FUNDAMENTAL GROUP OF PLANE CURVES AND RELATED INVARIANTS

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Abstract

In this paper we want present some problems related to the study of the topology of the complement of an algebraic curve in the complex projective plane $\mathbb{P}^2$. We present the relations between the study of such topology with surface singularities. We also present the classic tool to calculate the fundamental group of the complement to a plane curve: the braid monodromy. Besides the classical generic procedure, we present the asymptotic procedure, which makes the computation of the group more effective. Finally, we study the characteristic varieties, introduced by Libgober. They result in new, powerful and effective invariants of such a group.

The framework of this paper is the study of topological invariants of the pair $(\mathbb{P}^2, \mathcal{C})$ where $\mathcal{C}$ is a reduced algebraic curve (possibly singular) and $\mathbb{P}^2$ denotes the complex projective plane. For instance the first homology group of the complement $\mathbb{P}^2 \setminus \mathcal{C}$ only depends on two algebraic invariants: the number of irreducible components of $\mathcal{C}$ and their degrees. The other non-trivial homology group (note that $H_0(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}$) is $H_2(\mathbb{P}^2 \setminus \mathcal{C})$ and it depends on local topological invariants of the singularities of $\mathcal{C}$. Nevertheless, in the early 30's, Zariski showed an example of two irreducible curves of degree six with the same local topological invariants (six cusps each) but defining non-equivalent pairs. Zariski proved that their complements were not homeomorphic by showing that the fundamental groups of their complements were not isomorphic. Both sextics could be distinguished by the *position* of their singularities. Namely, one of them had their cusps lying on a conic, whereas the other one not. This second type of invariant, the fundamental group of the complement has been widely studied in the literature. Techniques for computing a finite presentation of this group were first carried out by Zariski [27] and van Kampen [14].
In a general setting one can be interested in studying fundamental groups of the complement of a complex quasi-projective subvariety W in $\mathbb{P}^n$, (i.e. W might not be closed). Let $\bar{W}$ be its closure. Then the following equalities hold, see e.g. [10] Proposition 4.1.1:

Proposition 0.1 *1.* $\pi_1(\mathbb{P}^n \setminus W) = 0$ *if* $\dim W < n-1$;

2. $\pi_1(\mathbb{P}^n \setminus W) = \pi_1(\mathbb{P}^n \setminus \bar{W})$ *if* $\dim W = n-1$.

This means that it is enough to study fundamental groups $\pi_1(\mathbb{P}^n \setminus \bar{W})$ of complements of hypersurfaces in the projective space.
One can go further and apply the following basic result which restricts our attention to the case of plane curves, see e.g. [7]:

Theorem 0.2 Zariski Theorem of Lefschetz type. *Let V be a hypersurface in $\mathbb{P}^n$. For almost every 2-plane $E \subset \mathbb{P}^n$ the following map*

$$\pi_1(E \setminus (E \cap V)) \to \pi_1(\mathbb{P}^n \setminus V)$$

induced by the inclusion is an isomorphism.

Therefore the fundamental group of the complement of a complex algebraic projective curve $\mathcal{C} = E \cap V$ in $E = \mathbb{P}^2$ is the most interesting case. The main tool to study this group is by means of braid theory (cf. section 2). Besides the classical Zariski-van Kampen method we present the asymptotic case, which makes the computation of the fundamental group more effective. Nevertheless, the practical complexity of using finitely presented groups makes this invariant difficult to work with.
A class of invariants of the complement of $\mathcal{C}$ in $\mathbb{P}^2$ could be the action of certain monodromies related to the curve $\mathcal{C}$. The simplest invariant of a map $h : X \to X$ of a topological space into itself is the zeta-function. In section 1, we show that in our case the zeta-function is as bad as the Euler characteristic of the complement.
In section 4, we present some finer invariants, namely, the characteristic varieties of $\mathcal{C}$ and its Alexander polynomial. These algebraic invariants were introduced by Libgober [16]. It was originally used to obtain information about all abelian covers of $\mathbb{P}^2$ ramified along $\mathcal{C}$. They consist of a finite number of translated tori in $(\mathbb{C}^*)^{b_1}$ where b_1 the first Betti number of $\mathbb{P}^2 \setminus \mathcal{C}$. They are invariants of the fundamental group and hence topological invariants of the pair $(\mathbb{P}^2, \mathcal{C})$. The advantage in their study is that they are simpler to compute than the fundamental group but still fine enough to be sensitive to the *position* of the curve singularities.

1 General Invariants

Let us start with a general situation. Assume the hypersurface V is given in $\mathbb{P}^n$ as the zero locus of a degree d homogeneous polynomial $f \in \mathbb{C}[x_0, x_1, \ldots, x_n]$. We can assume that $d > 1$ otherwise the complement is the n-dimensional affine space which is simply connected.
Since $d > 1$ the cone of the hypersurface V in the $n+1$-dimensional affine space has the origin as a singular point. Let $(f, 0) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be the germ at the origin of the corresponding holomorphic function. The Milnor Fibration Theorem states that $(f, 0)$ defines a C^∞-locally trivial fibration in a small neighborhood of the origin, any of its fibers is called **the Milnor fiber** $V_f = \{z \in \mathbb{C}^{n+1} : f(z) = \epsilon, \|z\| \leq \delta\}$, $(0 < |\epsilon| \ll \delta$, δ small enough) of the singularity f at $0 \in \mathbb{C}^{n+1}$, see Milnor's book, [21].
Since f is a homogeneous polynomial, then the local Milnor fibration

defined above is equivalent to the fibration $f : \{z \in \mathbb{C}^{n+1} : \|f(z)\| = 1\} \to S^1$; we choose $F := f^{-1}(1)$ as Milnor fiber.
The projection $\pi : F \to \mathbb{P}^n \setminus V : (x_0, x_1, \dots, x_n) \mapsto [x_0 : x_1 : \dots : x_n]$ is well defined because $V := f^{-1}(0)$. The map π is a local homeomorphism (F is a smooth manifold). The fiber of π at any point $P \in \mathbb{P}^n \setminus V$ consists of d different points.
All the above conditions together imply (F, π) is an unramified cyclic covering of degree d of $\mathbb{P}^n \setminus V$, and the monodromy $h : F \to F$ acting as $h(x_0, \dots, x_n) = (\zeta_d x_0, \dots, \zeta_d x_n)$, where $\zeta_d := \exp(2\pi i/d)$, coincides with the generator of the group $Aut_U(F) = \mathbb{Z}/(d)\mathbb{Z}$, where $U := \mathbb{P}^n \setminus V$. Let $y_1 \in F$ be a base point and $y_0 = \pi(y_1) \in \mathbb{P}^n \setminus V$, one has the short exact sequence

$$1 \to \pi_1(F, y_1) \to \pi_1(\mathbb{P}^n \setminus V, y_0) \to \mathbb{Z}/(d)\mathbb{Z} \to 1.$$

Milnor also proved for $n \geq 2$ that if $(V, 0) \subset (\mathbb{C}^{n+1}, 0)$ has an isolated singularity the Milnor fiber F is simply connected. It turns out that if the projective hypersurface $V \subset \mathbb{P}^n$ is nonsingular, the Milnor fiber F is simply connected and $\pi_1(\mathbb{P}^n \setminus V, y_0)$ is cyclic of order d.

Example 1.1 *If $\mathcal{C}$ is a nonsingular conic in $\mathbb{P}^2$, its corresponding fundamental group $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ is $\mathbb{Z}/(2)\mathbb{Z}$.*
If $\mathcal{C}$ is the union of two different lines $\mathcal{C} = L_1 \cup L_2$ the complement $\mathbb{P}^2 \setminus \mathcal{C}$ is $\mathbb{C} \times \mathbb{C}^*$ whose fundamental group is $\mathbb{Z}$.

Some invariants of the local Milnor fibration $(f, 0) : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$, where f is a homogeneous polynomial, are related to the fundamental group of the complement in the projective plane of the projective curve $\mathcal{C} = \{f = 0\}$. One of these invariants is the zeta function of the Milnor fibration. A second invariant, which is finer, is the Alexander polynomial of the Milnor fibration.

Definition 1.2 *Let $h : X \to X$ be a map from a topological space X (say, with finite dimensional homology) into itself. The* **zeta-function** *$\zeta_h(t)$* **of** *h is the rational function defined by*

$$\zeta_h(t) := \prod_{q \geq 0} \{\det [\, id - t\, h_*|_{H_q(X;\mathbb{C})}]\}^{(-1)^q}.$$

In the local Milnor fibration case the zeta-function of the germ of holomorphic function $(f,0)$ is the zeta-function $\zeta_f(t)$ associated to the *geometric monodromy* corresponding to a small loop around the singular value 0. The Alexander polynomial of the germ of holomorphic function $(f,0)$ is

$$\Delta_f(t) = \det\,[\,id - t\,h_*|_{H_n(F;\mathbb{C})}\,].$$

When f is homogeneous, the local Milnor fibration defined above is also equivalent to the global **affine Milnor fibration** defined by the polynomial function $f : \mathbb{C}^{n+1} \to \mathbb{C}$.

Let $P : \mathbb{C}^{n+1} \to \mathbb{C}$ be any polynomial map. It is well known that there exists a finite set $B(P) \subset \mathbb{C}$ such that the polynomial function P is a C^∞ locally trivial fibration over its complement. The monodromy transformation h of this fibration corresponding to the loop $z_0 \cdot \exp(2\pi i \tau)$ $(0 \leq \tau \leq 1)$ with $\|z_0\|$ big enough is called the **geometric monodromy at infinity** of the polynomial P. Let h_* be its action in the homology groups of the fiber (the level set) $\{P = t_0\}$. The **zeta-function of the monodromy at infinity** of the polynomial P is the rational function

$$\zeta_P^\infty(t) = \prod_{q \geq 0} \{\det\,[\,id - t\,h_*|_{H_q(\{P=t_0\};\mathbb{C})}\,]\}^{(-1)^q},$$

and the **Alexander polynomial at infinity** is

$$\Delta_P^\infty(t) = \det\,[\,id - t\,h_*|_{H_n(\{P=t_0\};\mathbb{C})}\,].$$

If f is homogenous, it turns out that $\zeta_f(t) = \zeta_f^\infty(t)$ and $\Delta_f(t) = \Delta_f^\infty(t)$.

Any polynomial function $P : \mathbb{C}^{n+1} \to \mathbb{C}$ defines also a meromorphic function P on the projective space $\mathbb{P}^{n+1}$. At each point x of the infinite hyperplane $\mathbb{P}^n_\infty$ the germ of the meromorphic function (P,x) has the form $\dfrac{F(u,x_1,\ldots,x_n)}{u^d}$ where $u, x_1, \ldots, x_n$ are local coordinates such that $\mathbb{P}^n_\infty = \{u = 0\}$, (F,x) is the germ of a holomorphic function, and d is the degree of the polynomial P.

In [12], for a meromorphic germ $\left(f = \frac{P}{Q}, 0\right) : (\mathbb{C}^{n+1}, 0) \to \mathbb{P}^1$, there were defined the infinite Milnor fiber, the infinite monodromy transformation and thus the infinite zeta-function $\zeta_f^\infty(t)$. Let $\zeta_{P,x}^\infty(t)$ be the corresponding zeta-function of the germ of the meromorphic function P at the point $x \in \mathbb{P}^n_\infty$.

For the aim of convenience, in [12] we considered only meromorphic germs $\left(f = \frac{P}{Q}, 0\right)$ with $P(0) = Q(0) = 0$. At a generic point of the infinite hyperplane $\mathbb{P}^n_\infty$ the meromorphic function P has the form $\frac{1}{u^d}$. For a germ of the form $\left(f = \frac{1}{Q}, 0\right)$ with $Q(0) = 0$, it is reasonable to give the following definition: its infinite Milnor fiber coincides with the (usual) Milnor fiber of the holomorphic germ $(Q, 0)$; thus $\zeta_f^\infty(t) = \zeta_Q(t)$. According to this definition, for the germ $\left(\frac{1}{u^d}, 0\right)$ its infinite zeta-function is equal to $(1 - t^d)$.
Let $\mathcal{S} = \{\Xi\}$ be a stratification of the infinite hyperplane $\mathbb{P}^n_\infty$ (that is a partitioning of $\mathbb{P}^n_\infty$ into semi-analytic subspaces without any regularity conditions) such that, for each stratum Ξ of $\mathcal{S}$, the infinite zeta-function $\zeta^\infty_{P,x}(t)$ does not depend on x, for $x \in \Xi$. Let us denote this zeta-function by $\zeta^\infty_\Xi(t)$.

Theorem 1.3 *[13]*

$$\zeta_P(t) = \prod_{\Xi \in \mathcal{S}} [\zeta^\infty_\Xi(t)]^{\chi(\Xi)},$$

where $\chi(\bullet)$ denotes the Euler-Poincaré characteristic.

In particular, in our case $f \in \mathbb{C}[x_0, x_1, x_2]$ is reduced (it has no multiple factors) and the formula above can be simplified. Let $\mathrm{Sing}(\mathcal{C}) \in \mathbb{P}^2$ consist of s points $Q_1, \ldots, Q_s$. One has the following natural stratification of the infinite hyperplane $\mathbb{P}^2_\infty$:

1. the 2-dimensional stratum $\Xi^2 = \mathbb{P}^2_\infty \setminus \{\mathcal{C}\}$;
2. the 1-dimensional stratum $\Xi^1 = \{\mathcal{C}\} \setminus \{Q_1, \ldots, Q_s\}$;
3. the 0-dimensional strata Ξ^0_i $(i = 1, \ldots, s)$, each consisting of one point Q_i.

The Euler characteristic of the stratum Ξ^2 is equal to

$$\chi(\mathbb{P}^2_\infty) - \chi(\mathcal{C}) = 3 - \chi(2, d) + (-1) \sum_{i=1}^{s} \mu_i,$$

where $\chi(2, d) = 3 + \frac{(1-d)^3 - 1}{d}$ is the Euler characteristic of a non-singular curve of degree d in the complex projective space $\mathbb{P}^2_\infty$, μ_i is the Milnor number of the germ of the hypersurface $\mathcal{C} \subset \mathbb{P}^2_\infty$ at the point Q_i. At each point of the stratum Ξ^2, the germ of the meromorphic function f has (in some local coordinates u, y_1, y_2) the form $\frac{1}{u^d}$ ($\mathbb{P}^2_\infty = \{u = 0\}$) and its infinity zeta-function $\zeta^\infty_{\Xi^1}(t)$ is equal to $(1 - t^d)$.

At each point of the stratum Ξ^1, the germ of the polynomial f has (in some local coordinates u, y_1, y_2) the form $\frac{y_1}{u^d}$. Its infinity zeta-function $\zeta^\infty_{\Xi^1}(t)$ is equal to 1 and thus it does not contribute a factor to the zeta-function of the polynomial f.

At a point Q_i ($i = 1, \ldots, s$), the germ of the meromorphic function f has the form $\varphi(u, y_1, y_2) = \dfrac{g_i(y_1, y_2)}{u^d}$, where g_i is a local equation of the curve $\mathcal{C} \subset \mathbb{P}^2_\infty$ at the point Q_i.

To compute the infinite zeta-function $\zeta^\infty_\varphi(t)$ of the meromorphic germ $(\varphi, 0)$, let us consider the embedded resolution $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ of the curve singularity g_i, i.e., a proper modification of $(\mathbb{C}^2, 0)$ which is an isomorphism outside the origin in $\mathbb{C}^2$ and such that, at each point of the exceptional divisor $\mathcal{D}$, the lifting $g_i \circ \pi$ of the function g_i to the space $\mathcal{X}$ of the modification has (in some local coordinates) the form $y_1^{m_1} y_2^{m_2}$ ($m_i \geq 0$).

Let us consider the modification $\widetilde{\pi} = id \times \pi : (\mathbb{C}_u \times \mathcal{X}, 0 \times \mathcal{D}) \to (\mathbb{C}^3, 0) = (\mathbb{C}_u \times \mathbb{C}^2, 0)$ of the space $(\mathbb{C}^3, 0)$ – the trivial extension: $(u, x) \mapsto (u, \pi(x))$. Let $\widetilde{\varphi} = \varphi \circ \widetilde{\pi}$ be the lifting of the meromorphic function φ to the space $\mathbb{C}_u \times \mathcal{X}$ of the modification $\widetilde{\pi}$. Let $\mathcal{M}^\infty_{\widetilde{\varphi}} = \widetilde{\pi}^{-1}(\mathcal{M}^\infty_\varphi)$ ($\mathcal{M}^\infty_\varphi$ is the infinite Milnor fiber of the germ $(\varphi, 0)$) be the local level set of the meromorphic function $\widetilde{\varphi}$ (close to the infinite one). In the natural way one has the (infinite) monodromy $h^\infty_{\widetilde{\varphi}}$ acting on $\mathcal{M}^\infty_{\widetilde{\varphi}}$ and its zeta-function $\zeta^\infty_{\widetilde{\varphi}}(t)$.

Theorem 1.4 *[13]*

$$\zeta^\infty_{\widetilde{\varphi}}(t) = \zeta^\infty_\varphi(t).$$

For $\bar{m} = (m_1, m_2)$ with integer $m_1 \geq m_2 \geq 0$, let $S_{\bar{m}}$ be the set of points of the exceptional divisor $\mathcal{D}$ of the resolution π at which the lifting of the germ g_i has the form $y_1^{m_1} \cdot y_2^{m_2}$; for $m \geq 1$, let S_m be $S_{(m,0)}$.

At a point $x \in \{0\} \times S_{\bar{m}} \subset \{0\} \times \mathcal{D}$, the lifting $\widetilde{\varphi} = \varphi \circ \widetilde{\pi}$ of the function φ has the local form $\dfrac{y_1^{m_1} \cdot y_2^{m_2}}{u^d}$. Thus, for fixed $\bar{m}$, the infinite zeta-function $\zeta^\infty_{\widetilde{\varphi},x}(t)$ of the germ of the meromorphic function $\widetilde{\varphi}$ at a

point x from $\{0\} \times S_{\bar{m}}$ is one and the same. It can be determined by the Varchenko type formula from [12]. In any case $\zeta^{\infty}_{\tilde{\varphi},x}(t) = 1$.
According to Theorem 1.3 $\zeta^{\infty}_{\tilde{\varphi}}(t) = 1$ and by Theorem 1.4 $\zeta^{\infty}_{\varphi}(t) = 1$.
It turns out that for any homogeneous polynomial f, its zeta-function at infinity is equal to

$$\zeta_f(t) = (1 - t^d)^{\chi(\Xi^2)},$$

where $\chi(\Xi^2) = \frac{1-(1-d)^3}{d} + \sum_{i=1}^{s} \mu(g_i)$ and g_i is a local equation of the curve $\mathcal{C} = \{f = 0\} \subset \mathbb{P}^2$ at its singular point Q_i.
In particular it implies that the zeta-function only depends on the topological type of the singularities of $\mathcal{C} \subset \mathbb{P}^2$.
Nevertheless, it is known that the Alexander polynomial of f is a finer invariant, in fact it is related with the Alexander polynomial of the curve $\mathcal{C} \subset \mathbb{P}^2$.
A common way to deal with the complement $\mathbb{P}^2 \backslash \mathcal{C}$ is passage to the affine case. Let $L_\infty \subset \mathbb{P}^2$ be a line and denote $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$ and $\mathcal{C}_{af} = \mathcal{C} \setminus L_\infty$. Then $\mathbb{C}^2 \backslash \mathcal{C}_{af} \hookrightarrow \mathbb{P}^2 \backslash \mathcal{C}$ induces a surjection $\pi_1(\mathbb{P}^2 \backslash (\mathcal{C} \cup L_\infty)) \twoheadrightarrow \pi_1(\mathbb{P}^2 \backslash \mathcal{C})$, a well-understood map (at least for L_∞ transverse to $\mathcal{C}$ cf. section 2).
A. Libgober [15] defined the Alexander polynomial $\Delta(\mathcal{C}_{af})(t)$ of $\mathcal{C}_{af}$ as the characteristic polynomial of the endomorphism of the first homology group of the infinite cyclic cover of $\mathbb{C}^2 \setminus \mathcal{C}_{af}$ induced by deck transformations, see also section 4.

Definition 1.5 *The Alexander polynomial $\Delta(\mathcal{C})(t)$ of a projective curve $\mathcal{C} \subset \mathbb{P}^2$ is the Alexander polynomial of an affine curve $\mathcal{C}_{af} \subset \mathbb{C}^2$, where L_∞ is a generic line.*

Theorem 1.6 *[24] If $\mathcal{C} \subset \mathbb{P}^2$ is a reduced curve defined by a homogeneous polynomial f then $\Delta(\mathcal{C})(t)$ is equal to $\Delta_f(t)$.*

The computation of $\Delta_f(t)$ using resolution of singularities has been done by several authors, see Esnault [11], Loeser-Vaquie [19] and Artal-Bartolo [1].
Nevertheless, in section 4 new invariants will be defined, the characteristic varieties, which are more sensitive to the topology of the complement.
A plane curve $\mathcal{C}$ in $\mathbb{P}^2$ defined by a degree d homogeneous polynomial $\{f_d(x, y, z) = 0\}$ appears also as a tangent cone of a germ of surface

singularity $(V,0) \subset (\mathbb{C}^3,0)$ defined by a holomorphic germ $f = f_d + f_{d+1} + \ldots$.
If the surface singularity $(V,0) \subset (\mathbb{C}^3,0)$ is isolated, some topological invariants of $(V,0) \subset (\mathbb{C}^3,0)$ are related to the curve $\mathcal{C}$. A typical example of this situation is the class of superisolated singularities introduced by Luengo in [20]. He proved that the link of a superisolated singularities only deends on the combinatorial type of $\mathcal{C}$. This was used by Artal in [2] (see also [3]) to give a counterexample to the following conjecture by Yau.

Conjecture 1.7 *The topological type of a surface singularity* $(V,0) \subset (\mathbb{C}^3,0)$ *is determined by its link and its characteristic polynomial.*

2 Zariski-van Kampen Theorem

In this section we will describe the so called Zariski-van Kampen method to calculate a presentation of the fundamental group of the complement to an plane algebraic curve. The first description was given by van Kampen in [14]. In fact, to prove this Theorem he uses the famous Seifert-van Kampen Theorem for the first time. Other descriptions of this method can be found in [27], [7], [10].

Let $\mathcal{C}$ be an algebraic curve in $\mathbb{P}^2$. Let P be a point not in $\mathcal{C}$ and L_∞ a line containing P such that L_∞ intersects $\mathcal{C}$ transversally.

Let us choose a suitable coordinate system so that $P = [0:1:0]$ and $L_\infty = \{Z = 0\}$. We will write the affine complex plane $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$ with coordinates $(x = X/Z, y = Y/Z)$ and denote the affine curve $\mathcal{C} \cap \mathbb{C}^2$ by $\mathcal{C}_{af}$. When no ambiguity seems likely to arise we will refer to $\mathcal{C}_{af}$ simply by $\mathcal{C}$. The following well-known result shows the relative importance of the particular choice of the line at infinity.

Lemma 2.1 *For a given projective plane curve $\mathcal{C}$, the topology of the pair $(\mathbb{C}^2, \mathcal{C})$ does not depend on the choice of the line at infinity L_∞ as long as L_∞ intersects $\mathcal{C}$ transversally. Therefore neither does $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$.*

Let us choose $f(x,y)$ an equation for $\mathcal{C}$. Let us consider $p : \mathbb{C}^2 \to \mathbb{C}$ the projection on the first coordinate. Let Δ be the set of critical values of $p|_{\mathcal{C}}$, that is, $\Delta := \{x \in \mathbb{C} | Discrim_y(f) = 0\} = \{x_1, x_2, \ldots, x_k\}$.

Lemma 2.2 *The restriction map $p| : \mathbb{C}^2 \setminus (\mathcal{C} \cup p^{-1}(\Delta)) \to \mathbb{C} \setminus \Delta$ is a locally trivial fibration.*

Let us consider a disk $F \subset \mathbb{C}$ so that $\Delta \subset F$ and another disk E so that $\mathcal{C} \cap (F \times \mathbb{C}) \subset F \times E$ (note that the existence of E is a consequence of the condition $P \notin \mathcal{C}$).

Let us fix a base point $* := (*_1, *_2)$ for the space $\mathbb{C}^2 \setminus (\mathcal{C} \cup p^{-1}(\Delta))$ such that $*_1 \in \partial F$ and $*_2 \in \partial E$. For the sake of simplicity we will refer to $*_1$,$*_2$ and $(*_1, *_2)$ simply by $*$ if no ambiguity seems likely to arise.

Let us denote by $L_c := \{x = c\}$ the fibers of p and by $Y_c := L_c \cap \mathcal{C}$ the intersection of L_c with $\mathcal{C}$. Using the exact sequence of homotopy for the generic fiber $L_* \setminus Y_*$ of $p|$ one has the following

Lemma 2.3 *Under the previous conditions*

$$1 \to \pi_1(L_* \setminus Y_*, *) \xrightarrow{i_*} \pi_1(\mathbb{C}^2 \setminus (\mathcal{C} \cup L_1 \cup L_2 \cup \ldots \cup L_k), *) \xrightarrow{p_*} \pi_1(\mathbb{C} \setminus \Delta, *) \to 1$$

is an exact sequence of groups.

Note that $\pi_1(L_* \setminus Y_*, *)$ and $\pi_1(\mathbb{C} \setminus \Delta, *)$ are both free groups generated by $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ and $\{\tau_1, \tau_2, \ldots, \tau_k\}$ respectively. The closed paths γ_i and τ_j can be chosen to be a good system of generators in the sense of [22] which basically corresponds to the following picture

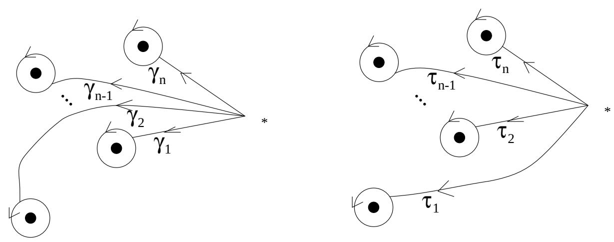

Figure 1.

Note that $\gamma_n \cdot \gamma_{n-1} \cdot \ldots \cdot \gamma_1$ and $\tau_k \cdot \tau_{k-1} \cdot \ldots \cdot \tau_1$ are homotopic to the boundary of the disks E and F respectively.

Proposition 2.4 *Under the previous notations there is a finite presentation of* $\pi_1(\mathbb{C}^2 \setminus (\mathcal{C} \cup L_1 \cup L_2 \cup \ldots \cup L_k), *)$ *as follows*

$$< \gamma_1, \gamma_2, \ldots, \gamma_n, \tilde{\tau}_1, \tilde{\tau}_2, \ldots, \tilde{\tau}_k | \tilde{\tau}_i^{-1} \cdot \gamma_j \cdot \tilde{\tau}_i = \beta_i(\gamma_j) >, \quad (1)$$

where $\tilde{\tau}_i$ *are liftings of* τ_i *by* $p|$ *and* $\beta_i(\gamma_j)$ *are words in the* γ_j*'s.*

Remark 2.5 Determination of $\beta_i(\gamma_j)$ and election of $\tilde{\tau}_i$

Since $p|$ *is a locally trivial fibration its monodromy associates to each* τ_i *a homeomorphism* h_{τ_i} *of the generic fiber* $L_* \setminus Y_*$*. One can choose such homeomorphisms so that they are the identity outside* E*. If we choose* $\tilde{\tau}$ *so that it lies on the line* $\{y = *\}$*, then the induced map on* $\pi_1(L_* \setminus Y_*, *)$ *coincides with the* β_i*'s given in (1). In this case, note that* h_{τ_i} *can be regarded as a braid in* E *with the set of strings* Y_* *(see [6]). By means of the geometric representation of such a braid one can easily find an expression for* $\beta_i(\gamma_j)$ *as a word in* $\gamma_1, \ldots, \gamma_n$ *(See figure 6).*

Note that for different choices of $\tilde{\tau}_i$ *one obtains different* β_i*'s. From now on we will consider the lifting of* τ_i *as in the previous paragraph.*

Definition 2.6 *Let $\mathcal{C}$ be an algebraic curve on a complex manifold X. Let $* \notin \mathcal{C}$ be the base point and p a smooth point of $\mathcal{C}$. Let us consider an analytic disk D intersecting $\mathcal{C}$ transversally at $\{p\} = D \cap \mathcal{C}$. Let α be a path joining $*$ and a point of ∂D. A closed path in $\pi_1(X \setminus \mathcal{C}, *)$ of the form $\alpha \partial D \alpha^{-1}$ (with ∂D positively oriented) will be called meridian of $\mathcal{C}$ in X.*

Note that $\tilde{\tau}_i$ as defined before is a meridian of L_{x_i} in $\mathbb{C}^2 \setminus \mathcal{C}$.

Proposition 2.7 *Let $\mathcal{C}$ be an irreducible curve in a complex manifold X. Any two meridians of $\mathcal{C}$ in X are conjugate in $\pi_1(X \setminus \mathcal{C}, *)$.*

The following is a well-known result.

Proposition 2.8 *Let $\mathcal{C}$ and X be as above, and let m be any meridian of $\mathcal{C}$ in X, then $\pi_1(X, *) = \pi_1(X \setminus \mathcal{C}, *)/ < m >$.*

This implies $\pi_1(\mathbb{C}^2 \setminus (\mathcal{C} \cup L_1 \cup L_2 \cup \ldots \cup L_k), *)/ < \tilde{\tau}_1, \ldots, \tilde{\tau}_k >= \pi_1(\mathbb{C}^2 \setminus \mathcal{C}, *)$ and the following result follows

Proposition 2.9 *The fundamental group of the complement of $\mathcal{C}$ in $\mathbb{C}^2$ is a finitely presented group given by*

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}, *) =< \gamma_1, \gamma_2, \ldots, \gamma_n | \gamma_j = \beta_i(\gamma_j) >$$

In the following we want to give a description of $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}, *)$. In order to do so we will use the following relationship $\pi_1(\mathbb{P}^2 \setminus \mathcal{C} \cup L_\infty, *) = \pi_1(\mathbb{C}^2 \setminus \mathcal{C}, *)$. By proposition 2.8, it will be enough to calculate a meridian of the line at infinity L_∞. This is given in the following

Proposition 2.10 *The closed path $(\gamma_n \cdot \ldots \cdot \gamma_2 \cdot \gamma_1)^{-1}$ is a meridian of the line at infinity L_∞*

Proof. Let us consider the point $P = [0 : 1 : 0]$ and the affine coordinate system $(x = X/Y, z = Z/Y)$ in $\mathbb{P}^2 \setminus \{Y = 0\}$. Since $P \notin \mathcal{C}$ one can find two disks F_∞ and E_∞ in such a way that the polydisk $F_\infty \times E_\infty \cap \mathcal{C}$ centered in P has an empty intersection with $\mathcal{C}$. Consider now the extension of the projection p, defined in Section 1, to the line at infinity L_∞. The restriction of such an extension to the boundary of the polydisk $\tilde{p} : \partial(F_\infty \times E_\infty) \to \mathbb{P}^1$ is a fibration on $\mathbb{P}^1$ with fiber S^1 and Euler number -1.

Recall the disks E and F defined in section 1. It is not difficult to see that one can choose E, F, F_∞, E_∞ so that

1. $\partial E \times \partial F = \partial E_\infty \times \partial F_\infty$.
2. $F \times \partial E = F_\infty \times \partial E_\infty$ where ∂E_∞ has the reversed orientation.
3. $\partial(F_\infty \times E_\infty) \setminus \tilde{p}^{-1}(F) = \partial F_\infty \times E_\infty$
4. $p(\partial F_\infty \times E_\infty) = \mathbb{P}^1 \setminus F$ and $\partial F_\infty \times \{cte.\}$ are fibers of $\tilde{p}$.

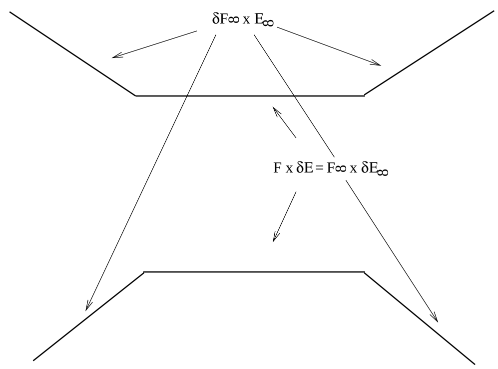

Figure 2.

Therefore, one can trivialize $\tilde{p}$ over $\mathbb{P}^1 \setminus F$ using $\partial F_\infty \times E_\infty \to \mathbb{P}^1 \setminus F$. The lifting of ∂F^{-1}, say $\tilde{\tau}_\infty$, is a meridian of the line L_∞. On the other hand, ∂F is homotopic to $\tau_n \cdot \ldots \tau_2 \cdot \tau_1$. Also note that the trivialization of $\tilde{p}$ over F given by $F \times \partial E = \tilde{p}^{-1}(F)$ allows to lift $\tau_n \cdot \ldots \tau_2 \cdot \tau_1$ as $\tilde{\tau}_n \cdot \ldots \tilde{\tau}_2 \cdot \tilde{\tau}_1$. Hence, the difference between both liftings (considered in one trivialization) is the path $\partial E_\infty^{-1} = \partial E$, that is, the (oriented) fiber to the power the Euler number of $\tilde{p}$ (-1 in our case).

Thus $(\tilde{\tau}_n \cdot \ldots \tilde{\tau}_2 \cdot \tilde{\tau}_1)^{-1} = \tilde{\tau}_\infty \cdot (\gamma_n \cdot \ldots \cdot \gamma_2 \cdot \gamma_1)$, and since $\tilde{\tau}_i = 1$ the result follows. $\square$

Remark 2.11 *The proof of proposition 2.10 might seem a little artificial, and it certainly can be simplified. The reason why we choose this method is by analogy to a more general case considered in the next section.*

As a consequence, one has the following classical result.

Theorem 2.12 *The fundamental group of the complement of $\mathcal{C}$ in $\mathbb{P}^2$ is a finitely presented group given by*

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}, *) =$$
$$=< \gamma_1, \gamma_2, \ldots, \gamma_n | \gamma_j = \beta_i(\gamma_j), \gamma_n \cdot \ldots \cdot \gamma_2 \cdot \gamma_1 = 1 >$$

3 Zariski-Van-Kampen Theorem with Vertical Asymptotes.

We recall that if the point of projection P is not in $\mathcal{C}$, then

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C} \cup (L_1 \cup L_2 \cup \ldots \cup L_k), *) =$$
$$=< \gamma_1, \gamma_2, \ldots, \gamma_n, \tilde{\tau}_1, \tilde{\tau}_2, \ldots, \tilde{\tau}_k | \tilde{\tau}_i^{-1} \cdot \gamma_j \cdot \tilde{\tau}_i = \beta_i(\gamma_j) >$$

Since $\tilde{\tau}_1, \tilde{\tau}_2, \ldots, \tilde{\tau}_k$ were chosen to be meridians of $L_1, L_2, \ldots, L_k$ one has

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}, *) =< \gamma_1, \gamma_2, \ldots, \gamma_n | \gamma_j = \beta_i(\gamma_j) >$$

Note that $\gamma_n \cdot \ldots \cdot \gamma_2 \cdot \gamma_1$ is the inverse of a meridian of the line at infinity. Hence:

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}, *) =< \gamma_1, \gamma_2, \ldots, \gamma_n | \gamma_j = \beta_i(\gamma_j), \gamma_n \cdot \ldots \cdot \gamma_2 \cdot \gamma_1 = 1 >$$

In many instances it might seem more natural (or easier in order to calculate $\beta_i(\gamma_j)$) to project from a point P on the curve $\mathcal{C}$. The construction above is no longer true since the $\tilde{\tau}_i$ described above are not meridians of the lines L_i. It is nevertheless possible to give a method to calculate the fundamental group from such a projection.

For the sake of simplicity, we will assume that L_∞ does not belong to the tangent cone of $\mathcal{C}$ at P.

There are two types of special vertical lines:

i) Those not belonging to the tangent cone of $\mathcal{C}$ at P and intersecting $\mathcal{C}_{af}$ non-transversally. We will denote them by $L_1, \ldots, L_m$ corresponding to fibers at the points $\{x_1, \ldots, x_m\}$.

ii) Those belonging to the tangent cone of $\mathcal{C}$ at P. They will be denoted by $L_{m+1}, \dots, L_k$, the fibers at the points $\{x_{m+1}, \dots, x_k\}$.

In this case the disks F and E do not exist anymore. However, one can find $D_{x_i}, i = m+1, \dots, k$ pairwise disjoint disks centered on x_i, F and E such that $\mathcal{C} \cap ((F \setminus \cup_{i=m+1}^{k} D_{x_i}) \times \mathbb{C}) \subset (F \setminus \cup_{i=m+1}^{k} D_{x_i}) \times E$.

One thing is still true, though: for any point $*$ in $\partial F \times \partial E$ and $\tilde{\tau}_i$ liftings in $(F \setminus \cup_{i=m+1}^{k} D_{x_i}) \times \{*\}$ one has

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C} \cup (L_1 \cup L_2 \cup \ldots \cup L_k), *) =$$

$$=< \gamma_1, \gamma_2, \dots, \gamma_n, \tilde{\tau}_1, \tilde{\tau}_2, \dots, \tilde{\tau}_k | \tilde{\tau}_i^{-1} \cdot \gamma_j \cdot \tilde{\tau}_i = \beta_i(\gamma_j) > .$$

The difference with the generic case is that, for $i = 1, ..., m$, the closed path $\tilde{\tau}_i$ is no longer a meridian of L_i. Our purpose now is to calculate meridians of L_i, $i = 1, ..., m$, see figure 4.

Example 3.1 *Let $\mathcal{C}$ be the curve of equation $(Z^2+YX)(Z^2-YX)Y = 0$. The equation of $\mathcal{C}$ in $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{z = 0\}$ with coordinates $x = X/Z, y = Y/Z$ is $(1+yx)(1-yx)y = 0$. Projecting from $[0:1:0]$ the only special fiber is $L := \{x = 0\}$. Let $* = (1, 2)$ be the base point. In figure 3 are considered the generators $\gamma_1, \gamma_2, \gamma_3$ on the generic line $\{x = 1\}$.*

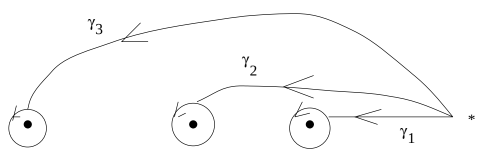

Figure 3.

Note that

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C} \cup L, *) =< \gamma_1, \gamma_2, \gamma_3, \tilde{\tau} | \tilde{\tau}^{-1} \cdot \gamma_i \cdot \tilde{\tau} = \beta(\gamma_i) > .$$

As in figure 4, let us consider a disk bounded by $\tilde{\tau}$ (we can think of it as placed on the line $y = 2$). Inside such disk let us take a meridian of L, say $\tilde{\tau}'$ and two meridians of $\mathcal{C}$, say a and b.

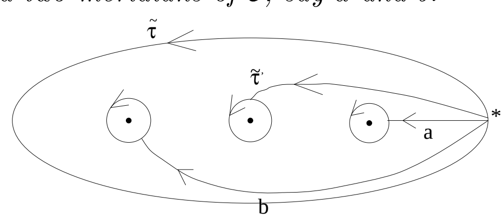

Figure 4.

Note that $a \cdot b \cdot \tilde{\tau}' = \tilde{\tau}$. Applying Lemma 2.8 to the meridian $\tilde{\tau}'$, one has

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}, *) =< \gamma_1, \gamma_2, \gamma_3, \tilde{\tau} | \tilde{\tau}^{-1} \cdot \gamma_i \cdot \tilde{\tau} = \beta(\gamma_i), \tilde{\tau} = a \cdot b > . \qquad (2)$$

One can easily calculate $\beta(\gamma_i), a$ and *b by means of the generators γ_1, γ_2 and γ_3 as (see figure 6):*

$$\beta(\gamma_i) = (\gamma_3 \cdot \gamma_2 \cdot \gamma_1)^{-1} \cdot \gamma_i \cdot (\gamma_3 \cdot \gamma_2 \cdot \gamma_1)$$

$$a = \gamma_1$$

$$b = (\gamma_2 \cdot \gamma_1)^{-1} \cdot \gamma_3 \cdot (\gamma_2 \cdot \gamma_1)$$

Therefore, the last relation in (2) becomes $\tilde{\tau} = \gamma_2^{-1} \cdot \gamma_3 \cdot \gamma_2 \cdot \gamma_1$

Note that the information needed to calculate a and b in the example reminds to that used to calculate $\beta(\gamma_i)$. In particular, a is γ_1 and b can be written as $\alpha \cdot \gamma_1' \cdot \alpha^{-1}$, where α represents "half way" of τ^{-1} and γ' is the closed path in $x = -1$ shown in figure 5.

Figure 5.

In general, one can see that, for $i = m+1, \ldots, k$, the following relation holds $\tilde{\tau}_i = \tilde{\tau}_i' \cdot w_i(\gamma_1, \ldots, \gamma_n) = \tilde{\tau}_i' \cdot w_i$, where $\tilde{\tau}_i'$ is a meridian of L_i. For the sake of simplicity we will consider $w_i = 1$ for $i = 1, ..., m$. Therefore, one obtains the following

$$\pi_1(\mathbb{C} \setminus \mathcal{C}, *) =< \gamma_1, \gamma_2, \ldots, \gamma_n, \tilde{\tau}_1, \tilde{\tau}_2, \ldots, \tilde{\tau}_k | \tilde{\tau}_i^{-1} \cdot \gamma_j \cdot \tilde{\tau}_i = \beta(\gamma_j), \tilde{\tau}_i = w_i >$$

$$=< \gamma_1, \gamma_2, \ldots, \gamma_n | w_i^{-1} \cdot \gamma_j \cdot w_i = \beta(\gamma_j) > .$$

Using the fact that L_∞ is not tangent to $\mathcal{C}$ at P, one has the analogue to the generic case, that is, $(\tilde{\tau}_1 \cdot \ldots \cdot \tilde{\tau}_k)^{-1} = \tilde{\tau}_\infty \cdot (\gamma_n \cdot \ldots \cdot \gamma_1)$. In this case, $\tilde{\tau}_i$ is not always trivial in $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}, *)$, but $\tilde{\tau}_i = w_i$. Thus,

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}, *) =$$

$$=<\gamma_1,\gamma_2,\ldots,\gamma_n | w_i^{-1}\cdot\gamma_j\cdot w_i=\beta(\gamma_j),(w_1\cdot w_2\cdot\ldots\cdot w_k)^{-1}=(\gamma_n\cdot\ldots\cdot\gamma_1)>.$$

In this last part of the section we will describe how to calculate the paths w_i as words of $\gamma_1,...,\gamma_n$. For simplicity purposes, we will assume that the base point $*$ is sufficiently close to L_i, that is $*\in\partial D_{x_i}$. Hence there exists a disk D with the properties $\mathcal{C}\cap(D_{x_i}\times\partial D)=\emptyset$ and $L_i\cap\mathcal{C}\subset\{x_i\}\times D$. In other words, all branches tangent to L_i at infinity intersect $D_{x_i}\times\mathbb{C}$ outside $D_{x_i}\times D$.

We can also choose E $(D\subset E)$ such that $\mathcal{C}\cap(D_{x_i}\times\mathbb{C})\subset D_{x_i}\times E$ and $*\in\partial E$. We will denote $\tilde\tau_i$, that is $\partial D_{x_i}\times\{*\}$, by $\tilde\tau$.

Let d be a loop in L_* based on $*$ of the form $\alpha\cdot\partial D\cdot\alpha^{-1}$, with α any path in $L_*\setminus\mathcal{C}$ joining $*$ with an arbitrary point of ∂D. The loop d can be written as a word in $\gamma_1,\ldots,\gamma_n$. In the previous example, the loop d is $\gamma_3^{-1}\cdot\gamma_2\cdot\gamma_3$, see figure 6.

Case 1: d is not trivial, that is, not all the branches of $\mathcal{C}$ along $L_{x_i}\in\mathbb{P}^2$ are tangent to L_{x_i} at infinity.

It is not difficult to calculate a word $\tilde w$ $(=\tilde w(\gamma_1,\ldots,\gamma_n))$ such that $\tilde\tau^{-1}\cdot d\cdot\tilde\tau=\tilde w\cdot d\cdot\tilde w^{-1}$. Such word can be computed from $\beta_i(d)$ analogously to how we proceeded with $\beta_i(\gamma_j)$. The following equalities take place in $G=\pi_1(\partial D_{x_i}\times E\setminus\mathcal{C},*)$ and, hence are also true in $\pi_1(\mathbb{C}^2\setminus\mathcal{C},*)$.

Let m be the meridian of L_i based on $\alpha(1)$ corresponding to $\partial D_{x_i}\times\{\alpha(1)\}$ and let α be as above. The following closed path $\tilde\tau'=\alpha\cdot m\cdot\alpha^{-1}$ satisfies $\tilde{\tau'}^{-1}\cdot d\cdot\tilde\tau'=d$.

The next step will be to determine w from $\tilde\tau=\tilde\tau'\cdot w$ as a function of d and $\tilde w$, which are already known.

Since $\tilde\tau^{-1}\cdot d\cdot\tilde\tau=\tilde w\cdot d\cdot\tilde w^{-1}$, $\tilde{\tau'}^{-1}\cdot d\cdot\tilde\tau'=d$, then $\tilde w\cdot d\cdot\tilde w^{-1}=w\cdot d\cdot w^{-1}$. One has the following

Lemma 3.2 *Under the conditions above $w=\tilde w\cdot d^k$ for an integer k.*

Proof. The result is a consequence of the following facts:

1. There is an exact sequence
$$1\to<\gamma_1,\ldots,\gamma_n>\to G\to<\tau>\to 1.$$
2. The commutator of an element a in a free group is the subgroup generated by the word b so that $b^r=a$ and r maximal. □

To calculate k it is enough to note that, in $\pi_1(\partial D_{x_i} \times E \setminus (\mathcal{C} \cap (D_{x_i} \times D)), *)$, the loops $\tilde{\tau}$ and $\tilde{\tau}'$ are homotopic. Moreover, $\pi_1(\partial D_{x_i} \times E \setminus (\mathcal{C} \cap (D_{x_i} \times D)), *)$ is the quotient of $\pi_1(\partial D_{x_i} \times E \setminus \mathcal{C}, *)$ by the normal subgroup generated by those γ_i around the points Y_* outside D. We will denote by $[\sigma]$ the class of σ in this quotient. Note that if σ can be written as a word in $\gamma_1, \ldots, \gamma_n$, then $[\sigma]$ can be obtained simply substituting γ_i by either $[\gamma_i]$, in case γ_i surrounds a point inside D, or 1 in case γ_i surrounds a points outside D. Hence, the expression $\tilde{\tau} = \tilde{\tau}' \cdot \tilde{w} \cdot d^k$ considered in the quotient, becomes $1 = [\tilde{w}][d]^k$. Therefore both $[\tilde{w}] = [d]^{-k}$ and k are perfectly determined.

Case 2: d is the trivial word. One can add the line $y = 0$ to the original curve and apply the previous case, where d is the non-trivial γ_{n+1}. After computing the word w' (as a word in $\gamma_1, \ldots, \gamma_n, \gamma_{n+1}$) one can obtain the original w by substituting $\gamma_{n+1} = 1$ in w'.

Exercise 3.3 *From the example 3.1 and using the construction above the reader should be able to determine the fundamental group of the curves:*

1. $(Z^2 - XY)Y = 0,$
2. $(Z^2 - XY)(Z^2 + XY) = 0.$

Note that both fundamental groups are non-isomorphic. The interest of this pair of curves stands in the fact that the braid associated to τ *in remark 2.5 coincide. Also note that the curve 1 belongs to the case 1 and the curve 2 to the second case in the previous paragraph.*

Exercise 3.4 *Calculate the fundamental group of the curves:*

1. $(Z^3 - XY^2) = 0,$
2. $(Z^2 - XY) = 0.$

using the projection from $[0 : 1 : 0]$.

In this case, note that the fundamental groups are again different, and the braids also coincide. The difference with respect to the previous example is that both curves belong to case 2.

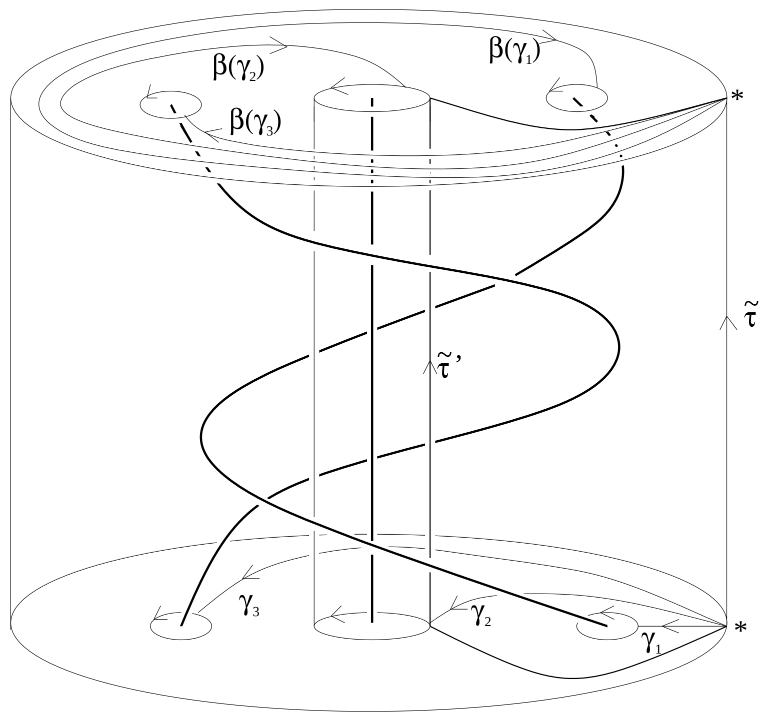

Figure 6.

4 Alexander Invariants

Let X be a connected CW-complex such that $H_1(X)$ has a surjective homomorphism ψ onto a group A (note A has to be abelian). Since $H_1(X)$ is the quotient of $\pi_1(X)$ by the commutator subgroup, then $\pi_1(X)$ also has a surjective homomorphism $\tilde{\psi}$ onto A and hence there is a short exact sequence

$$0 \to K \xrightarrow{\varphi} \pi_1(X) \xrightarrow{\tilde{\psi}} A \to 0.$$

Let $\tilde{X}_K$ be the covering corresponding to the normal subgroup $K \subset \pi_1(X)$. The group of covering transformations of φ is A and acts on $H_i(\tilde{X}_K)$ for any $i \geq 0$. Such an action makes $H_i(\tilde{X}_K)$ a module over the group algebra $\Lambda_{\mathbb{Z}} = \mathbb{Z}[A]$. The module $H_i(\tilde{X}_K)$ is called the **i-th Alexander module of X with respect to ψ (and A)**. Note the following facts:

1. We will assume (up to homotopy equivalence) that X has a single 0-simplex e_0 and therefore the 1-simplices of X can be considered as elements of $\pi_1(X)$. Note that this is possible since X is connected.
2. $\tilde{X}_K$ is a (possibly non-finite) CW-complex with $C_0(\tilde{X}_K) \cong \Lambda_{\mathbb{Z}}$.
3. If A is finitely generated, then $\Lambda_{\mathbb{Z}}$ is a Noetherian ring.

One can lift the generators of the cell complex $C_i(X)$ to the covering complex $\tilde{X}_K$ in a natural way so that the group A of deck transformations acts freely and transitively on the lifting. Extending this lifting to $C_i(X)$ one has $C_i(\tilde{X}_K) = C_i(X) \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}}$. Note that only the lifting map applied to $C_1(X)$ is *not* a homomorphism. For instance, if the lifting of the 1-cell $e \in C_1(X)$ is $e \otimes 1$, the lifting of $-e$ is $-e \otimes \psi(e)^{-1}$ which is not its opposite.
The corresponding boundary map is the following morphism

$$\tilde{\delta}_i(z \otimes a) = \begin{cases} \delta_i(z) \otimes a & \text{if } i > 2 \\ e_0 \otimes (\tilde{\psi}(z) - 1)\, a & \text{if } i = 1 \end{cases}$$

for $i \neq 1$, where $\tilde{\psi}(z)$ makes sense by 1.
In order to describe the boundary map for $i = 1$, let us fix for any 2-cell $e_2 \in C_2(X)$ a certain closed path ∂e_2 representing its boundary as

induced by the cell map on the boundary. It might happen that such map is constant. In that case $\tilde{\delta}_2(e_2 \otimes a) = 0$. Otherwise ∂e_2 can be written as a composition of closed paths (1-cells), say $\partial e_2 = x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \ldots \cdot x_n^{\varepsilon_n}$, where $\varepsilon_1 = \pm 1$. Then $\tilde{\delta}_2$ can be described recursively as a function of its boundary $\tilde{\delta}_2(e_2 \otimes a) = D(\partial e_2, a)$ where

$$D(x^{\varepsilon} \cdot y, a) = \begin{cases} x \otimes a + D\left(y, \psi(x)a\right) & \text{if } \varepsilon = 1 \\ -x \otimes \psi^{-1}(x)a + D\left(y, \psi^{-1}(x)a\right) & \text{if } \varepsilon = -1. \end{cases} \tag{3}$$

Therefore $H_i(C_*(\tilde{X}_K), \tilde{\delta}_*) = H_i(\tilde{X}_K)$. Moreover, by 3, if A is a finitely generated group, then the Alexander modules are finitely generated $\Lambda_{\mathbb{Z}}$-modules.

Let us consider for now the particular case where A has a single generator t (that is $A = \mathbb{Z}/k\mathbb{Z}$ and $k \geq 0$). If one uses multiplicative notation for the operation in A, then $\Lambda_{\mathbb{Z}}$ can be described as $\mathbb{Z}[t, t^{-1}]$, the ring of Laurent polynomials in one variable with integer coefficients (with the relationship $t^k = 1$). Tensoring the ring $\Lambda_{\mathbb{Z}}$ by $\mathbb{C}$ one obtains a Principal Ideal Domain $\Lambda_{\mathbb{C}} = \mathbb{C}[t, t^{-1}]$ and hence the extended modules $H_i(\tilde{X}_K, \mathbb{C})$ have a decomposition as finitely generated modules on $\Lambda_{\mathbb{C}}$.

$$H_i(\tilde{X}_K, \mathbb{C}) \cong \Lambda_{\mathbb{C}}^{r_i} \oplus \Lambda_{\mathbb{C}}/\lambda_1^i \oplus \ldots \oplus \Lambda_{\mathbb{C}}/\lambda_{n_i}^i \tag{4}$$

In the case where the first Alexander module is a torsion module (i.e. $r_1 = 0$) then the polynomial $\lambda = \lambda_1^1 \cdot \ldots \cdot \lambda_{n_1}^1$ from (4) is called the **Alexander polynomial of** X.

Let us consider the endomorphism of modules defined by multiplication by $t-1$. By definition one has the following short exact sequence

$$0 \longrightarrow C_i(\tilde{X}_K) \xrightarrow{t-1} C_i(\tilde{X}_K) \xrightarrow{j} \frac{C_i(\tilde{X}_K)}{(t-1)C_i(\tilde{X}_K)} \cong C_i(X) \longrightarrow 0.$$

Hence

$$H_1(\tilde{X}_K, \mathbb{C}) \xrightarrow{t-1} H_1(\tilde{X}_K, \mathbb{C}) \xrightarrow{j} H_1(X, \mathbb{C}) \xrightarrow{\delta} H_0(\tilde{X}_K, \mathbb{C}) \xrightarrow{t-1} H_0(\tilde{X}_K, \mathbb{C}). \tag{5}$$

Note that

$$H_0(\tilde{X}_K) = \frac{C_0(\tilde{X}_K)}{(t-1)C_0(\tilde{X}_K)} \cong C_0(X) = \mathbb{Z}.$$

Hence (5) becomes

$$H_1(\tilde{X}_K, \mathbb{C}) \xrightarrow{t-1} H_1(\tilde{X}_K, \mathbb{C}) \xrightarrow{j} H_1(X, \mathbb{C}) \xrightarrow{\delta} \mathbb{C} \xrightarrow{0} \mathbb{C}. \tag{6}$$

Also note that the *coker* of the map

$$\Lambda_{\mathbb{C}}/\lambda \xrightarrow{t-1} \Lambda_{\mathbb{C}}/\lambda$$

is non-trivial if and only if $\lambda(1) \neq 0$. Hence, if $H_1(\tilde{X}_K, \mathbb{C}) = \mathbb{C}$, then $H_1(\tilde{X}_K, \mathbb{C})$ is a torsion module and $t-1$ is not a factor of its Alexander polynomial.
Let us get back to the general case of a abelian group A. Our purpose now is to give a finite free presentation of the first Alexander invariant in the case where X is a finite CW-complex of dimension 2. This is the case for complements of links in S^3 or complements of curves in the projective complex plane (cf. [18]). In order to do so we will first calculate a presentation for the homology of $\tilde{X}_K$ relative to $\widetilde{e}_0$, the inverse image of the 0-cell in $\tilde{X}_K$. Note that $C_0(\tilde{X}_K) = \mathbb{Z}\widetilde{e}_0$ and hence

$$C_2(\tilde{X}_K, \widetilde{e}_0) \;\to\; C_1(\tilde{X}_K, \widetilde{e}_0) \;\to\; H_1(\tilde{X}_K, \widetilde{e}_0) \;\to\; 0 \tag{7}$$

provides a finite free presentation of $H_1(\tilde{X}_K, \widetilde{e_0})$. Also note that a presentation of $\pi_1(X)$ can be given by using its 1-cells as generators, say $x_1, ..., x_{b_1}$ and the boundaries of its 2-cells as relations, say $\partial y_1 = r_1, ..., \partial y_{b_2} = r_{b_2}$, where $b_i = \dim C_i(X)$. Therefore (7) becomes

$$\Lambda_{\mathbb{Z}}^{b_2} \xrightarrow{\phi} \Lambda_{\mathbb{Z}}^{b_1} \;\to\; H_1(\tilde{X}_K, \widetilde{e}_0) \;\to\; 0. \tag{8}$$

The matrix $\mathrm{Mat}(\phi)$ of ϕ can be determined as follows from (3)

$$\mathrm{Mat}(\phi) = (D_j(r_i))$$

where

$$\begin{array}{l} D_j(x_i) = \delta_{ij} \\ D_j(x_i^{-1}) = \begin{cases} -\tilde{\psi}(x_j)^{-1} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \\ D_j(xx') = D_j(x) + \tilde{\psi}(x) D_j(x'). \end{array} \tag{9}$$

Let us recall the definition of Fitting ideal of a finitely generated module M over a commutative noetherian ring with unity R. Let

$$R^m \xrightarrow{\phi} R^n \;\to\; M \;\to\; 0$$

be a finite free presentation of the R-module M. Let $\mathrm{Mat}(\phi)$ be the $(n \times m)$ R-matrix of ϕ.

Definition 4.1 *The k-th* **Fitting ideal** *of M is defined as the ideal generated by*

$$\begin{cases} 0 & \text{if } k \leq max\{0, n-m\} \\ 1 & \text{if } k > n \\ \text{minors of } \mathrm{Mat}(\phi) \text{ of order } (n-k+1) & \text{otherwise.} \end{cases}$$

Such an ideal does not depend on the presentation of M and it is denoted by $F_k(M)$ or simply F_k if no ambiguity seems likely to arise.

Since

$$D_j(xrx^{-1}) = \tilde{\psi}(x) D_j(r) \quad \forall j = 1, ..., b_1$$

$$D_j(1) = 0$$

$$D_j(x_i r_i) = \delta_{ij} \text{ if } r_i \text{ does not depend on } x_i$$

then the Fitting ideals of $H_1(\tilde{X}_K, \tilde{e}_0)$ do not depend on the isomorphism class of $\pi_1(X)$. Note that the previous paragraph shows that Fox calculus – defined by (9) – gives a finite free presentation of $H_1(\tilde{X}_K, \tilde{e}_0)$. Also note that if we had a finite free presentation of $H_1(\tilde{X}_K)$ and $A = \mathbb{Z}$, then the Alexander polynomial could be characterize as generator of $F_1(H_1(\tilde{X}_K))$. This remark allows for a more general invariant of the first Alexander module.

Definition 4.2 *In the above conditions the* **k-th characteristic variety** *of the R-module M can be defined as*

$$\mathrm{Char}_k(M) := \mathrm{Supp}_R\left(R/F_k(M)\right),$$

where $\mathrm{Supp}_R(N)$ is the set of prime ideals p in R such that $N_p \neq 0$.
In the case where $M = H_1(\tilde{X}_K, \mathbb{C})$ and $A = \mathbb{Z}$, we will define the i-th characteristic variety of X, denoted by $\mathrm{Char}_i(X)$, as $\mathrm{Char}_i(H_1(\tilde{X}_K, \mathbb{C}))$.

In other words, the set of zeroes of the Alexander polynomial coincides with Spec $\mathrm{Char}_1(X) \subset$ Spec $\Lambda_{\mathbb{C}} = \mathbb{C}^*$.
We are hence left to describe the relationship between both $\mathrm{Char}_i(X)$ and $\mathrm{Char}_i(H_1(\tilde{X}_K, \tilde{e}_0))$. One has the following result

Proposition 4.3 *The following equality holds except maybe for the augmentation ideal, generated by $\psi(H_1(\tilde{X}_K)) - 1$*

$$\mathrm{Char}_i(X) = \mathrm{Char}_{i+1}(H_1(\tilde{X}_K, \tilde{e}_0; \mathbb{C})). \tag{10}$$

Proof. see [8].
The characteristic varieties can also be calculated by means of a more algebraic point of view, as described in [16] and [4]. This calculation refers to the existence of certain linear systems of plane curves called superabundant (that is, with higher dimension than expected). For instance, as mentioned in the Introduction, Zariski found a sextic with 6 cusps of type $\mathbb{A}_2$ on a conic (recall that a singular point P is of type $\mathbb{A}_k$ if the equation of the curve can be written locally around P as $x^2 - y^3$). The superabundance of the linear system of conics passing through the six cusps results in a non-trivial characteristic varieties, and hence non-trivial Alexander polynomial.
As mentioned in the Introduction, the characteristic varieties are topological invariants of the complement of a plane curve, they can be used to distinguish non-isotopic curves. A typical example is given by a sextic with six cusps $\mathbb{A}_2$ not lying on a conic. The absence of superabundance in this case results in a trivial Alexander polynomial, and hence, the sextic of Zariski cannot be isotopic to this one.
Several examples of non-isotopic curves with the same combinatorics (basically same degree, number of components and type of singularities) have been given. They are called **Zariski pairs** (cf. [1]) and reveals the fact that not only the set of singularities, but their position affects the topology of the pair $(\mathbb{P}^2, \mathcal{C})$. As an example of this we can give the following

Proposition 4.4 *There exist two sextics $\mathcal{C}_1$ and $\mathcal{C}_2$ whose singularities are of type $\mathbb{A}_{17}$ and $\mathbb{A}_1$ so that*

$$\Delta(\mathcal{C}_1) = 1$$

$$\Delta(\mathcal{C}_2) = t^2 - t + 1.$$

Proof. Their construction can be found in [5]. For the curve $\mathcal{C}_2$ there exists a conic passing through the singular point with multiplicity 12, whereas for $\mathcal{C}_1$ there isn't. The existence (or not) of a superabundant linear system of conics with such multiplicity of intersection at the $\mathbb{A}_{17}$ produces a non-trivial (resp. trivial) first characteristic variety for $\mathcal{C}_2$ (resp. for $\mathcal{C}_1$).

The example is a counterexample to the following conjecture by Degtyarev [9], see also [25] and [26]

Conjecture 4.5 *Consider irreducible sextics with a fixed singular set of the form* $a\mathbb{A}_1 + \sum b_d \mathbb{A}_{3d-1} + c\mathbb{E}_6$, $\sum db_d + 2c = 6$, *and put* $a_{max} = 10 - \sum b_d \lfloor 3d/2 \rfloor - 3c$. *Then*

1. *any two sextics with* $a = a_{max}$ *are isotopic to each other and are abundant.*

2. *if* $a < a_{max}$, *then there are exactly two isotopy classes, one abundant and one non-abundant.*

The characteristic varieties of a curve $\mathcal{C}$ is a more sensitive invariant to the topology of the complement than just the Alexander polynomial. This is specially true in the reducible case. The following is an example of a Zariski pair with trivial Alexander polynomial.
Let us consider the following set of conditions on (complex) projective plane curves

1 $\mathbb{C}$ has two irreducible components $\mathbb{C}_1$ and $\mathbb{C}_2$.

2 $\mathbb{C}_1$ is a smooth conic.

3 $\mathbb{C}_2$ is a curve of degree 4 having two singular points of type $\mathbb{A}_3$ (tacnode) and $\mathbb{A}_1$ (node).

4 $\mathbb{C}_1 \cap \mathbb{C}_2 = \{P\}$, and P is a smooth point of $\mathbb{C}_2$. Then $(\mathbb{C}, P)$ is a singularity of type $\mathbb{A}_{15}$.

5 The tangent line to $\mathbb{C}$ at P does not pass through the tacnode.

5' The tangent line to $\mathbb{C}$ at P passes through the tacnode.

In [4], it is proven that there exists a sextic curve $\mathbb{C}^{(1)}$ satisfying conditions 1-5 and another sextic $\mathbb{C}^{(2)}$ satisfying 1-4 and 5'. The following proposition is also proven

Proposition 4.6 *1. The Alexander polynomials of* $\mathbb{C}^{(1)}$ *and* $\mathbb{C}^{(2)}$ *are both trivial.*

2.

$$Char_1(\mathbb{C}^{(1)}) = \{(1,1)\}$$

whereas

$$Char_1(\mathbb{C}^{(2)}) = \{(1,1), (-1,1), (\sqrt{-1}, -1), (-\sqrt{-1}, -1)\}.$$

Therefore $\mathbb{C}^{(1)}$ *and* $\mathbb{C}^{(2)}$ *constitute a Zariski pair.*

Thera are also examples showing that characteristic varieties do not contain as much information as the fundamental group does. For instance Artal and Carmona exhibit in [3] a Zariski pair with non-isomrphis fundamental groups, but both trivial characteristic varieties. This keeps open the problem of finding finer and finer invariants which can be effectively calculated but that keep the esential information about the topology of the complement of curves.

References

[1] E. Artal Bartolo, 'Sur les Couples de Zariski', *Journal of Algebraic Geometry* 3, (1994), 223–247.

[2] E. Artal Bartolo, 'Forme de Jordan de la Monodromie des Singularités Superisolées de Surfaces', *Memoirs of the A.M.S.* vol. 109, No. 525 (1994).

[3] E. Artal Bartolo, J. Carmona Ruber 'Zariski Pairs, Fundamental Groups and Alexander Polynomials', *Journal of Math. Soc. of Japan* 50, No. 3 (1998) 521–543.

[4] E. Artal Bartolo, J. Carmona Ruber, J. Cogolludo, H.-O Tokunaga, 'Sextics with Singular Points in Special Position', *preprint.*

[5] E. Artal Bartolo, J. Carmona Ruber, J. Cogolludo, 'On Sextic Curves with Big Milnor Number', *preprint.*

[6] J. S. Birman, 'Braids, Links and Mapping Class Groups', *Annals of Math Studies* 82, Princeton University Press, (1975).

[7] D. Chéniot, 'Une démonstration du théorème de Zariski sur les sections hyperplanes d'une hypersurface projective et du théorème de Van Kampen sur le groupe fondamental du complémentaire d'une courbe projective plane', *Compositio Math.* 27 (1973), 141–158.

[8] J. I. Cogolludo, 'Topological Invariants of the Complement to Arrangements of rational Plane Curves', *Thesis.*

[9] A. Degtyarev, 'Alexander Polynomial of a Curve of Degree Six', *Journal of Knot Theory and Its Ramifications,* Vol. 3, No. 4 (1994) 439–454.

[10] A. Dimca, 'Topology of Hypersurface Singularities', Springer-Verlag, New-York, (1992).

[11] H. Esnault, 'Fibre de Milnor d'un cône sur une courbe algébrique plane', *Inv. Math.* 68 (1982) 477–496.

[12] S.M. Gusein-Zade, I. Luengo, A. Melle-Hernández, 'Zeta Functions of Germs of Meromorphic Functions, and the Newton Diagram'. *Funct. Anal. Appl.* 32 (1998), No. 2, 93–9

[13] S.M. Gusein-Zade, I. Luengo, A. Melle-Hernández, 'On the Zeta-function of a Polynomial at Infinity'. *Bulletin des Sciences Mathematiques* Vol. 124, No. 3, (2000), 213–224.

[14] van Kampen, 'On the Fundamental Group of an Algebraic Curve', *American Journal of Math.* 55, (1933), 255–260.

[15] A. Libgober, 'Alexander Polynomial of Plane Algebraic Curves and Cyclic Multiple Planes'. *Duke Math. J.* 49 (1982), No. 4, 833–851.

[16] A. Libgober, 'Characteristic Varieties of Algebraic Curves', *preprint.*

[17] A. Libgober, 'On the Homology of Finite Abelian Coverings', *Topology and its Applications* 43, (1992), 157–166.

[18] A. Libgober, 'On the Homotopy Type of the Complement to Plane Algebraic Curves', *J. Reine Angew. Math.* 367, (1986), 103–114.

[19] F. Loeser, M. Vaquie, 'Le polynôme d'Alexander d'une courbe plane projective', *Topology* 20 (1990) 163–173.

[20] I. Luengo, 'The μ-constant stratum is not smooth'. *Inv. Math.* 90 (1987) 139–152

[21] J. Milnor, 'Singular Points of Complex Hypersurfaces'. *Annals of Mathematics Studies*, No. 61 Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo 1968.

[22] B. Moishezon, 'Stable Brached Curves and Braid Monodromies', *Lecture Notes in Math.* 862, Springer Verlag, Berlin-New York, 1981.

[23] M. Oka, 'Two Transforms of Plane Curves and Their Fundamental Groups', *Journal of Math. Sci. Univ. Tokyo* 3, No. 2, (1995), 399–443.

[24] R. Randell, 'Milnor Fibers and Alexander Polynomials of Plane Curves', *Singularities, Part 2 (Arcata, Calif., 1981),* Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, R.I., 1983, 415–419.

[25] H.-o Tokunaga, 'Some Examples of Zariski Pairs Arising from Certain Elliptic $K3$ Surfaces', *Math. Zeit.* 227, No. 3, (1998), 465–477.

[26] H.-o Tokunaga, 'Some Examples of Zariski Pairs Arising from Certain Elliptic $K3$ Surfaces, II: Degtyarev's Conjecture', *Math. Zeit.* 230 (1999) 389–400.

[27] O. Zariski, 'A Theorem on the Poincaré Group of an Algebraic Hypersurface'. *Annals of Maths.* 38, No. 1, (1937), 131–141.

[28] O. Zariski, 'The Topological Discriminant Group of a Riemann Surface of Genus p', *American Journal of Math.* 59, (1937), 335–358.

[29] O. Zariski, 'Algebraic Surfaces', Springer-Verlag, (1971).

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