{n_0}x^{n_0})}{(1-(uv)^2x)(1-uvx)} \right) \right].$$

Proof. By Corollary 8.3, the critical values are of the form $\sigma_c = \mu_1 - \mu_2 + n$ with $1 \leq n \leq [\mu_1 - \mu_2]$. Now $\sigma_m < \sigma_c < \sigma$ is equivalent to $n \leq [\sigma - \mu_1 + \mu_2]$ (note that $\sigma - \mu_1 + \mu_2 \notin \mathbb{Z}$ since σ is not critical). Therefore,

$$e(\mathcal{N}_\sigma) - e(\mathcal{N}_{\sigma_m^+}) = \sum_{\sigma_m < \sigma_c < \sigma} e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-}) = \\ = \sum_{n=1}^{n_0} \text{coeff}_{x^0} \left[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g((uv)^{g-1+n} - (uv)^{1-g+d_1-d_2})}{(1-uv)^2(1-x)(1-uvx)x^{[\mu_1-\mu_2]-n}} \right. \\ \left. \left(\frac{(uv)^{(d_1-d_2-1)/2-n}}{1-(uv)^{-1}x} - \frac{(uv)^{g+n}}{1-(uv)^2x} \right) \right] = \\ = \text{coeff}_{x^0} \left[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g}{(1-uv)^2(1-x)(1-uvx)x^{[\mu_1-\mu_2]}} \right. \\ \left(\frac{1}{1-(uv)^{-1}x} \sum_{n=1}^{n_0} (uv)^{g-1+(d_1-d_2-1)/2}x^n - \frac{1}{1-(uv)^{-1}x} \sum_{n=1}^{n_0} (uv)^{1-g+(3d_1-3d_2-1)/2-n}x^n \right. \\ \left. \left. - \frac{1}{1-(uv)^2x} \sum_{n=1}^{n_0} (uv)^{2g-1+2n}x^n + \frac{1}{1-(uv)^2x} \sum_{n=1}^{n_0} (uv)^{1+d_1-d_2+n}x^n \right) \right] =$$

$$\begin{aligned}
&= \text{coeff}_{x^0} \left[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g}{(1-uv)^2(1-x)(1-uvx)x^{[\mu_1-\mu_2]}} \right. \\
&\quad \left(\frac{(uv)^{g-1+(d_1-d_2-1)/2}x(1-x^{n_0})}{(1-(uv)^{-1}x)(1-x)} - \frac{(uv)^{1-g+(3d_1-3d_2-1)/2-1}x(1-(uv)^{-n_0}x^{n_0})}{(1-(uv)^{-1}x)^2} \right. \\
&\quad \left. \left. - \frac{(uv)^{2g-1+2}x(1-(uv)^{2n_0}x^{n_0})}{(1-(uv)^2x)^2} + \frac{(uv)^{1+d_1-d_2+1}x(1-(uv)^{n_0}x^{n_0})}{(1-(uv)^2x)(1-uvx)} \right) \right].
\end{aligned}$$

□

Theorem 9.2. *Suppose that d_1 is odd and d_2 is even. Then the large σ moduli space $\mathcal{N}_{\sigma_M^+} = \mathcal{N}_{\sigma_M^+}^s$ is smooth and compact. If $\mu_1 - \mu_2 > 2g - 2$, its Hodge polynomial is*

$$\begin{aligned}
e(\mathcal{N}_{\sigma_M^+}) &= \frac{(1+u)^{2g}(1+v)^{2g}}{(1-uv)^3(1-(uv)^2)^2} \left[(1+u^2v)^{2g}(1+uv^2)^{2g}(1-(uv)^{2N}) \right. \\
&\quad - N(1+u^2v)^g(1+uv^2)^g(1+u)^g(1+v)^g(uv)^{N+g-1}(1-(uv)^2) \\
&\quad + (1+u)^{2g}(1+v)^{2g}(1+uv)^2(uv)^{2g-2+(N+1)/2} \left((1-(uv)^{N+1}) - \frac{N+1}{2}(1-uv)(1+(uv)^N) \right) \\
&\quad \left. - g(1+u)^{2g-1}(1+v)^{2g-1}(1-(uv)^2)^2(uv)^{2g-2+(N+1)/2}(1-(uv)^N) \right],
\end{aligned}$$

where $N = d_1 - d_2 - 2g + 2$.

Proof. The first statement follows from Theorem 6.1. To compute $e(\mathcal{N}_{\sigma_M^+}) - e(\mathcal{N}_{\sigma_m^+})$ we use Proposition 9.1 for $\sigma = \sigma_M^+$. Note that in this case $n_0 = [\mu_1 - \mu_2]$. All the terms in the formula of Proposition 9.1 involving x^{n_0} yield positive powers of x , so they can be disregarded for computing coeff_{x^0} . Hence

$$\begin{aligned}
e(\mathcal{N}_{\sigma_M^+}) &= e(\mathcal{N}_{\sigma_m^+}) + \text{coeff}_{x^0} \left[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g}{(1-uv)^2(1-x)(1-uvx)x^{[\mu_1-\mu_2]}} \right. \\
&\quad \left. \left(\frac{(uv)^{g-1+(d_1-d_2-1)/2}x}{(1-(uv)^{-1}x)(1-x)} - \frac{(uv)^{(3d_1-3d_2-1)/2-g}x}{(1-(uv)^{-1}x)^2} - \frac{(uv)^{2g+1}x}{(1-(uv)^2x)^2} + \frac{(uv)^{d_1-d_2+2}x}{(1-(uv)^2x)(1-uvx)} \right) \right].
\end{aligned}$$

As $\mu_1 - \mu_2 > 2g - 2$, let $m \geq 0$ such that $[\mu_1 - \mu_2] = 2g - 2 + m$. Introduce the following function

$$F(a, b, c) = \text{coeff}_{x^0} \left(\frac{(1+ux)^g(1+vx)^g x^{3-2g-m}}{(1-ax)^2(1-bx)(1-cx)} \right) = \text{Res}_{x=0} \left(\frac{(1+ux)^g(1+vx)^g x^{2-2g-m}}{(1-ax)^2(1-bx)(1-cx)} \right),$$

where $a, b, c \neq 0$. So

$$\begin{aligned}
e(\mathcal{N}_{\sigma_M^+}) &= e(\mathcal{N}_{\sigma_m^+}) + \frac{(1+u)^{3g}(1+v)^{3g}}{(1-uv)^2} \left((uv)^{3g-3+m} F(1, uv, (uv)^{-1}) - (uv)^{5g-5+3m} F((uv)^{-1}, 1, uv) \right. \\
&\quad \left. - (uv)^{2g+1} F((uv)^2, 1, uv) + (uv)^{4g-1+2m} F(uv, 1, (uv)^2) \right)
\end{aligned} \tag{9.1}$$

using $d_1 - d_2 = 4g - 3 + 2m$.

The function

$$G(x) = \frac{(1+ux)^g(1+vx)^g x^{2-2g-m}}{(1-ax)^2(1-bx)(1-cx)}$$

is a meromorphic function on $\mathbb{C} \cup \{\infty\}$ with poles at $x = 0$, $x = 1/a$, $x = 1/b$ and $x = 1/c$. Note that there is no pole at ∞ . So

$$F(a, b, c) = -\text{Res}_{x=1/a} G(x) - \text{Res}_{x=1/b} G(x) - \text{Res}_{x=1/c} G(x).$$

An easy calculation, using that

$$\begin{aligned}\operatorname{Res}_{x=1/a}G(x) &= \frac{d}{dx}\Big|_{x=1/a} (G(x)(x-1/a)^2). \\ \operatorname{Res}_{x=1/b}G(x) &= G(x)(x-1/b)|_{x=1/b}, \\ \operatorname{Res}_{x=1/c}G(x) &= G(x)(x-1/c)|_{x=1/c},\end{aligned}$$

yields

$$\begin{aligned}F(a, b, c) &= \frac{a^{m-1}b(a+u)^g(a+v)^g}{(a-b)^2(c-a)} + \frac{a^{m-1}c(a+u)^g(a+v)^g}{(b-a)(c-a)^2} \\ &+ \frac{b^m(b+u)^g(b+v)^g}{(a-b)^2(b-c)} + \frac{c^m(c+u)^g(c+v)^g}{(c-a)^2(c-b)} \\ &+ \frac{a^{m-1}(a+u)^{g-1}(a+v)^{g-1}}{(a-b)(a-c)} \left(g a(2a+u+v) + (m-2)(a+u)(a+v) \right).\end{aligned}$$

Using this into (9.1) and Theorem 7.2, we have

$$\begin{aligned}e(\mathcal{N}_{\sigma_M^+}) &= \frac{(1+u)^{2g}(1+v)^{2g}}{(1-uv)^3(1-(uv)^2)^2} \left[(1+u^2v)^{2g}(1+uv^2)^{2g}(1-(uv)^{4g+4m-2}) \right. \\ &+ (1-2m-2g)(1+u^2v)^g(1+uv^2)^g(1+u)^g(1+v)^g(uv)^{3g+2m-2}(1-(uv)^2) \\ &+ (1+u)^{2g}(1+v)^{2g}(1+uv)^2(uv)^{3g+m-2} \left((1-(uv)^{2g+2m}) - (m+g)(1-uv)(1+(uv)^{2g+2m-1}) \right) \\ &\left. - g(1+u)^{2g-1}(1+v)^{2g-1}(1-(uv)^2)^2(uv)^{3g+m-2}(1-(uv)^{2g+2m-1}) \right].\end{aligned}$$

As $N = d_1 - d_2 - 2g + 2 = 2m + 2g - 1$, we get the formula in the statement. \square

Corollary 9.3. *Suppose that d_1 is even and d_2 is odd. Then the large σ moduli space $\mathcal{N}_{\sigma_M^+} = \mathcal{N}_{\sigma_M^+}^s$ is smooth and compact. If $\mu_1 - \mu_2 > 2g - 2$ its Hodge polynomial has the same formula as that of Theorem 9.2.*

Proof. Use the isomorphism $\mathcal{N}_\sigma(2, 2, d_1, d_2) \cong \mathcal{N}_\sigma(2, 2, -d_2, -d_1)$ together with Theorem 9.2. \square

Corollary 9.4. *Suppose that $d_1 + d_2$ is odd and $\mu_1 - \mu_2 > 2g - 2$. Then the Poincaré polynomial of $\mathcal{N}_{\sigma_M^+}$ is*

$$\begin{aligned}P_t(\mathcal{N}_{\sigma_M^+}) &= \frac{(1+t)^{4g}}{(1-t^2)^3(1-t^4)^2} \left[(1+t^3)^{4g}(1-t^{4N}) - N(1+t^3)^{2g}(1+t)^{2g}t^{2N+2g-2}(1-t^4) \right. \\ &+ (1+t)^{4g}(1+t^2)^2t^{N+4g-3} \left((1-t^{2N+2}) - \frac{N+1}{2}(1-t^2)(1+t^{2N}) \right) \\ &\left. - g(1+t)^{4g-2}(1-t^4)^2t^{N+4g-3}(1-t^{2N}) \right],\end{aligned}$$

where $N = d_1 - d_2 - 2g + 2$. \square

REFERENCES

- [1] ÁLVAREZ-CÓNSUL, L.; GARCÍA-PRADA, O.: Dimensional reduction, $SL(2, \mathbb{C})$ -equivariant bundles, and stable holomorphic chains. *Internat. J. Math.* **12** (2001) 159–201.
- [2] ÁLVAREZ-CÓNSUL, L.; GARCÍA-PRADA, O.; SCHMITT, A.: On the geometry of moduli spaces of holomorphic chains over compact Riemann surfaces. *Internat. Math. Res. Papers*, Art ID 73597 (2006) 1–82.
- [3] BRADLOW, S. B.; DASKALOPOULOS: Moduli of stable pairs for holomorphic bundles over Riemann surfaces. *Internat. J. Math.* **2** (1991) 477–513.
- [4] BRADLOW, S. B.; GARCÍA-PRADA, O.: Stable triples, equivariant bundles and dimensional reduction. *Math. Ann.* **304** (1996) 225–252.
- [5] BRADLOW, S. B.; GARCÍA-PRADA, O.; GOTHEN, P.B.: Moduli spaces of holomorphic triples over compact Riemann surfaces. *Math. Ann.* **328** (2004) 299–351.

- [6] BURILLO, J.: El polinomio de Poincaré–Hodge de un producto simétrico de variedades kählerianas compactas. *Collect. Math.* **41** (1990) 59–69.
- [7] DEL BAÑO, S.: On the motive of moduli spaces of rank two vector bundles over a curve. *Compositio Math.* **131** (2002) 1–30.
- [8] DELIGNE, P.: Théorie de Hodge I,II,III. In *Proc. I.C.M.*, vol. 1, 1970, pp. 425–430; in *Publ. Math. I.H.E.S.* **40** (1971) 5–58; *ibid.* **44** (1974) 5–77.
- [9] DELIGNE, P.: Théorème de Lefschetz et critères de dégénérescence de suites spectrales. *Publ. Math. I.H.E.S.* **35** (1968) 259–278.
- [10] DURFEE, A.H.: *Algebraic varieties which are a disjoint union of subvarieties*, *Lecture Notes in Pure Appl. Math.* **105**, Marcel Dekker, 1987, pp. 99–102.
- [11] DANIVOL, V.I.; KHOVANSKIĬ, A.G.: Newton polyhedra and an algorithm for computing Hodge-Deligne numbers, *Math. U.S.S.R. Izvestiya* **29** (1987) 279–298.
- [12] EARL, R.; KIRWAN, F.: The Hodge numbers of the moduli spaces of vector bundles over a Riemann surface. *Q. J. Math.* **51** (2000) 465–483.
- [13] FULTON, W.; MACPHERSON, R.: A compactification of configuration spaces. *Annals of Math.* **139** (1994) 183–225.
- [14] GARCÍA-PRADA, O.: Dimensional reduction of stable bundles, vortices and stable pairs. *Internat. J. Math.* **5** (1994) 1–52.
- [15] GARCÍA-PRADA, O.; GOTHEN, P.B.; MUÑOZ, V.: Betti numbers of the moduli space of rank 3 parabolic Higgs bundles. *Memoirs Amer. Math. Soc.* In press.
- [16] GOTHEN, P.B.: The Betti numbers of the moduli space of stable rank 3 Higgs bundles on a Riemann surface. *Internat. J. Math.* **5** (1994) 861–875.
- [17] GOTHEN, P.B., KING, A.D.: Homological algebra of quiver bundles. *J. London Math. Soc. (2)* **71** (2005) 85–99.
- [18] GRIFFITHS, P.: Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping. *Publ. Math. I.H.E.S.* **38** (1970) 125–180.
- [19] HITCHIN, N.J.: The self-duality equations on a Riemann surface, *Proc. London Math. Soc. (3)* **55** (1987) 59–126.
- [20] KIRWAN, F.: On the homology of compactifications of moduli spaces of vector bundles over a Riemann surface. *Proc. London Math. Soc. (3)* **53** (1986) 237–266.
- [21] MUÑOZ, V.; ORTEGA, D.; VÁZQUEZ-GALLO, M.-J.: Hodge polynomials of the moduli spaces of pairs. *Internat. J. Math.* In press.
- [22] NARASIMHAN, M. S.; RAMANAN, S.: Geometry of Hecke cycles. I. C. P. Ramanujam—a tribute, pp. 291–345, *Tata Inst. Fund. Res. Studies in Math.* **8**, Springer, Berlin-New York, 1978.
- [23] SCHMITT, A.: A universal construction for the moduli spaces of decorated vector bundles. *Transform. Groups* **9** (2004) 167–209.
- [24] THADDEUS, M.: Stable pairs, linear systems and the Verlinde formula. *Invent. Math.* **117** (1994) 317–353.

DEPARTAMENTO DE MATEMÁTICAS, CSIC, SERRANO 113 BIS, 28006 MADRID, SPAIN

INSTITUTE FOR ADVANCED STUDY, EINSTEIN DRIVE, PRINCETON, NEW JERSEY 08540, USA

E-mail address: vicente.munoz@imaff.cfmac.csic.es

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID, 28049 MADRID, SPAIN

E-mail address: daniel.ortega@uam.es

DEPARTAMENTO DE INGENIERÍA CIVIL: SERVICIOS URBANOS, UNIDAD DOCENTE: MATEMÁTICAS, ESCUELA DE INGENIERÍA DE OBRAS PÚBLICAS, UNIVERSIDAD POLITÉCNICA DE MADRID, ALFONSO XII 3 Y 5, 28014 MADRID, SPAIN

E-mail address: mariajesus.vazquez@upm.es