

NEARLY HYPO STRUCTURES AND COMPACT NEARLY KÄHLER 6-MANIFOLDS WITH CONICAL SINGULARITIES

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ABSTRACT. We prove that any totally geodesic hypersurface N^5 of a 6-dimensional nearly Kähler manifold M^6 is a Sasaki-Einstein manifold, and so it has a hypo structure in the sense of [12]. We show that any Sasaki-Einstein 5-manifold defines a nearly Kähler structure on the sin-cone $N^5 \times \mathbb{R}$, and a compact nearly Kähler structure with conical singularities on $N^5 \times [0, \pi]$ when N^5 is compact thus providing a link between Calabi-Yau structure on the cone $N^5 \times [0, \pi]$ and the nearly Kähler structure on the sin-cone $N^5 \times [0, \pi]$. We define the notion of *nearly hypo* structure that leads to a general construction of nearly Kähler structure on $N^5 \times \mathbb{R}$. We determine *double hypo* structure as the intersection of hypo and nearly hypo structures and classify double hypo structures on 5-dimensional Lie algebras with non-zero first Betti number. An extension of the concept of nearly Kähler structure is introduced, which we refer to as *nearly half flat* $SU(3)$ -structure, that leads us to generalize the construction of nearly parallel G_2 -structures on $M^6 \times \mathbb{R}$ given in [3]. For $N^5 = S^5 \subset S^6$ and for $N^5 = S^2 \times S^3 \subset S^3 \times S^3$, we describe explicitly a Sasaki-Einstein hypo structure as well as the corresponding nearly Kähler structures on $N^5 \times \mathbb{R}$ and $N^5 \times [0, \pi]$, and the nearly parallel G_2 -structures on $N^5 \times \mathbb{R}^2$ and $(N^5 \times [0, \pi]) \times [0, \pi]$.

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Date: 26th May 2008.

1. INTRODUCTION

Let N^5 be a 5-manifold with an $SU(2)$ -structure, that is, the frame bundle of N^5 has a reduction to the group $SU(2)$. Recently, Conti and Salamon [12] have proved that such a structure is determined by a quadruplet $(\eta, \omega_1, \omega_2, \omega_3)$ of differential forms, that we shall abbreviate as (η, ω_i) , where η is a 1-form and ω_i are 2-forms satisfying certain relations (see Section 2). An $SU(2)$ -structure (η, ω_i) is said to be *hypo* if the 2-form ω_1 and the 3-forms $\eta \wedge \omega_2$ and $\eta \wedge \omega_3$ are closed.

Hypo geometry is a generalization of Sasaki-Einstein geometry. In fact, any Sasaki-Einstein 5-manifold has an $SU(2)$ -structure (η, ω_i) , where η is the contact form, that satisfies the differential equations

$$(1.1) \quad d\eta = -2\omega_3, \quad d\omega_1 = 3\eta \wedge \omega_2, \quad d\omega_2 = -3\eta \wedge \omega_1,$$

and so is a hypo structure, after interchanging the form ω_1 with ω_3 . This is due to the following. A Sasaki-Einstein 5-manifold N^5 may be defined as a Riemannian manifold such that $N^5 \times \mathbb{R}$ with the cone metric is Kähler and Ricci flat [5], that is, it has holonomy contained in $SU(3)$ or, equivalently, its $SU(3)$ -structure is integrable. This means that there is an almost Hermitian structure, with Kähler form F , and a complex volume form $\Psi = \Psi_+ + i\Psi_-$ on $N^5 \times \mathbb{R}$ satisfying $dF = d\Psi_+ = d\Psi_- = 0$. But an integrable $SU(3)$ -structure on the cone $N^5 \times \mathbb{R}$ induces an $SU(2)$ -structure on N^5 satisfying (1.1) (see Section 2 for details).

Our goal in this paper is twofold: on the one hand, to show that Sasaki-Einstein (hypo) 5-manifolds are also closely related with nearly Kähler 6-manifolds (weak holonomy $SU(3)$ manifolds) giving a method to construct nearly Kähler manifolds from Sasaki-Einstein 5-manifolds; and on the other hand, to give a method of construction of nearly parallel G_2 -structures on $M^6 \times \mathbb{R}$ starting from certain $SU(3)$ -structures on M^6 , which we call *nearly half flat*, leading to a generalization of the construction given in [3].

To this end, in Section 2 it is shown that any totally geodesic hypersurface N^5 of a nearly Kähler 6-manifold M^6 has a natural Sasaki-Einstein $SU(2)$ -structure $(\eta, \omega_1, \omega_2, \omega_3)$ satisfying (1.1). Furthermore, the converse also holds. In fact, we prove that any Sasaki-Einstein $SU(2)$ -structure on N^5 satisfying (1.1) defines an $SU(3)$ -structure on the sin-cone $N^5 \times \mathbb{R}$ which is nearly Kähler (see Theorem 3.7 in Section 3). Actually, our result is slightly more general and it applies to *nearly hypo* $SU(2)$ -structures satisfying the *evolution nearly hypo equations* established in Proposition 3.2. Nearly hypo structures are the natural $SU(2)$ -structures induced on oriented hypersurfaces of nearly Kähler 6-manifolds. In particular, when N^5 is a compact Sasaki-Einstein $SU(2)$ -manifold, one gets a compact nearly Kähler structure with conical singularities on $N^5 \times [0, \pi]$.

Returning to a Sasaki-Einstein structure, it can be defined as a structure whose cone is Kähler and Ricci flat. We show (see Corollary 3.8) that in dimension 5 a Sasaki-Einstein structure could also be defined as a structure whose sin-cone is nearly Kähler (weak holonomy $SU(3)$ manifold). In this way, Sasaki-Einstein 5-manifolds provide a link between Calabi-Yau cones and nearly Kähler (weak holonomy $SU(3)$) sin-cones.

More general, in Section 4, we define a *double hypo structure* as an $SU(2)$ -structure which is hypo and nearly hypo; a diagram representing the relations among the classes of $SU(2)$ -structures is inserted. In Section 4, we show also that double hypo structures are precisely those $SU(2)$ -structures whose sin-cone carry a half-flat $SU(3)$ -structure. We describe all 5-dimensional Lie algebras with non-zero first Betti number which have a double hypo structure, and prove that solvable Lie groups cannot admit invariant double hypo structures. Double hypo structures give a relation between Calabi-Yau solution to Conti-Salamon evolution equations (2.6) and nearly Kähler solution to the nearly hypo evolution equations (3.2) discovered in Proposition 3.2 below.

In [3] it is proved that if M^6 is a nearly Kähler manifold, then the sin-cone $M^6 \times \mathbb{R}$ has a natural nearly parallel G_2 -structure. We generalize this construction of nearly parallel G_2 -structures proving (see Proposition 5.2) that any *nearly half flat* $SU(3)$ -structure (F, Ψ_+, Ψ_-) on M^6 , which means that $d\Psi_- = -2F \wedge F$, can be lifted to a nearly parallel G_2 -structure on $M^6 \times \mathbb{R}$ if and only if it satisfies the *evolution nearly half flat equation* (5.6) established in Section 5. In this section we insert two figures, one of them represents the relations among the classes of $SU(3)$ -structures, and the other illustrates how it is possible to get special G_2 -metrics by evolution from $SU(3)$ -structures.

In Section 6 we consider the oriented hypersurfaces $N^5 = S^5 \subset S^6$ and $N^5 = S^2 \times S^3 \subset S^3 \times S^3$. Since $S^5 \subset S^6$ is totally geodesic in S^6 with the metric of the nearly Kähler structure on S^6 , it induces a Sasaki-Einstein hypo structure on S^5 satisfying (1.1). We describe explicitly such a structure on S^5 as well as the nearly Kähler structure on $S^5 \times \mathbb{R}$ and the nearly parallel G_2 -structure on $S^5 \times \mathbb{R}^2$.

For $S^2 \times S^3 \subset S^3 \times S^3$ we notice that $S^2 \times S^3$ is not totally geodesic in $S^3 \times S^3$ with the metric of the nearly Kähler structure, and we see that the $SU(2)$ -structure induced on $S^2 \times S^3$ is hypo but it does not satisfy the first equation of (1.1). We modify it a little to obtain a Sasaki-Einstein $SU(2)$ -structure on $S^2 \times S^3$ satisfying equations (1.1), and then we describe the nearly Kähler structure on $S^2 \times S^3 \times \mathbb{R}$ and the nearly parallel G_2 -structure on $S^2 \times S^3 \times \mathbb{R}^2$.

Finally, we use the recently discovered in [16] infinite family of explicit compact Sasaki-Einstein 5-manifold $Y^{p,q}$ to construct infinite family of compact nearly Kähler manifold with conical singularities on $Y^{p,q} \times [0, \pi]$.

2. HYPO STRUCTURES ON 5-MANIFOLDS

In this section we show that any totally geodesic hypersurface of a nearly Kähler manifold has a Sasaki-Einstein $SU(2)$ -structure satisfying (1.1). First we need to recall some properties of $SU(2)$ -structures and, in particular, of hypo structures on 5-manifolds.

Consider a 5-manifold N^5 with an $SU(2)$ -structure $(\eta, \omega_1, \omega_2, \omega_3)$, that is to say, η is a 1-form and ω_i are 2-forms on M satisfying

$$(2.1) \quad \omega_i \wedge \omega_j = \delta_{ij}v, \quad v \wedge \eta \neq 0,$$

for some 4-form v , and

$$(2.2) \quad X \lrcorner \omega_1 = Y \lrcorner \omega_2 \Rightarrow \omega_3(X, Y) \geq 0,$$

where $X \lrcorner$ denotes the contraction by X . Then, it induces an $SU(3)$ -structure (F, Ψ_+, Ψ_-) on $N^5 \times \mathbb{R}$ defined by

$$(2.3) \quad F = \omega_1 + \eta \wedge dt, \quad \Psi = \Psi_+ + i\Psi_- = (\omega_2 + i\omega_3) \wedge (\eta + idt),$$

where t is a coordinate on \mathbb{R} .

Vice versa, let $f : N^5 \rightarrow M^6$ be an oriented hypersurface of a 6-manifold M^6 with an $SU(3)$ -structure (F, Ψ_+, Ψ_-) , and denote by \mathbb{N} the unit normal vector field. Then the $SU(3)$ -structure induces an $SU(2)$ -structure $(\eta, \omega_1, \omega_2, \omega_3)$ on N^5 defined by the equalities [12]

$$(2.4) \quad \eta = -\mathbb{N} \lrcorner F, \quad \omega_1 = f^*F, \quad \omega_2 = \mathbb{N} \lrcorner \Psi_-, \quad \omega_3 = -\mathbb{N} \lrcorner \Psi_+.$$

An $SU(2)$ -structure determined by (η, ω_i) is called *hypo* if it satisfies the equations [12]

$$(2.5) \quad d\omega_1 = 0, \quad d(\eta \wedge \omega_2) = 0, \quad d(\eta \wedge \omega_3) = 0.$$

Suppose that M^6 has holonomy contained in $SU(3)$, that is, the $SU(3)$ -structure (F, Ψ_+, Ψ_-) is integrable (i.e. Calabi-Yau) or, equivalently,

$$dF = d\Psi_+ = d\Psi_- = 0.$$

It is not hard to see that any oriented hypersurface N^5 of M^6 is naturally endowed with a hypo structure [12]. Indeed, the conditions $dF = d\Psi_+ = d\Psi_- = 0$ imply that the induced $SU(2)$ -structure on N^5 defined by (2.4) satisfies (2.5). Regarding the converse, Conti and Salamon [12] prove that a real analytic hypo structure on N^5 (that is, when N^5 and the reduction of the frame bundle of N^5 both are analytic) can be lifted to an integrable $SU(3)$ -structure on $N^5 \times \mathbb{R}$, that is, (η, ω_i) belongs to a one-parameter family of hypo structures $(\eta(t), \omega_i(t))$ satisfying the evolution equations

$$(2.6) \quad \begin{cases} \partial_t \omega_1 = -d\eta \\ \partial_t(\eta \wedge \omega_3) = d\omega_2 \\ \partial_t(\eta \wedge \omega_2) = -d\omega_3. \end{cases}$$

Next we study totally geodesic hypersurfaces of nearly Kähler 6-manifolds M^6 , that is, M^6 has an $SU(3)$ -structure (F, Ψ_+, Ψ_-) which satisfies the following differential equations [21]

$$(2.7) \quad dF = 3\Psi_+, \quad d\Psi_- = -2F \wedge F.$$

Lemma 2.1. *If $f : N^5 \rightarrow M^6$ is a totally geodesic hypersurface of a nearly Kähler manifold M^6 , then the induced $SU(2)$ -structure (2.4) on N^5 satisfies the differential equations (1.1).*

Proof. Let $(M^6, g, F, \Psi_+, \Psi_-)$ be a nearly Kähler 6-manifold. The Nijenhuis tensor N is a 3-form $N = -\Psi_-$ and it is parallel with respect to the Gray characteristic connection ∇ [22]. This connection was defined by Gray [19, 18, 20] and it turns out to be the unique linear connection preserving the nearly Kähler structure and having totally skew-symmetric torsion $T = N = -\Psi_-$ [15], i.e.

$$(2.8) \quad \nabla = \nabla^g + \frac{1}{2}T = \nabla^g - \frac{1}{2}\Psi_-, \quad \nabla\Psi_- = 0,$$

where ∇^g is the Levi-Civita connection of the metric g .

We calculate using (2.4) and (2.7) that

$$(2.9) \quad d\omega_1 = d(f^*F) = 3f^*\Psi_+ = 3\eta \wedge \omega_2,$$

$$(2.10) \quad d\eta = -d(\mathbb{N}\lrcorner F) = -(L_{\mathbb{N}}F) + \mathbb{N}\lrcorner dF = -(L_{\mathbb{N}}F) - 3\omega_3,$$

where L denotes the Lie derivative.

Further, $-(L_{\mathbb{N}}F) = -\nabla^g_{\mathbb{N}}F$, since N^5 is totally geodesic. Apply (2.8) to the latter equality, take into account $\nabla F = 0$ and (2.4) to derive

$$(2.11) \quad -\nabla^g_{\mathbb{N}}F = -\nabla_{\mathbb{N}}F - \frac{1}{2} \sum_{i=1}^6 e_i \lrcorner F \wedge e_i \lrcorner (\mathbb{N}\lrcorner \Psi_-) = -\mathbb{N}\lrcorner \Psi_+ = \omega_3,$$

where $\{e_1, \dots, e_6 = \mathbb{N}\}$ is an $SU(3)$ adapted basis. Substitute (2.11) into (2.10) to get the first equality in (1.1).

In view of (2.9), it remains to prove the third equality in (1.1). Similarly as above, applying (2.4), (2.8) and (2.7), we calculate

$$(2.12) \quad \begin{aligned} d\omega_2 &= d(\mathbb{N}\lrcorner \Psi_-) = L_{\mathbb{N}}\Psi_- - \mathbb{N}\lrcorner d\Psi_- = \nabla^g_{\mathbb{N}}\Psi_- + 2\mathbb{N}\lrcorner(F \wedge F) = \\ &= -\frac{1}{4}\mathbb{N}\lrcorner\left(\sum_{j=1}^6 e_j \lrcorner \Psi_- \wedge e_j \lrcorner \Psi_-\right) + 2\mathbb{N}\lrcorner(F \wedge F) = \\ &= -\frac{1}{2}\mathbb{N}\lrcorner(F \wedge F) + 2\mathbb{N}\lrcorner(F \wedge F) = \frac{3}{2}\mathbb{N}\lrcorner(F \wedge F) = -3\eta \wedge \omega_1, \end{aligned}$$

where we have used the identity

$$\sum_{j=1}^6 e_j \lrcorner T \wedge e_j \lrcorner T = 2F \wedge F$$

valid on any nearly Kähler 6-manifold [15]. \square

Theorem 2.2. *Any totally geodesic hypersurface N^5 of a nearly Kähler 6-manifold M^6 admits a Sasaki-Einstein hypo structure, and therefore the Conti-Salamon evolution equations (2.6) can be solved for $N^5 \times \mathbb{R}$.*

Proof. Clearly Lemma 2.1 implies that the induced $SU(2)$ -structure satisfies (2.5), i.e, it is a hypo structure. Moreover, Lemma 2.1 shows that the induced almost contact metric structure (η, ω_3) on N^5 is Sasaki-Einstein. Indeed, (1.1) implies that the conical $SU(3)$ -structure on $M = N^5 \times \mathbb{R}$ defined by

$$(2.13) \quad F = t^2\omega_3 + t\eta \wedge dt, \quad \Psi = t^2(\omega_2 + i\omega_1) \wedge (t\eta + idt)$$

satisfies $dF = d\Psi = 0$, i.e. it is an integrable $SU(3)$ -structure (see e.g. [5]) which clearly is a solution to the Conti-Salamon evolution equations (2.6). \square

Remark 2.3. We notice that any Sasaki-Einstein 5-manifold has a hypo $SU(2)$ -structure which satisfies (1.1). In fact, we know that a Sasaki-Einstein 5-manifold N^5 is such that the cone $N^5 \times \mathbb{R}$ is Kähler and Ricci flat, that is, its $SU(3)$ -structure is integrable, and so induces an $SU(2)$ -structure on N^5 satisfying (1.1) which is equivalent to equations (14) in [12], although the two forms ω_2, ω_3 are not given explicitly there since the $SU(3)$ -structure

on the cone is not explicit; we just know that such a structure does exist and is given by (2.13).

3. NEARLY HYPO STRUCTURES

Let (η, ω_i) be an $SU(2)$ -structure on N^5 and consider the $SU(3)$ -structure (F, Ψ_+, Ψ_-) on $N^5 \times \mathbb{R}$ defined by (2.3).

We look for sufficient conditions imposed on the $SU(2)$ -structure (η, ω_i) which imply that the induced $SU(3)$ -structure on $N^5 \times \mathbb{R}$ is nearly Kähler, i.e. it satisfies (2.7).

Definition 3.1. We call an $SU(2)$ -structure (η, ω_i) on a 5-manifold N^5 a *nearly hypo structure* if it satisfies the following two equations:

$$(3.1) \quad d\omega_1 = 3\eta \wedge \omega_2, \quad d(\eta \wedge \omega_3) = -2\omega_1 \wedge \omega_1.$$

Consider $SU(2)$ -structures $(\eta(t), \omega_i(t))$ on N^5 depending on a real parameter $t \in \mathbb{R}$, and the corresponding $SU(3)$ -structures $(F(t), \Psi_+(t), \Psi_-(t))$ on $N^5 \times \mathbb{R}$. We have

Proposition 3.2. *An $SU(2)$ -structure (η, ω_i) on N^5 can be lifted to a nearly Kähler structure $(F(t), \Psi_+(t), \Psi_-(t))$ on $N^5 \times \mathbb{R}$ defined by (2.3) if and only if it is a nearly hypo structure which generates an 1-parameter family of $SU(2)$ -structures $(\eta(t), \omega_i(t))$ satisfying the following evolution nearly hypo equations*

$$(3.2) \quad \begin{cases} \partial_t \omega_1 = -d\eta - 3\omega_3, \\ \partial_t(\eta \wedge \omega_3) = d\omega_2 + 4\eta \wedge \omega_1, \\ \partial_t(\eta \wedge \omega_2) = -d\omega_3. \end{cases}$$

Proof. Take the exterior derivatives in (2.3) to get that the equations (2.7) hold precisely when (3.1) and the first two equalities in (3.2) are fulfilled.

It remains to show that the equations (3.2) imply that (3.1) hold for each t . Indeed, using (3.2), we calculate

$$\partial_t(d\omega_1 - 3\eta \wedge \omega_2) = -3(d\omega_3 + \partial_t(\eta \wedge \omega_2)) = 0.$$

Hence, the first equality in (3.1) is independent on t and therefore is valid for all t since it holds in the beginning for $t = 0$. Further, using the already proved first equality in (3.1) as well as the defining equalities (2.1), we obtain

$$\partial_t[d(\eta \wedge \omega_3) + 2\omega_1 \wedge \omega_1] = -4\eta \wedge d\omega_1 = 0.$$

Hence, both equalities in (3.1) survive in time. \square

Remark 3.3. The assumption $(\eta(t), \omega_i(t))$ to be an $SU(2)$ -structure for all t in Proposition 3.2 can not be avoided as it is shown in the example described in the last Section 6.4.

Proposition 3.4. *Any $SU(2)$ -structure satisfying the two first equations of (1.1) is a nearly hypo structure.*

Proof. The two first equations of (1.1) together with (2.1) yield

$$d\omega_1 = 3\eta \wedge \omega_2, \quad d(\eta \wedge \omega_3) = -2\omega_3 \wedge \omega_3 = -2\omega_1 \wedge \omega_1.$$

\square

More generally, we have

Proposition 3.5. *Let $f : N^5 \rightarrow M^6$ be an immersion of an oriented 5-manifold into a 6-manifold with a nearly Kähler structure. Then the $SU(2)$ -structure induced on N^5 is a nearly hypo structure.*

Proof. It follows from (2.4) that [12]

$$\eta \wedge \omega_2 = f^* \Psi_+, \quad \eta \wedge \omega_3 = f^* \Psi_-.$$

Since f^* commutes with d , the above equality together with (2.4) and (2.7) imply (3.1). \square

Now, a question remains.

Question 1. Does the converse of Proposition 3.5 hold?, i.e. is it true that any (real analytic) nearly hypo structure on N^5 can be lifted to a nearly Kähler structure on $N^5 \times \mathbb{R}$?

Remark 3.6. The affirmative answer to this question is equivalent to showing the existence of a solution of the evolution nearly hypo equations (3.2). From a private communication with D. Conti [13], we know that the answer of Question 1 is affirmative, at least locally, for real analytic nearly hypo structures. In fact, if N^5 is compact, there is a solution to the nearly hypo evolution equations on N^5 , i.e. the real analytic nearly hypo structure on N^5 can be lifted to a nearly Kähler structure on $N^5 \times I$, for a sufficiently small interval I ; and if N^5 is non-compact, one always has a local solution to these equations, that is, there is an open set $U \subset N^5$ such that the real analytic nearly hypo structure on the 5-manifold N^5 can be lifted to a nearly Kähler structure on $U \times I$, for a sufficiently small interval I .

Now, we prove the main result in this section solving explicitly the equations (3.2) for Sasaki-Einstein 5-manifolds.

Theorem 3.7. *Let (N^5, η, ω_i) be a Sasaki-Einstein $SU(2)$ -structure satisfying (1.1). Then the $SU(3)$ -structure F, Ψ_+, Ψ_- on $N^5 \times \mathbb{R}$ defined for $0 \leq t \leq \pi$ by*

$$(3.3) \quad \begin{aligned} F &= \sin^2 t (\sin t \omega_1 + \cos t \omega_3) + \sin t \eta \wedge dt, \\ \Psi_+ &= \sin^3 t \eta \wedge \omega_2 - \sin^2 t (-\cos t \omega_1 + \sin t \omega_3) \wedge dt, \\ \Psi_- &= \sin^3 t (-\cos t \omega_1 + \sin t \omega_3) \wedge \eta + \sin^2 t \omega_2 \wedge dt, \end{aligned}$$

is a nearly Kähler structure on $N^5 \times \mathbb{R}$ generating the well known Einstein metric

$$g_6 = dt^2 + \sin^2 t g_5,$$

where g_5 is the Sasaki-Einstein metric on N^5 .

If (N^5, η, ω_i) is compact then $(N^5 \times [0, \pi], F, \Psi_+, \Psi_-)$ is a compact nearly Kähler 6-manifold with two conical singularities at $t = 0$ and $t = \pi$.

Proof. Consider the $SU(2)$ -structure $(\eta(t), \omega_i(t))$ depending on a real parameter t :

$$(3.4) \quad \begin{aligned} \eta(t) &= \sin t \eta, \\ \omega_1(t) &= \sin^2 t (\sin t \omega_1 + \cos t \omega_3), \\ \omega_2(t) &= \sin^2 t \omega_2, \\ \omega_3(t) &= \sin^2 t (-\cos t \omega_1 + \sin t \omega_3). \end{aligned}$$

Applying (1.1) and (2.1), we see that the structure defined by (3.4) satisfies the nearly hypo structure conditions (3.1) as well as the nearly hypo evolution equations (3.2). Consequently, (3.3) satisfies (2.7) and therefore it is a nearly Kähler structure on $N^5 \times \mathbb{R}$. \square

As a consequence of the proof of Theorem 3.7, we derive

Corollary 3.8. *An $SU(2)$ -manifold (N^5, η, ω_i) is Sasaki-Einstein if and only if the sin-cone $(N^5 \times \mathbb{R}, F, \Psi_+, \Psi_-)$ with the $SU(3)$ -structure defined by (3.3) is a nearly Kähler manifold for any $0 < t < \pi$.*

Proof. The equations (3.3) imply

$$dF = \sin t dt \wedge [3 \sin t \cos t \omega_1 - 3 \sin^2 t \omega_3 + 2\omega_3 + d\eta] + \sin^2 t (\sin t d\omega_1 + \cos t d\omega_3).$$

Consequently, $dF = 3\Psi_+ \Leftrightarrow d\omega_1 = 3\eta \wedge \omega_2, \quad d\eta = -2\omega_3$. Using this equivalence, we obtain

$$d\Psi_- + 2F \wedge F = \sin^3 t [\sin t \omega_3 \wedge (d\eta + 2\omega_3) - \cos t \omega_1 \wedge d\eta] + \sin^2 t (3\omega_1 \wedge \eta + d\omega_2) \wedge dt.$$

Hence, $d\Psi_- = -2F \wedge F \Leftrightarrow d\omega_2 = -3\eta \wedge \omega_1$. Thus, (1.1) are equivalent to (2.7) and the proof is complete. \square

Remark 3.9. Any Sasaki-Einstein 5-manifold generates, on one hand, a Calabi-Yau structure on the cone and, on the other hand, it generates a nearly Kähler (weak holonomy $SU(3)$) structure on the sin-cone, thus giving a link between these two structures in dimension six. Moreover, Lemma 2.1 shows that any totally geodesic hypersurface of a nearly Kähler 6-manifold carries a natural Sasaki-Einstein structure and therefore one gets a non-compact Calabi-Yau cone generated by that structure. Vice versa, any totally umbilic hypersurface in a Calabi-Yau 6-manifold with shape operator $A = id$ carries a natural Sasaki-Einstein structure which could be lifted to a nearly Kähler structure on the sin-cone according to Theorem 3.7. It seems that a (local) description of totally geodesic hypersurfaces of a nearly Kähler 6-manifold as well as the (local) description of totally umbilic hypersurfaces of a Calabi-Yau 6-manifold with shape operator equal to the identity will provide an explicit relation between Calabi-Yau 6-manifolds and nearly Kähler 6-manifolds.

Remark 3.10. There exist nonhomogeneous examples of Sasaki-Einstein 5-manifolds; for instance, there are known 14 nonhomogeneous Sasaki-Einstein metrics on $S^2 \times S^3$ [6, 7]. Using these structures we obtain examples of local nonhomogeneous nearly Kähler 6-manifolds constructed according to Theorem 3.7.

4. DOUBLE HYPO STRUCTURES

In this section we are interested in the class of $SU(2)$ -structures on a 5-manifold which are in the intersection class of hypo and nearly hypo structures.

Definition 4.1. An $SU(2)$ -structure (η, ω_i) on a 5-manifold is said to be *double hypo* if it is hypo and nearly hypo simultaneously.

The following picture illustrates the various classes of $SU(2)$ -structures.

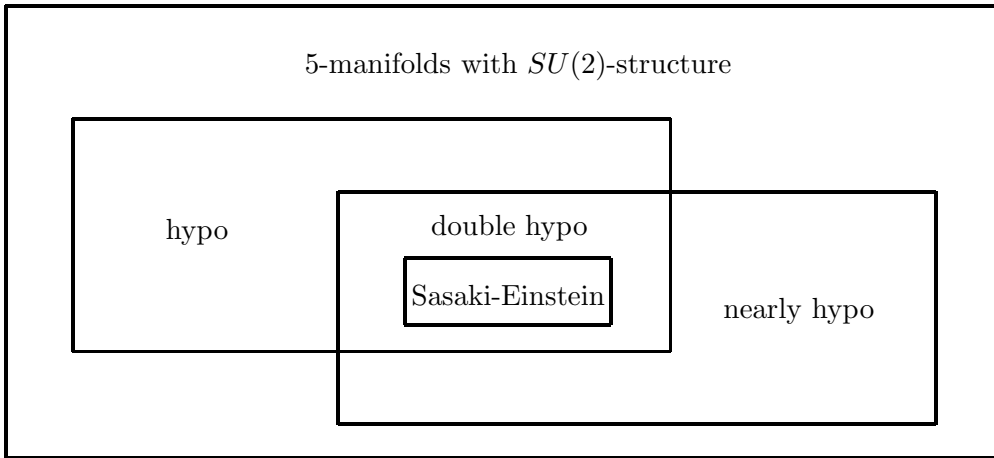


Figure 1: Classes of $SU(2)$ -structures

Double hypo structures can be lifted in the analytic case, on one hand, to an integrable $SU(3)$ -structure due to the Conti-Salamon result [12] and, on the other hand, taking account of Remark 3.6, to a nearly Kähler structure, which provides a relationship between these distinguished classes of 6-dimensional manifolds:

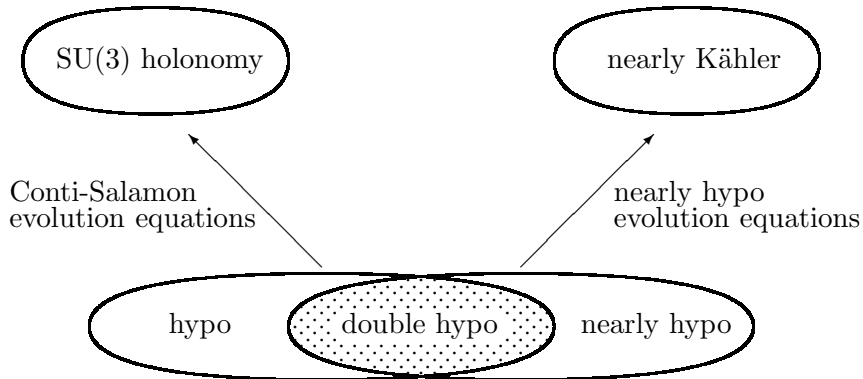


Figure 2: Special metrics obtained from evolution of $SU(2)$ -structures

In Figure 2, we write $SU(3)$ holonomy for $SU(3)$ -structures such that the holonomy of its metric is contained in $SU(3)$. Moreover, taking into account (3.3), we must notice that the sin-cone metric of a Sasaki-Einstein structure on a 5-manifold N^5 defines a nearly Kähler metric on $N^5 \times \mathbb{R}$ and, by (2.13), the cone metric of a Sasaki-Einstein structure on N^5 defines a metric on $N^5 \times \mathbb{R}$ whose holonomy is contained in $SU(3)$.

In order to give a characterization of double hypo structures, we first recall that an $SU(3)$ -structure (F, Ψ_+, Ψ_-) on a 6-manifold M^6 is called half-flat if it satisfies the conditions

$$(4.1) \quad dF \wedge F = d\Psi_+ = 0.$$

These structures become of recent interest mainly because Hitchin shows in [21] that an $SU(3)$ -structure on M^6 can be lifted to a G_2 -holonomy structure on $M^6 \times \mathbb{R}$, exactly when the underlying $SU(3)$ -structure is half flat.

Theorem 4.2. *Let (η, ω_i) be an $SU(2)$ -structure on a 5-manifold N^5 . The following conditions are equivalent:*

- i). (η, ω_i) is a double hypo structure;
- ii). (η, ω_i) satisfies the equations

$$(4.2) \quad d(\eta \wedge \omega_1) = 0, \quad d\omega_1 = 3\eta \wedge \omega_2, \quad d(\eta \wedge \omega_3) = -2\omega_1 \wedge \omega_1, \quad d\omega_3 = 0.$$

- iii). the sin-cone $(N^5 \times (0, \pi), F, \Psi_+, \Psi_-)$ with the $SU(3)$ -structure determined by (3.3) is half-flat.

Proof. The equivalence of i) and ii) is straightforward consequence from (2.5), (3.1) and (4.2).

We calculate from (3.3) that

$$d\Psi_+ = \sin^2 t dt \wedge [\cos t(3\eta \wedge \omega_2 - d\omega_1) + \sin t d\omega_3] + \sin^3 t d(\eta \wedge \omega_2).$$

Consequently, $d\Psi_+ = 0 \implies d\omega_1 = 3\eta \wedge \omega_2, \quad d\omega_3 = 0$. Using this equivalence, we obtain

$$d(F \wedge F) = 2\sin^3 t [\cos t(2\omega_1 \wedge \omega_1 + d(\eta \wedge \omega_3)) + \sin t \omega_1 \wedge d\eta] \wedge dt.$$

Hence, (4.2) are equivalent to (4.1) □

4.1. Double hypo structures on Lie groups. Next we determine the left-invariant double hypo structures on Lie groups G satisfying $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$, where \mathfrak{g} denotes the Lie algebra of G . In particular we show that solvable Lie groups cannot admit structures of this type.

Proposition 4.3. *Let G be a Lie group endowed with a left-invariant double hypo structure (η, ω_i) . If the Lie algebra \mathfrak{g} of G satisfies $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$, then there is a basis e^1, \dots, e^5 for \mathfrak{g}^* and a real number μ such that*

$$(4.3) \quad \eta = e^5, \quad \omega_1 = e^{12} + e^{34}, \quad \omega_2 = e^{13} + e^{42}, \quad \omega_3 = e^{14} + e^{23},$$

and

$$(4.4) \quad \begin{cases} de^1 = 0, \\ de^2 = \mu e^{34} - 3e^{35}, \\ de^3 = -\mu e^{24} + 3e^{25}, \\ de^4 = \mu e^{14}, \\ de^5 = -4e^{23} + \frac{\mu^2}{3}(e^{14} - e^{23}). \end{cases}$$

Proof. Let V be the subspace of \mathfrak{g}^* orthogonal to η , and let $\alpha \in \mathfrak{g}^*$ be nonzero and closed. Thus, $\alpha = \beta + \rho\eta$, where $\beta \in V$ and $\rho \in \mathbb{R}$. Now, $d\alpha = 0$ is equivalent to $d\beta = -\rho d\eta$. Therefore, $\gamma = \frac{1}{\|\beta\|} \beta$ is a unit element in $V = \langle \eta \rangle^\perp$ satisfying

$$d\gamma = \lambda d\eta,$$

with $\lambda = -\rho/\|\beta\|$. By [12, Corollary 3] there is a basis e^1, \dots, e^5 for \mathfrak{g}^* satisfying (4.3) with $e^1 = \gamma$. In terms of this basis the differentials of e^1, \dots, e^5 are given by

$$(4.5) \quad de^i = \sum_{1 \leq j < k \leq 5} c_{jk}^i e^{jk}, \quad 1 \leq i \leq 5,$$

where $c_{jk}^1 = \lambda c_{jk}^5$ for all j, k . The 41 remaining coefficients $\lambda, c_{jk}^2, \dots, c_{jk}^5$ must satisfy the Jacobi identity $d(de^i) = 0$, $1 \leq i \leq 5$, and the double hypo conditions (4.2).

First, a direct calculation shows that

$$\begin{aligned} d\omega_1 = de^{12} + de^{34} = & -(\lambda c_{13}^5 + c_{23}^2 + c_{12}^4)e^{123} - (\lambda c_{14}^5 + c_{24}^2 - c_{12}^3)e^{124} - (\lambda c_{15}^5 + c_{25}^2)e^{125} \\ & - (c_{34}^2 - c_{13}^3 - c_{14}^4)e^{134} - (c_{35}^2 - c_{15}^4)e^{135} - (c_{45}^2 + c_{15}^3)e^{145} + (\lambda c_{34}^5 + c_{23}^3 + c_{24}^4)e^{234} \\ & + (\lambda c_{35}^5 + c_{25}^4)e^{235} + (\lambda c_{45}^5 - c_{25}^3)e^{245} - (c_{35}^3 + c_{45}^4)e^{345}. \end{aligned}$$

Since $3\eta \wedge \omega_2 = 3e^{135} - 3e^{245}$, we have that $d\omega_1 = 3\eta \wedge \omega_2$ if and only if the coefficients λ, c_{jk}^i satisfy the following relations:

$$(4.6) \quad \begin{aligned} c_{25}^2 &= -\lambda c_{15}^5, \quad c_{12}^3 = \lambda c_{14}^5 + c_{24}^2, \quad c_{15}^3 = -c_{45}^2, \quad c_{25}^3 = 3 + \lambda c_{45}^5, \quad c_{12}^4 = -\lambda c_{13}^5 - c_{23}^2, \\ c_{14}^4 &= c_{34}^2 - c_{13}^3, \quad c_{15}^4 = 3 + c_{35}^2, \quad c_{24}^4 = -\lambda c_{34}^5 - c_{23}^3, \quad c_{25}^4 = -\lambda c_{35}^5, \quad c_{45}^4 = -c_{35}^3. \end{aligned}$$

On the other hand, since

$$\begin{aligned} d(\eta \wedge \omega_3) = de^{145} + de^{235} = & (c_{14}^5 + c_{23}^5)e^{1234} + (c_{15}^5 + c_{12}^2 + c_{13}^3 - c_{23}^4)e^{1235} \\ & + (\lambda c_{12}^5 - c_{25}^5 + \lambda c_{34}^5 + c_{14}^3 + c_{23}^3)e^{1245} + (\lambda c_{13}^5 - c_{35}^5 - c_{14}^2 - c_{34}^4)e^{1345} \\ & + (\lambda c_{23}^5 + c_{45}^5 - c_{24}^2 - c_{34}^3)e^{2345} \end{aligned}$$

and $\omega_1 \wedge \omega_1 = 2e^{1234}$, we conclude that $d(\eta \wedge \omega_3) = -2\omega_1 \wedge \omega_1$ if and only if

$$(4.7) \quad \begin{aligned} c_{23}^3 &= -\lambda c_{12}^5 + c_{25}^5 - \lambda c_{34}^5 - c_{14}^3, \quad c_{34}^3 = \lambda c_{23}^5 + c_{45}^5 - c_{24}^2, \quad c_{23}^4 = c_{15}^5 + c_{12}^2 + c_{13}^3, \\ c_{34}^4 &= \lambda c_{13}^5 - c_{35}^5 - c_{14}^2, \quad c_{23}^5 = -4 - c_{14}^5. \end{aligned}$$

A direct calculation shows that

$$\begin{aligned} d\omega_3 = de^{14} + de^{23} = & -c_{15}^5 e^{123} + c_{25}^5 e^{124} + (\lambda c_{35}^5 - c_{45}^2)e^{125} + c_{35}^5 e^{134} - (c_{15}^2 + c_{35}^4)e^{135} \\ & - (\lambda c_{15}^5 - c_{35}^3)e^{145} - c_{45}^5 e^{234} + (\lambda c_{15}^5 - c_{35}^3)e^{235} - (\lambda c_{25}^5 + c_{45}^3)e^{245} - (\lambda c_{35}^5 - c_{45}^2)e^{345}, \end{aligned}$$

which implies that ω_3 is closed if and only if

$$(4.8) \quad c_{45}^2 = c_{35}^3 = c_{45}^3 = c_{15}^5 = c_{25}^5 = c_{35}^5 = c_{45}^5 = 0, \quad c_{35}^4 = -c_{15}^2.$$

Moreover, the 3-form $\eta \wedge \omega_1$ is closed if and only if the additional relation

$$(4.9) \quad c_{34}^5 = -c_{12}^5$$

is satisfied.

Notice also that $0 = d^2(\omega_1) = 3d(\eta \wedge \omega_2) = 3(d\eta \wedge \omega_2 - \eta \wedge d\omega_2)$ implies $d\eta \wedge \omega_2 = \eta \wedge d\omega_2$, which is equivalent to the conditions

$$(4.10) \quad c_{23}^2 = -c_{14}^2, \quad c_{24}^2 = -4\lambda + c_{13}^2, \quad c_{24}^3 = -c_{12}^2 - c_{34}^2 + c_{13}^3, \quad c_{13}^4 = \lambda c_{12}^5 + c_{14}^3, \quad c_{24}^5 = c_{13}^5.$$

Now, it is easy to see that the coefficient of e^{245} in the 3-form $d(de^5)$ vanishes if and only if

$$(4.11) \quad c_{12}^5 = 0.$$

From (4.6)–(4.11) we get that the structure equations (4.5) reduce to

$$(4.12) \quad \begin{cases} de^1 = \lambda de^5, \\ de^2 = c_{12}^2 e^{12} + c_{13}^2 e^{13} + c_{14}^2 e^{14} + c_{15}^2 e^{15} - c_{14}^2 e^{23} - (4\lambda - c_{13}^2) e^{24} + c_{34}^2 e^{34} + c_{35}^2 e^{35}, \\ de^3 = -(4\lambda - \lambda c_{14}^5 - c_{13}^2) e^{12} + c_{13}^3 e^{13} + c_{14}^3 e^{14} - c_{14}^3 e^{23} - (c_{12}^2 + c_{34}^2 - c_{13}^3) e^{24} \\ \quad + 3e^{25} - (\lambda c_{14}^5 + c_{13}^2) e^{34}, \\ de^4 = -(\lambda c_{13}^5 - c_{14}^2) e^{12} + c_{14}^3 e^{13} + (c_{34}^2 - c_{13}^3) e^{14} + (3 + c_{35}^2) e^{15} + (c_{12}^2 + c_{13}^3) e^{23} \\ \quad + c_{14}^3 e^{24} + (\lambda c_{13}^5 - c_{14}^2) e^{34} - c_{15}^2 e^{35}, \\ de^5 = c_{13}^5 e^{13} + c_{14}^5 e^{14} - (4 + c_{14}^5) e^{23} + c_{13}^5 e^{24}, \end{cases}$$

where the 11 coefficients $\lambda, c_{12}^2, c_{13}^2, c_{14}^2, c_{15}^2, c_{34}^2, c_{35}^2, c_{13}^3, c_{14}^3, c_{13}^5$ and c_{14}^5 must satisfy the Jacobi identity $d(de^i) = 0$, for $1 \leq i \leq 5$.

For the rest of the proof we follow a decision tree depending on the nullity of the coefficients to conclude that the Jacobi identity is satisfied if and only if $c_{14}^5 = (c_{34}^2)^2/3$ and the remaining coefficients vanish. The proof of this fact is rather long but straightforward, so we omit details. \square

It is easy to see that the Lie group determined by (4.4) is isomorphic to $SU(2) \times A^2$ for $\mu = 0$ and $SU(2) \times \text{Aff}(\mathbb{R})$ for $\mu \neq 0$, where A^2 denotes a 2-dimensional abelian Lie group and $\text{Aff}(\mathbb{R})$ is the group of affine transformations of \mathbb{R} . As a consequence of Proposition 4.3, the Lie group $SU(2) \times A^2$ has a unique (up to equivalence) left-invariant double hypo structure. Moreover:

Corollary 4.4. *Let $\{X_1, X_2, X_3\}$ and $\{Y_1, Y_2\}$ be the standard basis of left-invariant vector fields on $SU(2)$ and A^2 , respectively, that is,*

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \quad [Y_1, Y_2] = 0,$$

and let us denote by $\{\alpha^i, \beta^j\}$ the dual basis of $\{X_i, Y_j\}$. For each $\rho \in \mathbb{R}$, the $SU(2)$ -structure on the Lie group $SU(2) \times A^2$ given by

$$\begin{aligned} \eta &= \frac{1}{3}\alpha^1, & \omega_1 &= \frac{1}{2\sqrt{3}}(-\alpha^2\beta^1 + \frac{\rho}{6\sqrt{3}}\alpha^2\alpha^3 + \frac{1}{3}\alpha^3\beta^2), \\ \omega_2 &= -\frac{1}{2\sqrt{3}}(\alpha^3\beta^1 + \frac{1}{3}\alpha^2\beta^2), & \omega_3 &= \frac{\rho}{6\sqrt{3}}\alpha^2\beta^1 + \frac{1}{3}\beta^1\beta^2 + \frac{1}{12}\alpha^2\alpha^3, \end{aligned}$$

is nearly hypo, and it is double hypo if and only if $\rho = 0$.

Proof. In terms of the new basis $\{e^i\}$ defined by

$$\alpha^1 = 3e^5, \quad \alpha^2 = 2\sqrt{3}e^2, \quad \alpha^3 = 2\sqrt{3}e^3, \quad \beta^1 = e^1, \quad \beta^2 = \rho e^2 + 3e^4,$$

the structure equations of the Lie group are

$$de^1 = 0, \quad de^2 = -3e^{35}, \quad de^3 = 3e^{25}, \quad de^4 = \rho e^{35}, \quad de^5 = -4e^{23},$$

and the $SU(2)$ -structure is given by (4.3). Therefore, when $\rho = 0$ the structure is precisely the one given in Proposition 4.3 for $\mu = 0$. Finally, it is easy to check that for each $\rho \neq 0$ the structure is nearly hypo but the form ω_3 is not closed. \square

From our discussion above and Remark 3.6 it follows in particular that $S^3 \times \mathbb{T}^2$ is a real analytic manifold having an analytic double hypo structure, therefore:

Corollary 4.5. *The double hypo structure on $S^3 \times \mathbb{T}^2$ given by*

$$\eta = \frac{1}{3}\alpha^1, \quad \omega_1 = -\frac{1}{2\sqrt{3}}(\alpha^2\beta^1 - \frac{1}{3}\alpha^3\beta^2), \quad \omega_2 = -\frac{1}{2\sqrt{3}}(\alpha^3\beta^1 + \frac{1}{3}\alpha^2\beta^2), \quad \omega_3 = \frac{1}{3}\beta^1\beta^2 + \frac{1}{12}\alpha^2\alpha^3,$$

can be lifted both to a nearly Kähler structure and to a Calabi-Yau structure.

From Proposition 4.3 we get that solvable Lie groups do not admit left-invariant double hypo structures. Since there exist nilpotent Lie groups having left-invariant hypo structures [12], the class of manifolds with double hypo structures is a proper subclass of that consisting of hypo manifolds. Moreover, Corollary 4.4 shows the existence of nearly hypo structures which are not double hypo.

Corollary 4.6. *The Lie group $SU(2) \times \text{Aff}(\mathbb{R})$ admits a 1-parametric family of left-invariant double hypo structures. More precisely, if $\{X_1, X_2, X_3\}$ and $\{Z_1, Z_2\}$ are the standard basis of left-invariant vector fields on $SU(2)$ and $\text{Aff}(\mathbb{R})$, respectively, that is,*

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \quad [Z_1, Z_2] = Z_2,$$

then (up to equivalence) any left-invariant double hypo structure (η, ω_i) on $SU(2) \times \text{Aff}(\mathbb{R})$ belongs to the family

$$\begin{aligned} \eta &= \frac{1}{3}(\alpha^1 + \mu\gamma^2), & \omega_1 &= \frac{1}{\mu(\mu^2+12)^{\frac{1}{2}}}(\alpha^2\gamma^1 + \mu\alpha^3\gamma^2), \\ \omega_2 &= \frac{1}{\mu(\mu^2+12)^{\frac{1}{2}}}(\alpha^3\gamma^1 - \mu\alpha^2\gamma^2), & \omega_3 &= -\frac{1}{\mu}\gamma^1\gamma^2 + \frac{1}{\mu^2+12}\alpha^2\alpha^3, \end{aligned}$$

for some $\mu \in \mathbb{R} - \{0\}$, where $\{\alpha^i, \gamma^j\}$ denotes the dual basis of $\{X_i, Z_j\}$.

Proof. It follows directly from Proposition 4.3 taking

$$\alpha^1 = -\mu e^4 + 3e^5, \quad \alpha^2 = (\mu^2 + 12)^{\frac{1}{2}} e^2, \quad \alpha^3 = (\mu^2 + 12)^{\frac{1}{2}} e^3, \quad \gamma^1 = -\mu e^1, \quad \gamma^2 = e^4.$$

\square

We finish this section by showing that the double hypo structures of Proposition 4.3 can be deformed into hypo structures. In fact, it is easy to see that for each $r \in \mathbb{R} - \{0\}$ and

$\mu, \tau \in \mathbb{R}$, the Lie group G determined by the equations

$$(4.13) \quad \begin{cases} de^1 = 0, \\ de^2 = \mu e^{34} + r e^{35}, \\ de^3 = -\mu e^{24} - r e^{25}, \\ de^4 = \mu e^{14}, \\ de^5 = (\tau - \frac{\mu^2}{r})e^{23} - \frac{\mu^2}{r}(e^{14} - e^{23}), \end{cases}$$

is isomorphic to the product $H \times K$, where $H = A^2$ for $\mu = 0$ and $H = \text{Aff}(\mathbb{R})$ for $\mu \neq 0$, and $K = \text{SU}(2)$ if $r\tau > 0$, $K = \text{SL}(2, \mathbb{R})$ if $r\tau < 0$ and $H = \text{E}(2)$ if $\tau = 0$, $\text{E}(2)$ being the group of rigid motions of Euclidean 2-space. Moreover, a direct calculation shows that the $\text{SU}(2)$ -structure given by (4.3) is always hypo, and it is double hypo if and only if $r = -3$ and $\tau = -4 - \frac{\mu^2}{3}$.

5. NEARLY HALF FLAT STRUCTURES ON 6-MANIFOLDS

In this section we generalize the construction of nearly parallel G_2 -structures on $M^6 \times \mathbb{R}$ induced from a nearly Kähler structure on M^6 described in [3]. For general results on G_2 -manifolds, see [14].

Let (F, Ψ_+, Ψ_-) be an $\text{SU}(3)$ -structure on a 6-manifold M^6 . We consider the G_2 -structure ϕ on $M^6 \times \mathbb{R}$ defined by the 3-form ϕ given by

$$(5.1) \quad \phi = F \wedge dq - \Psi_-,$$

where dq is the standard 1-form on \mathbb{R} . We also have a 4-form

$$(5.2) \quad *_7 \phi = \frac{1}{2}F \wedge F + \Psi_+ \wedge dq,$$

where $*_7$ denotes the Hodge star operator on $M^6 \times \mathbb{R}$.

Vice versa, let $f : M^6 \rightarrow P^7$ be a hypersurface of a G_2 -manifold (P^7, ϕ) and denote by \mathbb{N} the unit normal. Then the G_2 -structure ϕ induces an $\text{SU}(3)$ -structure (F, Ψ_+, Ψ_-) on M^6 defined by the equalities

$$(5.3) \quad F = \mathbb{N} \lrcorner \phi, \quad \Psi_+ = -\mathbb{N} \lrcorner *_7 \phi, \quad \Psi_- = -f^* \phi.$$

The types of the induced $U(3)$ -structures are investigated in [11, 17, 8] while the types of the induced $\text{SU}(3)$ -structures are studied recently in [10].

We recall that a G_2 -structure is called *nearly parallel* if

$$(5.4) \quad d\phi = 4*\phi.$$

It is well known that nearly parallel G_2 -structures are Einstein with positive scalar curvature $s = 54 \cdot 7 \cdot 16 = 6048$.

Hitchin shows in [21] that an $\text{SU}(3)$ -structure on M^6 can be lifted to a *parallel* G_2 -structure (5.1) on $M^6 \times \mathbb{R}$, i.e. [14], a G_2 -structure satisfying $d\phi = d*\phi = 0$ (or, equivalently, $M^6 \times \mathbb{R}$ has a metric whose holonomy is contained in G_2), exactly when the underlying $\text{SU}(3)$ -structure is half flat (note that the half-flat condition compatible with (5.1) reads $dF \wedge F = d\Psi_- = 0$).

Thus, any double hypo structure on a 5-manifold could produce a G_2 -holonomy metric by solving Hitchin's flow equations (compatible with (5.1))

$$\partial_q \Psi_- = -dF, \quad d\Psi_+ = -\frac{1}{2} \partial_q (F \wedge F)$$

since its sin-cone is half-flat due to Theorem 4.2.

Next, we search for sufficient conditions imposed on an $SU(3)$ -structure (F, Ψ_+, Ψ_-) which imply that the G_2 -structure on $M^6 \times \mathbb{R}$ determined by (5.1) is nearly parallel, i.e. it satisfies (5.4).

Definition 5.1. We call an $SU(3)$ -structure (F, Ψ_+, Ψ_-) on a 6-manifold M^6 *nearly half flat* if it satisfies the equation

$$(5.5) \quad d\Psi_- = -2F \wedge F.$$

In particular, any nearly Kähler 6-manifold carries a nearly half flat structure.

The following diagram represents the relations among $SU(3)$ -structures on 6-manifolds:

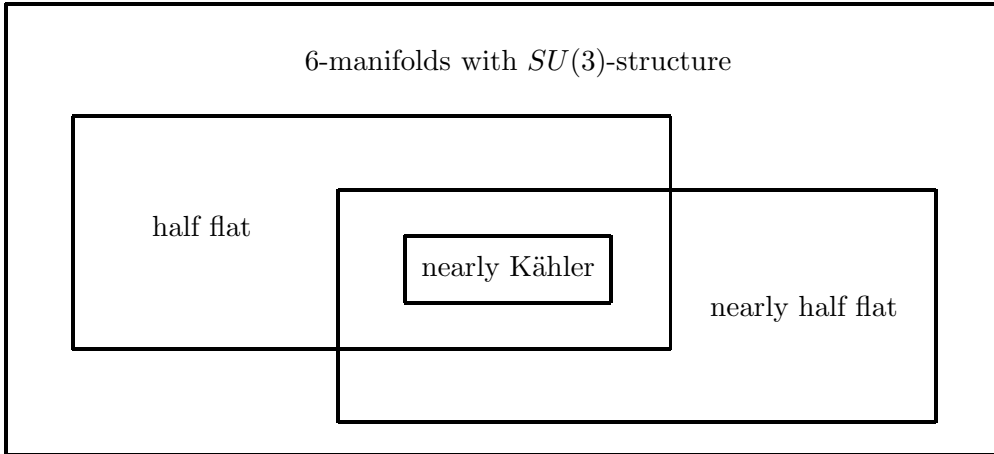


Figure 3: Classes of $SU(3)$ -structures

Consider $SU(3)$ -structures $(F(q), \Psi_+(q), \Psi_-(q))$ on M^6 depending on a real parameter $q \in \mathbb{R}$ and the corresponding G_2 -structure $\phi(q)$ on $M^6 \times \mathbb{R}$. We have

Proposition 5.2. *An $SU(3)$ -structure (F, Ψ_+, Ψ_-) on M^6 can be lifted to a nearly parallel G_2 -structure $\phi(q)$ on $M^6 \times \mathbb{R}$ defined by (5.1) if and only if it is a nearly half flat structure and the following evolution nearly half flat equations hold*

$$(5.6) \quad \begin{aligned} \partial_q \Psi_- &= 4\Psi_+ - dF, \\ d\Psi_+ &= -\frac{1}{2} \partial_q (F \wedge F). \end{aligned}$$

Proof. Take the exterior derivative in (5.1) and use (5.2) to get that the equation (5.4) holds precisely when (5.5) and (5.6) are fulfilled. Moreover, (5.6) imply that (5.5) holds for any time q due to the equality $\partial_q (d\Psi_- + 2F \wedge F) = 4d\Psi_+ + 2\partial_q (F \wedge F)$. \square

As a consequence of the above considerations, we can recover one of the main results in [3].

Theorem 5.3. [3] *Let (M^6, F, Ψ_+, Ψ_-) be a nearly Kähler structure. Then the G_2 -structure ϕ on $M^6 \times \mathbb{R}$ defined for $0 \leq q \leq \pi$ by*

$$(5.7) \quad \phi = \sin^2 q F \wedge dq - \sin^3 q (-\cos q \Psi_+ + \sin q \Psi_-)$$

is a nearly parallel G_2 -structure on $M^6 \times \mathbb{R}$ generating the well known Einstein metric

$$g_7 = dq^2 + \sin^2 q g_6,$$

where g_6 is the nearly Kähler metric on M^6 .

If (M^6, F, Ψ_+, Ψ_-) is compact then $(M^6 \times [0, \pi], \phi)$ is a compact nearly parallel G_2 -manifold with two conical singularities at $q = 0$ and $q = \pi$.

Proof. Consider the $SU(3)$ -structure $(F(q), \Psi_+(q), \Psi_-(q))$ depending on a real parameter q :

$$(5.8) \quad \begin{aligned} F(q) &= \sin^2 q F \\ \Psi_+(q) &= \sin^3 q (\sin q \Psi_+ + \cos q \Psi_-), \\ \Psi_-(q) &= \sin^3 q (-\cos q \Psi_+ + \sin q \Psi_-). \end{aligned}$$

Applying (2.7), we see that the structure defined by (5.8) satisfies the nearly half flat conditions (5.5) as well as the evolution nearly parallel equation (5.6). Consequently, the structure (5.7) satisfies (5.4) and therefore it is a nearly parallel G_2 -structure on $M^6 \times \mathbb{R}$. \square

As a consequence of the proof of Theorem 5.3, we obtain

Corollary 5.4. *An $SU(3)$ -manifold (N^6, F, Ψ_+, Ψ_-) is nearly Kähler if and only if the sine cone $(N^6 \times \mathbb{R}, \phi)$ with the G_2 -structure defined by (5.7) is a nearly parallel G_2 -manifold for any $0 < t < \pi$.*

Proof. The equations (5.7) imply

$$d\phi = \sin^3 q \cos q d\Psi_+ - \sin^4 q d\Psi_- + [\sin^2 q dF - (3\sin^2 q \cos^2 q - \sin^4 q)\Psi_+ + 4\sin^3 q \cos q \Psi_-] \wedge dq.$$

Consequently, $d\phi = 4 * \phi \Leftrightarrow d\omega_1 = 3\eta \wedge \omega_2, \quad d\eta = -2\omega_3$. Using this equivalence, we obtain

$$d\phi - 4 * \phi = \sin^3 q [\cos q d\Psi_+ - \sin q (d\Psi_- + 2F \wedge F)] + \sin^2 q (dF - 3\Psi_+) \wedge dq.$$

Hence, $d\phi = 4 * \phi \Leftrightarrow dF = 3\Psi_+, \quad d\Psi_- = -2F \wedge F$. Thus, (2.7) are equivalent to (5.4) and the proof is complete. \square

More generally we have

Proposition 5.5. *Let $f : M^6 \rightarrow P^7$ be an immersion of an oriented 6-manifold into a 7-manifold with a nearly parallel G_2 -structure. Then the $SU(3)$ -structure induced on M^6 is a nearly half flat $SU(3)$ -structure.*

Proof. Since f^* commutes with d , the equalities (5.3) substituted into (5.4) yield (5.5). \square

Question 2. Does the converse of Proposition 5.5 hold? i.e. is it true that any (real analytic) nearly half flat structure on M^6 can be lifted to a nearly parallel G_2 -structure on $M^6 \times \mathbb{R}$? This is equivalent to prove the existence of a solution of the evolution nearly half flat equation (5.6).¹

Notice that nearly Kähler structures can be lifted, on one hand, to a metric with holonomy contained in G_2 (that is, to a parallel G_2 -structure) due to Hitchin result [21] and, on the other hand, taking account Corollary 5.4, to a nearly parallel G_2 -structure, providing a relation between these special classes on 7-dimensional manifolds:

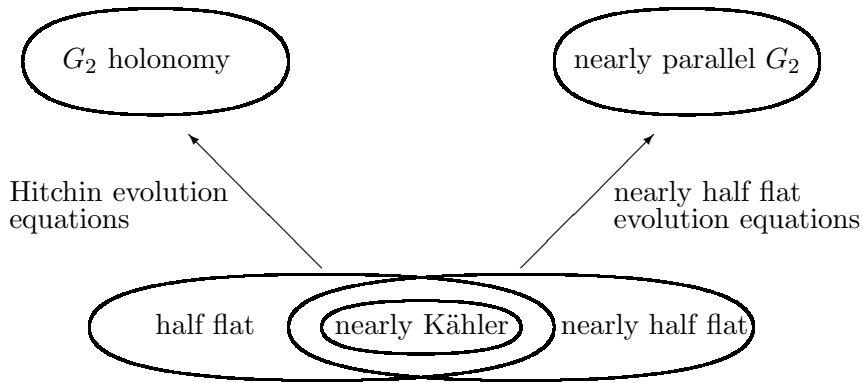


Figure 4: Special metrics obtained from evolution of $SU(3)$ -structures

6. EXAMPLES

For $N^5 = S^5 \subset S^6$ and for $N^5 = S^2 \times S^3 \subset S^3 \times S^3$, we give an explicit description of the Sasaki-Einstein hypo $SU(2)$ -structure on N^5 which generates a new nearly Kähler structure with two conical singularities on $S^2 \times S^3 \times [0, \pi]$ as well as a nearly parallel G_2 -structure on $N^5 \times [0, \pi] \times [0, \pi]$ according to Theorem 3.7 and Theorem 5.3. We also apply our results to the new compact Sasaki-Einstein manifolds $Y^{p,q}$, which are diffeomorphic to $S^2 \times S^3$ and were constructed recently in [16], to obtain a new nearly Kähler structure with two conical singularities on $Y^{p,q} \times \mathbb{R}$ and a nearly parallel G_2 -structure on $Y^{p,q} \times \mathbb{R}^2$.

Finally, we give an example of an analytic double hypo structure and a solution to the Conti-Salamon hypo evolution equations (2.6) as well as a solution to the nearly hypo evolution equations (3.2) which is an $SU(2)$ -structure only in the beginning for $t = 0$. This shows a difference between Hitchin theorem [21] which says that any solution to the Hitchin flow equations starting with a half-flat $SU(3)$ -structure is automatically a half-flat $SU(3)$ -structure for all t .

6.1. The Nearly Kähler structure on $S^5 \times \mathbb{R}$. We begin with an explicit description of

¹Recently we learned that Stock proves in Theorem 2.5 of [24] that Question 2 has an affirmative answer for nearly half flat structures on closed 6-manifolds M^6 , i.e. they can be lifted to a nearly parallel G_2 -structure on $M^6 \times I$, for a sufficiently small interval I .

6.1.1. *The standard $SU(3)$ -structure on S^6 .* Using the stereographic projection of $S^6 - \{p\}$ on \mathbb{R}^6 from the point $p = (0, \dots, 0, 1) \in \mathbb{R}^7$, one can check that a basis for the vector fields on $S^6 - \{p\}$ consists of $\{E_i; 1 \leq i \leq 6\}$ with

$$\begin{aligned} (E_1)_x &= (1 - x_7 - x_1^2, -x_1x_2, -x_1x_3, -x_1x_4, -x_1x_5, -x_1x_6, x_1(1 - x_7)), \\ (E_2)_x &= (-x_1x_2, 1 - x_7 - x_2^2, -x_2x_3, -x_2x_4, -x_2x_5, -x_2x_6, x_2(1 - x_7)), \\ (E_3)_x &= (-x_1x_3, -x_2x_3, 1 - x_7 - x_3^2, -x_3x_4, -x_3x_5, -x_3x_6, x_3(1 - x_7)), \\ (E_4)_x &= (-x_1x_4, -x_2x_4, -x_3x_4, 1 - x_7 - x_4^2, -x_4x_5, -x_4x_6, x_4(1 - x_7)), \\ (E_5)_x &= (-x_1x_5, -x_2x_5, -x_3x_5, -x_4x_5, 1 - x_7 - x_5^2, -x_5x_6, x_5(1 - x_7)), \\ (E_6)_x &= (-x_1x_6, -x_2x_6, -x_3x_6, -x_4x_6, -x_5x_6, 1 - x_7 - x_6^2, x_6(1 - x_7)), \end{aligned}$$

for any arbitrary point $x \in S^6 - \{p\}$. (Notice that this basis is orthogonal and $\|E_i\|^2 = (1 - x_7)^2$.) The basis $\{\alpha_i; 1 \leq i \leq 6\}$ for the 1-forms on $S^6 - \{p\}$ dual to $\{E_i; 1 \leq i \leq 6\}$ is given by

$$\alpha_i = \frac{1}{1 - x_7} dx_i + \frac{x_i}{(1 - x_7)^2} dx_7.$$

From now on, we write $x_{ij} = x_i x_j$, $x_{ijk} = x_i x_j x_k$, $dx_{ij} = dx_i \wedge dx_j$, and so forth. We will need also the expressions of α_{ij} and α_{ijk} in terms of dx_{ij} and dx_{ijk} , respectively;

$$\alpha_{ij} = \frac{1}{(1 - x_7)^2} dx_{ij} + \frac{1}{(1 - x_7)^3} (x_j dx_{i7} - x_i dx_{j7}),$$

for $1 \leq i < j \leq 6$, and

$$\alpha_{ijk} = \frac{1}{(1 - x_7)^3} dx_{ijk} + \frac{1}{(1 - x_7)^4} (x_i dx_{jk7} - x_j dx_{ik7} + x_k dx_{ij7}),$$

for $1 \leq i < j < k \leq 6$. Let $U = \sum_{i=1}^7 x_i \frac{\partial}{\partial x_i}$ be the unit normal vector field to $S^6 - \{p\}$. We identify \mathbb{R}^7 with the imaginary part of the space of Cayley numbers, and define a vector cross product $x \times y$, where $x, y \in \mathbb{R}^7$, by the imaginary part of the Cayley number xy . Then, the standard almost complex structure on S^6 is defined by $J(X) = U \times X$ for any vector field X

on S^6 . A simple calculation shows that

$$\begin{aligned}
(JE_1)_x &= (-x_{16}, x_{15} + x_3(1 - x_7), x_{14} - x_2(1 - x_7), -x_{13} + x_5(1 - x_7), \\
&\quad -x_{12} - x_4(1 - x_7), x_1^2 - x_7(1 - x_7), x_6(1 - x_7)), \\
(JE_2)_x &= (-x_{26} - x_3(1 - x_7), x_{25}, x_{24} + x_1(1 - x_7), -x_{23} + x_6(1 - x_7), \\
&\quad -x_2^2 + x_7(1 - x_7), x_{12} - x_4(1 - x_7), -x_5(1 - x_7)), \\
(JE_3)_x &= (-x_{36} + x_2(1 - x_7), x_{35} - x_1(1 - x_7), x_{34}, -x_3^2 + x_7(1 - x_7), \\
&\quad -x_{23} - x_6(1 - x_7), x_{13} + x_5(1 - x_7), -x_4(1 - x_7)), \\
(JE_4)_x &= (-x_{46} - x_5(1 - x_7), x_{45} - x_6(1 - x_7), x_4^2 - x_7(1 - x_7), -x_{34}, \\
&\quad -x_{24} + x_1(1 - x_7), x_{14} + x_2(1 - x_7), x_3(1 - x_7)), \\
(JE_5)_x &= (-x_{56} + x_4(1 - x_7), x_5^2 - x_7(1 - x_7), x_{45} + x_6(1 - x_7), \\
&\quad -x_{35} - x_1(1 - x_7), -x_{25}, x_{15} - x_3(1 - x_7), x_2(1 - x_7)), \\
(JE_6)_x &= (-x_6^2 + x_7(1 - x_7), x_{56} + x_4(1 - x_7), x_{46} - x_5(1 - x_7), -x_{36} - x_2(1 - x_7), \\
&\quad -x_{26} + x_3(1 - x_7), x_{16}, -x_1(1 - x_7)).
\end{aligned}$$

Now we take the natural metric g on $S^6 - \{p\}$. Thus, $(S^6 - \{p\}, g, J)$ is a nearly Kähler manifold and hence has an $SU(3)$ -structure. The Kähler form, $F(X, Y) = g(JX, Y)$, for any X, Y vector fields on $S^6 - \{p\}$, has the form

$$F = \left(\sum_{i=1}^6 x_i dx_i \right) \wedge (-\beta)/(1 - x_7) + \beta_1 + \beta \wedge dx_7/(1 - x_7),$$

where β is the 1-form

$$\beta = x_6 dx_1 - x_1 dx_6 + x_2 dx_5 - x_5 dx_2 - x_4 dx_3 + x_3 dx_4,$$

and β_1 is the 2-form given by

$$\begin{aligned}
\beta_1 &= x_7(-dx_{16} + dx_{25} + dx_{34}) + x_1 dx_{23} + x_3 dx_{12} - x_2 dx_{13} + x_1 dx_{45} \\
&\quad + x_5 dx_{14} - x_4 dx_{15} + x_2 dx_{46} + x_6 dx_{24} - x_4 dx_{26} - x_3 dx_{56} - x_6 dx_{35} + x_5 dx_{36}.
\end{aligned}$$

Now, using that $\sum_{i=1}^7 x_i dx_i = 0$, it follows that

$$F = \beta \wedge dx_7 + \beta_1.$$

Then it is easy to obtain

$$dF = 3(dx_{257} + dx_{347} - dx_{167} + dx_{123} + dx_{145} + dx_{246} - dx_{356}).$$

Now, a long calculation shows that JdF is expressed, in terms of the α_{ijk} , as

$$\begin{aligned}
JdF = & 3(1-x_7)^2(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2+x_7^2) \left((-x_{34}-x_{25}+x_{16})\alpha_{123} \right. \\
& - (x_1^2+x_2^2+x_4^2-x_7(1-x_7))\alpha_{124} + (x_{23}-x_{45}-x_6(1-x_7))\alpha_{125} \\
& - (x_{13}+x_{46}-x_5(1-x_7))\alpha_{126} - (x_{23}-x_{45}+x_6(1-x_7))\alpha_{134} \\
& + (x_1^2+x_3^2+x_5^2-x_7(1-x_7))\alpha_{135} + (x_{12}+x_{56}+x_4(1-x_7))\alpha_{136} \\
& + (x_{34}+x_{25}+x_{16})\alpha_{145} - (x_{15}-x_{26}+x_3(1-x_7))\alpha_{146} \\
& + (x_{14}-x_{36}-x_2(1-x_7))\alpha_{156} + (x_{13}+x_{46}+x_5(1-x_7))\alpha_{234} \\
& + (x_{12}+x_{56}-x_4(1-x_7))\alpha_{235} + (x_2^2+x_3^2+x_6^2-x_7(1-x_7))\alpha_{236} \\
& - (x_{15}-x_{26}-x_3(1-x_7))\alpha_{245} - (-x_{34}+x_{25}+x_{16})\alpha_{246} \\
& - (-x_{24}-x_{35}-x_1(1-x_7))\alpha_{256} - (x_{14}-x_{36}+x_2(1-x_7))\alpha_{345} \\
& - (x_{24}+x_{35}-x_1(1-x_7))\alpha_{346} + (x_{34}-x_{25}+x_{16})\alpha_{356} \\
& \left. + (x_4^2+x_5^2+x_6^2-x_7(1-x_7))\alpha_{456} \right).
\end{aligned}$$

The 3-forms Ψ_+ and Ψ_- of the $SU(3)$ -structure on $S^6 - \{p\}$ are given by

$$\begin{aligned}
\Psi_+ &= \frac{1}{3}dF = dx_{257} + dx_{347} - dx_{167} + dx_{123} + dx_{145} + dx_{246} - dx_{356}, \\
\Psi_- &= \frac{1}{3}JdF = \frac{1}{(1-x_7)} \left(-x_4dx_{127} + x_5dx_{137} \right. \\
& \quad + x_2dx_{147} - x_3dx_{157} + x_6dx_{237} - x_1dx_{247} - x_3dx_{267} + x_1dx_{357} \\
& \quad \left. + x_2dx_{367} + x_6dx_{457} - x_5dx_{467} + x_4dx_{567} \right) + \text{terms not containing } dx_7.
\end{aligned}$$

6.1.2. *The $SU(2)$ -structure on S^5 .* Let us consider $S^5 = \{(x_1, \dots, x_6) \in \mathbb{R}^6 \mid \sum_{i=1}^6 x_i^2 = 1\} \subset S^6$, and $\mathbb{N} = \frac{\partial}{\partial x_7}$ the unit normal vector field to S^5 . Then, using (2.4), the $SU(2)$ -structure (η, ω_i) on S^5 is given by

$$\begin{aligned}
(6.1) \quad \eta &= -\frac{\partial}{\partial x_7} \lrcorner F = x_6dx_1 - x_1dx_6 + x_2dx_5 - x_5dx_2 + x_3dx_4 - x_4dx_3, \\
\omega_1 &= f^*(F) = x_3dx_{12} - x_2dx_{13} + x_1dx_{23} + x_5dx_{14} - x_4dx_{15} + x_1dx_{45} \\
& \quad + x_6dx_{24} - x_4dx_{26} + x_2dx_{46} + x_5dx_{36} - x_6dx_{35} - x_3dx_{56}, \\
\omega_2 &= \frac{\partial}{\partial x_7} \lrcorner \Psi_- = -x_4dx_{12} + x_5dx_{13} + x_2dx_{14} - x_3dx_{15} + x_6dx_{23} - x_1dx_{24} \\
& \quad - x_3dx_{26} + x_1dx_{35} + x_2dx_{36} + x_6dx_{45} - x_5dx_{46} + x_4dx_{56}, \\
\omega_3 &= -\frac{\partial}{\partial x_7} \lrcorner \Psi_+ = dx_{16} - dx_{34} - dx_{25}.
\end{aligned}$$

Next we show that the $SU(2)$ -structure on S^5 defined by (6.1) satisfies Lemma 2.1. First, we see that

$$d\eta = -2(dx_{16} - dx_{34} - dx_{25}) = -2\omega_3.$$

The expression of ω_1 gives

$$d\omega_1 = 3(dx_{123} + dx_{246} + dx_{145} - dx_{356}).$$

Using that $\sum_{i=1}^6 x_i^2 = 1$, so $\sum_{i=1}^6 x_i dx_i = 0$ on S^5 , we verify

$$3\eta \wedge \omega_2 = d\omega_1, \quad -3\eta \wedge \omega_1 = d\omega_2$$

Now, we apply Theorem 3.7 and Theorem 5.3 to get

Theorem 6.1. *Let $(S^5, \eta, \omega_i, g_5)$ be the standard Sasaki-Einstein 5-sphere endowed with the $SU(2)$ -structure determined by (6.1). Then*

- i) *The $SU(3)$ -structure on $S^5 \times [0, \pi]$ defined by (3.3) is a nearly Kähler structure generating the round metric on the 6-sphere, $g_6 = dt^2 + \sin^2 t g_5$, with two conical singularities at $t = 0, t = \pi$.*
- ii) *The G_2 -structure on $(S^5 \times [0, \pi]) \times [0, \pi]$ defined by (5.7) is a nearly parallel G_2 -structure generating the round metric on the 7-sphere, $g_7 = dq^2 + \sin^2 q(dt^2 + \sin^2 t g_5)$ with singularities at $t = 0, t = \pi, q = 0, q = \pi$.*

6.2. The Nearly Kähler structure on $S^2 \times S^3 \times \mathbb{R}$. As in the previous example, first we describe explicitly

6.2.1. The standard $SU(3)$ -structure on $S^3 \times S^3$. Let us consider the sphere S^3 , viewed as the Lie group $SU(2)$, with the basis of left-invariant 1-forms $\{\alpha_1, \alpha_2, \alpha_3\}$ satisfying

$$d\alpha_1 = -\alpha_2 \wedge \alpha_3, \quad d\alpha_2 = \alpha_1 \wedge \alpha_3, \quad d\alpha_3 = -\alpha_1 \wedge \alpha_2.$$

Denote by $\{\beta_1, \beta_2, \beta_3\}$ another basis on a second sphere S^3 satisfying the same relations. Then, a nearly Kähler structure on $S^3 \times S^3$ is given by ([1], [9])

$$F = \frac{i}{2}(\mu_1 \wedge \bar{\mu}_1 + \mu_2 \wedge \bar{\mu}_2 + \mu_3 \wedge \bar{\mu}_3), \quad \Psi = i(\mu_1 \wedge \mu_2 \wedge \mu_3),$$

where $\mu_j = \frac{1}{3}(\alpha_j + e^{\frac{2\pi i}{3}} \beta_j)$, for $j = 1, 2, 3$.

In terms of the real forms $\{\alpha_j, \beta_j\}$, the forms F , Ψ_+ and Ψ_- are expressed as

$$\begin{aligned} F &= \frac{\sqrt{3}}{18}(\alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 + \alpha_3 \wedge \beta_3), \\ \Psi_+ &= \frac{\sqrt{3}}{54}(-\alpha_{12} \wedge \beta_3 + \alpha_{13} \wedge \beta_2 - \alpha_{23} \wedge \beta_1 + \alpha_1 \wedge \beta_{23} - \alpha_2 \wedge \beta_{13} + \alpha_3 \wedge \beta_{12}), \\ \Psi_- &= \frac{1}{54}(2\alpha_{123} - \alpha_{12} \wedge \beta_3 + \alpha_{13} \wedge \beta_2 - \alpha_{23} \wedge \beta_1 - \alpha_1 \wedge \beta_{23} + \alpha_2 \wedge \beta_{13} - \alpha_3 \wedge \beta_{12} + 2\beta_{123}). \end{aligned}$$

It is easy to check that the corresponding metric on $S^3 \times S^3$ is

$$(6.2) \quad g = \frac{1}{9}(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 - \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3).$$

6.2.2. *The $SU(2)$ -structure on $S^2 \times S^3$.* In order to show explicitly the induced $SU(2)$ -structure on the hypersurface $S^2 \times S^3$, we first describe $S^3 \times S^3$ as the submanifold of \mathbb{R}^8 ,

$$S^3 \times S^3 = \{(x_1, \dots, x_4, x_5, \dots, x_8) \in \mathbb{R}^8 \mid x_1^2 + \dots + x_4^2 = x_5^2 + \dots + x_8^2 = 1\}.$$

With this description, we can identify

$$\begin{aligned} \alpha_1 &= 2x_4dx_1 + 2x_3dx_2 - 2x_2dx_3 - 2x_1dx_4, & \beta_1 &= 2x_8dx_5 + 2x_7dx_6 - 2x_6dx_7 - 2x_5dx_8, \\ \alpha_2 &= -2x_3dx_1 + 2x_4dx_2 + 2x_1dx_3 - 2x_2dx_4, & \beta_2 &= -2x_7dx_5 + 2x_8dx_6 + 2x_5dx_7 - 2x_6dx_8, \\ \alpha_3 &= 2x_2dx_1 - 2x_1dx_2 + 2x_4dx_3 - 2x_3dx_4, & \beta_3 &= 2x_6dx_5 - 2x_5dx_6 + 2x_8dx_7 - 2x_7dx_8. \end{aligned}$$

We shall denote by $\{U_j, V_j\}_{j=1}^3$ the basis of vector fields on $S^3 \times S^3$ dual to $\{\alpha_j, \beta_j\}_{j=1}^3$.

Let us consider the hypersurface $S^2 \times S^3 \subset S^3 \times S^3$ given by $x_4 = 0$. Then, with respect to the metric (6.2), the vector field

$$\mathbb{N} = -\sqrt{3}(2x_1U_1 + 2x_2U_2 + 2x_3U_3 + x_1V_1 + x_2V_2 + x_3V_3)$$

is a unit normal vector field along $S^2 \times S^3$.

Next, we describe explicitly the induced $SU(2)$ -structure (2.4), taking f as the inclusion map.

A direct calculation, using that $x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 \equiv 0$ on $S^2 \times S^3$, shows that the form η is expressed as

$$(6.3) \quad \begin{aligned} \eta = -\mathbb{N} \lrcorner F &= \frac{1}{3}(x_1\beta_1 + x_2\beta_2 + x_3\beta_3) = \frac{2}{3}((x_{18} - x_{27} + x_{36})dx_5 \\ &+ (x_{17} + x_{28} - x_{35})dx_6 + (-x_{16} + x_{25} + x_{38})dx_7 + (-x_{15} - x_{26} - x_{37})dx_8). \end{aligned}$$

Since f is the inclusion, taking $x_4 = 0$ in the expressions of α_j above, we get

$$\begin{aligned} \omega_1 = f^*F &= \frac{2\sqrt{3}}{9} \left((x_{26} + x_{37})dx_{15} + (-x_{25} - x_{38})dx_{16} + (x_{28} - x_{35})dx_{17} \right. \\ &+ (-x_{27} + x_{36})dx_{18} + (-x_{16} + x_{38})dx_{25} + (x_{15} + x_{37})dx_{26} \\ &+ (-x_{18} - x_{36})dx_{27} + (x_{17} - x_{35})dx_{28} + (-x_{17} - x_{28})dx_{35} \\ &\left. + (x_{18} - x_{27})dx_{36} + (x_{15} + x_{26})dx_{37} + (-x_{16} + x_{25})dx_{38} \right). \end{aligned}$$

For computing ω_2 and ω_3 , take into account the equality $x_3\alpha_1 \wedge \alpha_2 - x_2\alpha_1 \wedge \alpha_3 + x_1\alpha_2 \wedge \alpha_3 = 4(x_3dx_{12} - x_2dx_{13} + x_1dx_{23})$, to get

$$\begin{aligned}\omega_2 &= \mathbb{N} \lrcorner \Psi_- = \frac{2\sqrt{3}}{9} \left(-x_3dx_{12} + x_2dx_{13} - x_1dx_{23} + x_8dx_{15} + x_7dx_{16} - x_6dx_{17} - x_5dx_{18} \right. \\ &\quad \left. - x_7dx_{25} + x_8dx_{26} + x_5dx_{27} - x_6dx_{28} + x_6dx_{35} - x_5dx_{36} + x_8dx_{37} - x_7dx_{38} \right); \\ \omega_3 &= -\mathbb{N} \lrcorner \Psi_+ = \frac{2}{9} \left(-x_3dx_{12} + x_2dx_{13} - x_1dx_{23} - x_8dx_{15} - x_7dx_{16} + x_6dx_{17} + x_5dx_{18} \right. \\ &\quad \left. + x_7dx_{25} - x_8dx_{26} - x_5dx_{27} + x_6dx_{28} - x_6dx_{35} + x_5dx_{36} - x_8dx_{37} + x_7dx_{38} \right) \\ &\quad + \frac{4}{9} \left(x_3dx_{56} - x_2dx_{57} + x_1dx_{58} + x_1dx_{67} + x_2dx_{68} + x_3dx_{78} \right).\end{aligned}$$

Notice that $S^2 \times S^3$ is *not* a totally geodesic hypersurface of $S^3 \times S^3$; for example, for $T = x_2U_1 - x_1U_2$ which is tangent to $S^2 \times S^3$, we have

$$g(\nabla_T \mathbb{N}, V_3) = -\frac{\sqrt{3}}{36}(x_1^2 + x_2^2),$$

which is non-zero on $S^2 \times S^3$, and thus the second fundamental form does not vanish identically. Therefore, we cannot apply Lemma 2.1 to establish that the $SU(2)$ -structure (η, ω_i) induced on $S^2 \times S^3$ from the nearly Kähler structure of $S^3 \times S^3$ is hypo. To solve this problem, we proceed as follows. We have

$$\begin{aligned}d\eta &= \frac{1}{3}(dx_1 \wedge \beta_1 - x_1 \wedge \beta_{23} + dx_2 \wedge \beta_2 + x_2 \wedge \beta_{13} + dx_3 \wedge \beta_3 - x_3 \wedge \beta_{12}) \\ &= \frac{2}{3} \left(x_8dx_{15} + x_7dx_{16} - x_6dx_{17} - x_5dx_{18} - x_7dx_{25} \right. \\ &\quad \left. + x_8dx_{26} + x_5dx_{27} - x_6dx_{28} + x_6dx_{35} - x_5dx_{36} + x_8dx_{37} - x_7dx_{38} \right. \\ &\quad \left. - 2x_3dx_{56} + 2x_2dx_{57} - 2x_1dx_{58} - 2x_1dx_{67} - 2x_2dx_{68} - 2x_3dx_{78} \right),\end{aligned}$$

and so we can write

$$\omega_3 = -\frac{1}{3}d\eta + \frac{2}{9}(-x_3dx_{12} + x_2dx_{13} - x_1dx_{23}),$$

which implies that

$$d\eta \neq -2\omega_3,$$

since the form $-x_3dx_{12} + x_2dx_{13} - x_1dx_{23}$ is the standard volume form on S^2 , and

$$d\omega_3 = 0,$$

because $d(-x_3dx_{12} + x_2dx_{13} - x_1dx_{23}) = 0$ on $S^2 \times S^3$. Moreover, we get

$$\begin{aligned}d\omega_1 &= 3\eta \wedge \omega_2, \\ d\omega_2 &= -3\eta \wedge \omega_1.\end{aligned}$$

The previous equalities show that (η, ω_i) is hypo on $S^2 \times S^3$, but it does not satisfy equations (1.1) because $d\eta \neq -2\omega_3$.

On the other hand, a direct calculation shows that

$$\eta \wedge (d\eta)^2 = -\frac{2}{27}(x_3 dx_{12} - x_2 dx_{13} + x_1 dx_{23}) \wedge \beta_{123} \neq 0,$$

so η is a contact form on $S^2 \times S^3$.

Remark 6.2. Let us see that $3\eta = x_1\beta_1 + x_2\beta_2 + x_3\beta_3 \in \Omega^1(S^2 \times S^3)$ is the natural contact form on $S^2 \times S^3$ seen as the tangent sphere bundle over S^3 (see [4]). As S^3 is parallelizable, the tangent bundle to S^3 is isomorphic to $\mathbb{R}^3 \times S^3$. Let V_1, V_2, V_3 be an orthonormal basis of left-invariant vector fields, and let $\beta_1, \beta_2, \beta_3$ be the dual basis of left-invariant 1-forms. The isomorphism $\mathbb{R}^3 \times S^3 \cong TS^3$ is given by $((a_1, a_2, a_3), p) \mapsto \sum a_i V_i(p)$. The metric of S^3 is $g = \beta_1^2 + \beta_2^2 + \beta_3^2$. Consider the unit sphere in the tangent bundle $T_1 S^3 \cong S^2 \times S^3$. If x_1, x_2, x_3 are the natural coordinates in the \mathbb{R}^3 factor of $\mathbb{R}^3 \times S^3$, then $T_1 S^3$ is given by the equation $x_1^2 + x_2^2 + x_3^2 = 1$.

The natural 1-form of T^*S^3 (the Liouville form) is given as $\lambda \in \Omega^1(T^*S^3)$, $\lambda_\alpha(v) = \alpha(d\pi(v))$, where $\pi : T^*S^3 \rightarrow S^3$. Using the metric, we identify $g : TS^3 \cong T^*S^3$. Then $g^*\lambda|_{T_1 S^3}$ is the natural contact form for $T_1 S^3$ (see [4]).

It is easy to see that $3\eta = g^*\lambda|_{T_1 S^3}$. Actually, $x_1\beta_1 + x_2\beta_2 + x_3\beta_3 = g^*\lambda \in \Omega^1(TS^3)$. Equivalently, we need to see that $y_1\beta_1 + y_2\beta_2 + y_3\beta_3 = \lambda \in \Omega^1(T^*S^3)$, where y_1, y_2, y_3 are the coordinates of the \mathbb{R}^3 factor of $T^*S^3 \cong \mathbb{R}^3 \times S^3$. But take $\alpha = \sum a_i \beta_i(p) \in T_p^*S^3$. Then

$$\lambda_\alpha(v_1, v_2) = \alpha(v_2) = \sum a_i \beta_i(p)(v_2) = \left(\sum y_i \beta_i \right) ((a_1, a_2, a_3), p)(v_1, v_2),$$

for $(v_1, v_2) \in T_\alpha(\mathbb{R}^3 \times S^3) = T_\alpha(T^*S^3)$, identifying β_i in S^3 with its pull-back to $\mathbb{R}^3 \times S^3$.

The following result shows how the hypo structure on $S^2 \times S^3$ described above can be deformed into a double hypo structure, and even into a Sasaki-Einstein structure.

Proposition 6.3. *Let (η, ω_i) be the hypo structure on $S^2 \times S^3$ given above. For each $\lambda < 0$ and $\mu > \frac{\lambda}{3}$, the quadruplet*

$$(6.4) \quad (\tilde{\eta} = \eta, \quad \tilde{\omega}_1 = \sqrt{3\lambda(\lambda - 3\mu)} \omega_1, \quad \tilde{\omega}_2 = \sqrt{3\lambda(\lambda - 3\mu)} \omega_2, \quad \tilde{\omega}_3 = \lambda d\eta + \mu \text{vol}_{S^2})$$

defines a hypo structure on $S^2 \times S^3$, which is double hypo if and only if $\lambda < -\frac{1}{4}$ and $\mu = \frac{\lambda(4\lambda+2)}{3(4\lambda+1)}$. Moreover, the $SU(2)$ -structure (6.4) is Sasaki-Einstein only for $\lambda = -\frac{1}{2}$ and $\mu = 0$.

Proof. Since (η, ω_i) is a $SU(2)$ -structure and $d\eta \wedge d\eta = -\frac{2}{3}d\eta \wedge \text{vol}_{S^2}$, we have that

$$\tilde{\omega}_i \wedge \tilde{\omega}_i = \lambda(\lambda - 3\mu)d\eta \wedge d\eta$$

for $i = 1, 2, 3$. Moreover, $\omega_i \wedge \text{vol}_{S^2} = 0$ and $\omega_i \wedge d\eta = 0$ for $i = 1, 2$, so the quadruplet (6.4) satisfies (2.1).

In order to see that (6.4) also satisfies condition (2.2), let $X = \sum_{i=1}^3 (f_i U_i + a_i V_i), Y = \sum_{i=1}^3 (g_i U_i + b_i V_i)$ be vector fields such that $X \lrcorner \tilde{\omega}_1 = Y \lrcorner \tilde{\omega}_2$. This condition implies that

$$\begin{aligned} x_2 g_3 - x_3 g_2 &= x_3(x_3 f_1 - x_1 f_3) - x_2(x_1 f_2 - x_2 f_1), \\ x_3 g_1 - x_1 g_3 &= x_1(x_1 f_2 - x_2 f_1) - x_3(x_2 f_3 - x_3 f_2), \\ x_1 g_2 - x_2 g_1 &= x_2(x_2 f_3 - x_3 f_2) - x_1(x_3 f_1 - x_1 f_3), \end{aligned}$$

and

$$\begin{aligned} b_1 &= x_2 a_3 - x_3 a_2 + x_3(x_3 g_1 - x_1 g_3) - x_2(x_1 g_2 - x_2 g_1), \\ b_2 &= x_3 a_1 - x_1 a_3 + x_1(x_1 g_2 - x_2 g_1) - x_3(x_2 g_3 - x_3 g_2), \\ b_3 &= x_1 a_2 - x_2 a_1 + x_2(x_2 g_3 - x_3 g_2) - x_1(x_3 g_1 - x_1 g_3). \end{aligned}$$

Then, on $S^2 \times S^3$ we have that

$$d\eta(X, Y) = -\frac{1}{6} (b_1^2 + b_2^2 + b_3^2 + (x_2 a_3 - x_3 a_2)^2 + (x_3 a_1 - x_1 a_3)^2 + (x_1 a_2 - x_2 a_1)^2)$$

and

$$\text{vol}_{S^2}(X, Y) = \frac{1}{4} ((b_1 - x_2 a_3 + x_3 a_2)^2 + (b_2 - x_3 a_1 + x_1 a_3)^2 + (b_3 - x_1 a_2 + x_2 a_1)^2).$$

Therefore, $(\lambda d\eta + \mu \text{vol}_{S^2})(X, Y) \geq 0$ when $\lambda < 0$ and $\mu > \lambda/3$.

The $SU(2)$ -structure (6.4) clearly satisfies that $d\tilde{\omega}_1 = 3\tilde{\eta} \wedge \tilde{\omega}_2$, $d\tilde{\omega}_2 = -3\tilde{\eta} \wedge \tilde{\omega}_3$ and $d\tilde{\omega}_3 = 0$. Moreover, using again that $d\eta \wedge d\eta = -\frac{2}{3}d\eta \wedge \text{vol}_{S^2}$, the structure is double hypo if and only if $2\lambda^2 + \lambda - 6\lambda\mu - \frac{3}{2}\mu = 0$. Therefore, $\lambda \neq -\frac{1}{4}$ and $\mu = \frac{\lambda}{3}(1 + \frac{1}{4\lambda+1})$ in order the latter relation be satisfied. Since $\mu > \lambda/3$, we must have $\lambda < -1/4$. Finally, the $SU(2)$ -structure (6.4) satisfies equations (1.1), i.e. it is a Sasaki-Einstein hypo structure, only for $\lambda = -1/2$ and $\mu = 0$. \square

Now, we apply Proposition 6.3 and Theorems 3.7 and 5.3 to get

Theorem 6.4. *Let $(S^2 \times S^3, \tilde{\eta}, \tilde{\omega}_i, g)$ be the Sasaki-Einstein manifold endowed with the $SU(2)$ -structure determined by (6.4) for $\lambda = -1/2$ and $\mu = 0$. Then*

- i) *The $SU(3)$ -structure on $S^2 \times S^3 \times [0, \pi]$ defined by (3.3) is a nearly Kähler structure generating the metric $g_6 = dt^2 + \sin^2 t g$ with two conical singularities at $t = 0, t = \pi$.*
- ii) *The G_2 -structure on $(S^2 \times S^3 \times S^1) \times [0, \pi]$ defined by (5.7) is a nearly parallel G_2 -structure generating the metric $g_7 = dq^2 + \sin^2 q(dt^2 + \sin^2 t g)$ with singularities at $t = 0, t = \pi, q = 0, q = \pi$.*

6.3. The Nearly Kähler structure on $Y^{p,q} \times \mathbb{R}$. We start with the recently discovered in [16] infinite family of Sasaki-Einstein metric on $S^2 \times S^3$, labeled by two coprime integers $p > 1, q < p$ and refered as $Y^{p,q}$. Geometrically they are all $U(1)$ -bundles over an axially squashed S^2 bundle over a round S^2 . We take the explicit local description of the Sasaki-Einstein $SU(2)$ -structure presented in [23]. In terms of local coordinates $y, \beta, \theta, \phi, \psi$ they can be described as follows:

$$\begin{aligned} \eta &= \frac{1}{3} (d\psi - \cos \theta d\phi + y(d\beta + c \cos \theta d\phi)), \\ \omega_3 &= \frac{1}{6} ((cy - 1) \sin \theta d\theta \wedge d\phi - dy \wedge (d\beta + c \cos \theta d\phi)), \\ \omega_2 &= \sqrt{\frac{1 - cy}{6w(y)r(y)}} \left(d\theta \wedge dy - \frac{w(y)r(y) \sin \theta}{6} d\theta \wedge d\beta \right), \\ \omega_1 &= \sqrt{\frac{1 - cy}{6w(y)r(y)}} \left(\sin \theta d\phi \wedge dy + \frac{w(y)r(y)}{6} d\theta \wedge (d\beta + c \cos \theta d\phi) \right), \end{aligned} \tag{6.5}$$

where

$$w(y) = \frac{2(a - y^2)}{1 - cy}, \quad r(y) = \frac{a - 3y^2 + 2cy^3}{a - y^2},$$

and a, c are constants. If $c = 0$ one can obtain the known homogeneous metric on $S^2 \times S^3$ and for $c = 1 = a$ one can recover the round metric on 5-sphere S^5 . However, for $c \neq 0, 0 < a < 1$ one can get irregular Sasaki-Einstein structures, i.e. the orbits of the Killing vector field dual to η are non-compact [16].

It is easy to check that the $SU(2)$ -structure (6.5) satisfies (1.1). Apply Theorem 3.7 and Theorem 5.3 to get

Theorem 6.5. *Let $(Y^{p,q}, \eta, \omega_i, g)$ be the Sasaki-Einstein manifold endowed with the $SU(2)$ -structure determined by (6.5). Then*

- i) *The $SU(3)$ -structure on $Y^{p,q} \times [0, \pi]$ defined by (3.3) is a nearly Kähler structure generating the metric $g_6 = dt^2 + \sin^2 t g$ with two conical singularities at $t = 0, t = \pi$.*
- ii) *The G_2 -structure on $(Y^{p,q} \times [0, \pi]) \times [0, \pi]$ defined by (5.7) is a nearly parallel G_2 -structure generating the metric $g_7 = dq^2 + \sin^2 q(dt^2 + \sin^2 t g)$ with singularities at $t = 0, t = \pi, q = 0, q = \pi$.*

6.4. Evolution which is not an $SU(2)$ -structure. For half flat $SU(3)$ -structures Hitchin shows [21] that if his evolution equations are satisfied, and for $t = 0$ the structure is half-flat, then the half flat $SU(3)$ condition is preserved in time provided some non-degeneracy condition for the evolved $SU(3)$ -structure holds.

For a hypo and nearly hypo $SU(2)$ -structure we find an example which solves the Conti-Salamon and our nearly hypo evolution equations but there exists a solution to the evolution equations which is not an $SU(2)$ -structure, i.e. the situation is a little bit different.

We take the double hypo structure on the Lie group isomorphic to $SU(2) \times A^2$ defined in Proposition 4.3 by (4.3) and (4.4) for $\mu = 0$.

We find the following solution to the Conti-Salamon hypo evolution equations (2.6)

$$\eta(t) = \eta, \quad \omega_1(t) = \omega_1 - t d\eta, \quad \omega_3(t) = -\sinh 3t \omega_1 + \omega_3, \quad \omega_2(t) = \cosh 3t \omega_2$$

which is not an $SU(2)$ -structure since $\omega_2^2 = \omega_3^2 = \cosh^2 3t$ while $\omega_1^2 = 1$.

We obtain the following solution to the nearly-hypo evolution equations (3.2)

$$\begin{aligned} \eta(t) &= \eta, \quad \omega_2(t) = \cos \sqrt{3} t \omega_2, \\ \omega_1(t) &= \cos \sqrt{3} t \omega_1 - \frac{\sqrt{3}}{2} \sin 2\sqrt{3} t e^{14} + \frac{1}{2\sqrt{3}} \sin 2\sqrt{3} t e^{23}, \\ \omega_3(t) &= \frac{1}{\sqrt{3}} \sin \sqrt{3} t \omega_1 + \cos 2\sqrt{3} t e^{14} + \left(\frac{4}{3} - \frac{1}{3} \cos 2\sqrt{3} t \right) e^{23} \end{aligned}$$

which is not an $SU(2)$ -structure again.

Acknowledgments. This work has been partially supported through grants MCyT (Spain) MTM2004-07090-C03-01, MTM2005-08757-C04-02 and Project UPV 00127.310-E-15909/2004. We would like to thank Simon Salamon for stimulating discussions on the existence of nearly

parallel G_2 -structures with conical singularities, and Diego Conti for informing us on his answer to Question 1. We also thank Anna Fino for helpful discussions, and the referee for useful comments. In particular, the referee suggested the inclusion of diagrams which improved greatly the exposition.

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